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More information

1

Equations of Motion

1.1 The fluid state

Consider a fluid that can be regarded as continuous and locally homogeneous at all levels of subdivision. At any time t and position $\mathbf{x} = (x_1, x_2, x_3)$ the state of the fluid is defined when the velocity $\mathbf{v}$ and any two thermodynamic variables are specified. A fluid in unsteady motion, in which temperature and pressure vary with position and time, cannot strictly be in thermodynamic equilibrium, and it will be necessary to discuss how to define the thermodynamic properties of the small individual *fluid particles* of which the fluid may be supposed to consist.

The distinctive fluid property possessed by both liquids and gases is that these fluid particles can move freely relative to one another under the influence of applied forces or other externally imposed changes at the boundaries of the fluid. Five scalar partial differential equations are required for determining these motions. They are statements of conservation of mass, momentum, and energy, and they are to be solved subject to appropriate *boundary* and *initial* conditions, dependent on the problem at hand. This book is concerned with the use of these equations to formulate and analyse a wide range of model problems whose solutions will help the reader to understand the intricacies of fluid motion.

1.2 The material derivative

Let v_i denote the component of the fluid velocity $\mathbf{v}$ in the x_i direction of the fixed rectangular coordinate system (x_1, x_2, x_3) and consider the rate at which any function $F(\mathbf{x}, t)$ varies *following the motion* of a fluid particle. Suppose the particle is at $\mathbf{x}$ at time t, and at $\mathbf{x} + \delta\mathbf{x}$ a short time later at time $t + \delta t$, where $\delta\mathbf{x} = \mathbf{v}(\mathbf{x}, t)\delta t +, \ldots,$ where the terms omitted vanish more rapidly than δt as $\delta t \to 0$. Then the value of F at the new position of the fluid particle is

$$F(\mathbf{x} + \delta\mathbf{x}, t + \delta t) = F(\mathbf{x}, t) + v_j \delta t \frac{\partial F}{\partial x_j}(\mathbf{x}, t) + \delta t \frac{\partial F}{\partial t}(\mathbf{x}, t) + \cdots +,$$

where the repeated suffix j implies summation over $j = 1, 2, 3$. The limiting value of the ratio

$$\frac{F(\mathbf{x}+\delta\mathbf{x}, t+\delta t) - F(\mathbf{x}, t)}{\delta t} \quad \text{as} \quad \delta t \to 0$$

is called the material (or 'Lagrangian') derivative of F. It is denoted by DF/Dt, and

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + v_j \frac{\partial F}{\partial x_j} \equiv \frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F. \tag{1.2.1}$$

DF/Dt measures the time rate of change of F as seen by an observer moving with the fluid particle that occupies position $\mathbf{x}$ at the current time t.

1.3 Conservation of mass: Equation of continuity

A fluid particle of volume V and mass density ρ has a total mass of ρV. This cannot change as the particle moves around in the fluid, and therefore satisfies

$$\frac{D(\rho V)}{Dt} = 0,$$

so that

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{V}\frac{DV}{Dt} = 0. \tag{1.3.1}$$

Now DV/Dt is the rate at which the volume of the fluid particle increases and is ultimately equal to $V \operatorname{div} \mathbf{v}$ when $V \to 0$. This is a consequence of the following integral definition of the divergence:

$$\operatorname{div} \mathbf{v} = \lim_{V \to 0} \frac{1}{V} \oint_{\mathrm{S}} \mathbf{v} \cdot d\mathbf{S} \tag{1.3.2}$$

where the integration is over the closed material surface S forming the boundary of V, on which the vector surface element $d\mathbf{S}$ is directed *out* of V. Hence, using definition (1.2.1) of D/Dt, mass conservation equation (1.3.1) can be transformed into any of the following equivalent forms of the *equation of continuity*

$$\left.\begin{aligned} \tfrac{1}{\rho}\tfrac{D\rho}{Dt} + \operatorname{div} \mathbf{v} &= 0, \\ \tfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ \tfrac{\partial \rho}{\partial t} + \tfrac{\partial}{\partial x_j}(\rho v_j) &= 0. \end{aligned}\right\} \tag{1.3.3}$$

In the special case of an *incompressible* fluid, the density ρ of a fluid particle cannot change, although it may be different for different fluid particles. Therefore both $D\rho/Dt = 0$ and $DV/Dt = 0$, and the continuity equation reduces to

$$\operatorname{div} \mathbf{v} = 0. \tag{1.3.4}$$

This represents a kinematic (or *geometric*) constraint on possible motions of the fluid.

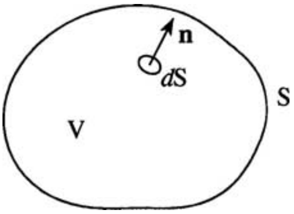

Figure 1.4.1

1.4 Momentum equation

The momentum equation is derived by consideration of the rate of change of momentum of a fluid particle (Figure 1.4.1) subject to the effects of the normally applied pressure p on its bounding surface S, the normal and tangential viscous stresses on S, and any body force (such as gravity) $\mathbf{F}$ per unit volume. Let σ_{ij} denote the *viscous stress tensor*, defined such that the *traction* force per unit area on a surface element of S with unit normal n_i is $\sigma_{ij}n_j$. Then the i component of the momentum equation for a small fluid element of volume V becomes

$$\rho V \frac{Dv_i}{Dt} = \oint_{\mathrm{S}} \left(- p\delta_{ij} + \sigma_{ij} \right) n_j d\mathrm{S} + VF_i,$$

where the unit normal $\mathbf{n}$ is directed out of V. The surface integral can be transformed into an integral over the interior volume of the fluid element by application of the divergence theorem,

$$\oint_{\mathrm{S}} \mathcal{F}(\mathbf{x}) n_j d\mathrm{S} = \int_{\mathrm{V}} \frac{\partial \mathcal{F}}{\partial x_j}(\mathbf{x}) d^3\mathbf{x}, \tag{1.4.1}$$

where $\mathcal{F}(\mathbf{x})$ is any scalar or vector field. Thus, as $V \to 0$,

$$\oint_{\mathrm{S}} \left(- p\delta_{ij} + \sigma_{ij} \right) n_j d\mathrm{S} \to V \frac{\partial}{\partial x_j} \left(- p\delta_{ij} + \sigma_{ij} \right),$$

and the momentum equation becomes

$$\rho \frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + F_i. \tag{1.4.2}$$

The viscous stress is caused by the molecular diffusion of momentum between neighbouring fluid particles and is non-zero only when neighbouring particles are in relative motion; σ_{ij} must therefore depend on the velocity gradient.

1.4.1 Relative motion of neighbouring fluid elements

Let v_i denote the velocity at $\mathbf{x}$. The velocity $v_i + \delta v_i$ at a neighbouring point $\mathbf{x} + \delta\mathbf{x}$ at the same time is given to first order in $\delta\mathbf{x}$ by

$$\delta v_i = \delta x_j \frac{\partial v_i}{\partial x_j} \equiv \delta x_j e_{ij} + \delta x_j \xi_{ij}, \tag{1.4.3}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \xi_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{1.4.4}$$

are, respectively, the symmetric and antisymmetric components of $\partial v_i/\partial x_j$.

The diagonal elements $(i = j)$ of the 3×3 antisymmetric tensor ξ_{ij} are zero; the remaining six elements satisfy $\xi_{ij} = -\xi_{ji}$, and are therefore determined by three independent quantities $\omega_1,\ \omega_2,$ and ω_3, say. We can then write

$$\xi_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k, \tag{1.4.5}$$

where ϵ_{ijk} is the *alternating tensor* whose components are zero unless $i,\ j,$ and k are all different, and then $\epsilon_{ijk} = \pm 1$ according to whether $i,\ j,$ and k are or are not in cyclic order. We obtain an explicit representation of ξ_{ij} by identifying the $i,\ j$ component of ξ_{ij} with the element in the ith row and jth column of a 3×3 matrix, i.e.,

$$\xi_{ij} = -\frac{1}{2}\begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

By equating corresponding terms on the two sides of Eq. (1.4.5), we see that ω_i is just the ith component of the *vorticity* vector $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$,

$$\omega_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \quad \omega_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \quad \omega_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2},$$

and that

$$\delta x_j \xi_{ij} = -\frac{1}{2}\epsilon_{ijk}\delta x_j \omega_k = \frac{1}{2}(\boldsymbol{\omega} \wedge \delta\mathbf{x})_i.$$

The symmetric array e_{ij} is called the *rate of strain* tensor. The sum of the diagonal elements,

$$e_{11} + e_{22} + e_{33} \equiv e_{kk} = \operatorname{div} \mathbf{v},$$

is independent of the orientation of the coordinate axes. The contribution of e_{ij} to the relative velocity δv_i of Eq. (1.4.3) can be written as

$$\delta x_j e_{ij} = \frac{1}{2}\frac{\partial}{\partial \delta x_i}\left(e_{jk}\delta x_j \delta x_k\right),$$

where the differentiation is with respect to the displacement δx_i (in terms of which e_{jk} is constant).

Hence the velocity of the fluid at $\mathbf{x} + \delta\mathbf{x}$ relative to that at $\mathbf{x}$ can be written as

$$\delta\mathbf{v} = \frac{1}{2}\nabla\left(e_{ij}\delta x_i \delta x_j\right) + \frac{1}{2}\boldsymbol{\omega} \wedge \delta\mathbf{x}, \tag{1.4.6}$$

where the gradient is taken with respect to $\delta\mathbf{x}$. The term in $\boldsymbol{\omega}$ represents relative motion as a *rigid-body rotation*, at angular velocity $\frac{1}{2}\boldsymbol{\omega}$, with no distortion of the fluid particle.

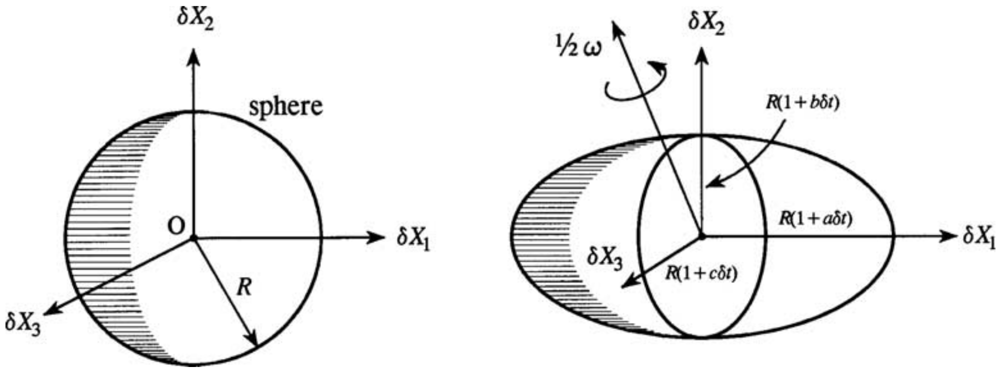

Figure 1.4.2

The gradient term, however, represents an *irrotational* distortion of the fluid element, in the direction of the normal at $\delta\mathbf{x}$ to the quadric surface:

$$\frac{1}{2}e_{ij}\delta x_i \delta x_j = C,$$

where C is a constant whose value is chosen to make the surface pass through the point $\delta\mathbf{x}$. By means of a suitable rotation of the local coordinate axes at $\mathbf{x}$, from δx_i to δX_i, say, we can transform the quadric to the normal form:

$$\frac{1}{2}\left[a(\delta X_1)^2 + b(\delta X_2)^2 + c(\delta X_3)^2\right] = C, \tag{1.4.7}$$

where a, b, and c are called the principal rates of strain; their sum is an invariant of the coordinate transformation that satisfies

$$a + b + c = e_{kk} \equiv \operatorname{div}\mathbf{v}.$$

The distortion produced by e_{ij} can now be seen to be one of *pure strain*. For, if the fluid element at $\mathbf{x}$ was initially a sphere of radius R (Figure 1.4.2), after time δt it is deformed without rotation into an ellipsoid with semi-axes $R(1 + a\delta t)$, $R(1 + b\delta t)$, and $R(1 + c\delta t)$, respectively, along the directions of the principal axes δX_1, δX_2, δX_3, with change in volume equal to

$$\frac{4}{3}\pi R^3(a + b + c)\delta t \equiv \frac{4}{3}\pi R^3 \operatorname{div}\mathbf{v}\delta t,$$

which vanishes when the fluid is incompressible.

1.4.2 Viscous stress tensor

There is no relative motion between neighbouring points of a fluid particle in solid-body rotation. Therefore there can be no viscous force $\sigma_{ij}n_j$ on a surface element with normal $\mathbf{n}$ that separates these points. Thus in a first approximation we assume that σ_{ij} is a *linear* combination of those gradients $\partial v_i/\partial x_j$ of the velocity that represent a purely straining motion of the fluid, that is, of relative motion defined by the strain tensor e_{ij}.

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Put

$$e_{ij} = \left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right) + \frac{1}{3}e_{kk}\delta_{ij}.$$

The first term on the right-hand side represents a straining motion involving no net change in volume [with principal axes of strain a, b, c of the corresponding quadric (1.4.7) satisfying $a + b + c = 0$], whereas the second term describes an isotropic dilatation of a fluid element (so that a spherical fluid particle remains spherical but expands or contracts to a new size). Such physically different straining motions might be expected to make essentially distinct contributions to the viscous stress tensor. If the fluid properties are assumed to be locally isotropic (independent of the orientation of local coordinate axes at any point in the fluid), we can therefore set

$$\sigma_{ij} = 2\eta\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right) + \eta' e_{kk}\delta_{ij}, \tag{1.4.8}$$

where η and η' are called, respectively, the **shear** and **bulk** coefficients of viscosity. They generally vary with both the pressure and temperature and with position in the fluid. The bulk coefficient of viscosity η' vanishes for monatomic gases, and in this case (and for most liquids, such as water) the fluid is said to be ‘Stokesian’, with

$$\sigma_{ij} = 2\eta\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right). \tag{1.4.9}$$

When velocity gradients are present the fluid cannot be in strict thermodynamic equilibrium, and thermodynamic variables, such as the pressure and density, require special interpretation. For a fluid in non-uniform motion, it is usual to define the density ρ and internal energy e per unit mass in the usual way, such that ρ and ρe are the mass and internal energy per unit volume, respectively. The pressure and all other thermodynamic quantities are then defined by means of the same functions of ρ and e that would be used for a system in thermal equilibrium. However, the *thermodynamic pressure* $p = p(\rho, e)$ so defined is then no longer the sole source of normal stress in the fluid. We obtain the mean normal stress at **x** by averaging $(p\delta ij - \sigma_{ij})n_i n_j$ over all possible orientations of a unit vector **n** at **x**. We do this by evaluating the following integral over the surface of the unit sphere,

$$\frac{1}{4\pi}\oint n_i n_j dS = \frac{1}{3}\delta_{ij},$$

and multiplying by $(p\delta ij - \sigma_{ij})$, to find

$$\text{mean normal stress} = p - \eta' e_{kk} \equiv p - \eta' \operatorname{div}\mathbf{v}.$$

The mean normal stress therefore differs from the thermodynamic pressure p if the bulk coefficient of viscosity η' is non-zero. This happens in a fluid whose molecules possess rotational (or other internal) degrees of freedom whose relaxation time (during which thermal equilibrium is re-established after, say, a compression of the fluid) is large relative to the equilibration time of the translational degrees of freedom.

For example, when a diatomic gas is compressed ($\text{div}\,\mathbf{v} < 0$) the temperature must rise, but the corresponding increase in the rotational energy lags slightly behind that of the translational energy; the thermodynamic pressure $p = (\gamma - 1)\rho e$ (γ = specific-heat ratio) accordingly is smaller than the actual pressure $p - \eta' \text{div}\,\mathbf{v}$ by an amount equal to $-\eta' \text{div}\,\mathbf{v}$.

It may be shown (Landau & Lifshitz 1987) that, whereas the thermodynamic pressure differs from the mean normal stress by a term linear in $\text{div}\,\mathbf{v}$, the corresponding departure of the thermodynamic entropy s (per unit mass) from the true entropy is proportional at least to the square of such gradients, and the difference is usually small in practice. This can be deduced from a consideration of thermodynamic relation (1.5.4) given in the next section.

1.4.3 Navier–Stokes equation

When the variations of η and η' can be neglected, the substitution of Eq. (1.4.8) into momentum equation (1.4.2) yields the *Navier–Stokes* equation:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \eta \nabla^2 \mathbf{v} + \left(\eta' + \frac{1}{3}\eta\right) \nabla\, \text{div}\,\mathbf{v} + \mathbf{F}. \tag{1.4.10}$$

By means of the vector identity curl curl $= \nabla\,\text{div} - \nabla^2$, we can also write

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - \eta\, \text{curl}\, \boldsymbol{\omega} + \left(\eta' + \frac{4}{3}\eta\right) \nabla\, \text{div}\,\mathbf{v} + \mathbf{F}. \tag{1.4.11}$$

1.4.4 The Reynolds equation and Reynolds stress

We obtain an equation for the rate of change of momentum density ρv_i by adding continuity equation (1.3.3) multiplied by v_i to momentum equation (1.4.2), and writing the result in the form

$$\frac{\partial(\rho v_i)}{\partial t} = -\frac{\partial \pi_{ij}}{\partial x_j} + F_i, \tag{1.4.12}$$

where

$$\pi_{ij} = p\delta_{ij} + \rho v_i v_j - \sigma_{ij} \tag{1.4.13}$$

is the called the *momentum flux tensor*.

Equation (1.4.12) is the *Reynolds equation*. By integrating it over the interior volume V of a *fixed* control surface S and applying the divergence theorem, we can write

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathrm{V}} \rho v_i d^3\mathbf{x} &= -\oint_{\mathrm{S}} \pi_{ij} dS_j + \int_{\mathrm{V}} F_i d^3\mathbf{x} \\ &\equiv \oint_{\mathrm{S}} \left(-p dS_i - \rho v_i v_j dS_j + \sigma_{ij} dS_j \right) + \int_{\mathrm{V}} F_i d^3\mathbf{x}, \end{aligned} \tag{1.4.14}$$

where the surface element dS_i is directed *out* of V. The terms in the surface integral on the second line respectively represent the flux of i momentum through S into V produced by the surface pressure p, by the *Reynolds stress* $-\rho v_i v_j$ (by the convection of momentum ρv_i per unit volume by the normal component of the velocity v_j), and by the action of frictional forces on S.

1.5 The energy equation

The energy equation is derived from a consideration of the total energy of the fluid: the kinetic energy of the gross fluid motions and the thermodynamic 'internal' energy. The equation governs the dissipation of mechanical energy and its transformation into heat.

Consider a small fluid element of volume V bounded by a surface S with unit *outward* normal **n** (Figure 1.4.1). The kinetic and internal energies per unit volume are equal respectively to $\frac{1}{2}\rho v^2$ and ρe, and the total energy of the fluid in V is $E = \rho V(\frac{1}{2}v^2 + e)$. Changes in E are produced by the work done by the pressure and viscous frictional forces on the boundary S, by the flux of heat energy through S by molecular diffusion, and by the work performed by the body force **F** within V. Because the mass in V is conserved $[D(\rho V)/Dt = 0]$, we can write

$$V\rho \frac{D}{Dt}\left(\frac{1}{2}v^2 + e\right) = \oint_S (-pn_i + \sigma_{ij}n_j)\, v_i dS + \oint_S \kappa \frac{\partial T}{\partial x_j} n_j dS + VF_i v_i,$$

where T is the temperature and κ is the thermal conductivity of the fluid. Using the divergence theorem (for small V), dividing through by V, and expanding divergence derivatives on the right-hand side by the product rule, we find

$$\rho \frac{D}{Dt}\left(\frac{1}{2}v^2\right) + \rho \frac{De}{Dt} = -v_i \frac{\partial p}{\partial x_i} - p \operatorname{div} \mathbf{v} + v_i \frac{\partial \sigma_{ij}}{\partial x_j} + \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right) + F_i v_i.$$

This is greatly simplified by subtraction of the product of v_i and momentum equation (1.4.2) to obtain

$$\rho \frac{De}{Dt} = -p \operatorname{div} \mathbf{v} + \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right). \qquad (1.5.1)$$

We obtain a more useful form of this equation by first noting, from continuity equation (1.3.3) and from definitions (1.4.4) of e_{ij} and (1.4.8) of σ_{ij}, that

$$\operatorname{div} \mathbf{v} = -\frac{1}{\rho}\frac{D\rho}{Dt}, \quad \sigma_{ij}\frac{\partial v_i}{\partial x_j} = \sigma_{ij} e_{ij} \equiv 2\eta \left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right)^2 + \eta'(\operatorname{div} \mathbf{v})^2,$$

so that Eq. (1.5.1) becomes

$$\rho \frac{De}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = 2\eta \left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right)^2 + \eta'(\operatorname{div} \mathbf{v})^2 + \operatorname{div}\left(\kappa \nabla T\right). \qquad (1.5.2)$$

The left-hand side can be expressed in terms of the specific entropy s of the fluid by application of the first law of thermodynamics to unit mass of fluid:

$$de = Tds - pdV.$$

If V is the volume occupied by unit mass, then $\rho V = 1$,

$$dV = d\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}d\rho,$$

and therefore

$$Tds = de - \frac{p}{\rho^2}d\rho. \tag{1.5.3}$$

Hence energy equation (1.5.2) becomes

$$\rho T\frac{Ds}{Dt} = 2\eta\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right)^2 + \eta'(\operatorname{div}\mathbf{v})^2 + \operatorname{div}(\kappa\nabla T). \tag{1.5.4}$$

The quantity $\rho T Ds/Dt$ is the time rate of change following the fluid particles of the heat gained per unit volume of fluid. The term $2\eta(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij})^2 + \eta'(\operatorname{div}\mathbf{v})^2 > 0$ is the rate of production of heat by frictional dissipation of macroscopic motions, i.e., the rate at which mechanical energy is dissipated per unit volume of the fluid; $\operatorname{div}(\kappa\nabla T)$ is the rate at which heat energy is gained per unit volume by molecular diffusion.

1.5.1 Alternative treatment of the energy equation

Let us use the identity

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \boldsymbol{\omega}\wedge\mathbf{v} + \nabla\left(\frac{1}{2}v^2\right) \tag{1.5.5}$$

to write momentum equation (1.4.2) in the form

$$\rho\frac{\partial v_i}{\partial t} + \rho\frac{\partial}{\partial x_i}\left(\frac{1}{2}v^2\right) + \frac{\partial p}{\partial x_i} = -\rho(\boldsymbol{\omega}\wedge\mathbf{v})_i + \frac{\partial\sigma_{ij}}{\partial x_j} + F_i.$$

Take the scalar product with v_i and use continuity equation (1.3.3) to obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \operatorname{div}\left(\rho\mathbf{v}\frac{1}{2}v^2\right) + \mathbf{v}\cdot\nabla p = v_i\frac{\partial\sigma_{ij}}{\partial x_j} + F_i v_i. \tag{1.5.6}$$

This result is further transformed by introduction of the *enthalpy* w, defined by

$$w = e + \frac{p}{\rho}, \tag{1.5.7}$$

in terms of which the first law (1.5.3) becomes

$$dw = Tds + \frac{dp}{\rho}. \tag{1.5.8}$$

Then a simple calculation shows that

$$\mathbf{v}\cdot\nabla p = \frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho\mathbf{v}w) - \rho T\frac{Ds}{Dt},$$

and therefore that Eq. (1.5.6) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho e\right)+\frac{\partial}{\partial x_j}\left[\rho v_j\left(w+\frac{1}{2}v^2\right)-v_i\sigma_{ij}\right]=\rho T\frac{Ds}{Dt}-\sigma_{ij}\frac{\partial v_i}{\partial x_j}+F_i v_i. \quad (1.5.9)$$

This equation shows how the overall energy of the fluid is coupled to the production of heat within the fluid and the work done by viscous stresses and the body force **F**. For an isentropic, inviscid fluid,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho e\right)+\operatorname{div}\left[\rho\mathbf{v}\left(w+\frac{1}{2}v^2\right)\right]=\mathbf{F}\cdot\mathbf{v}. \quad (1.5.10)$$

Let this equation be integrated over the interior V of a *fixed* control surface S:

$$\frac{\partial}{\partial t}\int_{\mathrm{V}}\left(\frac{1}{2}\rho v^2+\rho e\right)d^3\mathbf{x}=-\oint_{\mathrm{S}}\left[\mathbf{v}\left(\frac{1}{2}\rho v^2+\rho e\right)+p\mathbf{v}\right]\cdot d\mathbf{S}+\int_{\mathrm{V}}\mathbf{F}\cdot\mathbf{v}\,d^3\mathbf{x},$$

where the surface element $d\mathbf{S}$ is directed out of V. This equates the rate of increase of energy inside S to the sum of its rate of convection across S by the flow velocity **v**, and to the rates of working of the ambient pressure on S and the body force in V. In a viscous fluid the surface integral is augmented by the contribution

$$\oint_{\mathrm{S}} v_i\sigma_{ij}n_j\,dS,$$

which represents the rate of working by frictional forces on the boundary S. In addition, the remaining terms on the right-hand side of Eq. (1.5.9) (other than the body force **F**) represent the net energy gain within S by heat addition. Indeed,

$$\begin{aligned}\sigma_{ij}\frac{\partial v_i}{\partial x_j}&=2\eta\left(e_{ij}-\frac{1}{3}e_{kk}\delta_{ij}\right)^2+\eta'(\operatorname{div}\mathbf{v})^2\\&=\text{rate of frictional heating per unit volume.}\end{aligned}$$

Therefore, if $\mathbf{Q}=-\kappa\nabla T$ is the *heat flux* vector, so that the rate at which heat flows into S is just $-\oint_{\mathrm{S}}\mathbf{Q}\cdot d\mathbf{S}$, then

$$\rho T\frac{Ds}{Dt}=\sigma_{ij}\frac{\partial v_i}{\partial x_j}-\operatorname{div}\mathbf{Q}. \quad (1.5.11)$$

This is just Equation (1.5.4).

1.5.2 Energy equation for incompressible flow

When the flow is incompressible, the energy equation is merely a linear combination of the continuity and momentum equations. However, the special case of flow subject to a conservative body force $\mathbf{F}=\rho\nabla\Phi(\mathbf{x})$ is of particular interest. By using the third form of continuity equation (1.3.3), we can write Eq. (1.5.6) as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho\Phi\right)+\frac{\partial}{\partial x_j}\left[v_j\left(p-\rho\Phi+\frac{1}{2}\rho v^2\right)-v_i\sigma_{ij}\right]=-2\eta e_{ij}^2. \quad (1.5.12)$$

For an incompressible fluid of uniform density $\rho=\rho_o=$ constant, the term $\rho\Phi$ in the time derivative can be omitted.