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Mathematical toolbox

This book is about mathematical models coming from several fields of science and economics that are described by difference or differential equations. Therefore we begin by presenting basic concepts and tools from the theory of difference and differential equations, which will allow us to understand and analyse these models. The multitude of problems that can be dealt with using so few techniques is a testimony to the unifying power of mathematics.

1.1 Difference equations

Since difference equations are conceptually simpler, we begin with them. The reader should be aware that we present here a bare minimum of results that are necessary to analyse of the examples in this book. A comprehensive theory of difference equations can be found for example in (Elaydi, 2005).

We consider difference equations which can be written in the form

$$x_{n+k} = F(n, x_n, \dots, x_{n+k-1}), \qquad n \in \mathbb{N}_0, \tag{1.1}$$

where $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ is a fixed number and F is a given function of $k+1$ variables. Such an equation is called a difference equation of order k. If F does not explicitly depend on n, then we say that the equation is autonomous. Furthermore, if F depends linearly on $x_n, \dots, x_{n+k-1}$, then we say that (1.1) is a linear equation. Otherwise, we say that it is nonlinear.

If we are given k initial values $x_1, \ldots, x_k$, then the term x_{k+1} is uniquely determined by (1.1) and then all other terms can be found by successive iterations. These terms form a sequence $(x_n)_{n\in\mathbb{N}_0}$ which we call *a solution* to (1.1). Thus the problems of existence and uniqueness of solutions, which play an essential role in the theory of differential equations, are here largely irrelevant. The problem, however, is to find a closed form of the solution; that is, a formula defining the terms of the sequence $(x_n)_{n\in\mathbb{N}_0}$ explicitly in terms of the variable n. While, in general, finding such an explicit solution is impossible, we shall discuss several cases when it can be accomplished. In more difficult situations we have to confine ourselves to qualitative analysis which will be discussed in Chapter 4.

1.1.1 First-order linear difference equations

The general first-order difference equation has the form

$$x_{n+1} = a_n x_n + g_n, \quad n \geq 0, \tag{1.2}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(g_n)_{n\in\mathbb{N}_0}$ are given sequences. It is clear that using (1.2) we may calculate any element x_n provided we know only one initial point, so that we supplement (1.2) with an initial value x_0. It is easy to check, by induction, that the solution is given by

$$x_n = x_0 \prod_{k=0}^{n-1} a_k + \sum_{k=0}^{n-1} g_k \prod_{i=k+1}^{n-1} a_i, \tag{1.3}$$

where we adopt the convention that $\prod\limits_{n}^{n-1} = 1$. Similarly, to simplify notation, we put $\sum\limits_{k=j+1}^{j} = 0$.

Exercise 1.1 Show that if in (1.2) we have $a_n = a$ for all $n \geq 0$, then (1.3) takes the form

$$x_n = a^n x_0 + \sum_{k=0}^{n-1} a^{n-k-1} g_k. \tag{1.4}$$

If, moreover $g_n = g$ for $n \geq 0$, then

$$x_n = \begin{cases} a^n x_0 + g\frac{a^n-1}{a-1} & \text{if} \quad a \neq 1, \\ x_0 + gn & \text{if} \quad a = 1. \end{cases} \tag{1.5}$$

1.1.2 Linear difference equations of higher order

Though the book mainly is concerned with equations of first order, in some examples we will need solutions to higher-order linear equations with constant coefficients; that is, equations of the form

$$x_{n+k} + a_1 x_{n+k-1} + \cdots + a_k x_n = 0, \qquad n \in \mathbb{N}_0, \tag{1.6}$$

where k is a fixed number, called the order of the equation, and $a_1, \ldots, a_k$ are known numbers. This equation determines the values of x_m, $m > k$, by k preceding values. Thus, we need k initial values $x_0, x_1, \ldots, x_{k-1}$ to start iterations. The general theory of such equations requires tools from linear algebra, which are beyond the scope of this book, see (Elaydi, 2005). Therefore we only will present basic results which easily can be checked to hold true in particular examples.

To find the general solution to (1.6), we build the so-called *characteristic equation*

$$\lambda^k + a_1 \lambda^{k-1} + \cdots + a_k = 0. \tag{1.7}$$

If this equation has k distinct roots $\lambda_1, \ldots, \lambda_k$, then the general solution is given by

$$x_n = C_1 \lambda_1^n + \cdots + C_k \lambda_k^n, \quad n \geq k, \tag{1.8}$$

where $C_1, \ldots, C_k$ are constants that are to be determined so that $(y_n)_{n \in \mathbb{N}_0}$ satisfies the initial conditions for $n = 0, \ldots k-1$. If, however, there is a multiple root, say λ_i, of multiplicity n_i, then in the expansion (1.8) we must use n_i terms $\{\lambda_i^n, n\lambda_i^n, \ldots, n^{n_i-1}\lambda_i^n\}$.

1.1.3 Nonlinear equations

As we said earlier, most difference equations cannot be solved explicitly. In some cases, however, a smart substitution could reduce them to a simpler form. In this subsection we present two classes of solvable nonlinear equations, which will be used later. Some other cases are discussed in Section 2.4.

The homogeneous Ricatti equation. Consider the equation

$$x_{n+1}x_n + a_n x_{n+1} + b_n x_n = 0, \quad n \in \mathbb{N}_0, \tag{1.9}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are given sequences with non-zero elements. Then the substitution

$$y_n = \frac{1}{x_n}$$

transforms (1.9) into

$$b_n y_{n+1} + a_n y_n + 1 = 0, \tag{1.10}$$

which is a first-order linear equation. We note that in the above transformation we had to assume $x_n \neq 0$. If, however, $x_n = 0$ for some n, then $x_m = 0$ for $m > n$.

The inhomogeneous Ricatti equation. The inhomogeneous Riccati equation is

$$x_{n+1}x_n + a_n x_{n+1} + b_n x_n = c_n, \quad n \in \mathbb{N}_0, \tag{1.11}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ and $(c_n)_{n\in\mathbb{N}_0}$ are given sequences. Upon the substitution

$$x_n = \frac{y_{n+1}}{y_n} - a_n,$$

it becomes

$$\left(\frac{y_{n+2}}{y_{n+1}} - a_{n+1}\right)\left(\frac{y_{n+1}}{y_n} - a_n\right) + a_n\left(\frac{y_{n+2}}{y_{n+1}} - a_{n+1}\right) + b_n\left(\frac{y_{n+1}}{y_n} - a_n\right) = c_n.$$

Simplifying, we obtain the second-order linear equation

$$y_{n+2} + (b_n - a_{n+1})y_{n+1} - (c_n + a_n b_n)y_n = 0. \tag{1.12}$$

In particular, if the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ and $(c_n)_{n\in\mathbb{N}_0}$ are constant, then the above equation is explicitly solvable by the method described in Section 1.1.2.

1.2 Differential equations – an introduction

The present book is mostly about applying differential equations to concrete models, thus we refer the reader to dedicated texts, such as (Braun, 1983; Glendinning, 1994; Schroers, 2011; Strogatz, 1994), to learn more about the theory of differential equations. However, to make the presentation self-consistent, we provide some basic facts and ideas.

In this book we shall be solely concerned with ordinary differential equations (ODEs) that can be written in the form

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}) = 0, \tag{1.13}$$

where F is a given scalar function of $n+1$ variables and $y^{(k)}$, for $k = 1, \ldots, n$, denotes the derivative of order k with respect to t. For lower order derivatives we will use the more conventional notation $y^{(1)} = y', y^{(2)} = y''$, etc. As with the difference equations, we say that (1.13) is autonomous if F does not depend on t and it is linear if F is linear in $y, y', \ldots, y^{(n-1)}$. The order of the equation is the order of the highest derivative appearing in it.

To solve the ODE (1.13) means to find an n-times continuously differentiable function $y(t)$ such that for any t (from some interval), (1.13) becomes an identity. Thus, if we are given a function y, it is easy to check whether it is a solution of (1.13) or not. However, in contrast to difference equations, finding a solution to (1.13) is a difficult, and often impossible, task. A quick reflection brings to mind three questions relevant to solving a differential equation:

(i) can we be sure that a given equation possesses a solution at all?
(ii) if we know that there is a solution, are there systematic methods for finding it?
(iii) having found a solution, can we be sure that there are no other solutions?

Question (i) is usually referred to as the **existence problem** for differential equations, and Question (iii) as the **uniqueness problem**. Unless we deal with very simple situations, these two questions should be addressed before attempting to find a solution.

After all, what is the point of trying to solve an equation if we do not know whether the solution exists, or whether the one we found is unique. Let us discuss briefly Question (i) first. Roughly speaking, we can come across the following situations:

(a) no function exists which satisfies the equation;
(b) the equation has a solution but no one knows what it looks like;
(c) the equation can be solved in a closed form.

Case (a) is not very common in mathematics and it should never happen in mathematical modelling. Indeed, if a given equation was an exact reflection of a real life phenomenon, then the fact that this phenomenon exists would ensure that this equation can be solved. However, models are imperfect reflections of the reality and therefore it may happen that in the modelling process we missed some crucial facts, rendering the final equation unsolvable. Thus, establishing solvability of the equation constructed in the modelling process serves as an important first step in validating the model. Unfortunately, these problems are usually very difficult and require quite advanced mathematics that is beyond the scope of this course. We shall, however, provide basic theorems pertaining to this question that are sufficient for the discussed problems.

Case (b) may look somewhat enigmatic but, as we said above, there are advanced theorems allowing us to ascertain the existence of solutions without actually displaying them. Actually, many of the most interesting equations appearing in applications do not have known explicit solutions. It is important to realize that even if we do not know a formula for the solution, the fact that one does exist means we can find its numerical or graphical representation to any reasonable accuracy. Also, very often we can find important features of the solution without knowing its explicit formula. These features include e.g., long time behaviour; that is, whether it settles at a certain equilibrium value or oscillates, whether it is monotonic or periodic, etc. These questions will be studied in the final part of the book.

Some examples, when the situation described in (c) occurs and which thus also partially address Question (ii), are discussed in Section 1.3 below.

Having dealt with Questions (i) and (ii) let us move to the problem of uniqueness. Typically (1.13) determines a family of solutions, parametrised by several constants, rather than a single function. Such a class is called the *general solution* of the equation. By imposing an appropriate number of *side conditions* we specify the constants thus obtaining a *special solution* – ideally one member of the class.

A side condition may take all sorts of forms, such as 'at $t = 15$, y must have the value of 0.4' or 'the area under the curve $y = y(t)$ between $t = 0$ and $t = 24$ must be 100'. Very often, however, it specifies the initial value $y(0)$ of the solution and the derivatives $y^{(k)}(0)$ for $k = 1, \ldots, n-1$. In this case the side conditions are called the *initial conditions*. Problems consisting of (1.13) with initial conditions are called *initial value problems* or *Cauchy problems*

1.3 Some equations admitting closed form solutions

In this section we shall provide a brief overview of methods for solving differential equations which will appear in this book. This shows that in some situations the answer to Question (ii) of the previous section is affirmative. It is important to understand, however, that there is a deeper theory behind each method and due caution should be exercised when applying the formulae listed below, see (Braun, 1983; Schroers, 2011; Strogatz, 1994).

1.3.1 Separable equations

Separable equations are equations which can be written as

$$y' = g(t)h(y), \tag{1.14}$$

where g and h are known functions. Constant functions $y \equiv \bar{y}$, such that $h(\bar{y}) = 0$, are solutions to (1.14). They are called *stationary or equilibrium solutions*.

To find the general solution, we assume that $h(y)$ is finite and nowhere zero, and divide both sides of (1.14) by $h(y)$ to get

$$\frac{1}{h(y)}y' = g(t). \tag{1.15}$$

Denoting $H(y) = \int dy/h(y)$, (1.15) can be written as

$$(H(y(t)))' = g(t).$$

Integrating, we obtain the solution in the implicit form,

$$H(y(t)) = \int g(t)dt + c, \tag{1.16}$$

where c is an arbitrary constant. Since, by assumption, $H'(y) = h^{-1}(y) \neq 0$, we can use the inverse function theorem (Courant and John, 1999) to claim that the function H is locally invertible and thus the explicit solution can be found, at least locally, as

$$y(t) = H^{-1}\left(\int g(t)dt + c\right), \tag{1.17}$$

with c depending on the side conditions.

1.3.2 First-order linear differential equations

The general *first-order linear differential equation* is of the form

$$y' + a(t)y = b(t), \tag{1.18}$$

where a and b are known continuous functions of t. One method of solving (1.18) is to multiply both sides of (1.18) by the so-called *integrating factor* μ which is a solution to

$$\mu' = \mu a(t),$$

i.e., $\mu(t) = e^{\int a(t)dt}$. Then

$$\mu(t)y' + \mu(t)a(t)y = \mu(t)b(t)$$

can be written as

$$(\mu(t)y(t))' = \mu(t)b(t),$$

and thus

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)b(t)dt + c\right) \\ &= \exp\left(-\int a(t)dt\right)\left(\int b(t)\exp\left(\int a(t)dt\right)dt + c\right), \end{aligned} \tag{1.19}$$

where c is a constant of integration which is to be determined from the initial conditions. It is worthwhile noting that the solution is the sum of the general solution to the homogeneous equation (that is, with $b(t) \equiv 0$),

$$c\exp\left(-\int a(t)dt\right),$$

and a particular solution to the full equation (1.18).

1.3.3 Equations of homogeneous type

A differential equation that can be written in the form

$$y' = f\left(\frac{y}{t}\right), \tag{1.20}$$

where f is a function of the single variable $z = y/t$ is said to be of *homogeneous type*. To solve (1.20), let us make the substitution

$$y = tz, \tag{1.21}$$

where z is the new unknown function. Then, by the product rule,

$$y' = z + tz'$$

and (1.20) becomes

$$tz' = f(z) - z. \tag{1.22}$$

Equation (1.22) is a separable equation and so it can be solved as in Section 1.3.1.

1.3.4 Equations that can be reduced to first-order equations

Some higher-order equations can be reduced to first-order equations. We shall discuss two such cases for second-order equations.

Equations that do not contain the unknown function. If we have an equation of the form

$$F(t, y', y'') = 0, \tag{1.23}$$

then the substitution $z = y'$ reduces this equation to the first-order equation

$$F(t, z, z') = 0. \tag{1.24}$$

If

$$z = \phi(t, C)$$

is the general solution to (1.24), where C is an arbitrary constant, then y is the solution of

$$y' = \phi(t, C),$$

so that

$$y(t) = \int \phi(t, C)dt + C_1.$$

Equations that do not contain the independent variable. Let us consider the equation

$$F(y, y', y'') = 0, \tag{1.25}$$

that does not involve the independent variable t. Such an equation also can be reduced to a first-order equation as long as $y' \neq 0$; that is, if there are no turning points of the solution. Then the derivative y' locally is a function of y; that is, we can write $y' = g(y)$ for some function g. Indeed, by the inverse function theorem, see (Courant and John, 1999), the function $y = y(t)$ is locally invertible provided $y' \neq 0$ and, writing $t = t(y)$, we can define $g(y) = y'(t(y))$. Using the chain rule we obtain

$$y'' = \frac{d}{dt}y' = \frac{dg}{dy}\frac{dy}{dt} = y'\frac{dg}{dy} = g(y)\frac{dg}{dy}. \tag{1.26}$$

Substituting (1.26) into (1.25) gives a first-order equation with y as an independent variable,

$$F\left(y, g, g\frac{dg}{dy}\right) = 0. \tag{1.27}$$