Free Boundary Problems and Asymptotic Behavior of Singularly Perturbed Partial Differential Equations

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Chapter 2
Uniqueness, Stability and Uniform Lipschitz Estimates

Abstract In this chapter, we first prove the uniqueness of solutions to the Dirichlet boundary value problem (1.4) by the sub- and super-solution method. In Sect. 2.2, we use the same method to prove the stability of solutions to the corresponding parabolic initial-boundary value problem. Finally, by the same idea, we prove the uniform Lipschitz estimates for solutions to these two problems, under suitable boundary conditions.

2.1 A Uniqueness Result for the Elliptic System

In this section, we prove the uniqueness of solutions to the following Dirichlet boundary value problem.

\[
\begin{aligned}
\Delta u_i &= \kappa u_i \sum_{j \neq i} b_{ij} u_j, \quad \text{in } \Omega, \\
u_i &= \varphi_i, \quad \text{on } \partial \Omega.
\end{aligned}
\] (2.1)

Here \(b_{ij} \geq 0\) are constants, satisfying \(b_{ij} = b_{ji}\). \(\varphi_i\) are given nonnegative Lipschitz continuous functions on \(\partial \Omega\). We prove the following theorem.

**Theorem 2.1.1** For any \(\kappa \geq 0\), there exists a unique solution to the problem (2.1).

We use the following iteration scheme to prove the uniqueness of solutions for (2.1). First, we know the following harmonic extension is possible:

\[
\begin{aligned}
\Delta u_{i,0} &= 0, \quad \text{in } \Omega, \\
u_{i,0} &= \varphi_i, \quad \text{on } \partial \Omega,
\end{aligned}
\] (2.2)

that is, this equation has a unique positive solution \(u_{i,0} \in C^2(\Omega) \cap C^0(\bar{\Omega})\) by Theorem 4.3 of [25].

Then the iteration can be defined as:

\[
\begin{aligned}
\Delta u_{i,m+1} &= \kappa u_{i,m+1} \sum_{j \neq i} u_{j,m}, \quad \text{in } \Omega, \\
u_{i,m+1} &= \varphi_i \quad \text{on } \partial \Omega.
\end{aligned}
\] (2.3)
This is a linear equation. It satisfies the maximum principle, so the existence and uniqueness of the solution \( u_{i,m+1} \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) is clear (cf. Theorem 6.13 in [25]).

Now concerning these \( u_{i,m} \), we have the following result.

**Proposition 2.1.2** In \( \Omega \)

\[
\begin{align*}
   u_{i,0}(x) > u_{i,2}(x) > \cdots > u_{i,2m}(x) > \cdots > u_{i,2m+1}(x) > \cdots > u_{i,3}(x) > u_{i,1}(x).
\end{align*}
\]

**Proof** We divide the proof into several claims.

**Claim 1** \( \forall i, m, u_{i,m} > 0 \) in \( \Omega \).

Because \( \sum_{j \neq i} u_{j,0} > 0 \) in \( \Omega \), the equation (2.3) satisfies the maximum principle. Because the boundary value \( \phi_i \geq 0 \), \( u_{i,1} > 0 \) in \( \Omega \). By induction, we see the claim holds true for all \( u_{i,m} \).

**Claim 2** \( u_{i,1} < u_{i,0} \) in \( \Omega \).

From the equation, now we have

\[
\begin{align*}
   &\begin{cases}
      \Delta u_{i,1} \geq 0, & \text{in } \Omega, \\
      u_{i,1} = u_{i,0}, & \text{on } \partial \Omega,
   \end{cases}
\end{align*}
\]

so we get \( u_{i,1} < u_{i,0} \) by the comparison principle.

In the following, we assume the conclusion of the proposition is valid until \( 2m + 1 \), that is in \( \Omega \)

\[
\begin{align*}
   u_{i,0} > \cdots > u_{i,2m} > u_{i,2m+1} > u_{i,2m-1} > \cdots > u_{i,1}.
\end{align*}
\]

Then we have the following.

**Claim 3** \( u_{i,2m+1} \leq u_{i,2m+2} \).

By (2.3), we have

\[
\begin{align*}
   \Delta u_{i,2m+2} &\leq \kappa u_{i,2m+2} \sum_{j \neq i} u_{j,2m}. \quad (2.4) \\
   \Delta u_{i,2m+1} &\leq \kappa u_{i,2m+1} \sum_{j \neq i} u_{j,2m}. \quad (2.5)
\end{align*}
\]

Because \( u_{i,2m+1} \) and \( u_{i,2m+2} \) have the same boundary value, comparing (2.4) and (2.5), by the comparison principle again we obtain that \( u_{i,2m+1} \leq u_{i,2m+2} \).

**Claim 4** \( u_{i,2m+2} \leq u_{i,2m} \).
This can be seen by comparing the equations they satisfy:

\[
\begin{align*}
\Delta u_{i,2m+2} &= \kappa u_{i,2m+2} \sum_{j \neq i} u_{j,2m+1}, \\
\Delta u_{i,2m+1} &= \kappa u_{i,2m+1} \sum_{j \neq i} u_{j,2m}. 
\end{align*}
\]

By assumption, we have \( u_{j,2m+1} \geq u_{j,2m-1} \), so the claim follows from the comparison principle again.

**Claim 5** \( u_{i,2m+3} \geq u_{i,2m+1} \).

This can be seen by comparing the equations they satisfy:

\[
\begin{align*}
\Delta u_{i,2m+3} &= \kappa u_{i,2m+3} \sum_{j \neq i} u_{j,2m+2}, \\
\Delta u_{i,2m+1} &= \kappa u_{i,2m+1} \sum_{j \neq i} u_{j,2m}. 
\end{align*}
\]

By Claim 4, we have \( u_{j,2m} \geq u_{j,2m+2} \), so the claim follows from the comparison principle again.

Now we know that there exist two family of functions \( u_i \) and \( v_i \), such that \( \lim_{m \to \infty} u_{j,2m}(x) = u_j(x) \) and \( \lim_{m \to \infty} u_{j,2m+1}(x) = v_j(x) \), \( \forall x \in \Omega \). Moreover, by standard elliptic estimates, we know this convergence is smooth in \( \Omega \) and uniformly on \( \overline{\Omega} \). So by taking the limit in (2.3), we obtain the following equations:

\[
\begin{align*}
\Delta u_i &= \kappa u_i \sum_{j \neq i} v_j, \\
\Delta v_i &= \kappa v_i \sum_{j \neq i} u_j.
\end{align*}
\]  

(2.6)

Because \( u_{i,2m+1} \leq u_{j,2m} \), by taking limit, we also have

\[
v_i \leq u_i.
\]  

(2.7)

Now summing (2.6), we have

\[
\begin{align*}
\Delta \left( \sum_i u_i \right) &= \kappa \sum_i \left( u_i \sum_{j \neq i} v_j \right), \\
\Delta \left( \sum_i v_i \right) &= \kappa \sum_i \left( v_i \sum_{j \neq i} u_j \right).
\end{align*}
\]  

(2.8)

It is easily seen that

\[
\sum_i \left( u_i \sum_{j \neq i} v_j \right) = \sum_i v_i \left( \sum_{j \neq i} u_j \right),
\]
so we must have \( \sum_i u_i \equiv \sum_i v_i \) because they have the same boundary value. This means, by (2.7), \( u_i \equiv v_i \in C^2(\Omega) \cap C^0(\overline{\Omega}) \). In particular, they satisfy (2.1). This proves the existence part of Theorem 2.1.1. \( \square \)

**Proposition 2.1.3** If there exist another positive solution \( w_i \) of (2.1), we must have \( u_i \equiv w_i \).

**Proof** We will prove \( u_{i,2m} \geq w_i \geq u_{i,2m+1}, \forall m \), and then the proposition follows immediately. We divide the proof into several claims.

**Claim 1** \( w_i \leq u_{i,0} \).

This is because

\[
\begin{cases}
\Delta w_i \geq 0, & \text{in } \Omega, \\
w_i = u_{i,0}, & \text{on } \partial \Omega.
\end{cases}
\]

**Claim 2** \( w_i \geq u_{i,1} \).

This is because

\[
\begin{cases}
\Delta w_i = \kappa w_i \sum_{j \neq i} w_j, \\
\Delta u_{i,1} = \kappa u_{i,1} \sum_{j \neq i} u_j, 0.
\end{cases}
\]

Noting that we have \( w_j < u_{j,0} \), so we can apply the comparison principle to get the claim.

In the following, we assume that our claim is valid until \( 2m + 1 \), that is

\[ u_{i,2m} \geq w_i \geq u_{i,2m+1}. \]

Then we have the following.

**Claim 3** \( u_{i,2m+2} \geq w_i \).

This can be seen by comparing the equations they satisfy:

\[
\begin{cases}
\Delta w_i = \kappa w_i \sum_{j \neq i} w_j, \\
\Delta u_{i,2m+2} = \kappa u_{i,2m+3} \sum_{j \neq i} u_{j,2m+1}.
\end{cases}
\]

By assumption, we have \( u_{j,2m+1} \leq w_j \), so the claim follows from the comparison principle again.

**Claim 4** \( u_{i,2m+3} \leq w_i \).
This can be seen by comparing the equations they satisfy:

\[
\begin{cases}
\Delta w_i = \kappa w_i \sum_{j \neq i} w_j, \\
\Delta u_{i,2m+3} = \kappa u_{i,2m+3} \sum_{j \neq i} u_{j,2m+2}.
\end{cases}
\]

By Claim 3, we have \( u_{j,2m+2} \geq w_j \), so the claim follows from the comparison principle again. \[\square\]

**Remark 2.1.4** From our proof, we know that the uniqueness result still holds for equations of more general form:

\[
\begin{cases}
\Delta u_i = u_i \sum_{j \neq i} b_{ij}(x) u_j, & \text{in } \Omega \\
u_i = \varphi_i & \text{on } \partial \Omega,
\end{cases}
\]

where \( b_{ij}(x) \) are positive (and smooth enough) functions defined in \( \overline{\Omega} \), which satisfy \( b_{ij} \equiv b_{ji} \).

### 2.2 Asymptotics in the Parabolic Case

The method in the previous section can also be used to prove the stability of solutions to the following parabolic initial-boundary value problem.

\[
\begin{cases}
\frac{\partial u_i}{\partial t} - \Delta u_i = -\kappa u_i \sum_{j \neq i} b_{ij} u_j, & \text{in } \Omega \times (0, +\infty), \\
u_i = \varphi_i, & \text{on } \partial \Omega \times (0, +\infty), \\
u_i = \phi_i, & \text{on } \Omega \times \{0\}.
\end{cases}
\]

Here \( b_{ij} > 0 \) and \( \varphi_i \) are those given in Theorem 2.1.1. \( \phi_i \) are given nonnegative Lipschitz continuous functions in \( \Omega \), such that \( \phi_i = \varphi_i \) on \( \partial \Omega \). We prove the following theorem.

**Theorem 2.2.1** For any \( \kappa \geq 0 \), there exists a unique global solution \( U \) of (2.9). As \( t \to +\infty \), \( U(t) \) converges to the solution of (2.1) in \( C(\overline{\Omega}) \).

**Proof** Let us consider the iteration scheme analogous to (2.3). First, we consider

\[
\begin{cases}
\frac{\partial u_{i,0}}{\partial t} - \Delta u_{i,0} = 0, & \text{in } \Omega \times (0, +\infty), \\
u_{i,0} = \varphi_i & \text{on } \partial \Omega \times (0, +\infty), \\
u_{i,0} = \phi_i & \text{on } \Omega \times \{0\}.
\end{cases}
\]
We know this equation has a unique positive solution $u_{i,0}(x,t)$. We also have

$$
\lim_{t \to +\infty} u_{i,0}(x,t) = u_{i,0}(x),
$$

where the convergence is (for example), in the space of $C^0(\overline{\Omega})$ and $u_{i,0}(x)$ is the solution of (2.2). In fact, we can prove that

$$
\int_{\Omega} \left| \frac{\partial u_{i,0}}{\partial t} \right|^2 dx \leq C_1 e^{-C_2 t}
$$

for some positive constants $C_1$ and $C_2$.

Now the iteration can be defined as:

$$
\begin{cases}
\frac{\partial u_{i,m+1}}{\partial t} + \frac{\Delta u_{i,m+1}}{\partial t} = -\kappa u_{i,m+1} \sum_{j \neq i} u_{j,m}, & \text{in } \Omega \times (0, +\infty), \\
u_{i,m+1} = \varphi_i & \text{on } \partial\Omega \times (0, +\infty), \\
u_{i,m+1} = \phi_i & \text{on } \Omega \times \{0\}.
\end{cases}
$$

(2.10)

This is just a linear parabolic equation, and there exists a unique global solution $u_{i,m+1}(x,t)$. Differentiating (2.10) in time $t$, we get

$$
\frac{\partial}{\partial t} \frac{\partial u_{i,m+1}}{\partial t} - \frac{\Delta u_{i,m+1}}{\partial t} = -\kappa \frac{\partial u_{i,m+1}}{\partial t} \sum_{j \neq i} u_{j,m} - \kappa u_{i,m+1} \sum_{j \neq i} \frac{\partial u_{j,m}}{\partial t}.
$$

(2.11)

By the induction assumption and maximum principle, we know there exist constants $C'_m$, $C_{m,1}$ and $C_{m,2}$ such that for $t > 1$,

$$
\sum_{j \neq i} u_{j,m+1} \leq C'_m, \quad (2.12)
$$

$$
\int_{\Omega} \left| \frac{\partial u_{i,m}}{\partial t} \right|^2 dx \leq C_{m,1} e^{-C_{m,2} t}. \quad (2.13)
$$

Multiplying (2.11) by $\frac{\partial u_{i,m+1}}{\partial t}$, with the help of (2.12), we get (note that we have the boundary condition $\frac{\partial u_{i,m+1}}{\partial t} = 0$ on $\partial\Omega$)

$$
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \left| \frac{\partial u_{i,m+1}}{\partial t} \right|^2 + \int_{\Omega} \left| \frac{\partial u_{i,m+1}}{\partial t} \right|^2 \leq \kappa C'_m \int_{\Omega} \sum_{j \neq i} \left| \frac{\partial u_{j,m}}{\partial t} \right| \left| \frac{\partial u_{i,m+1}}{\partial t} \right|.
$$
Using Cauchy inequality, we get
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \left| \frac{\partial u_{i,m+1}}{\partial t} \right|^2 + \int_{\Omega} \left| \nabla \frac{\partial u_{i,m+1}}{\partial t} \right|^2 \leq \kappa C'_m \left( \int_{\Omega} \sum_{j \neq i} \left| \frac{\partial u_{j,m}}{\partial t} \right|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} \left| \frac{\partial u_{i,m+1}}{\partial t} \right|^2 \right)^{\frac{1}{2}}.
\]

By (2.13) and the Poincare inequality, we get
\[
\int_{\Omega} \left| \frac{\partial u_{i,m+1}}{\partial t} \right|^2 \, dx \leq C_{m+1,1} e^{-C_{m+1,2} t},
\]
for some positive constants $C_{m+1,1}$ and $C_{m+1,2}$.

By standard parabolic estimate, this also imply
\[
\sup_{\Omega} \left| \frac{\partial u_{i,m+1}}{\partial t} \right| \leq C_{m+1,1} e^{-C_{m+1,2} t},
\]
for another two constants $C_{m+1,1}$ and $C_{m+1,2}$. This implies
\[
\lim_{t \to +\infty} u_{i,m+1}(x,t) = u_{i,m+1}(x),
\]
where $u_{i,m+1}(x)$ is the solution of (2.3). Furthermore, the convergence can be taken (for example) in the space of $C^0(\Omega)$.

The same method of Sect. 2.2 gives, in $\Omega \times (0, +\infty)$
\[
u i,0 > \cdots > u_i,2m > u_i,2m+2 > \cdots > u_i > \cdots > u_i,2m+1 > u_i,2m-1 > \cdots > u_i,1.
\]
Now our Theorem 2.2.1 can be easily seen. In fact, $\forall \varepsilon > 0$, there exists a $m$, such that
\[
\max_{\bar{\Omega}} \left| u_{i,2m}(x) - u_{i}(x) \right| < \varepsilon
\]
and
\[
\max_{\bar{\Omega}} \left| u_{i,2m+1}(x) - u_{i}(x) \right| < \varepsilon.
\]
We also have that there exists a $T > 0$, depending on $m$ only, such that, $\forall t > T$,
\[
\max_{\bar{\Omega}} \left| u_{i,2m}(x,t) - u_{i,2m}(x) \right| < \varepsilon,
\]
and
\[
\max_{\bar{\Omega}} \left| u_{i,2m+1}(x,t) - u_{i,2m+1}(x) \right| < \varepsilon.
\]
Combing these together, we get \( \forall t > T \),

\[
\max_{\overline{\Omega}} \left| u_i(x, t) - u_i(x) \right| < 4\epsilon.
\]

This implies that \( u_i(x, t) \) converges to the solution \( u_i(x) \) of (2.1) as \( t \to +\infty \), uniformly on \( \overline{\Omega} \). (If the boundary values are sufficiently smooth, the convergence in Theorem 2.2.1 can be improved to be smooth enough.) \( \square \)

### 2.3 A Uniform Lipschitz Estimate

Finally, by the same idea as in the previous sections, we prove the uniform Lipschitz estimates for solutions to the above two problems (2.9) and (2.2).

**Theorem 2.3.1** There exists a constant \( C > 0 \) independent of \( \kappa \), such that for any \( \kappa \geq 0 \) and solution \((u_{i,\kappa})\) of (2.1), we have

\[
\sup_{\Omega} |\nabla u_{i,\kappa}| \leq C.
\]

**Theorem 2.3.2** There exists a constant \( C > 0 \) independent of \( \kappa \), such that for any \( \kappa \geq 0 \) and solution \((u_{i,\kappa})\) of (2.9), we have

\[
\sup_{\Omega \times [0, +\infty)} \text{Lip}(u_{i,\kappa}) \leq C.
\]

We will only treat the parabolic case. The elliptic case is similar.

We need an additional assumption on the initial-boundary values here. Let \( \Phi_i \) be the solution of

\[
\begin{aligned}
\frac{\partial \Phi_i}{\partial t} - \Delta \Phi_i &= 0, \quad \text{in } \Omega \times (0, +\infty), \\
\Phi_i &= \varphi_i, \quad \text{on } \partial \Omega \times (0, +\infty), \\
\Phi_i &= \phi_i, \quad \text{on } \Omega \times \{0\}.
\end{aligned}
\]

(2.14)

We assume that \( \Phi_i \) are Lipschitz continuous on the closure of \( \Omega \times (0, +\infty) \). Note that by comparison principle, we have (see [11] for the proof in the elliptic case)

\[
\begin{aligned}
\Phi_i &\geq u_{i,\kappa}, \\
\Phi_i - \sum_{j \neq i} \Phi_j &\leq u_{i,\kappa} - \sum_{j \neq i} u_{j,\kappa}.
\end{aligned}
\]

(2.15)

First differentiating (2.9) in a space direction \( e \) we obtain an equation for \( D_e u := e \cdot \nabla u \):

\[
\left( \frac{\partial}{\partial t} - \Delta \right) D_e u_{i,\kappa} = -\kappa D_e u_{i,\kappa} \sum_{j \neq i} u_{j,\kappa} - \kappa u_{i,\kappa} \sum_{j \neq i} D_e u_{j,\kappa}.
\]
Now using the Kato inequality for smooth functions $\phi$

$$|\nabla|\phi| = |\nabla \phi| \text{ a.e., } |\Delta|\phi| \geq |\Delta \phi|,$$

we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)|D_e u_{i, \kappa}| \leq -\kappa|D_e u_{i, \kappa}| \sum_{j \neq i} u_{j, \kappa} + \kappa u_{i, \kappa} \sum_{j \neq i} |D_e u_{j, \kappa}|.$$

Summing these in $i$, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right)\sum_{i} |D_e u_{i, \kappa}| \leq 0.$$

By the assumption on $\Phi_i$ and (2.15), we have

$$\sup_{\partial \Omega \times (0, +\infty)} \frac{\partial u_{i, \kappa}}{\partial \nu} \leq C,$$

for all $i$, where $\nu$ is the outward unit normal vector and $C$ is independent of $\kappa$. With the assumption of Lipschitz continuity of the boundary values on $\partial \Omega \times (0, +\infty)$, we in fact have

$$\sup_{\partial \Omega \times (0, +\infty)} |\nabla u_{i, \kappa}| \leq C,$$

with a constant $C$ independent of $\kappa$ again. Next, we also have at $t = 0, u_{i, \kappa} = \phi_i$, so

$$\sup_{\Omega \times \{0\}} |\nabla u_{i, \kappa}| = \sup_{\Omega} |\nabla \phi_i|.$$

Now the maximum principle implies a global uniform bound:

$$\sup_{\Omega \times (0, +\infty)} |\nabla u_{i, \kappa}| \leq C.$$

Then by a standard method we can get the uniform Lipschitz bound with respect to the parabolic distance.

**Remark 2.3.3** Without the boundary regularity, we can still get an interior uniform bound. Multiplying the equation by $u_{i, \kappa}$ and integrating by parts, we can get a $L^2$ bound for any $T > 0$

$$\sum_{i} \int_{T}^{T+1} \int_{\Omega} |\nabla u_{i, \kappa}|^2 \leq C,$$

with $C$ independent of $\kappa$ and $T$. Then we can use the mean value property for subcaloric (or subharmonic function) to give a uniform upper bound of $|\nabla u_{i, \kappa}|$. 
Remark 2.3.4 If we consider the original Lotka–Volterra system
\[ \frac{\partial u_i}{\partial t} - \Delta u_i = a_i u_i - u_i^2 - \kappa u_i \sum_{j \neq i} u_j, \]
with homogeneous Dirichlet boundary condition, the above results still hold. In fact, we only need to prove a boundary gradient estimate, which can be guaranteed by the following argument: if we define \( v_i \) to be the solution of
\[ \frac{\partial v_i}{\partial t} - \Delta v_i = a_i v_i - v_i^2, \]
with the same initial value, then by the maximum principle we have for each \( \kappa \)
\[ u_i,\kappa \leq v_i, \]
which, together with the boundary condition, implies
\[ \left| \frac{\partial u_i,\kappa}{\partial \nu} \right| \leq \left| \frac{\partial v_i}{\partial \nu} \right|, \]
where \( \nu \) is the unit outward normal vector to \( \partial \Omega \); using the boundary condition once again we get on the boundary
\[ |\nabla u_i,\kappa| \leq |\nabla v_i|, \]
where the right hand side is independent of \( \kappa \).