1

Rotating fluid bodies in equilibrium: fundamental notions and equations

1.1 The concept of an isolated body

An important and successful approach to solving problems throughout physics is to split the world into a system to be considered, its 'surroundings' and the 'rest of the universe', where the influence of the latter on the system being considered is neglected. The applicability of this concept to general relativity is not a trivial matter, since the spacetime structure at every point depends on the overall energy-momentum distribution.

Our aim is to find a description of a single fluid body (modelling a celestial body, e.g. a neutron star) under the influence of its own gravitational field. Fortunately, one often encounters such a body surrounded by a vacuum, where the closest other bodies are so far away that an intermediate region with a weak gravitational field exists. In such a situation (see Fig. 1.1) one can discuss the *far field* of the body. If the distant outside world (the 'rest of the universe') is *isotropic*, which it is according to astronomical observations and the standard cosmological models, then the line element corresponding to the far field of an arbitrary stationary body can be written as follows (see Stephani 2004):

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta + 2g_{\alpha 4}\mathrm{d}x^\alpha\mathrm{d}t + g_{44}\mathrm{d}t^2,$$

with

$$\begin{aligned} g_{\alpha\beta} &= (1 + 2M/r)\eta_{\alpha\beta} + \mathcal{O}(r^{-2}), \\ g_{\alpha 4} &= 2r^{-3}\epsilon_{\alpha\beta\gamma}x^\beta J^\gamma + \mathcal{O}(r^{-3}), \\ g_{44} &= -(1 - 2M/r) + \mathcal{O}(r^{-2}), \end{aligned} \tag{1.1}$$

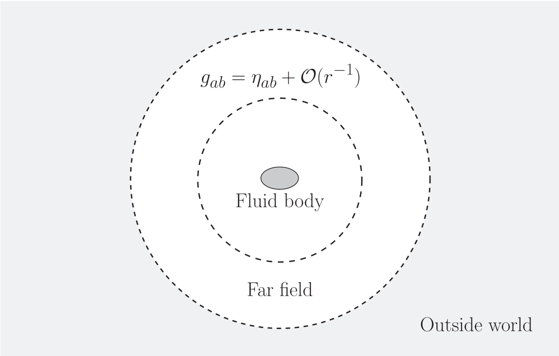

Fig. 1.1. The far field of an isolated body (adapted from Stephani 2004).

where $r^2 = \eta_{\alpha\beta}x^\alpha x^\beta = x^2+y^2+z^2$. For $r \to \infty$ the metric acquires the Minkowski form, i.e. the spacetime is 'asymptotically flat'. We stress that the condition of asymptotic flatness as discussed here is a consequence of the assumption of an isotropic outside world.[1]

M is the gravitational mass of the body and J^α its angular momentum. The $g_{\alpha 4}$-term represents the famous Lense–Thirring effect of a rotating source on the gravitational field, also called the 'gravitomagnetic' effect – in analogy to the magnetic field generated by a rotating electric charge distribution in Maxwell's electrodynamics.

In the next section, we shall provide arguments suggesting that the metric of a rotating fluid body in equilibrium is axially symmetric. Therefore, throughout this book, we shall deal with stationary and axisymmetric spacetimes. Under these conditions, the exterior (vacuum) Einstein equations can be reduced to the so-called Ernst equation, which can be attacked by analytic solution methods from soliton theory. However, the full rotating body problem requires the simultaneous solution of the inner equations, including the correct matching conditions. Note that the shape of the body's surface is not known in advance! The final result must be a globally regular and asymptotically flat solution to the Einstein equations, which can only be found by numerical methods in general (see Chapter 3). But, fortunately, there are a few interesting limiting cases that can be solved completely analytically (see Chapter 2).

[1] For an anisotropic outside world, it would be necessary to add a series with increasing powers of r to (1.1). The expressions (1.1), without these extra terms, would nevertheless be a good approximation to the body's far field as long as r is not too large ('local inertial system' on cosmic scales). However, for an isotropic outside world, the notion of a body's rotation with respect to the local inertial system coincides with the notion of rotation with respect to the external environment (the 'fixed stars'). Later, we shall simply speak of a rotation 'with respect to infinity'.

1.2 Fluid bodies in equilibrium

We want to consider configurations that are strictly stationary, thus implying thermodynamic equilibrium and the absence of gravitational radiation. This leads us, more or less stringently, to the conditions of

(i) zero temperature,
(ii) rigid rotation, and
(iii) axial symmetry.

Thermodynamic equilibrium would also permit a non-zero constant temperature.[2] However, as discussed for example in Landau and Lifshitz (1980), such configurations are unrealistic. Normal stars are hot, but not in global thermal equilibrium: their central temperature is much higher than their surface temperature and they emit a significant amount of electromagnetic radiation. Fortunately, neutron stars – the most interesting stars from the general relativistic point of view – can indeed be considered to be ‘cold matter’ objects, since their temperature is much lower than the Fermi temperature. Hence, our idealized assumption of zero temperature fits very well for neutron stars.

Provided that some (arbitrarily small) viscosity is present, any deviation from rigid rotation will vanish in an equilibrium state of a rotating star. For the calculation of the rigidly rotating equilibrium state itself, we may then adopt the model of a perfect fluid, since viscosity has no effect in the absence of any shear or expansion. It will, however, affect stability properties.

Moreover, within general relativity, any deviation of a uniformly rotating star from axial symmetry will result in gravitational radiation, which is also incompatible with a strict equilibrium state. For a more in-depth discussion of points (ii) and (iii), see Lindblom (1992).

Therefore, in the next sections, we shall treat stationary and axisymmetric, uniformly rotating, cold, perfect fluid bodies.

1.3 The metric of an axisymmetric perfect fluid body in stationary rotation

In accordance with our assumptions of axisymmetry and stationarity, we shall use coordinates t (time) and φ (azimuthal angle) adapted to the corresponding Killing vectors:

$$\boldsymbol{\xi} = \frac{\partial}{\partial t}, \quad \boldsymbol{\eta} = \frac{\partial}{\partial \varphi}, \tag{1.2}$$

[2] Note that in general relativity, the equilibrium condition of constant temperature T is replaced by the Tolman condition $T(-g'_{tt})^{1/2} = \text{constant}$ (Tolman 1934), where the prime denotes a corotating frame of reference.

Cambridge University Press
978-0-521-86383-4 - Relativistic Figures of Equilibrium
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where $\boldsymbol{\xi}$ is normalized according to

$$\xi^i \xi_i \to -1 \quad \text{at spatial infinity} \tag{1.3}$$

and the orbits of the spacelike Killing vector $\boldsymbol{\eta}$ are closed, with periodicity 2π. The symmetry axis is characterized by

$$\boldsymbol{\eta} = 0 \quad \text{along the symmetry axis.} \tag{1.4}$$

It can be shown that the metric of an axisymmetric perfect fluid body in stationary rotation is orthogonally transitive, i.e. it admits 2-spaces orthogonal to the Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ (Kundt and Trümper 1966). This allows us to write the metric in the following form (Lewis 1932, Papapetrou 1966):

$$\mathrm{d}s^2 = \mathrm{e}^{-2U}\left[\mathrm{e}^{2k}(\mathrm{d}\varrho^2 + \mathrm{d}\zeta^2) + W^2\mathrm{d}\varphi^2\right] - \mathrm{e}^{2U}(\mathrm{d}t + a\,\mathrm{d}\varphi)^2, \tag{1.5}$$

or, equivalently,

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha}(\mathrm{d}\varrho^2 + \mathrm{d}\zeta^2) + W^2\mathrm{e}^{-2\nu}(\mathrm{d}\varphi - \omega\,\mathrm{d}t)^2 - \mathrm{e}^{2\nu}\mathrm{d}t^2, \tag{1.6}$$

where the functions U, a, k and W as well as ν, ω and α depend only on the coordinates ϱ and ζ. It can easily be verified that these functions are interrelated according to

$$\alpha = k - U, \quad W^{-1}\mathrm{e}^{2\nu} \pm \omega = \left(W\,\mathrm{e}^{-2U} \mp a\right)^{-1}. \tag{1.7}$$

We also note that U, a (or ν, ω) and W can be related to the scalar products of the Killing vectors, thus providing a coordinate independent characterization:

$$\xi^i \xi_i = -\,\mathrm{e}^{2U} = -\mathrm{e}^{2\nu} + \omega^2 W^2 \mathrm{e}^{-2\nu}, \tag{1.8a}$$

$$\eta^i \eta_i = W^2\mathrm{e}^{-2U} - a^2\mathrm{e}^{2U} = W^2\mathrm{e}^{-2\nu}, \tag{1.8b}$$

$$\xi^i \eta_i = -\,a\mathrm{e}^{2U} = -\omega W^2 \mathrm{e}^{-2\nu}. \tag{1.8c}$$

We call U the ‘generalized Newtonian potential’ and a the ‘gravitomagnetic potential’. Without loss of generality, the symmetry axis can be identified with the ζ-axis, i.e. it is characterized by $\varrho = 0$ and we have

$$0 \le \varrho < \infty, \quad -\infty < \zeta < \infty. \tag{1.9}$$

On the axis, the following conditions hold, see Stephani *et al.* (2003):

$$\varrho \to 0: \quad a \to 0,\ W \to 0,\ W/(\varrho\,\mathrm{e}^k) \to 1. \tag{1.10}$$

At spatial infinity, i.e. for $\varrho^2+\zeta^2 \to \infty$, the line element approaches the Minkowski metric in cylindrical coordinates ϱ, ζ and φ:

$$ds^2 = d\varrho^2 + d\zeta^2 + \varrho^2 d\varphi^2 - dt^2, \tag{1.11}$$

which means that

$$U \to 0,\ a \to 0,\ k \to 0,\ W \to \varrho \quad \text{as} \quad \varrho^2 + \zeta^2 \to \infty \tag{1.12}$$

as well as

$$\nu \to 0,\ \omega \to 0,\ \alpha \to 0 \quad \text{as} \quad \varrho^2 + \zeta^2 \to \infty. \tag{1.13}$$

Sometimes we shall use a 'corotating coordinate system' characterized by

$$\varrho' = \varrho, \quad \zeta' = \zeta, \quad \varphi' = \varphi - \Omega t, \quad t' = t, \tag{1.14}$$

where Ω is the constant angular velocity of the fluid body with respect to infinity. It can easily be verified that the line element retains its form (1.5) or (1.6) with

$$e^{2U'} = e^{2U}[(1 + \Omega a)^2 - \Omega^2 W^2 e^{-4U}], \tag{1.15a}$$

$$(1 - \Omega a')e^{2U'} = (1 + \Omega a)e^{2U}, \tag{1.15b}$$

$$k' - U' = k - U, \quad W' = W \tag{1.15c}$$

and

$$\nu' = \nu, \quad \omega' = \omega - \Omega, \quad \alpha' = \alpha. \tag{1.16}$$

Note that

$$\frac{\partial}{\partial t'} = \boldsymbol{\xi} + \Omega \boldsymbol{\eta}, \quad \frac{\partial}{\partial \varphi'} = \boldsymbol{\eta}. \tag{1.17}$$

We shall call the primed quantities U', a', etc. 'corotating potentials'.

1.4 Einstein's field equations inside and outside the body

The stationary and rigid rotation of the fluid is characterized by the 4-velocity field

$$u^i = e^{-V}(\xi^i + \Omega\, \eta^i), \quad \Omega = \text{constant}, \tag{1.18}$$

where $\Omega = d\varphi/dt = u^\varphi/u^t$ is the constant angular velocity with respect to infinity. Using $u^i u_i = -1$, the factor $e^{-V} = u^t$ is given by

$$(\xi^i + \Omega\, \eta^i)(\xi_i + \Omega\, \eta_i) = -e^{2V}. \tag{1.19}$$

Note that V is equal to the corotating potential U',

$$V \equiv U', \tag{1.20}$$

as defined in (1.15a). The energy-momentum tensor of a perfect fluid is

$$T_{ik} = (\epsilon + p)\, u_i u_k + p\, g_{ik}, \tag{1.21}$$

where the mass-energy density ϵ and the pressure p, according to our assumptions as discussed in Section 1.2, are related by a 'cold' equation of state $\epsilon = \epsilon(p)$ following from

$$\epsilon = \epsilon(\mu_{\mathrm{B}}), \quad p = p(\mu_{\mathrm{B}}) \tag{1.22}$$

at zero temperature, with the baryonic mass-density μ_{B}. Examples will be given in Section 1.5.

The specific enthalpy[3]

$$h = \frac{\epsilon + p}{\mu_{\mathrm{B}}} \tag{1.23}$$

can be calculated from $\epsilon(p)$ via the thermodynamic relation

$$\mathrm{d}h = \frac{1}{\mu_{\mathrm{B}}}\,\mathrm{d}p \qquad \text{(zero temperature)} \tag{1.24}$$

leading to

$$\frac{\mathrm{d}h}{h} = \frac{\mathrm{d}p}{\epsilon + p} \quad \Rightarrow \quad h(p) = h(0)\,\exp\left[\int_0^p \frac{\mathrm{d}p'}{\epsilon(p') + p'}\right]. \tag{1.25}$$

Note that $h(0) = 1$ for ordinary baryonic matter.[4] From $T^{ik}{}_{;k} = 0$ (a semicolon denotes the covariant derivative), we obtain, as a first integral of the equations inside the body,

$$h(p)\,\mathrm{e}^{V} = h(0)\,\mathrm{e}^{V_0} = \text{constant}. \tag{1.26}$$

This means that surfaces of constant p coincide with surfaces of constant V. The boundary of the fluid body is defined by $p = 0$, hence

$$V = V_0 \quad \text{along the boundary of the fluid.} \tag{1.27}$$

[3] Note that $\epsilon = \mu_{\mathrm{B}} + u_{\mathrm{int}}$, where u_{int} denotes the internal energy-density. Hence $h = 1 + h_{\mathrm{N}}$ with h_{N} being the specific enthalpy as it is usually defined in the non-relativistic (Newtonian) theory.
[4] An exception is strange quark matter as described by the MIT bag model, see Section 1.5.

The constant V_0 is related to the relative redshift z of zero angular momentum photons[5] emitted from the surface of the fluid and received at infinity via

$$z = e^{-V_0} - 1. \tag{1.28}$$

Equilibrium models, for a given equation of state, are fixed by two parameters, for example Ω and V_0.

The full set of equations that follows from Einstein's field equations $R_{ik} - \frac{1}{2}Rg_{ik} = 8\pi T_{ik}$ for the metric in the form (1.6), with (1.18) and (1.21), can be written in the following way, see e.g. Bardeen and Wagoner (1971):

$$\nabla \cdot (B\nabla\nu) - \frac{1}{2}\varrho^2 B^3 e^{-4\nu}(\nabla\omega)^2 = 4\pi e^{2\alpha} B\left[(\epsilon + p)\frac{1+v^2}{1-v^2} + 2p\right], \tag{1.29a}$$

$$\nabla \cdot (\varrho^2 B^3 e^{-4\nu}\nabla\omega) = -16\pi\varrho B^2 e^{2\alpha-2\nu}(\epsilon + p)\frac{v}{1-v^2}, \tag{1.29b}$$

$$\nabla \cdot (\varrho\nabla B) = 16\pi\varrho B e^{2\alpha} p \tag{1.29c}$$

with

$$B := W/\varrho \quad \text{and} \quad v := \varrho B e^{-2\nu}(\Omega - \omega), \tag{1.29d}$$

together with two equations, which provide the possibility of determining α via a line integral if the other three functions ν, ω and B are considered as given,

$$\begin{aligned}\varrho^{-1}(\alpha+\nu)_{,\varrho} &+ B^{-1}[B_{,\varrho}(\alpha+\nu)_{,\varrho} - B_{,\zeta}(\alpha+\nu)_{,\zeta}] - \frac{1}{2}\varrho^{-2}B^{-1}(\varrho^2 B_{,\varrho})_{,\varrho}\\ &+ \frac{1}{2}B^{-1}B_{,\zeta\zeta} - (\nu_{,\varrho})^2 + (\nu_{,\zeta})^2 + \frac{1}{4}\varrho^2 B^2 e^{-4\nu}[(\omega_{,\varrho})^2 - (\omega_{,\zeta})^2] = 0,\end{aligned} \tag{1.30a}$$

$$\begin{aligned}\varrho^{-1}(\alpha+\nu)_{,\zeta} &+ B^{-1}[B_{,\varrho}(\alpha+\nu)_{,\zeta} + B_{,\zeta}(\alpha+\nu)_{,\varrho}] - \frac{1}{2}\varrho^{-2}B^{-1}(\varrho^2 B_{,\zeta})_{,\varrho}\\ &- \frac{1}{2}B^{-1}B_{,\varrho\zeta} - 2\nu_{,\varrho}\,\nu_{,\zeta} + \frac{1}{2}\varrho^2 B^2 e^{-4\nu}\omega_{,\varrho}\,\omega_{,\zeta} = 0,\end{aligned} \tag{1.30b}$$

and (1.26), which allows us to express p and ϵ, via (1.25) and the equation of state, in terms of

$$e^V \equiv e^{U'} = e^\nu\sqrt{1-v^2}. \tag{1.31}$$

[5] Zero angular momentum means $\eta_i p^i = 0$ (p^i: 4-momentum of the photon), i.e. the (conserved) component of the orbital angular momentum with respect to the symmetry axis vanishes. In particular, this is satisfied for all photons emitted from the poles of a body of spheroidal topology, since $\boldsymbol{\eta}$ vanishes on the axis of symmetry. For other points on the surface, the condition $\eta_i p^i = 0$ places a restriction on the directions of emission.

Cambridge University Press
978-0-521-86383-4 - Relativistic Figures of Equilibrium
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In (1.29), the operator ∇ has the same meaning as in a Euclidean 3-space in which ϱ, ζ and φ are cylindrical coordinates. Note that v as defined in (1.29d) is the linear velocity of rotation with respect to 'locally non-rotating observers'.[6] Its invariant definition is given by

$$\frac{v}{\sqrt{1-v^2}} = \frac{\eta_i u^i}{\sqrt{\eta_k \eta^k}}. \tag{1.32}$$

In (1.30), we have made use of the comma notation for partial derivatives, e.g. $\partial\nu/\partial\varrho = \nu_{,\varrho}$. Note that instead of (1.30), the second order equation for α

$$\begin{aligned}\alpha_{,\varrho\varrho} + \alpha_{,\zeta\zeta} - \frac{1}{\varrho}\nu_{,\varrho} + \nabla\nu\left(\nabla\nu - B^{-1}\nabla B\right) - \frac{1}{4}\varrho^2 B^2 \mathrm{e}^{-4\nu}(\nabla\omega)^2 \\ = -4\pi \mathrm{e}^{2\alpha}(\epsilon + p),\end{aligned} \tag{1.33}$$

which follows from (1.29), (1.30) and (1.26), see Trümper (1967), can be used.

For the metric in the form (1.5), the equations take a simpler form if one uses the *corotating* potentials U', a', k' and W'. With $W' = W$, see (1.15c), they read

$$\nabla^2 U' - \frac{1}{\varrho}U'_{,\varrho} + \frac{\nabla U' \cdot \nabla W}{W} + \frac{\mathrm{e}^{4U'}(\nabla a')^2}{2W^2} = 4\pi(\epsilon + 3p)\mathrm{e}^{2k'-2U'}, \tag{1.34a}$$

$$(W^{-1}\mathrm{e}^{4U'}a'_{,\varrho})_{,\varrho} + (W^{-1}\mathrm{e}^{4U'}a'_{,\zeta})_{,\zeta} = 0, \tag{1.34b}$$

$$W_{,\varrho\varrho} + W_{,\zeta\zeta} = 16\pi p W \mathrm{e}^{2k'-2U'} \tag{1.34c}$$

together with

$$\begin{aligned}W_{,\varrho}k'_{,\varrho} - W_{,\zeta}k'_{,\zeta} &= \frac{1}{2}(W_{,\varrho\varrho} - W_{,\zeta\zeta}) + W[(U'_{,\varrho})^2 - (U'_{,\zeta})^2] \\ &\quad + \frac{\mathrm{e}^{4U'}}{4W}[(a'_{,\zeta})^2 - (a'_{,\varrho})^2],\end{aligned} \tag{1.35a}$$

$$W_{,\zeta}k'_{,\varrho} + W_{,\varrho}k'_{,\zeta} = W_{,\varrho\zeta} + 2WU'_{,\varrho}U'_{,\zeta} - \frac{\mathrm{e}^{4U'}}{2W}a'_{,\varrho}a'_{,\zeta} \tag{1.35b}$$

and (1.26).

[6] Locally non-rotating observers (also called 'zero angular momentum observers') have a 4-velocity field $u^i_{\text{zamo}} = \mathrm{e}^{-\nu}(\xi^i + \omega\eta^i)$. They rotate with the angular velocity ω with respect to infinity, but their angular momentum $\eta_i u^i_{\text{zamo}}$ vanishes, see Bardeen *et al.* (1972). This provides a nice interpretation for the metric functions ω and ν.

The vacuum case: the Ernst equation

Outside the body, the source terms on the right hand sides of Equations (1.34) vanish. Equation (1.34c) becomes a two-dimensional Laplace equation:

$$W_{,\varrho\varrho} + W_{,\zeta\zeta} = 0. \tag{1.36}$$

By means of a conformal transformation in ϱ-ζ space, it is always possible to choose

$$W \equiv \varrho. \tag{1.37}$$

In these 'canonical Weyl coordinates' the remaining field equations, written down for the functions U, a and k, are[7]

$$\nabla^2 U = -\frac{e^{4U}}{2\varrho^2}(\nabla a)^2, \tag{1.38}$$

$$(\varrho^{-1} e^{4U} a_{,\varrho})_{,\varrho} + (\varrho^{-1} e^{4U} a_{,\zeta})_{,\zeta} = 0 \tag{1.39}$$

together with the two equations

$$k_{,\varrho} = \varrho[(U_{,\varrho})^2 - (U_{,\zeta})^2] + \frac{e^{4U}}{4\varrho}[(a_{,\zeta})^2 - (a_{,\varrho})^2], \tag{1.40a}$$

$$k_{,\zeta} = 2\varrho U_{,\varrho} U_{,\zeta} - \frac{e^{4U}}{2\varrho} a_{,\varrho} a_{,\zeta}, \tag{1.40b}$$

which allow us to calculate k via a path-independent[8] line integral.

Equation (1.39) implies that a function b can be introduced according to

$$a_{,\varrho} = \varrho e^{-4U} b_{,\zeta}, \quad a_{,\zeta} = -\varrho e^{-4U} b_{,\varrho}, \tag{1.41}$$

satisfying the equation

$$(\varrho e^{-4U} b_{,\varrho})_{,\varrho} + (\varrho e^{-4U} b_{,\zeta})_{,\zeta} = 0. \tag{1.42}$$

It can easily be verified that the two Equations (1.38) and (1.42) can be combined into the Ernst equation (Ernst 1968, Kramer and Neugebauer 1968).

$$\Re f \, \nabla^2 f = (\nabla f)^2 \tag{1.43}$$

[7] As a consequence of the form invariance of the line element (1.5) under a coordinate transformation (1.14), the vacuum equations for U, a, k and W are the same as those for U', a', k' and W' ($= W$), and can be read off from Equations (1.34) and (1.35) for $\epsilon = p = 0$.
[8] The integrability condition is satisfied by virtue of (1.38) and (1.39).

for the complex 'Ernst potential'

$$f := \mathrm{e}^{2U} + \mathrm{i}b. \tag{1.44}$$

The Ernst equation (1.43), together with (1.41), (1.40) and (1.37), is equivalent to the vacuum Einstein equations in the stationary and axisymmetric case.

As already mentioned, the vacuum equations for the corotating potentials U' and a' have the same form as those for U and a. Therefore, the Ernst potential can also be introduced in the corotating system and the Ernst equation retains its form as well. This remarkable fact will be used later.

The global problem

For genuine fluid body problems, we shall not make use of canonical Weyl coordinates and the Ernst formalism in the exterior region. It is of greater advantage to have a global coordinate system ϱ, ζ in which all metric functions and their first derivatives are continuous at the surface of the body. In particular, this requirement leads to a unique solution $W(\varrho, \zeta)$, which differs from $W \equiv \varrho$ in the vacuum region.[9] The global problem consists in finding a regular, asymptotically flat solution to Equations (1.29) and (1.30) with source terms inside the fluid and without source terms in the vacuum region. We stress that the shape of the surface, characterized by $p = 0$, is not known from the outset.

1.5 Equations of state

In this section, we shall provide some examples of equations of state $\epsilon = \epsilon(p)$, which will be used in this book. The relation to the baryonic mass-density μ_B, consistent with Equations (1.23) and (1.25), will also be given. Note that in our units (with $c = 1$), there is no difference between energy-density ϵ and (total) mass-density μ, i.e. $\epsilon = \mu = \mu_\mathrm{B} + u_\mathrm{int}$, where u_int is the internal energy-density.

Homogeneous fluids

This simple model is characterized by the equation of state (EOS)

$$\epsilon = \text{constant}. \tag{1.45}$$

Assuming $h(0) = 1$, we obtain from (1.23) and (1.25) that $\epsilon = \mu = \mu_\mathrm{B}$, i.e. the internal energy density is zero.

[9] An important exception is given by the disc limit, where it turns out that $W \equiv \varrho$ holds globally, see Subsection 1.7.3. Another application of the Ernst formalism will be the derivation of the Kerr metric in Section 2.4.