1

Preliminaries

1.1 Linearization of the equations of fluid flow

Referring to a frame of reference which is either fixed or translated in space, let us consider the flow of an incompressible Newtonian fluid. The motion of the fluid is governed by the continuity equation for the flow velocity $\mathbf{u}$:

$$\nabla \cdot \mathbf{u} = 0 \tag{1.1.1}$$

which expresses conservation of mass, and the Navier–Stokes equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla P + \mu \nabla^2 \mathbf{u} + \rho \mathbf{b} \tag{1.1.2}$$

which expresses Newton's second law for a small parcel of fluid. In (1.1.2) ρ and μ are the density and the viscosity of the fluid, and $\mathbf{b}$ is a body force which, for simplicity, we shall assume to be constant. Inspecting the flow we find a characteristic length L related to the size of the boundaries, a characteristic velocity U determined by the particular mechanism driving the flow, and a characteristic time T that is either imposed by external forcing or simply defined as L/U. We scale the velocity by U, all lengths by L, time by T, and each term on the right-hand side of (1.1.2) by $\mu U/L^2$. Then, we introduce the dimensionless variables $\mathbf{u}' = \mathbf{u}/U$, $\mathbf{x}' = \mathbf{x}/L$, $t' = t/T$, $P' = PL/\mu U$, and write (1.1.2) in the dimensionless form

$$\beta \frac{\partial \mathbf{u}'}{\partial t'} + Re\, \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\nabla' P' + \nabla'^2 \mathbf{u}' + \frac{Re}{Fr} \frac{\mathbf{b}}{|\mathbf{b}|} \tag{1.1.3}$$

We identify the first term on the left-hand side of (1.1.3) as the inertial acceleration term, and the second term as the inertial convective term.

Three dimensionless numbers appear in (1.1.3): the frequency parameter $\beta = L^2/\nu T$, the Reynolds number $Re = UL/\nu$, and the Froude number $Fr = U^2/|\mathbf{b}|L$, where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. The frequency parameter β expresses the magnitude of inertial acceleration forces relative to viscous forces, or equivalently, the ratio between the characteristic time of diffusion of vorticity L^2/ν and the characteristic time of flow T. The Reynolds number expresses the magnitude of inertial convective forces relative to viscous forces, or equivalently, the ratio between the characteristic time of diffusion of vorticity L^2/ν and the convective time L/U. The Froude number expresses the magnitude of inertial convective forces relative to body forces. Finally, the group $Re/Fr = L^2|\mathbf{b}|/\nu U$ expresses the magnitude of body forces relative to viscous forces. It should be noted that in the absence of external forcing, T may be defined as L/U in which case β reduces to Re, and the dimensionless Navier–Stokes equation (1.1.3) involves only two independent parameters, namely Re and Fr.

Now, when $Re, \beta \ll 1$, all terms on the left-hand side of (1.1.3) are small compared with those on the right-hand side and thus may be neglected. Reverting to dimensional variables we find that the flow is governed by the *Stokes equation*

$$-\nabla P + \mu\nabla^2\mathbf{u} + \rho\mathbf{b} = 0 \tag{1.1.4}$$

which states that pressure, viscous, and body forces balance at any instant in time even though the flow may be unsteady. The instantaneous structure of the flow depends solely on the present boundary configuration and boundary conditions, and is independent of the history of motion. To be more precise, the history of motion enters the problem only by determining the current location of the boundaries.

When $Re \ll 1$ but $\beta \sim 1$, the inertial convective term on the left-hand side of (1.1.3) is small compared with the rest of the terms and thus may be neglected. Again reverting to dimensional variables we find that the flow is governed by the *unsteady Stokes equation* or *linearized Navier–Stokes equation*

$$\rho\frac{\partial\mathbf{u}}{\partial t} = -\nabla P + \mu\nabla^2\mathbf{u} + \rho\mathbf{b} \tag{1.1.5}$$

Because of the presence of the acceleration term, the instantaneous structure of the flow depends not only on the instantaneous boundary configuration and boundary conditions but also on the history of motion. Physically, the unsteady Stokes equation is valid for flows that are characterized by sudden acceleration or deceleration, such as those occurring during hydrodynamic braking, during the impact of a particle on a solid surface, or during the initial stages of the flow due to a particle settling from rest.

Problem

1.1.1 Consider a rigid body that is convected by a steady flow, and write the Navier–Stokes equation in an accelerating but not rotating frame of reference that is attached to the body. Identify β, Re, and Fr, and establish conditions for the steady or unsteady Stokes equation to be valid in the vicinity of the body.

1.2 The equations of Stokes flow

We have seen that the flow of a Newtonian fluid at small values of the frequency parameter β and Reynolds number Re is governed by the continuity equation $\nabla \cdot \mathbf{u} = 0$ and the Stokes equation

$$-\nabla P + \mu \nabla^2 \mathbf{u} + \rho \mathbf{b} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = 0 \tag{1.2.1}$$

where $\boldsymbol{\sigma}$ is the stress tensor defined as follows:

$$\sigma_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = -P\delta_{ij} + 2\mu e_{ij} \tag{1.2.2}$$

and

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.2.3}$$

is the rate of deformation tensor. Taking the divergence of (1.2.1) and using the continuity equation we find that the pressure is a harmonic function, i.e.

$$\nabla^2 P = 0 \tag{1.2.4}$$

Taking the Laplacian of the Stokes equation and using (1.2.4) we find that the velocity satisfies the vectorial biharmonic equation, i.e.

$$\nabla^4 \mathbf{u} = 0 \tag{1.2.5}$$

Finally, taking the curl of the Stokes equation and using the vector identity $\nabla \times \nabla \mathrm{F} = 0$, valid for any twice differentiable function F, we find that the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is a harmonic function, i.e.

$$\nabla^2 \boldsymbol{\omega} = 0 \tag{1.2.6}$$

Invoking the general properties of harmonic functions we deduce that there is no intense concentration of vorticity in Stokes flow and furthermore, that the vorticity attains extreme values at the boundaries of the flow. The onset of regions of recirculating fluid does not imply the presence of compact vortices, as it does in the case of inviscid flow.

Integrating (1.2.1) over a volume V that resides within the domain of flow and is bounded by the closed surface D, and applying the divergence theorem to convert the volume integral involving the stress into a surface integral, we find

$$\mathbf{F} = \int_D \boldsymbol{\sigma} \cdot \mathbf{n} \, dS \equiv \int_D \mathbf{f} \, dS = -\int_V \rho \mathbf{b} \, dV \tag{1.2.7}$$

where $\mathbf{F}$ is the *hydrodynamic force* exerted on D, $\mathbf{f} = \boldsymbol{\sigma} \cdot \mathbf{n}$ is the *surface force* or *traction* exerted on D, and $\mathbf{n}$ is the unit normal vector pointing outside V. Working in a similar manner we obtain

$$\mathbf{L} = \int_D \mathbf{x} \times (\boldsymbol{\sigma} \cdot \mathbf{n}) \, \mathrm{d}S \equiv \int_D \mathbf{x} \times \mathbf{f} \, \mathrm{d}S = -\int_V \rho \mathbf{x} \times \mathbf{b} \, \mathrm{d}V \qquad (1.2.8)$$

where $\mathbf{L}$ is the *hydrodynamic torque* exerted on D. Equations (1.2.7) and (1.2.8) imply that in the absence of a body force, the force and torque exerted on any volume of pure fluid are equal to zero, and furthermore, the force and torque exerted on any two reducible surfaces have the same values. It will be noted that the force exerted on a surface that encloses a boundary or a singular point may be finite, but must be equal to that exerted on the boundary or on a small surface that encloses the singular point; similarly for the torque.

For convenience, in the ensuing discussion we shall incorporate the effect of the body force $\mathbf{b}$ into a modified pressure defined as

$$P^{\mathrm{MOD}} = P - \rho \mathbf{b} \cdot \mathbf{x} \qquad (1.2.9)$$

and thus, we shall consider the body-force-free Stokes equation written in terms of P^{MOD}. The distinction between the regular and modified pressure will be relevant only when we consider boundary conditions for the surface force.

To facilitate the analysis of two-dimensional flow it is often helpful to introduce the stream function Ψ, defined by the equation

$$\mathbf{u} = \nabla \times (\mathbf{k} \Psi) \qquad (1.2.10)$$

where $\mathbf{k}$ is the unit vector perpendicular to the (x, y) plane of the flow. The underlying motivation for introducing the stream function is that the continuity equation is satisfied for any choice of Ψ and thus may be overlooked. Explicitly, the x and y Cartesian components of the velocity are given by

$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x} \qquad (1.2.11)$$

The radial and angular components of the velocity are given by

$$u_r = \frac{1}{r} \frac{\partial \Psi}{\partial \theta} \qquad u_\theta = -\frac{\partial \Psi}{\partial r} \qquad (1.2.12)$$

In terms of the stream function, the vorticity is equal to

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = -\mathbf{k} \nabla^2 \Psi \qquad (1.2.13)$$

Recalling that the vorticity is a harmonic function, we find that the stream function satisfies the biharmonic equation

$$\nabla^4 \Psi = 0 \qquad (1.2.14)$$

Now, using the identity $\nabla^2\mathbf{u} = -\nabla \times \boldsymbol{\omega}$ and the Stokes equation $\nabla^2\mathbf{u} = (1/\mu)\nabla P$, we find $\mu\nabla \times \boldsymbol{\omega} = -\nabla P$, which suggests that the magnitude of the vorticity and the pressure satisfy the Cauchy–Riemann equations

$$\frac{\partial \omega}{\partial x} = \frac{1}{\mu}\frac{\partial P}{\partial y} \qquad \frac{\partial \omega}{\partial y} = -\frac{1}{\mu}\frac{\partial P}{\partial x} \tag{1.2.15}$$

As a result, the complex function

$$f(z) = \omega + \frac{\mathrm{i}}{\mu}P \tag{1.2.16}$$

is an analytic function of $z = x + \mathrm{i}y$, where i is the square root of minus one. This observation allows us to study two-dimensional Stokes flow within the general framework of the theory of complex functions (Mikhlin 1957, Chapter 5; Langlois 1964, Chapter 7).

In an alternative formulation, we express a two-dimensional Stokes flow in terms of the Airy stress function Φ, defined by the equations

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} \qquad \sigma_{xy} = \sigma_{yx} = -\frac{\partial^2 \Phi}{\partial x \partial y} \qquad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} \tag{1.2.17}$$

It will be noted that the Stokes equation is satisfied for any choice of Φ. Using the continuity equation and recalling that the pressure is a harmonic function, we deduce that Φ satisfies the biharmonic equation. Writing out the three independent components of the stress tensor in terms of the pressure and the stream function, and using (1.2.17) to eliminate the pressure, we obtain

$$\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} = -4\mu\frac{\partial^2 \Psi}{\partial x \partial y} \qquad \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2} = \frac{1}{\mu}\frac{\partial^2 \Phi}{\partial x \partial y} \tag{1.2.18}$$

Furthermore, using (1.2.18), we find that the complex function $\chi = \Phi - \mathrm{i}\,2\mu\Psi$ satisfies the equation

$$\left(\frac{\partial^2 x}{\partial z^{*2}}\right)_z = 0 \tag{1.2.19}$$

where $z = x + \mathrm{i}y$, and an asterisk indicates the complex conjugate (Coleman 1980). Integrating (1.2.19) we find

$$\chi(z) = z^*\chi_1(z) + \chi_2(z) \tag{1.2.20}$$

where χ_1 and χ_2 are two arbitrary analytic functions of z. Different selections for χ_1 and χ_2 produce various types of two-dimensional flow (problem 1.2.3).

Next, we switch to cylindrical polar coordinates (x, σ, ϕ) and consider an axisymmetric flow in which neither the velocity nor the pressure depend on the azimuthal angle ϕ. To facilitate the analysis, it is convenient to introduce the Stokes stream function ψ defined by the equation

$$\mathbf{u} = \nabla \times \left(0, 0, \frac{\psi}{\sigma}\right) \tag{1.2.21}$$

It will be noted that the continuity equation is satisfied for any choice of ψ. The x and σ components of the velocity are given explicitly by

$$u_x = \frac{1}{\sigma}\frac{\partial \psi}{\partial \sigma} \qquad u_\sigma = -\frac{1}{\sigma}\frac{\partial \psi}{\partial x} \tag{1.2.22}$$

Switching temporarily to spherical polar coordinates (r, θ, ϕ), we find that the radial and meridional components of the velocity are given by

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \qquad u_\theta = -\frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r} \tag{1.2.23}$$

where $\sigma = r\sin\theta$ and $x = r\cos\theta$. The vorticity is equal to

$$\boldsymbol{\omega} = -\frac{1}{\sigma}E^2\psi\,\mathbf{e}_\phi \tag{1.2.24}$$

where $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction, and E^2 is a differential operator defined as

$$E^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \sigma^2} - \frac{1}{\sigma}\frac{\partial}{\partial \sigma} = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right) = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} \tag{1.2.25}$$

Noting that the vorticity is a harmonic function, we deduce that ψ satisfies the fouth-order differential equation

$$\mathrm{E}^4\psi = 0 \tag{1.2.26}$$

Turning our attention next to axisymmetric swirling flow (such as that produced by the axial rotation of a prolate spheroid), we find it convenient to introduce the *swirl* $\Omega(x, \sigma)$, defined by the equation

$$u_\phi = \frac{\Omega}{\sigma} \tag{1.2.27}$$

The vorticity associated with the swirling flow is given by

$$\boldsymbol{\omega} = \frac{1}{\sigma}\frac{\partial \Omega}{\partial \sigma}\mathbf{i} \tag{1.2.28}$$

where $\mathbf{i}$ is the unit vector in the x direction, and the pressure is constant. Substituting (1.2.27) into the Stokes equation, we find that the swirl satisfies the second-order differential equation

$$E^2\Omega = 0 \tag{1.2.29}$$

In an alternative formulation, we express a swirling flow in terms of a single scalar function χ defined by the equations

$$\sigma_{x\phi} = \mu \frac{1}{\sigma^2} \frac{\partial \chi}{\partial \sigma} \qquad \sigma_{\sigma\phi} = -\mu \frac{1}{\sigma^2} \frac{\partial \chi}{\partial x} \tag{1.2.30}$$

(Love 1944, p. 325). It will be noted that the Stokes equation is satisfied for any choice of χ. The reader may verify that χ is constant along a line of vanishing surface force. Recalling that

$$\sigma_{x\phi} = \mu \frac{\partial u_\phi}{\partial x} \qquad \sigma_{\sigma\phi} = \mu\sigma \frac{\partial}{\partial \sigma}\left(\frac{u_\phi}{\sigma}\right) \tag{1.2.31}$$

and using (1.2.30) we find that χ satisfies the differential equation

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial \sigma^2} - \frac{3}{\sigma}\frac{\partial \chi}{\partial \sigma} = 0 \tag{1.2.32}$$

Furthermore, using (1.2.32) we find that the function $\Phi = \chi \cos 2\phi/\sigma^2$ satisfies the three-dimensional Laplace equation

$$\nabla^2 \Phi = 0. \tag{1.2.33}$$

Problems

1.2.1 Show that the integral of the stress tensor over a volume V that encloses pure fluid is

$$\int_V \sigma_{ij}\,\mathrm{d}V = \int_D \sigma_{ik} n_k x_j\,\mathrm{d}S - \int_V \frac{\partial \sigma_{ik}}{\partial x_k} x_j\,\mathrm{d}V$$

where D is the boundary of V and $\mathbf{n}$ is the normal vector pointing outside V.

1.2.2 Establish the validity of the following reciprocal identity

$$\int_D \left(u_i' \frac{\partial u_k}{\partial x_i} - u_i \frac{\partial u_k'}{\partial x_i}\right) n_k\,\mathrm{d}S = 0$$

where $\mathbf{u}$ and $\mathbf{u}'$ are two incompressible vector fields and D is the boundary of an arbitrary volume of fluid V.

1.2.3 Show that setting

$$\chi = \tfrac{1}{2}\mu[z^2(z^* - \tfrac{1}{3}z) - 4z]$$

in (1.2.20) produces two-dimensional Poiseuille flow through a slot of unit half-width (Coleman 1980).

1.2.4 Show that the derivative of the function χ normal to a line C that rotates as a rigid body, i.e. $u_\phi = \Omega\sigma$ over C where Ω is the angular velocity of rotation, is equal to zero.

1.2.5 Prove (1.2.33).

1.3 Reversibility of Stokes flow

Let us assume that $\mathbf{u}$ and P form a pair of velocity and pressure fields that satisfy the equations of Stokes flow. Clearly, $-\mathbf{u}$ and $-P$ satisfy the equations of Stokes flow as well, thereby implying that reversed flow is a mathematically acceptable and physically viable solution. It should be noted that the direction of the force and torque acting on any surface are also reversed when the signs of $\mathbf{u}$ and P are switched. The property of reversibility is not shared by flow at finite Reynolds numbers for in that case the non-linear term $\mathbf{u}\cdot\nabla\mathbf{u}$ maintains its sign when the sign of the velocity is reversed.

The reversibility of Stokes flow may be invoked to derive a number of interesting and useful results. Consider, for instance, a solid sphere moving under the action of a shear flow in the vicinity of a plane wall. In principle, the hydrodynamic force acting on the sphere may have a component perpendicular to the wall and a component parallel to the wall. Let us assume for a moment that the component of the force perpendicular to the wall pushes the sphere away from the wall. Reversing the direction of the shear flow must reverse the direction of this force and thus must push the sphere towards the wall. Such an anisotropy, however, is physically unacceptable in view of the fore-and-aft symmetry of the domain of flow. We must conclude that the normal component of the force on the sphere is equal to zero, implying that the sphere must keep moving parallel to the wall.

Using the concept of reversibility we may infer that the streamline pattern around an axisymmetric and fore-and-aft symmetric object that moves along its axis must also be axisymmetric and fore-and-aft symmetric. The streamline pattern over a two-dimensional rectangular cavity must be symmetric with respect to the mid-plane of the cavity. A neutrally buoyant spherical particle that is convected by a parabolic flow in a cylindrical pipe may not move towards the center of the pipe or the wall, but must maintain its initial radial position. As an example of a more subtle situation, consider a buoyant drop, in the shape of a ring, rising under the action of gravity in an infinite ambient fluid. Reversibility requires that if the diameter of the ring is increasing, the cross-sectional shape of the ring may not have a fore-and-aft symmetry (Kojima, Hinch & Acrivos 1984).

Problem

1.3.1 Show that the force on a solid sphere that is rotating in the vicinity of a plane wall does not have a component perpendicular to the wall.

1.4 The reciprocal identity

Let us assume that $\mathbf{u}$ and $\mathbf{u}'$ are two solutions of the equations of Stokes flow with associated stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ respectively, and compute

$$u_i'\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i'\sigma_{ij}) - \sigma_{ij}\frac{\partial u_i'}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i'\sigma_{ij}) - \left[-P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]\frac{\partial u_i'}{\partial x_j}$$
$$= \frac{\partial}{\partial x_j}(u_i'\sigma_{ij}) - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i'}{\partial x_j} \qquad (1.4.1)$$

Note that we have used the continuity equation to eliminate the pressure. Interchanging the roles of $\mathbf{u}$ and $\mathbf{u}'$ we obtain

$$u_i\frac{\partial \sigma'_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i\sigma'_{ij}) - \mu\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} \qquad (1.4.2)$$

Subtracting (1.4.2) from (1.4.1) we find

$$\frac{\partial}{\partial x_j}(u_i'\sigma_{ij} - u_i\sigma'_{ij}) = u_i'\frac{\partial \sigma_{ij}}{\partial x_j} - u_i\frac{\partial \sigma'_{ij}}{\partial x_j} \qquad (1.4.3)$$

If the flows u and u' are regular, i.e. they contain no singular points, the right-hand side of (1.4.3) vanishes, yielding the reciprocal identity

$$\nabla\cdot(\mathbf{u}'\cdot\boldsymbol{\sigma} - \mathbf{u}\cdot\boldsymbol{\sigma}') = 0 \qquad (1.4.4)$$

due to Lorentz (1907). In problems 1.4.1 and 1.4.2 we shall discuss a generalization of the reciprocal identity for fluids with different viscosities, and an alternative expression in terms of the velocity and the pressure (Happel & Brenner 1973, pp. 80, 85). To place the reciprocal identity into a more general perspective, it will be useful to note that (1.4.4) is the counterpart of Green's second identity in the theory of potential flow (Kellogg 1954, p. 215), and Betti's formula in the theory of linear elastostatics (Love 1944, p. 173).

A useful form of the reciprocal identity emerges by integrating (1.4.4) over a volume of fluid V that is bounded by the closed surface D and then using the divergence theorem to convert the volume integral into a surface integral over D. In this manner we obtain

$$\int_D \mathbf{u}'\cdot\mathbf{f}\,dS = \int_D \mathbf{f}'\cdot\mathbf{u}\,dS \qquad (1.4.5)$$

where $\mathbf{f} = \boldsymbol{\sigma}\cdot\mathbf{n}$ and $\mathbf{f}' = \boldsymbol{\sigma}'\cdot\mathbf{n}$ are the surface forces exerted on D, and $\mathbf{n}$ is the unit normal vector pointing outside V.

The major strength of the reciprocal identity is that it allows us to obtain information about a flow without having to solve the equations of motion explicitly, but merely by using information about another flow.

To illustrate the resulting simplifications, we proceed now to discuss several applications in the field of particulate flows.

Let us consider a solid particle that is held stationary in an infinite incident ambient flow $\mathbf{u}^\infty$. The presence of the particle causes a disturbance flow $\mathbf{u}^D$ which is added to the ambient flow to give the total flow $\mathbf{u} = \mathbf{u}^\infty + \mathbf{u}^D$. Turning to the reciprocal theorem, we identify $\mathbf{u}'$ with the velocity produced when the particle translates with velocity $\mathbf{V}$. Exploiting the linearity of the Stokes equation we write the corresponding surface force exerted on the particle in the form

$$\mathbf{f}^T = -\mu \mathcal{G}^T \cdot \mathbf{V} \tag{1.4.6}$$

where $\mathcal{G}^T$ is the *translational surface force resistance matrix*, and the superscript T indicates translation. We select a control volume V that is enclosed by the surface of the particle S_P and by a surface S_∞ of large radius, and apply (1.4.5) for the pair $\mathbf{u}^T$ and $\mathbf{u}^D$ obtaining

$$\int_{S_\infty, S_P} \mathbf{u}^T \cdot \mathbf{f}^D \, dS = \int_{S_\infty, S_P} \mathbf{u}^D \cdot \mathbf{f}^T \, dS \tag{1.4.7}$$

Letting the radius of S_∞ tend to infinity we find that the surface integrals over S_∞ vanish, for the velocity at infinity decays at least as fast as the inverse of the distance r from the particle, and the surface force decays at least as fast as r^{-2} (see discussion of unbounded flow in section 2.3). Equation (1.4.7) then reduces to

$$\mathbf{V} \cdot \mathbf{F}^D = \int_{S_P} \mathbf{u}^D \cdot \mathbf{f}^T \, dS \tag{1.4.8}$$

where $\mathbf{F}^D$ is the disturbance force exerted on the particle, defined as

$$\mathbf{F}^D = \int_{S_P} \mathbf{f}^D \, dS \tag{1.4.9}$$

Applying the boundary condition $\mathbf{u} = 0$ or $\mathbf{u}^D = -\mathbf{u}^\infty$ on S_P, substituting (1.4.6) into (1.4.8), and noting that in the absence of a body force the disturbance force $\mathbf{F}^D$ is equal to the total force $\mathbf{F}$, we finally obtain

$$\mathbf{F} = \mu \int_{S_P} \mathbf{u}^\infty \cdot \mathcal{G}^T dS \tag{1.4.10}$$

(Brenner 1964b). Equation (1.4.10) provides us with an expression for the force simply in terms of the values of the incident velocity $\mathbf{u}^\infty$ over the surface of the particle and the surface force resistance matrix for translation. It will be noted that if the resistance matrix $\mathcal{G}^T$ happens to be constant, as is the case for a spherical particle, the force on the particle is simply proportional to the average value of the incident velocity over the surface of the particle.