CHAPTER ONE

The Inverse Trigonometric Functions

A proper understanding of how to solve trigonometric equations requires a theory of inverse trigonometric functions. This theory is complicated by the fact that the trigonometric functions are periodic functions — they therefore fail the horizontal line test quite seriously, in that some horizontal lines cross their graphs infinitely many times. Understanding inverse trigonometric functions therefore requires further discussion of the procedures for restricting the domain of a function so that the inverse relation is also a function. Once the functions are established, the usual methods of differential and integral calculus can be applied to them.

This theory gives rise to primitives of two purely algebraic functions

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x \quad (\text{or} -\cos^{-1} x) \qquad \text{and} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x,$$

which are similar to the earlier primitive $\int \frac{1}{x}\,dx = \log x$ in that in all three cases, a purely algebraic function has a primitive which is non-algebraic.

Study Notes: Inverse relations and functions were first introduced in Section 2H of the Year 11 volume. That material is summarised in Section 1A in preparation for more detail about restricted functions, but some further revision may be necessary. Sections 1B–1E then develop the standard theory of inverse trigonometric functions and their graphs, and the associated derivatives and integrals. In Section 1F these functions are used to establish some formulae for the general solutions of trigonometric equations.

1 A Restricting the Domain

Section 2H of the Year 11 volume discussed how the inverse relation of a function may or may not be a function, and briefly mentioned that if the inverse is not a function, then the domain can be restricted so that the inverse of this restricted function is a function. This section revisits those ideas and develops a more systematic approach to restricting the domain.

Inverse Relations and Inverse Functions: First, here is a summary of the basic theory of inverse functions and relations. The examples given later will illustrate the various points. Suppose that $f(x)$ is a function whose inverse relation is being considered.

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1 INVERSE FUNCTIONS AND RELATIONS:

- The graph of the inverse relation is obtained by reflecting the original graph in the diagonal line $y = x$.
- The inverse relation of a given relation is a function if and only if no horizontal line crosses the original graph more than once.
- The domain and range of the inverse relation are the range and domain respectively of the original function.
- To find the equations and conditions of the inverse relation, write x for y and y for x every time each variable occurs.
- If the inverse relation is also a function, the inverse function is written as $f^{-1}(x)$. Then the composition of the function and its inverse, in either order, leaves every number unchanged:
$$f^{-1}\big(f(x)\big) = x \qquad \text{and} \qquad f\big(f^{-1}(x)\big) = x.$$
- If the inverse is not a function, then the domain of the original function can be restricted so that the inverse of the restricted function is a function.

The following worked exercise illustrates the fourth and fifth points above.

WORKED EXERCISE: Find the inverse function of $f(x) = \dfrac{x-2}{x+2}$. Then show directly that $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(x)\big) = x$.

SOLUTION: Let $y = \dfrac{x-2}{x+2}$.

Then the inverse relation is $x = \dfrac{y-2}{y+2}$ (writing y for x and x for y)

$$xy + 2x = y - 2$$
$$y(x-1) = -2x - 2$$
$$y = \frac{2+2x}{1-x}.$$

Since there is only one solution for y, the inverse relation is a function,

and $f^{-1}(x) = \dfrac{2+2x}{1-x}$.

Then
$$f\big(f^{-1}(x)\big) = f\left(\frac{2+2x}{1-x}\right) = \frac{\frac{2+2x}{1-x} - 2}{\frac{2+2x}{1-x} + 2} \times \frac{1-x}{1-x} = \frac{(2+2x) - 2(1-x)}{(2+2x) + 2(1-x)} = \frac{4x}{4} = x, \text{ as required.}$$

and
$$f^{-1}\big(f(x)\big) = f^{-1}\left(\frac{x-2}{x+2}\right) = \frac{2 + \frac{2(x-2)}{x+2}}{1 - \frac{x-2}{x+2}} \times \frac{x+2}{x+2} = \frac{2(x+2) + 2(x-2)}{(x+2) - (x-2)} = \frac{4x}{4} = x \text{ as required.}$$

Increasing and Decreasing Functions: Increasing means getting bigger, and we say that a function $f(x)$ is an *increasing function* if $f(x)$ increases as x increases:

$$f(a) < f(b), \quad \text{whenever } a < b.$$

For example, if $f(x)$ is an increasing function, then provided $f(x)$ is defined there, $f(2) < f(3)$, and $f(5) < f(10)$. In the language of coordinate geometry, this means that *every chord slopes upwards*, because the ratio $\dfrac{f(b) - f(a)}{b - a}$ must be positive, for all pairs of distinct numbers a and b. *Decreasing functions* are defined similarly.

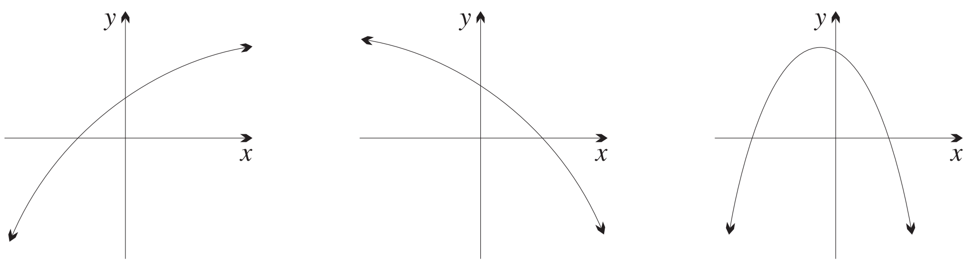
2 INCREASING AND DECREASING FUNCTIONS: Suppose that $f(x)$ is a function.

- $f(x)$ is called an *increasing function* if every chord slopes upwards, that is,

  $f(a) < f(b)$, whenever $a < b$.

- $f(x)$ is called a *decreasing function* if every chord slopes downwards, that is,

  $f(a) > f(b)$, whenever $a < b$.



An increasing function A decreasing function Neither of these

NOTE: These are *global* definitions, looking at the graph of the function as a whole. They should be contrasted with the *pointwise* definitions introduced in Chapter Ten of the Year 11 volume, where a function $f(x)$ was called *increasing at* $x = a$ if $f'(a) > 0$, that is, if the tangent slopes upwards at the point.

Throughout our course, a tangent describes the behaviour of a function at a particular point, whereas a chord relates the values of the function at two different points.

The exact relationship between the global and pointwise definitions of *increasing* are surprisingly difficult to state, as the examples in the following paragraphs demonstrate, but in this course it will be sufficient to rely on the graph and common sense.

The Inverse Relation of an Increasing or Decreasing Function: When a horizontal line crosses a graph twice, it generates a horizontal chord. But every chord of an increasing function slopes upwards, and so an increasing function cannot possibly fail the horizontal line test. This means that the inverse relation of every increasing function is a function. The same argument applies to decreasing functions.

3 INCREASING OR DECREASING FUNCTIONS AND THE INVERSE RELATION:

- The inverse of an increasing or decreasing function is a function.
- The inverse of an increasing function is increasing, and the inverse of a decreasing function is decreasing.

To justify the second remark, notice that reflection in $y = x$ maps lines sloping upwards to lines sloping downwards, and maps lines sloping downwards to lines sloping downwards.

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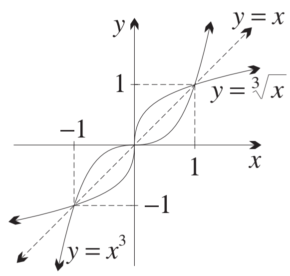
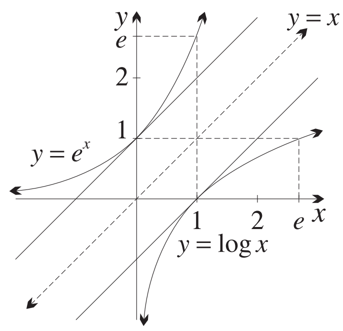
Example — The Cube and Cube Root Functions: The function $f(x) = x^3$ and its inverse function $f^{-1}(x) = \sqrt[3]{x}$ are graphed to the right.

- $f(x) = x^3$ is an increasing function, because every chord slopes upwards. Hence it passes the horizontal line test, and its inverse is a function, which is also increasing.
- $f(x)$ is not, however, increasing at every point, because the tangent at the origin is horizontal. Correspondingly, the tangent to $y = \sqrt[3]{x}$ at the origin is vertical.
- For all x, $\sqrt[3]{x^3} = x$ and $(\sqrt[3]{x}\,)^3 = x$.



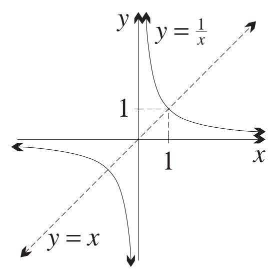
Example — The Logarithmic and Exponential Functions: The two functions $f(x) = e^x$ and $f^{-1}(x) = \log x$ provide a particularly clear example of a function and its inverse.

- $f(x) = e^x$ is an increasing function, because every chord slopes upwards. Hence it passes the horizontal line test, and its inverse is a function, which is also increasing.
- $f(x) = e^x$ is also increasing at every point, because its derivative is $f'(x) = e^x$ which is always positive.
- For all x, $\log e^x = x$, and for $x > 0$, $e^{\log x} = x$.



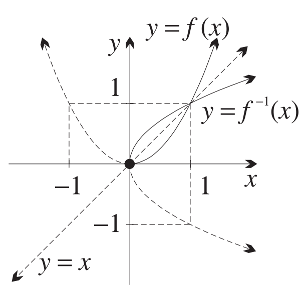
Example — The Reciprocal Function: The function $f(x) = 1/x$ is its own inverse, because the reciprocal of the reciprocal of any nonzero number is always the original number. Correspondingly, its graph is symmetric in $y = x$.

- $f(x) = 1/x$ is neither increasing nor decreasing, because chords joining points on the same branch slope downwards, and chords joining points on different branches slope upwards. Nevertheless, it passes the horizontal line test, and its inverse (which is itself) is a function.
- $f(x) = 1/x$ is decreasing at every point, because its derivative is $f'(x) = -1/x^2$, which is always negative.



Restricting the Domain — The Square and Square Root Functions: The two functions $y = x^2$ and $y = \sqrt{x}$ give our first example of restricting the domain so that the inverse of the restricted function is a function.

- $y = x^2$ is neither increasing nor decreasing, because some of its chords slope upwards, some slope downwards, and some are horizontal. Its inverse $x = y^2$ is not a function — for example, the number 1 has two square roots, 1 and -1.
- Define the restricted function $f(x)$ by $f(x) = x^2$, where $x \geq 0$. This is the part of $y = x^2$ shown undotted in the diagram on the right. Then $f(x)$ is an increasing function, and so has an inverse which is written as $f^{-1}(x) = \sqrt{x}$, and which is also increasing.
- For all $x > 0$, $\sqrt{x^2} = x$ and $(\sqrt{x}\,)^2 = x$.



Further Examples of Restricting the Domain: These two worked exercises show the process of restricting the domain applied to more general functions. Since $y = x$ is the mirror exchanging the graphs of a function and its inverse, and since points on a mirror are reflected to themselves, it follows that if the graph of the function intersects the line $y = x$, then it intersects the inverse there too.

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WORKED EXERCISE: Explain why the inverse relation of $f(x) = (x-1)^2 + 2$ is not a function. Define $g(x)$ to be the restriction of $f(x)$ to the largest possible domain containing $x = 0$ so that $g(x)$ has an inverse function. Write down the equation of $g^{-1}(x)$, then sketch $g(x)$ and $g^{-1}(x)$ on one set of axes.

SOLUTION: The graph of $y = f(x)$ is a parabola with vertex $(1, 2)$.
This fails the horizontal line test, so the inverse is not a function.
(Alternatively, $f(0) = f(2) = 3$, so $y = 3$ meets the curve twice.)
Restricting $f(x)$ to the domain $x \le 1$ gives the function

$$g(x) = (x-1)^2 + 2, \quad \text{where } x \le 1,$$

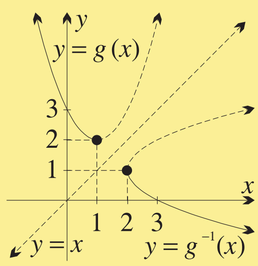
which is sketched opposite, and includes the value at $x = 0$.
Since $g(x)$ is a decreasing function, it has an inverse with equation

$$x = (y-1)^2 + 2, \quad \text{where } y \le 1.$$

Solving for y, $(y-1)^2 = x - 2$, where $y \le 1$,

$$y = 1 + \sqrt{x-2} \text{ or } 1 - \sqrt{x-2}, \quad \text{where } y \le 1.$$

Hence $\quad g(x) = 1 - \sqrt{x-2}$, since $y \le 1$.



WORKED EXERCISE: Use calculus to find the turning points and points of inflexion of $y = (x-2)^2(x+1)$, then sketch the curve. Explain why the restriction $f(x)$ of this function to the part of the curve between the two turning points has an inverse function. Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on one set of axes, and write down an equation satisfied by the x-coordinate of the point M where the function and its inverse intersect.

SOLUTION: For $\quad y = (x-2)^2(x+1) = x^3 - 3x^2 + 4,$

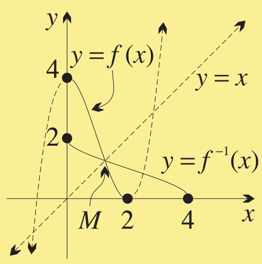
$$y' = 3x^2 - 6x = 3x(x-2),$$

and $\quad y'' = 6x - 6 = 6(x-1).$

So there are zeroes at $x = 2$ and $x = -1$, and (after testing) turning points at $(0, 4)$ (a maximum) and $(2, 0)$ (a minimum), and a point of inflexion at $(1, 2)$.
The part of the curve between the turning points is decreasing, so the function $f(x) = (x-2)^2(x+1)$, where $0 \le x \le 2$, has an inverse function $f^{-1}(x)$, which is also decreasing.
The curves $y = f(x)$ and $y = f^{-1}(x)$ intersect on $y = x$, and substituting $y = x$ into the function,

$$x = x^3 - 3x^2 + 4,$$

so the x-coordinate of M satisfies the cubic $x^3 - 3x^2 - x + 4 = 0$.



Exercise 1A

1. Consider the functions $f = \{(0, 2), (1, 3), (2, 4)\}$ and $g = \{(0, 2), (1, 2), (2, 2)\}$.

(a) Write down the inverse relation of each function.

(b) Graph each function and its inverse relation on a number plane, using separate diagrams for f and g.

(c) State whether or not each inverse relation is a function.

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2. The function $f(x) = x + 3$ is defined over the domain $0 \le x \le 2$.
(a) State the range of $f(x)$.
(b) State the domain and range of $f^{-1}(x)$.
(c) Write down the rule for $f^{-1}(x)$.

3. The function F is defined by $F(x) = \sqrt{x}$ over the domain $0 \le x \le 4$.
(a) State the range of $F(x)$.
(b) State the domain and range of $F^{-1}(x)$.
(c) Write down the rule for $F^{-1}(x)$.
(d) Graph F and F^{-1}.

4. Sketch the graph of each function. Then use reflection in the line $y = x$ to sketch the inverse relation. State whether or not the inverse is a function, and find its equation if it is. Also, state whether $f(x)$ and $f^{-1}(x)$ (if it exists) are increasing, decreasing or neither.
(a) $f(x) = 2x$
(b) $f(x) = x^3 + 1$
(c) $f(x) = \sqrt{1 - x^2}$
(d) $f(x) = x^2 - 4$
(e) $f(x) = 2^x$
(f) $f(x) = \sqrt{x - 3}$

5. Consider the functions $f(x) = 3x + 2$ and $g(x) = \frac{1}{3}(x - 2)$.
(a) Find $f\left(g(x)\right)$ and $g\left(f(x)\right)$.
(b) What is the relationship between $f(x)$ and $g(x)$?

6. Each function $g(x)$ is defined over a restricted domain so that $g^{-1}(x)$ exists. Find $g^{-1}(x)$ and write down its domain and range. (Sketches of g and g^{-1} will prove helpful.)
(a) $g(x) = x^2,\ x \ge 0$
(b) $g(x) = x^2 + 2,\ x \le 0$
(c) $g(x) = -\sqrt{4 - x^2}\,,\ 0 \le x \le 2$

7. (a) Write down $\dfrac{dy}{dx}$ for the function $y = x^3 - 1$.
(b) Make x the subject and hence find $\dfrac{dx}{dy}$.
(c) Hence show that $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$.

8. Repeat the previous question for $y = \sqrt{x}$.

DEVELOPMENT

9. The function $F(x) = x^2 + 2x + 4$ is defined over the domain $x \ge -1$.
(a) Sketch the graphs of $F(x)$ and $F^{-1}(x)$ on the same diagram.
(b) Find the equation of $F^{-1}(x)$ and state its domain and range.

10. (a) Solve the equation $1 - \ln x = 0$.
(b) Sketch the graph of $f(x) = 1 - \ln x$ by suitably transforming the graph of $y = \ln x$.
(c) Hence sketch the graph of $f^{-1}(x)$ on the same diagram.
(d) Find the equation of $f^{-1}(x)$ and state its domain and range.
(e) Classify $f(x)$ and $f^{-1}(x)$ as increasing, decreasing or neither.

11. (a) Carefully sketch the function defined by $g(x) = \dfrac{x + 2}{x + 1}$, for $x > -1$.
(b) Find $g^{-1}(x)$ and sketch it on the same diagram. Is $g^{-1}(x)$ increasing or decreasing?
(c) Find any values of x for which $g(x) = g^{-1}(x)$. [HINT: The easiest way is to solve $g(x) = x$. Why does this work?]

12. The previous question seems to imply that the graphs of a function and its inverse can only intersect on the line $y = x$. This is not always the case.
(a) Find the equation of the inverse of $y = -x^3$.
(b) At what points do the graphs of the function and its inverse meet?
(c) Sketch the situation.

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13. (a) Explain how the graph of $f(x) = x^2$ must be transformed to obtain the graph of $g(x) = (x+2)^2 - 4$.
(b) Hence sketch the graph of $g(x)$, showing the x and y intercepts and the vertex.
(c) What is the largest domain containing $x = 0$ for which $g(x)$ has an inverse function?
(d) Let $g^{-1}(x)$ be the inverse function corresponding to the domain of $g(x)$ in part (c). What is the domain of $g^{-1}(x)$? Is $g^{-1}(x)$ increasing or decreasing?
(e) Find the equation of $g^{-1}(x)$, and sketch it on your diagram in part (b).
(f) Classify $g(x)$ and $g^{-1}(x)$ as either increasing, decreasing or neither.

14. (a) Show that $F(x) = x^3 - 3x$ is an odd function.
(b) Sketch the graph of $F(x)$, showing the x-intercepts and the coordinates of the two stationary points. Is $F(x)$ increasing or decreasing?
(c) What is the largest domain containing $x = 0$ for which $F(x)$ has an inverse function?
(d) State the domain of $F^{-1}(x)$, and sketch it on the same diagram as part (b).

15. (a) State the domain of $f(x) = \dfrac{e^x}{1+e^x}$. (b) Show that $f'(x) = \dfrac{e^x}{(1+e^x)^2}$.
(c) Hence explain why $f(x)$ is increasing for all x.
(d) Explain why $f(x)$ has an inverse function, and find its equation.

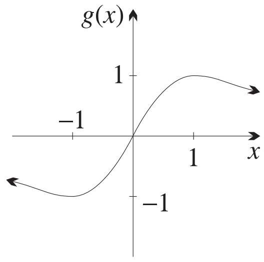
16. (a) Sketch $y = 1 + x^2$ and hence sketch $f(x) = \dfrac{1}{1+x^2}$. Is $f(x)$ increasing or decreasing?
(b) What is the largest domain containing $x = -1$ for which $f(x)$ has an inverse function?
(c) State the domain of $f^{-1}(x)$, and sketch it on the same diagram as part (a).
(d) Find the rule for $f^{-1}(x)$.
(e) Is $f^{-1}(x)$ increasing or decreasing?

17. (a) Show that any linear function $f(x) = mx + b$ has an inverse function if $m \neq 0$.
(b) Does the constant function $F(x) = b$ have an inverse function?

18. The function $f(x)$ is defined by $f(x) = x - \frac{1}{x}$, for $x > 0$.
(a) By considering the graphs of $y = x$ and $y = \frac{1}{x}$ for $x > 0$, sketch $y = f(x)$.
(b) Sketch $y = f^{-1}(x)$ on the same diagram.
(c) By completing the square or using the quadratic formula, show that

$$f^{-1}(x) = \tfrac{1}{2}\left(x + \sqrt{4 + x^2}\right).$$

19. The diagram shows the function $g(x) = \dfrac{2x}{1+x^2}$, whose domain is all real x.



(a) Show that $g(\frac{1}{a}) = g(a)$, for all $a \neq 0$.
(b) Hence explain why the inverse of $g(x)$ is not a function.
(c) (i) What is the largest domain of $g(x)$ containing $x = 0$ for which $g^{-1}(x)$ exists?
(ii) Sketch $g^{-1}(x)$ for this domain of $g(x)$.
(iii) Find the equation of $g^{-1}(x)$ for this domain of $g(x)$.
(d) Repeat part (c) for the largest domain of $g(x)$ that does not contain $x = 0$.
(e) Show that the two expressions for $g^{-1}(x)$ in parts (c) and (d) are reciprocals of each other. Why could we have anticipated this?

20. Consider the function $f(x) = \frac{1}{6}(x^2 - 4x + 24)$.
(a) Sketch the parabola $y = f(x)$, showing the vertex and any x- or y-intercepts.
(b) State the largest domain containing only positive numbers for which $f(x)$ has an inverse function $f^{-1}(x)$.
(c) Sketch $f^{-1}(x)$ on your diagram from part (a), and state its domain.
(d) Find any points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$.
(e) Let N be a negative real number. Find $f^{-1}(f(N))$.

21. (a) Prove, both geometrically and algebraically, that if an odd function has an inverse function, then that inverse function is also odd.
(b) What sort of even functions have inverse functions?

22. [The hyperbolic sine function] The function $\sinh x$ is defined by $\sinh x = \frac{1}{2}(e^x - e^{-x})$.
(a) State the domain of $\sinh x$.
(b) Find the value of $\sinh 0$.
(c) Show that $y = \sinh x$ is an odd function.
(d) Find $\frac{d}{dx}(\sinh x)$ and hence show that $\sinh x$ is increasing for all x.
(e) To which curve is $y = \sinh x$ asymptotic for large values of x?
(f) Sketch $y = \sinh x$, and explain why the function has an inverse function $\sinh^{-1} x$.
(g) Sketch the graph of $\sinh^{-1} x$ on the same diagram as part (f).
(h) Show that $\sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$, by treating the equation $x = \frac{1}{2}(e^y - e^{-y})$ as a quadratic equation in e^y.
(i) Find $\frac{d}{dx}(\sinh^{-1} x)$, and hence find $\int \frac{dx}{\sqrt{1 + x^2}}$.

EXTENSION

23. Suppose that f is a one-to-one function with domain D and range R. Then the function g with domain R and range D is the inverse of f if

$$f(g(x)) = x \text{ for every } x \text{ in } R \quad \text{and} \quad g(f(x)) = x \text{ for every } x \text{ in } D.$$

Use this characterisation to prove that the functions

$$f(x) = -\tfrac{2}{3}\sqrt{9 - x^2}\,, \text{ where } 0 \le x \le 3, \quad \text{and} \quad g(x) = \tfrac{3}{2}\sqrt{4 - x^2}\,, \text{ where } -2 \le x \le 0,$$

are inverse functions.

24. THEOREM: If f is a differentiable function for all real x and has an inverse function g, then $g'(x) = \dfrac{1}{f'(g(x))}$, provided that $f'(g(x)) \ne 0$.
(a) It is known that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and that $y = e^x$ is the inverse function of $y = \ln x$.
Use this information and the above theorem to prove that $\frac{d}{dx}(e^x) = e^x$.
(b) (i) Show that the function $f(x) = x^3 + 3x$ is increasing for all real x, and hence that it has an inverse function, $f^{-1}(x)$. (ii) Use the theorem to find the gradient of the tangent to the curve $y = f^{-1}(x)$ at the point $(4, 1)$.
(c) Prove the theorem in general.

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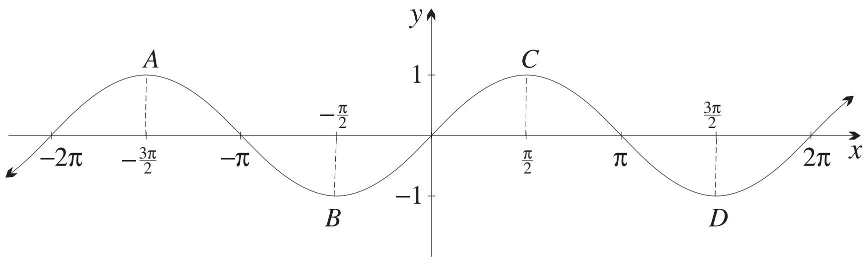
1 B Defining the Inverse Trigonometric Functions

Each of the six trigonometric functions fails the horizontal line test completely, in that there are horizontal lines which cross each of their graphs infinitely many times. For example, $y = \sin x$ is graphed below, and clearly every horizontal line between $y = 1$ and $y = -1$ crosses it infinitely many times.

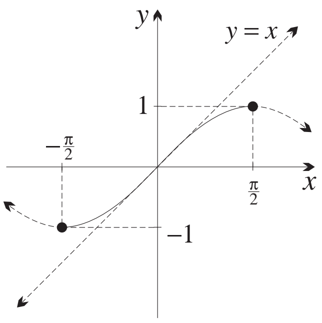


To create an inverse function from $y = \sin x$, we need to restrict the domain to a piece of the curve between two turning points. For example, the pieces AB, BC and CD all satisfy the horizontal line test. Since acute angles should be included, the obvious choice is the arc BC from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

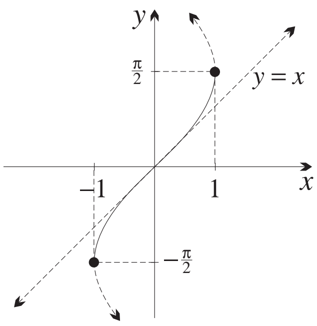
The Definition of $\sin^{-1} x$: The function $y = \sin^{-1} x$ (which is read as 'inverse sine ex') is accordingly defined to be the inverse function of the restricted function

$$y = \sin x, \quad \text{where } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}.$$

The two curves are sketched below. Notice, when sketching the graphs, that $y = x$ is a tangent to $y = \sin x$ at the origin. Thus when the graph is reflected in $y = x$, the line $y = x$ does not move, and so it is also the tangent to $y = \sin^{-1} x$ at the origin. Notice also that $y = \sin x$ is horizontal at its turning points, and hence $y = \sin^{-1} x$ is vertical at its endpoints.



$y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



$y = \sin^{-1} x$

4 THE DEFINITION OF $y = \sin^{-1} x$:
- $y = \sin^{-1} x$ is not the inverse relation of $y = \sin x$, it is the inverse function of the restriction of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- $y = \sin^{-1} x$ has domain $-1 \le x \le 1$ and range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
- $y = \sin^{-1} x$ is an increasing function.
- $y = \sin^{-1} x$ has tangent $y = x$ at the origin, and is vertical at its endpoints.

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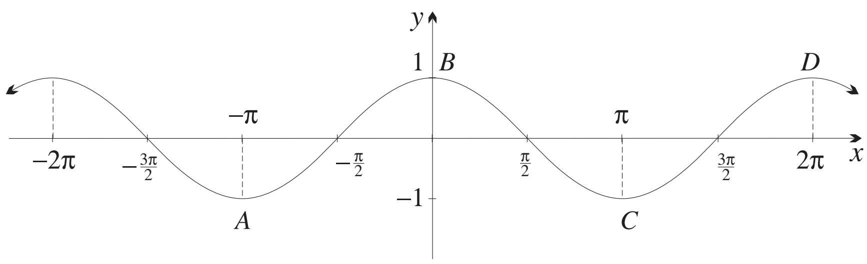
Bill Pender, David Sadler, Julia Shea and Derek Ward
Excerpt
More information

NOTE: In this course, radian measure is used exclusively when dealing with the inverse trigonometric functions. Calculations using degrees should be avoided, or at least not included in the formal working of problems.

5 RADIAN MEASURE: Use radians when dealing with inverse trigonometric functions.

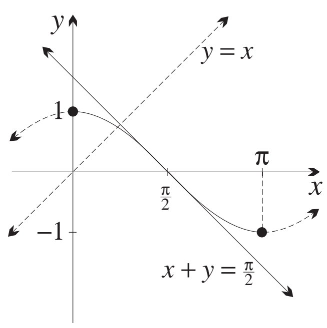
The Definition of $\cos^{-1} x$: The function $y = \cos x$ is graphed below. To create a satisfactory inverse function from $y = \cos x$, we need to restrict the domain to a piece of the curve between two turning points. Since acute angles should be included, the obvious choice is the arc BC from $x = 0$ to $x = \pi$.



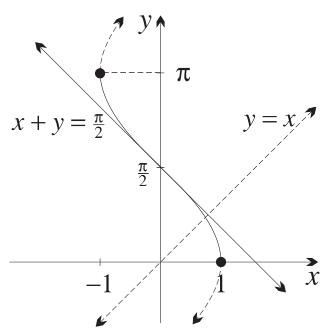
Thus the function $y = \cos^{-1} x$ (read as 'inverse cos ex') is defined to be the inverse function of the restricted function

$$y = \cos x, \quad \text{where } 0 \le x \le \pi,$$

and the two curves are sketched below. Notice that the tangent to $y = \cos x$ at its x-intercept $(\frac{\pi}{2}, 0)$ is the line t: $x + y = \frac{\pi}{2}$ with gradient -1. Reflection in $y = x$ reflects this line onto itself, so t is also the tangent to $y = \cos^{-1} x$ at its y-intercept $(0, \frac{\pi}{2})$. Like $y = \sin^{-1} x$, the graph is vertical at its endpoints.



$y = \cos x$, $0 \le x \le \pi$



$y = \cos^{-1} x$

6 THE DEFINITION OF $y = \cos^{-1} x$:
- $y = \cos^{-1} x$ is not the inverse relation of $y = \cos x$, it is the inverse function of the restriction of $y = \cos x$ to $0 \le x \le \pi$.
- $y = \cos^{-1} x$ has domain $-1 \le x \le 1$ and range $0 \le y \le \pi$.
- $y = \cos^{-1} x$ is a decreasing function.
- $y = \cos^{-1} x$ has gradient -1 at its y-intercept, and is vertical at its endpoints.

The Definition of $\tan^{-1} x$: The graph of $y = \tan x$ on the next page consists of a collection of disconnected branches. The most satisfactory inverse function is formed by choosing the branch in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.