Chapter 2
Quantized H_∞ Control for Time-Delay Systems with Missing Measurements

In this chapter, we aim to investigate the quantized H_∞ control problem for a class of nonlinear stochastic time-delay network-based systems with probabilistic data missing. A nonlinear stochastic system with state delays is employed to model the networked control systems where the measured output and input signals are quantized by two logarithmic quantizers, respectively. Moreover, the data missing phenomena are modeled by introducing a diagonal matrix composed of Bernoulli-distributed stochastic variables taking values 1 and 0, which describes that the data from different sensors may be lost with different missing probabilities. Subsequently, a sufficient condition is first derived in virtue of the method of sector-bounded uncertainties, which guarantees that the closed-loop system is stochastically stable and the controlled output satisfies H_∞ performance constraint for all nonzero exogenous disturbances under the zero initial condition. Then, the sufficient condition is decoupled into some inequalities for the convenience of practical verification. Based on that, quantized H_∞ controllers are designed successfully for some special classes of nonlinear stochastic time-delay systems by using Matlab LMI toolbox. Finally, some numerical simulation examples are exploited to show the effectiveness and applicability of the results derived.

2.1 Problem Formulation

Consider the networked nonlinear stochastic control system with two quantizers shown in Fig. 2.1.

The plant under consideration is assumed to be of the following form:

$$\begin{cases} x_{k+1} = f_1(x_k, x_{k-d}) + h_1(x_k)v_k + g_1(x_k)u_k + f_w(x_k, x_{k-d})w_k, \\ z_k = f_2(x_k, x_{k-d}) + h_2(x_k)v_k + g_2(x_k)u_k, \\ x_k = \varphi_k, \quad k = -d, -d+1, \dots, 0, \end{cases} \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^p$ is the control input, $z_k \in \mathbb{R}^l$ is the controlled output, w_k is a one-dimensional, zero-mean Gaussian white noise sequence

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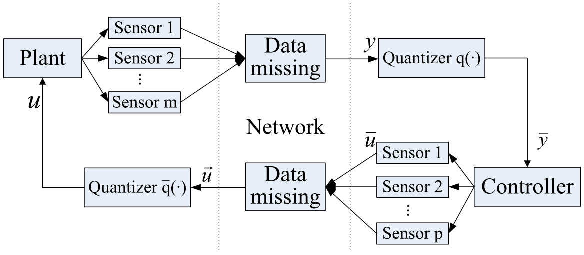
Fig. 2.1 Structure of a networked control system with two quantizers



on a probability space $(\Omega, \mathcal{F}, \text{Prob})$ with $\mathbb{E}w_k^2 = \theta$, and v_k is the exogenous disturbance input belonging to $l_2([0, \infty), \mathbb{R}^q)$.

The nonlinear functions $f_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$, $f_w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $h_1 : \mathbb{R}^n \to \mathbb{R}^{n\times q}$, $h_2 : \mathbb{R}^n \to \mathbb{R}^{l\times q}$, $g_1 : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, and $g_2 : \mathbb{R}^n \to \mathbb{R}^{l\times p}$ are smooth matrix-valued functions with $f_1(0,0) = 0$, $f_2(0,0) = 0$, and $f_w(0,0) = 0$. φ_k is a real-valued initial function on $[-d, 0]$.

The measurement with probabilistic sensor data missing is described as

$$y_k = \Gamma_k l(x_k) + k(x_k)v_k, \tag{2.2}$$

where $y_k \in \mathbb{R}^m$ is the measurement received at the node quantizer $q(\cdot)$. The nonlinear functions $l : \mathbb{R}^n \to \mathbb{R}^m$ and $k : \mathbb{R}^n \to \mathbb{R}^{m\times q}$ are also smooth matrix-valued functions with $l(0) = 0$. $\Gamma_k = \text{diag}\{\gamma_k^1, \ldots, \gamma_k^m\}$ is a diagonal matrix that accounts for the different missing rate of the individual channel. For any $1 \le r \le m$, γ_k^r is a Bernoulli-distributed stochastic variable taking values 1 and 0 with

$$\begin{aligned} &\text{Prob}\{\gamma_k^r = 1\} = \bar{\gamma}^r, \\ &\text{Prob}\{\gamma_k^r = 0\} = 1 - \bar{\gamma}^r, \end{aligned} \tag{2.3}$$

where $\bar{\gamma}^r \in [0, 1]$ is a known constant.

As shown in Fig. 2.1, before entering into the controller, the signal $y_k \in \mathbb{R}^m$ is quantized by the quantizer $q(\cdot)$ defined as

$$\bar{y}_k = q(y_k) = \begin{bmatrix} q_1(y_k^{(1)}) & q_2(y_k^{(2)}) & \cdots & q_m(y_k^{(m)}) \end{bmatrix}^T,$$

where $\bar{y}_k \in \mathbb{R}^m$ is the signal transmitted into the controller after the quantization. In this chapter, the quantizer $q(\cdot)$ is assumed to be of the logarithmic type. That is, for each $q_j(\cdot)$ $(1 \le j \le m)$, the set of quantization levels is described by

$$\begin{aligned} &\mathcal{U}_j = \left\{\pm\chi_i^{(j)}, \chi_i^{(j)} = \rho_j^i \chi_0^{(j)},\ i = 0,\ \pm1,\ \pm2,\ \ldots\right\} \cup \{0\}, \\ &0 < \rho_j < 1,\ \chi_0^{(j)} > 0. \end{aligned}$$

Each of the quantization level corresponds to a segment such that the quantizer maps the whole segment to this quantization level. The logarithmic quantizer $q_j(\cdot)$

is defined as

$$q_j\big(y_k^{(j)}\big)=\begin{cases}\chi_i^{(j)}, & \frac{1}{1+\delta_j}\chi_i^{(j)}<y_k^{(j)}\le\frac{1}{1-\delta_j}\chi_i^{(j)},\\ 0, & y_k^{(j)}=0,\\ -q_j(-y_k^{(j)}), & y_k^{(j)}<0,\end{cases}$$

with $\delta_j=(1-\rho_j)/(1+\rho_j)$.

By the results derived in [47], it follows that $q_j(y_k^{(j)})=(1+\Delta_k^{(j)})y_k^{(j)}$ such that $|\Delta_k^{(j)}|\le\delta_j$. Defining $\Delta_k=\mathrm{diag}\{\Delta_k^{(1)},\ldots,\Delta_k^{(m)}\}$, the measurements after quantization can be expressed as

$$\bar{y}_k=(I+\Delta_k)y_k. \tag{2.4}$$

Therefore, the quantizing effects have been transformed into sector bound uncertainties described above.

The dynamic observer-based control scheme for the plant (2.1) is described by

$$\begin{cases}\hat{x}_{k+1}=f_c(\hat{x}_k)+g_c(\hat{x}_k)\bar{y}_k,\\ \bar{u}_k=u_c(\hat{x}_k), \qquad f_c(0)=0, \qquad u_c(0)=0,\\ \hat{x}_k=0, \quad k=-d,-d+1,\ldots,0,\end{cases} \tag{2.5}$$

where $\hat{x}_k\in\mathbb{R}^n$ is the state estimate of the plant (2.1), $\bar{u}_k\in\mathbb{R}^p$ is the control input without transmission missing, and the matrix-valued nonlinear functions $f_c:\mathbb{R}^n\to\mathbb{R}^n$, $g_c:\mathbb{R}^n\to\mathbb{R}^{n\times m}$, $u_c:\mathbb{R}^n\to\mathbb{R}^p$ are controller parameters to be determined.

When the control signal $\bar{u}_k$ is transmitted on the network from the controller to the quantizer $\bar{q}(\cdot)$, the data missing phenomenon will probably occur again owing to the limited bandwidth of the communication channel. Therefore, the data missing model is applied to $\bar{u}_k$ again. Here, we introduce another diagonal matrix $\Xi_k=\mathrm{diag}\{\xi_k^1,\ldots,\xi_k^p\}$ where ξ_k^r is also assumed to be a Bernoulli-distributed stochastic variable satisfying

$$\begin{aligned}&\mathrm{Prob}\big\{\xi_k^r=1\big\}=\bar{\xi}^r,\\ &\mathrm{Prob}\big\{\xi_k^r=0\big\}=1-\bar{\xi}^r.\end{aligned} \tag{2.6}$$

Then, the control input with data missing $\mathbf{u}_k\in\mathbb{R}^p$ can be described as

$$\mathbf{u}_k=\Xi_k\bar{u}_k. \tag{2.7}$$

Similar to the signal $y_k\in\mathbb{R}^m$, the control signal $\mathbf{u}_k\in\mathbb{R}^p$ is also quantized by the quantizer $\bar{q}(\cdot)$ before entering the plant (2.1). Here, the quantizer $\bar{q}(\cdot)$ is also assumed to be of the logarithmic type and has the same form as the quantizer $q(\cdot)$. Specifically, the quantizer $\bar{q}(\cdot)$ is defined as

$$u_k=\bar{q}(\mathbf{u}_k)=\Big[\bar{q}_1(\mathbf{u}_k^{(1)}) \quad \bar{q}_2(\mathbf{u}_k^{(2)}) \quad \cdots \quad \bar{q}_p(\mathbf{u}_k^{(p)})\Big]^T,$$

where $u_k \in \mathbb{R}^p$ is the control input actually entering the plant (2.1). For each $\bar{q}_j(\cdot)$ $(1 \le j \le p)$, the set of quantization levels is described by

$$\bar{\mathscr{U}}_j = \left\{\pm\bar{\chi}_i^{(j)}, \bar{\chi}_i^{(j)} = \bar{\rho}_j^i \bar{\chi}_0^{(j)},\ i = 0,\ \pm 1,\ \pm 2,\ \ldots\right\} \cup \{0\},$$
$$0 < \bar{\rho}_j < 1,\ \bar{\chi}_0^{(j)} > 0,$$

and the quantizer $\bar{q}_j(\cdot)$ is defined as

$$\bar{q}_j\left(\mathbf{u}_k^{(j)}\right) = \begin{cases} \bar{\chi}_i^{(j)}, & \frac{1}{1+\bar{\delta}_j}\bar{\chi}_i^{(j)} < \mathbf{u}_k^{(j)} \le \frac{1}{1-\bar{\delta}_j}\bar{\chi}_i^{(j)}, \\ 0, & \mathbf{u}_k^{(j)} = 0, \\ -\bar{q}_j(-\mathbf{u}_k^{(j)}), & \mathbf{u}_k^{(j)} < 0, \end{cases}$$

with $\bar{\delta}_j = (1-\bar{\rho}_j)/(1+\bar{\rho}_j)$. To the end, the control input u_k can be expressed as

$$u_k = (I + \bar{\Delta}_k)\mathbf{u}_k, \tag{2.8}$$

where $\bar{\Delta}_k = \mathrm{diag}\{\bar{\Delta}_k^{(1)}, \ldots, \bar{\Delta}_k^{(p)}\}$ with $\bar{\Delta}_k^{(j)}$ satisfying $|\bar{\Delta}_k^{(j)}| \le \bar{\delta}_j$ for each $1 \le j \le p$.

For the sake of easy manipulation, we introduce two matrices

$$C_p^r := \mathrm{diag}\{\underbrace{\overbrace{0, \ldots, 0}^{r-1}, 1, 0, \ldots, 0}_{p}\} \quad \text{and} \quad C_m^r := \mathrm{diag}\{\underbrace{\overbrace{0, \ldots, 0}^{r-1}, 1, 0, \ldots, 0}_{m}\}, \tag{2.9}$$

and then rewrite the signals $\bar{y}_k \in \mathbb{R}^m$ and $u_k \in \mathbb{R}^p$ as

$$\bar{y}_k = (I + \Delta_k)\sum_{r=1}^{m} \gamma_k^r C_m^r l(x_k) + (I + \Delta_k)k(x_k)v_k \tag{2.10}$$

and

$$u_k = (I + \bar{\Delta}_k)\sum_{r=1}^{p} \xi_k^r C_p^r u_c(\hat{x}_k), \tag{2.11}$$

respectively.

Setting $\eta_k = [x_k^T\ \hat{x}_k^T]^T$, $\eta_{k-d} = [x_{k-d}^T\ \hat{x}_{k-d}^T]^T$ and substituting (2.10)–(2.11) into (2.1) and (2.5), we obtain the following closed-loop system:

$$\begin{cases} \eta_{k+1} = \mathscr{F}_1(\eta_k, \eta_{k-d}) + \mathscr{H}_1(\eta_k)v_k + \mathscr{F}_w(\eta_k, \eta_{k-d})w_k \\ \qquad + \sum_{r=1}^{p}\left(\xi_k^r - \bar{\xi}^r\right)\mathscr{G}_1^r(\eta_k) + \sum_{r=1}^{m}\left(\gamma_k^r - \bar{\gamma}^r\right)\mathscr{G}_2^r(\eta_k), \\ z_k = \mathscr{F}_2(\eta_k, \eta_{k-d}) + \mathscr{H}_2(\eta_k)v_k + \sum_{r=1}^{p}\left(\xi_k^r - \bar{\xi}^r\right)\mathscr{G}_3^r(\eta_k), \end{cases} \tag{2.12}$$

where

$$
\begin{aligned}
\mathscr{F}_1(\eta_k,\eta_{k-d}) &= \begin{bmatrix} f_1(x_k,x_{k-d}) + g_1(x_k)(I+\bar{\Delta}_k)\bar{\Xi}u_c(\hat{x}_k) \\ f_c(\hat{x}_k) + g_c(\hat{x}_k)(I+\Delta_k)\bar{\Gamma}l(x_k) \end{bmatrix}, \\
\mathscr{H}_1(\eta_k) &= \begin{bmatrix} h_1(x_k) \\ g_c(\hat{x}_k)(I+\Delta_k)k(x_k) \end{bmatrix}, \\
\mathscr{G}_1^r(\eta_k) &= \begin{bmatrix} g_1(x_k)(I+\bar{\Delta}_k)C_p^r u_c(\hat{x}_k) \\ 0 \end{bmatrix}, \\
\mathscr{G}_2^r(\eta_k) &= \begin{bmatrix} 0 \\ g_c(\hat{x}_k)(I+\Delta_k)C_m^r l(x_k) \end{bmatrix}, \\
\mathscr{F}_w(\eta_k,\eta_{k-d}) &= \begin{bmatrix} f_w(x_k,x_{k-d}) \\ 0 \end{bmatrix}, \\
\mathscr{F}_2(\eta_k,\eta_{k-d}) &= f_2(x_k,x_{k-d}) + g_2(x_k)(I+\bar{\Delta}_k)\bar{\Xi}u_c(\hat{x}_k), \\
\mathscr{G}_3^r(\eta_k) &= g_2(x_k)(I+\bar{\Delta}_k)C_p^r u_c(\hat{x}_k), \qquad \mathscr{H}_2(\eta_k) = h_2(x_k), \\
\bar{\Gamma} &= \operatorname{diag}\{\bar{\gamma}^1,\dots,\bar{\gamma}^m\}, \qquad \bar{\Xi} = \operatorname{diag}\{\bar{\xi}^1,\dots,\bar{\xi}^p\}.
\end{aligned}
\tag{2.13}
$$

Throughout this chapter, we assume that all the stochastic variables v_k, w_k, ξ_k^i ($i=1,2,\dots,p$) and γ_k^j ($i=1,2,\dots,m$) are uncorrelated each other.

Definition 2.1 The zero solution of the closed-loop system (2.12) with $v_k=0$ is said to be stochastically stable if, for any $\varepsilon>0$, there exists a $\delta>0$ such that

$$\mathbb{E}\{\|\eta_k\|\} < \varepsilon, \tag{2.14}$$

whenever $k \in \mathbb{I}^+$ and $\max_{k\in\{-d,-d+1,\dots,0\}} \|\bar{\varphi}_k\| < \delta$ where $\bar{\varphi}_k = [\varphi_k^T \ 0]^T$ for $k = -d, -d+1, \dots, 0$.

In Definition 2.1, the notion of stochastic stability is proposed for the stochastic discrete time-delayed system (2.12). Other definitions of stability for different kinds of stochastic systems can be found in [77, 100, 137].

The purpose of the problem addressed in this chapter is to design the parameters $f_c(\hat{x}_k)$, $g_c(\hat{x}_k)$, and $u_c(\hat{x}_k)$ of the nonlinear controller such that the following requirements are satisfied simultaneously for the given system (2.1) and the quantizers $q(\cdot)$ and $\bar{q}(\cdot)$:

(a) The zero solution of the closed-loop system (2.12) with $v_k=0$ is stochastically stable.
(b) Under the zero initial condition, the controlled output z_k satisfies

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|z_k\|^2\} \le \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|v_k\|^2\} \tag{2.15}$$

for all nonzero v_k, where $\gamma>0$ is a given disturbance attenuation level.

The aim of this chapter is to establish *a theoretical framework* for solving the H_∞ control problem for *general* nonlinear stochastic systems. Such a control problem stems from the networked control systems with data dropouts, quantization errors, and delays. There are different ways to define the Quality-of-Service (QoS) for NCS [28]. In this chapter, we consider two of the most popular QoS measures: (1) the point-to-point network allowable data dropout rate that is used to indicate the probability of data packet dropout in data transmission and (2) the point-to-point network throughput that is used to indicate how fast the signal can be sampled and sent as a packet through the network. Obviously, for the model considered in this chapter, the sampling period h and the data dropout rate ρ determine the control performance. We assume that the data are single-packet transmitted, different data packets have the same length L, and the network throughput distributed by packet scheduler is Q_{i_k} in $t \in [i_k h, i_{k+1} h)$. The network allowable data dropout rate is related to the packet scheduler, backlog controller, and algorithm complex of loss dropper policy. As discussed in the problem formulation, we use different data dropout rates (measurement missing probabilities) to quantify the random packet losses in the sensor-to-controller channel and the controller-to-actuator channel. On the other hand, the sampling period h is decided by the network throughput Q_{i_k} and the number of sensors. Small sampling period can have good control performance but can induce network congest and raise the data dropout rate. Since there has been a rich body of literature studying appropriate sampling method [65], in this chapter, the models are put forward based on the assumption that a sampled-data model can be obtained through online measurement such as sending probing data packet to measure network characteristics and QoS scheduling.

2.2 Main Results

To state the main results, we need the following lemma.

Lemma 2.1 *Suppose that there exist a Lyapunov functional* $V(\zeta) \in C^1(\mathbb{R}^{2(d+1)n})$ *and a function* $a(r) \in CK$ *satisfying the following conditions*:

$$V(0) = 0, \tag{2.16a}$$

$$a\big(\|\zeta\|\big) \le V(\zeta), \tag{2.16b}$$

$$\mathbb{E}\big\{V(\zeta_{k+1})\big\} \le \mathbb{E}\big\{V(\zeta_k)\big\}, \quad k \in \mathbb{I}^+, \tag{2.16c}$$

where $\zeta_k = [\eta_k^T \ \eta_{k-1}^T \ \cdots \ \eta_{k-d}^T]^T$. *Then the zero solution of closed-loop system* (2.12) *with* $v_k = 0$ *is stochastically stable.*

Proof First of all, note that $V(0) = 0$ and $V(\zeta)$ is continuous. Therefore, for any $\varepsilon > 0$, there exists a scalar $\delta > 0$ such that $V(\zeta_0) < a(\varepsilon)$ when $\|\zeta_0\| < \delta$. We aim to prove that $\mathbb{E}\{\|\eta_k\|\} < \varepsilon$ whenever $k \in \mathbb{I}^+$ and $\max_{k\in\{-d,-d+1,\dots,0\}} \|\bar{\varphi}_k\| < \delta$. By considering $\|\bar{\varphi}_k\| \le \|\zeta_0\|$ for all $k = -d, -d+1, \dots, 0$, we only need to prove that every solution η_k with $\|\zeta_0\| < \delta$ implies $\mathbb{E}\{\|\eta_k\|\} < \varepsilon$ for all $k \in \mathbb{I}^+$. Let us now prove the latter by contradiction. Suppose that, for a solution η_k satisfying $\|\zeta_0\| < \delta$, there exists a $k_1 \in \mathbb{I}^+$ such that $\mathbb{E}\{\|\eta_{k_1}\|\} \ge \varepsilon$. Noting that $\|\eta_k\| \le \|\zeta_k\|$, one has $\mathbb{E}\{\|\eta_{k_1}\|\} \le \mathbb{E}\{\|\zeta_{k_1}\|\}$. In addition, by using the Jensen inequality and considering the property of function $a(r)$, it follows from (2.16b) and (2.16c) that

$$\begin{aligned} a(\varepsilon) &\le a\big(\mathbb{E}\big\{\|\eta_{k_1}\|\big\}\big) \le a\big(\mathbb{E}\big\{\|\zeta_{k_1}\|\big\}\big) \\ &\le \mathbb{E}\big\{a\big(\|\zeta_{k_1}\|\big)\big\} \le \mathbb{E}\big\{V(\zeta_{k_1})\big\} \le \mathbb{E}\big\{V(\zeta_0)\big\} < a(\varepsilon), \end{aligned}$$

which is a contradiction. Therefore, it follows easily from Definition 2.1 that the zero solution of the augmented system (2.12) with $v_k = 0$ is stochastically stable. The proof is complete. □

The following theorem provides a sufficient condition under which the closed-loop system (2.12) is stochastically stable and the controlled output z_k satisfies the H_∞ criterion (2.15) under the zero initial condition for the given quantizers $q(\cdot)$ and $\bar{q}(\cdot)$.

Theorem 2.1 *Let the disturbance attenuation level $\gamma > 0$ be given. Suppose that there exist two real-valued functionals $V_1(\eta) \in C^2(\mathbb{R}^{2n})$ and $V_2(\eta) \in C^1(\mathbb{R}^{2n})$ satisfying*

$$V_1(0) = 0, \qquad V_2(0) = 0, \tag{2.17}$$

$$a\big(\|\eta\|\big) \le V_1(\eta), \qquad a\big(\|\eta\|\big) \le V_2(\eta), \quad \textit{where } a(r) \in CK, \tag{2.18}$$

and the following inequalities for any $\eta, \eta_\alpha, \eta_d \in \mathbb{R}^{2n}$:

$$\mathcal{A}(\eta,\eta_\alpha) = \gamma^2 I - \frac{1}{2}\mathscr{H}_1^T(\eta)V_{1\eta\eta}(\eta_\alpha)\mathscr{H}_1(\eta) - \mathscr{H}_2^T(\eta)\mathscr{H}_2(\eta) > 0, \tag{2.19}$$

$$\begin{aligned} \mathcal{J}(\eta,\eta_\alpha,\eta_d) := {}& \mathcal{B}(\eta,\eta_\alpha,\eta_d)\mathcal{A}^{-1}(\eta,\eta_\alpha)\mathcal{B}^T(\eta,\eta_\alpha,\eta_d) \\ &+ \frac{1}{2}\mathscr{F}_1^T(\eta,\eta_d)V_{1\eta\eta}(\eta_\alpha)\mathscr{F}_1(\eta,\eta_d) \\ &+ \frac{1}{2}\theta\mathscr{F}_w^T(\eta,\eta_d)V_{1\eta\eta}(\eta_\alpha)\mathscr{F}_w(\eta,\eta_d) + \frac{1}{2}\eta^T V_{1\eta\eta}(\eta_\alpha)\eta \\ &+ \mathscr{F}_2^T(\eta,\eta_d)\mathscr{F}_2(\eta,\eta_d) - \mathscr{F}_1^T(\eta,\eta_d)V_{1\eta\eta}(\eta_\alpha)\eta \\ &+ V_{1\eta}^T(\eta)\mathscr{F}_1(\eta,\eta_d) - V_{1\eta}^T(\eta)\eta + V_2(\eta) - V_2(\eta_d) \\ &+ \frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathscr{G}_1^{rT}(\eta)V_{1\eta\eta}(\eta_\alpha)\mathscr{G}_1^r(\eta) \end{aligned}$$

$$+\frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathscr{G}_2^{rT}(\eta)V_{1\eta\eta}(\eta_\alpha)\mathscr{G}_2^r(\eta)$$

$$+\sum_{r=1}^{p}\alpha_r^2\mathscr{G}_3^{rT}(\eta)\mathscr{G}_3^r(\eta)\leq 0, \tag{2.20}$$

where

$$\begin{aligned}\mathcal{B}(\eta,\eta_\alpha,\eta_d) = &\frac{1}{2}V_{1\eta}^T(\eta)\mathscr{H}_1(\eta)+\frac{1}{2}\mathscr{F}_1^T(\eta,\eta_d)V_{1\eta\eta}(\eta_\alpha)\mathscr{H}_1(\eta)\\ &-\frac{1}{2}\eta^T V_{1\eta\eta}(\eta_\alpha)\mathscr{H}_1(\eta)+\mathscr{F}_2^T(\eta,\eta_d)\mathscr{H}_2(\eta)\end{aligned} \tag{2.21}$$

with $\alpha_r=\sqrt{\bar{\xi}^r(1-\bar{\xi}^r)}$ *and* $\beta_r=\sqrt{\bar{\gamma}^r(1-\bar{\gamma}^r)}$. *Then system* (2.12) *with* $v_k=0$ *is stochastically stable, and the controlled output* z_k *satisfies the* H_∞ *criterion* (2.15) *for all nonzero* v_k *under the zero initial condition.*

Proof Choose the Lyapunov functional $V(\zeta_k)$ as

$$V(\zeta_k)=V_1(\eta_k)+\sum_{i=k-d}^{k-1}V_2(\eta_i), \tag{2.22}$$

where ζ_k is defined in Lemma 2.1. Note that the first term in (2.22) corresponds to the stability conditions for the discrete-time nonlinear stochastic systems *without* delays, and the second term in (2.22) corresponds to delay-independent stability conditions that account for the delay effects.

Obviously, the Lyapunov functional $V(\zeta_k)$ constructed as (2.22) satisfies (2.16a) and (2.16b). By Taylor's formula, there exists a scalar $\bar{\alpha}_k\in(0,1)$ such that

$$\begin{aligned}&V_1(\eta_{k+1})-V_1(\eta_k)\\ &\quad=V_{1\eta}^T(\eta_k)(\eta_{k+1}-\eta_k)+\frac{1}{2}(\eta_{k+1}-\eta_k)^T V_{1\eta\eta}(\eta_{\alpha_k})(\eta_{k+1}-\eta_k),\end{aligned} \tag{2.23}$$

where $\eta_{\alpha_k}=\eta_k+\bar{\alpha}_k(\eta_{k+1}-\eta_k)$.

Now, we first prove the stochastic stability of the closed-loop system (2.12) with $v_k=0$. By noting $\mathbb{E}w_k^2=\theta$,

$$\mathbb{E}\big\{\big(\xi_k^i-\bar{\xi}^i\big)\big(\xi_k^j-\bar{\xi}^j\big)\big\}=\begin{cases}\bar{\xi}^i(1-\bar{\xi}^i), & i=j,\\ 0, & i\neq j,\end{cases} \tag{2.24}$$

for $1\leq i\leq p,\ 1\leq j\leq p$ and

$$\mathbb{E}\big\{\big(\gamma_k^i-\bar{\gamma}^i\big)\big(\gamma_k^j-\bar{\gamma}^j\big)\big\}=\begin{cases}\bar{\gamma}^i(1-\bar{\gamma}^i), & i=j,\\ 0, & i\neq j,\end{cases} \tag{2.25}$$

for $1 \le i \le m$, $1 \le j \le m$, it can be calculated along the closed-loop system (2.12) with $v_k = 0$ that

$$\begin{aligned}
&\mathbb{E}\big\{V(\zeta_{k+1})\big\} - \mathbb{E}\big\{V(\zeta_k)\big\} \\
&\quad = \mathbb{E}\big\{V_1(\eta_{k+1}) - V_1(\eta_k) + V_2(\eta_k) - V_2(\eta_{k-d})\big\} \\
&\quad = \mathbb{E}\Bigg\{V_{1\eta}^T(\eta_k)(\eta_{k+1} - \eta_k) + \frac{1}{2}(\eta_{k+1} - \eta_k)^T V_{1\eta\eta}(\eta_{\alpha_k})(\eta_{k+1} - \eta_k) \\
&\qquad + V_2(\eta_k) - V_2(\eta_{k-d})\Bigg\} \\
&\quad = \mathbb{E}\Bigg\{V_{1\eta}^T(\eta_k)\Bigg(\mathscr{F}_1(\eta_k, \eta_{k-d}) + \mathscr{F}_w(\eta_k, \eta_{k-d})w_k + \sum_{r=1}^{p}\big(\xi_k^r - \bar{\xi}^r\big)\mathscr{G}_1^r(\eta_k) \\
&\qquad + \sum_{r=1}^{m}\big(\gamma_k^r - \bar{\gamma}^r\big)\mathscr{G}_2^r(\eta_k) - \eta_k\Bigg) + \frac{1}{2}\Bigg(\mathscr{F}_1(\eta_k, \eta_{k-d}) + \mathscr{F}_w(\eta_k, \eta_{k-d})w_k \\
&\qquad + \sum_{r=1}^{p}\big(\xi_k^r - \bar{\xi}^r\big)\mathscr{G}_1^r(\eta_k) + \sum_{r=1}^{m}\big(\gamma_k^r - \bar{\gamma}^r\big)\mathscr{G}_2^r(\eta_k) - \eta_k\Bigg)^T V_{1\eta\eta}(\eta_{\alpha_k}) \\
&\qquad \times \Bigg(\mathscr{F}_1(\eta_k, \eta_{k-d}) + \mathscr{F}_w(\eta_k, \eta_{k-d})w_k + \sum_{r=1}^{p}\big(\xi_k^r - \bar{\xi}^r\big)\mathscr{G}_1^r(\eta_k) \\
&\qquad + \sum_{r=1}^{m}\big(\gamma_k^r - \bar{\gamma}^r\big)\mathscr{G}_2^r(\eta_k) - \eta_k\Bigg) + V_2(\eta_k) - V_2(\eta_{k-d})\Bigg\} \\
&\quad = \mathbb{E}\Bigg\{\frac{1}{2}\mathscr{F}_1^T(\eta_k, \eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{F}_1(\eta_k, \eta_{k-d}) + \frac{1}{2}\theta\mathscr{F}_w^T(\eta_k, \eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k}) \\
&\qquad \times \mathscr{F}_w(\eta_k, \eta_{k-d}) + \frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathscr{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_1^r(\eta_k) + \frac{1}{2}\eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\eta_k \\
&\qquad + \frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathscr{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_2^r(\eta_k) - \mathscr{F}_1^T(\eta_k, \eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\eta_k \\
&\qquad + V_{1\eta}^T(\eta_k)\mathscr{F}_1(\eta_k, \eta_{k-d}) - V_{1\eta}^T(\eta_k)\eta_k + V_2(\eta_k) - V_2(\eta_{k-d})\Bigg\} \\
&\quad \le \mathfrak{J}(\eta_k, \eta_{\alpha_k}, \eta_{k-d}) \le 0,
\end{aligned} \tag{2.26}$$

which, by Lemma 2.1, confirms that system (2.12) with $v_k = 0$ is stochastically stable.

Next, let us show that the closed-loop system (2.12) satisfies the H_∞ performance constraint for all nonzero exogenous disturbances under the zero initial condition.

From (2.12) it follows

$$
\begin{aligned}
&\mathbb{E}\big\{V(\zeta_{k+1})-V(\zeta_k)+\|z_k\|^2-\gamma^2\|v_k\|^2\big\}\\
&=\mathbb{E}\big\{V_1(\eta_{k+1})-V_1(\eta_k)+V_2(\eta_k)-V_2(\eta_{k-d})+\|z_k\|^2-\gamma^2\|v_k\|^2\big\}\\
&=\mathbb{E}\Bigg\{V_{1\eta}^T(\eta_k)(\eta_{k+1}-\eta_k)+\frac{1}{2}(\eta_{k+1}-\eta_k)^T V_{1\eta\eta}(\eta_{\alpha_k})(\eta_{k+1}-\eta_k)\\
&\quad+V_2(\eta_k)-V_2(\eta_{k-d})+\|z_k\|^2-\gamma^2\|v_k\|^2\Bigg\}\\
&=\mathbb{E}\Bigg\{V_{1\eta}^T(\eta_k)\Bigg(\mathscr{F}_1(\eta_k,\eta_{k-d})+\mathscr{H}_1(\eta_k)v_k+\mathscr{F}_w(\eta_k,\eta_{k-d})w_k\\
&\quad+\sum_{r=1}^{p}\big(\xi_k^r-\bar{\xi}^r\big)\mathscr{G}_1^r(\eta_k)+\sum_{r=1}^{m}\big(\gamma_k^r-\bar{\gamma}^r\big)\mathscr{G}_2^r(\eta_k)-\eta_k\Bigg)\\
&\quad+\frac{1}{2}\Bigg(\mathscr{F}_1(\eta_k,\eta_{k-d})+\mathscr{H}_1(\eta_k)v_k+\mathscr{F}_w(\eta_k,\eta_{k-d})w_k\\
&\quad+\sum_{r=1}^{p}\big(\xi_k^r-\bar{\xi}^r\big)\mathscr{G}_1^r(\eta_k)+\sum_{r=1}^{m}\big(\gamma_k^r-\bar{\gamma}^r\big)\mathscr{G}_2^r(\eta_k)-\eta_k\Bigg)^T V_{1\eta\eta}(\eta_{\alpha_k})\\
&\quad\times\Bigg(\mathscr{F}_1(\eta_k,\eta_{k-d})+\mathscr{H}_1(\eta_k)v_k+\mathscr{F}_w(\eta_k,\eta_{k-d})w_k\\
&\quad+\sum_{r=1}^{p}\big(\xi_k^r-\bar{\xi}^r\big)\mathscr{G}_1^r(\eta_k)+\sum_{r=1}^{m}\big(\gamma_k^r-\bar{\gamma}^r\big)\mathscr{G}_2^r(\eta_k)-\eta_k\Bigg)+V_2(\eta_k)\\
&\quad-V_2(\eta_{k-d})-\gamma^2\|v_k\|^2+\Bigg(\mathscr{F}_2(\eta_k,\eta_{k-d})+\mathscr{H}_2(\eta_k)v_k\\
&\quad+\sum_{r=1}^{p}\big(\xi_k^r-\bar{\xi}^r\big)\mathscr{G}_3^r(\eta_k)\Bigg)^T\Bigg(\mathscr{F}_2(\eta_k,\eta_{k-d})+\mathscr{H}_2(\eta_k)v_k\\
&\quad+\sum_{r=1}^{p}\big(\xi_k^r-\bar{\xi}^r\big)\mathscr{G}_3^r(\eta_k)\Bigg)\Bigg\}\\
&=\mathbb{E}\Bigg\{V_{1\eta}^T(\eta_k)\mathscr{F}_1(\eta_k,\eta_{k-d})+\frac{1}{2}\mathscr{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{F}_1(\eta_k,\eta_{k-d})\\
&\quad+V_{1\eta}^T(\eta_k)\mathscr{H}_1(\eta_k)v_k-V_{1\eta}^T(\eta_k)\eta_k+\frac{1}{2}v_k^T\mathscr{H}_1^T(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{H}_1(\eta_k)v_k
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\theta\mathscr{F}_w^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{F}_w(\eta_k,\eta_{k-d})+\frac{1}{2}\eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\eta_k\\
&+\mathscr{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{H}_1(\eta_k)v_k-\eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{H}_1(\eta_k)v_k\\
&-\mathscr{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\eta_k+\frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathscr{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_1^r(\eta_k)\\
&+\frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathscr{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_2^r(\eta_k)+\sum_{r=1}^{p}\alpha_r^2\mathscr{G}_3^{rT}(\eta_k)\mathscr{G}_3^r(\eta_k)\\
&+\mathscr{F}_2^T(\eta_k,\eta_{k-d})\mathscr{F}_2(\eta_k,\eta_{k-d})+2\mathscr{F}_2^T(\eta_k,\eta_{k-d})\mathscr{H}_2(\eta_k)v_k\\
&+v_k^T\mathscr{H}_2^T(\eta_k)\mathscr{H}_2(\eta_k)v_k+V_2(\eta_k)-V_2(\eta_{k-d})-\gamma^2\|v_k\|^2\Bigg\}\\
=\mathbb{E}\Bigg\{&-v_k^T\mathcal{A}(\eta_k,\eta_{\alpha_k})v_k+\frac{1}{2}\mathscr{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{F}_1(\eta_k,\eta_{k-d})\\
&+2\mathcal{B}(\eta_k,\eta_{\alpha_k},\eta_{k-d})v_k+\frac{1}{2}\theta\mathscr{F}_w^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{F}_w(\eta_k,\eta_{k-d})\\
&+\frac{1}{2}\eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\eta_k+\mathscr{F}_2^T(\eta_k,\eta_{k-d})\mathscr{F}_2(\eta_k,\eta_{k-d})\\
&-\mathscr{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\eta_k+V_{1\eta}^T(\eta_k)\mathscr{F}_1(\eta_k,\eta_{k-d})\\
&-V_{1\eta}^T(\eta_k)\eta_k+V_2(\eta_k)-V_2(\eta_{k-d})+\frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathscr{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_1^r(\eta_k)\\
&+\frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathscr{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_2^r(\eta_k)+\sum_{r=1}^{p}\alpha_r^2\mathscr{G}_3^{rT}(\eta_k)\mathscr{G}_3^r(\eta_k)\Bigg\}.
\end{aligned}
\tag{2.27}
$$

Applying the "completing the square" rule, it can be easily seen that (2.27) is equal to

$$
\begin{aligned}
\mathbb{E}\Bigg\{&-\big(v_k-v_k^*\big)^T\mathcal{A}(\eta_k,\eta_{\alpha_k})\big(v_k-v_k^*\big)\\
&+\mathcal{B}(\eta_k,\eta_{\alpha_k},\eta_{k-d})\mathcal{A}^{-1}(\eta_k,\eta_{\alpha_k})\mathcal{B}^T(\eta_k,\eta_{\alpha_k},\eta_{k-d})\\
&+\frac{1}{2}\mathscr{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{F}_1(\eta_k,\eta_{k-d})\\
&+\frac{1}{2}\theta\mathscr{F}_w^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{F}_w(\eta_k,\eta_{k-d})\\
&+\frac{1}{2}\eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\eta_k+\mathscr{F}_2^T(\eta_k,\eta_{k-d})\mathscr{F}_2(\eta_k,\eta_{k-d})
\end{aligned}
$$

$$
\begin{aligned}
&- \mathscr{F}_1^T(\eta_k, \eta_{k-d}) V_{1\eta\eta}(\eta_{\alpha_k})\eta_k \\
&+ V_{1\eta}^T(\eta_k)\mathscr{F}_1(\eta_k, \eta_{k-d}) - V_{1\eta}^T(\eta_k)\eta_k + \frac{1}{2}\sum_{r=1}^{p}\alpha_r^2 \mathscr{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_1^r(\eta_k) \\
&+ V_2(\eta_k) - V_2(\eta_{k-d}) + \frac{1}{2}\sum_{r=1}^{m}\beta_r^2 \mathscr{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_2^r(\eta_k) \\
&+ \left.\sum_{r=1}^{p}\alpha_r^2 \mathscr{G}_3^{rT}(\eta_k)\mathscr{G}_3^r(\eta_k)\right\},
\end{aligned}
\tag{2.28}
$$

where $v_k^* = \mathcal{A}^{-1}(\eta_k, \eta_{\alpha_k})\mathcal{B}^T(\eta_k, \eta_{\alpha_k}, \eta_{k-d})$. Noticing (2.19), it follows from (2.28) that

$$
\begin{aligned}
&\mathbb{E}\left\{V(\zeta_{k+1}) - V(\zeta_k) + \|z_k\|^2 - \gamma^2\|v_k\|^2\right\} \\
&\le \mathbb{E}\Bigg\{\mathcal{B}(\eta_k, \eta_{\alpha_k}, \eta_{k-d})\mathcal{A}^{-1}(\eta_k, \eta_{\alpha_k})\mathcal{B}^T(\eta_k, \eta_{\alpha_k}, \eta_{k-d}) \\
&\quad + \frac{1}{2}\mathscr{F}_1^T(\eta_k, \eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{F}_1(\eta_k, \eta_{k-d}) + \frac{1}{2}\theta\mathscr{F}_w^T(\eta_k, \eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k}) \\
&\quad \times \mathscr{F}_w(\eta_k, \eta_{k-d}) + \frac{1}{2}\eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\eta_k \\
&\quad + \mathscr{F}_2^T(\eta_k, \eta_{k-d})\mathscr{F}_2(\eta_k, \eta_{k-d}) - \mathscr{F}_1^T(\eta_k, \eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\eta_k \\
&\quad + V_{1\eta}^T(\eta_k)\mathscr{F}_1(\eta_k, \eta_{k-d}) - V_{1\eta}^T(\eta_k)\eta_k + V_2(\eta_k) - V_2(\eta_{k-d}) \\
&\quad + \frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathscr{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_1^r(\eta_k) \\
&\quad + \frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathscr{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathscr{G}_2^r(\eta_k) + \sum_{r=1}^{p}\alpha_r^2\mathscr{G}_3^{rT}(\eta_k)\mathscr{G}_3^r(\eta_k)\Bigg\} \\
&= \mathbb{E}\left\{\mathcal{J}(\eta_k, \eta_{\alpha_k}, \eta_{k-d})\right\},
\end{aligned}
$$

and then it can be seen from (2.20) that

$$
\mathbb{E}\left\{V(\zeta_{k+1}) - V(\zeta_k) + \|z_k\|^2 - \gamma^2\|v_k\|^2\right\} \le 0. \tag{2.29}
$$

Under the zero initial condition, summing up (2.29) from 0 to ∞ with respect to k and considering $\mathbb{E}\{V(\zeta_\infty)\} \ge 0$, we obtain

$$
\sum_{k=0}^{\infty}\mathbb{E}\left\{\|z_k\|^2\right\} \le \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\left\{\|v_k\|^2\right\},
$$

which means that the desired H_∞ performance requirement is achieved, and therefore the proof of Theorem 2.1 is complete. □

In Theorem 2.1, a very general condition described by a second-order nonlinear inequality has been derived to guarantee the H_∞ performance and the stochastic stability of the closed-loop system (2.12). Such a nonlinear inequality, although it is difficult to solve, will play a theoretically significant role in the analysis and synthesis of H_∞ control problems. Based on Theorem 2.1, the corresponding H_∞ control problems for some special classes of nonlinear systems can be solved effectively. Take the polynomial nonlinear system as an example. One just needs to choose the Lyapunov function as a positive homogeneous polynomial. Then, by using the result in Theorem 2.1 together with the technique of complete square matrix representation (SMR) [25, 26], the existence condition of the desired H_∞ controllers can be formulated in terms of the feasibility problem for an LMI, which can be readily verified by the available SOS (sum of squares) solvers [26].

In order to derive more tractable sufficient conditions, in the sequel, we take the real-valued functions as $V_1(\eta) = \eta^T P\eta$ and $V_2(\eta) = \eta^T Q\eta$ where P and Q are positive definite matrices. The following corollary is obtained directly from Theorem 2.1.

Corollary 2.1 *Let the disturbance attenuation level $\gamma > 0$ be given. Suppose that there exist two positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$ satisfying the following conditions for all nonzero $\eta, \eta_d \in \mathbb{R}^{2n}$:*

$$A(\eta) = \gamma^2 I - \mathscr{H}_1^T(\eta)P\mathscr{H}_1(\eta) - \mathscr{H}_2^T(\eta)\mathscr{H}_2(\eta) > 0, \tag{2.30}$$

$$\begin{aligned} \mathcal{H}(\eta, \eta_d) := {} & B(\eta, \eta_d)A^{-1}(\eta)B^T(\eta, \eta_d) + \mathscr{F}_1^T(\eta, \eta_d)P\mathscr{F}_1(\eta, \eta_d) \\ & + \theta\mathscr{F}_w^T(\eta, \eta_d)P\mathscr{F}_w(\eta, \eta_d) + \mathscr{F}_2^T(\eta, \eta_d)\mathscr{F}_2(\eta, \eta_d) \\ & + \sum_{r=1}^{p} \alpha_r^2 \mathscr{G}_1^{rT}(\eta)P\mathscr{G}_1^r(\eta) + \sum_{r=1}^{m} \beta_r^2 \mathscr{G}_2^{rT}(\eta)P\mathscr{G}_2^r(\eta) \\ & + \sum_{r=1}^{p} \alpha_r^2 \mathscr{G}_3^{rT}(\eta)\mathscr{G}_3^r(\eta) + \eta^T(Q-P)\eta - \eta_d^T Q\eta_d \le 0, \end{aligned} \tag{2.31}$$

where

$$B(\eta, \eta_d) = \mathscr{F}_1^T(\eta, \eta_d)P\mathscr{H}_1(\eta) + \mathscr{F}_2^T(\eta, \eta_d)\mathscr{H}_2(\eta), \tag{2.32}$$

with α_r and β_r defined in Theorem 2.1. Then system (2.12) with $v_k = 0$ is stochastically stable, and the controlled output z_k satisfies the H_∞ criterion (2.15) for all nonzero v_k under the zero initial condition.

From (2.30)–(2.32) it can be observed that the inequalities of Corollary 2.1 are dependent on both the missing probability and the quantization effects Δ_k and $\bar{\Delta}_k$. If the quantization effects are taken as $\Delta_k = 0$ and $\bar{\Delta}_k = 0$, one can immediately obtain a sufficient condition to guarantee that the system without quantization effect (when $v_k = 0$) is stochastically stable while achieving the H_∞ performance constraint for all admissible missing observations and nonzero exogenous disturbances under the zero initial condition. Such a problem for *linear deterministic* system has been investigated in [156, 173], where the data missing phenomena have been modeled by one stochastic variable only. Obviously, Corollary 2.1 generalizes the results in [156, 173].

If $\bar{\gamma}^i = 1$ $(1 \le i \le m)$ and $\bar{\xi}^j = 1$ $(1 \le j \le p)$, i.e., the data missing phenomena do not arise, then a sufficient condition is easily obtained from Corollary 2.1 to make sure that the system without data missing (when $v_k = 0$) is stochastically stable with a guaranteed H_∞ performance index for nonzero exogenous disturbances under the zero initial condition. Similar results for *linear deterministic* system can be found in [47].

Corollary 2.1 provides a sufficient condition which guarantees the H_∞ performance and the stochastic stability of the closed-loop system (2.12). However, it should be pointed that the condition in Corollary 2.1 is dependent on the quantization effects Δ_k and $\bar{\Delta}_k$, which results in significant difficulty in checking such a sufficient condition in practice. Fortunately, the quantization effects of the logarithmic-type quantizers can be transformed into sector bound uncertainties. In fact, by defining $\bar{\Lambda} = \mathrm{diag}\{\bar{\delta}_1, \ldots, \bar{\delta}_p\}$, $\Lambda = \mathrm{diag}\{\delta_1, \ldots, \delta_m\}$, and $F_k = \mathrm{diag}\{\bar{\Delta}_k \bar{\Lambda}^{-1}, \Delta_k \Lambda^{-1}\}$, we can obtain an unknown real-valued time-varying matrix F_k satisfying $F_k F_k^T = F_k^T F_k \le I$. In what follows, we are devoted to eliminating the quantization effects and establishing some conditions that can be solved effectively. For this purpose, the coefficients of system (2.12) are rewritten as follows:

$$\begin{aligned}
\mathscr{F}_1(\eta_k, \eta_{k-d}) &= \mathscr{A}_1(\eta_k, \eta_{k-d}) + \big(\mathscr{S}_1(\eta_k) + \mathscr{S}_2(\eta_k)\big) F_k \mathscr{T}_1(\eta_k), \\
\mathscr{H}_1(\eta_k) &= \mathscr{B}_1(\eta_k) + \mathscr{S}_2(\eta_k) F_k \mathscr{T}_2(\eta_k), \\
\mathscr{F}_2(\eta_k, \eta_{k-d}) &= \mathscr{A}_2(\eta_k, \eta_{k-d}) + \mathscr{S}_3(\eta_k) F_k \mathscr{T}_3(\eta_k), \\
\mathscr{G}_1^r(\eta_k) &= \mathscr{C}_1^r(\eta_k) + \mathscr{S}_1(\eta_k) F_k \mathscr{T}_4^r(\eta_k), \\
\mathscr{G}_2^r(\eta_k) &= \mathscr{C}_2^r(\eta_k) + \mathscr{S}_2(\eta_k) F_k \mathscr{T}_5^r(\eta_k), \\
\mathscr{G}_3^r(\eta_k) &= \mathscr{C}_3^r(\eta_k) + \mathscr{S}_3(\eta_k) F_k \mathscr{T}_4^r(\eta_k),
\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}
&\mathscr{A}_1(\eta_k, \eta_{k-d}) = \begin{bmatrix} f_1(x_k, x_{k-d}) + g_1(x_k)\bar{\Xi} u_c(\hat{x}_k) \\ f_c(\hat{x}_k) + g_c(\hat{x}_k)\bar{\Gamma} l(x_k) \end{bmatrix}, \qquad \mathscr{B}_1(\eta_k) = \begin{bmatrix} h_1(x_k) \\ g_c(\hat{x}_k) k(x_k) \end{bmatrix}, \\
&\mathscr{T}_1(\eta_k) = \begin{bmatrix} \bar{\Lambda}\bar{\Xi} u_c(\hat{x}_k) \\ \Lambda \bar{\Gamma} l(x_k) \end{bmatrix}, \qquad \mathscr{T}_2(\eta_k) = \begin{bmatrix} 0 \\ \Lambda k(x_k) \end{bmatrix}, \qquad \mathscr{T}_3(\eta_k) = \begin{bmatrix} \bar{\Lambda}\bar{\Xi} u_c(\hat{x}_k) \\ 0 \end{bmatrix}, \\
&\mathscr{T}_4^r(\eta_k) = \begin{bmatrix} \bar{\Lambda} C_p^r u_c(\hat{x}_k) \\ 0 \end{bmatrix}, \qquad \mathscr{T}_5^r(\eta_k) = \begin{bmatrix} 0 \\ \Lambda C_m^r l(x_k) \end{bmatrix}, \\
&\mathscr{C}_1^r(\eta_k) = \begin{bmatrix} g_1(x_k) C_p^r u_c(\hat{x}_k) \\ 0 \end{bmatrix}, \qquad \mathscr{C}_2^r(\eta_k) = \begin{bmatrix} 0 \\ g_c(\hat{x}_k) C_m^r l(x_k) \end{bmatrix},
\end{aligned} \tag{2.34}$$

$$\mathscr{A}_2(\eta_k, \eta_{k-d}) = f_2(x_k, x_{k-d}) + g_2(x_k)\bar{\Xi} u_c(\hat{x}_k),$$

$$\mathscr{C}_3^r(\eta_k) = g_2(x_k)C_p^r u_c(\hat{x}_k), \qquad \mathscr{S}_1(\eta_k) = \mathrm{diag}\{g_1(x_k), 0\},$$

$$\mathscr{S}_2(\eta_k) = \mathrm{diag}\{0, g_c(\hat{x}_k)\}, \qquad \mathscr{S}_3(\eta_k) = \begin{bmatrix} g_2(x_k) & 0 \end{bmatrix}.$$

Before giving the next theorem, we first recall some well-known lemmas.

Lemma 2.2 (Matrix Inverse Lemma) *Let X, Y, B, and C be given matrices of appropriate dimensions with X, Y, and $Y^{-1} + CX^{-1}B$ being invertible. Then*

$$(X + BYC)^{-1} = X^{-1} - X^{-1}B\big(Y^{-1} + CX^{-1}B\big)^{-1}CX^{-1}.$$

Lemma 2.3 ([166]) *For any matrices A, H, E, and $U = U^T$ of appropriate dimensions, there exists a positive definite matrix X such that for all F satisfying $F^T F \le I$,*

$$(A + HFE)^T X(A + HFE) + U < 0$$

if and only if there exists a positive constant $\alpha > 0$ such that

$$\alpha^{-1}I - H^T XH > 0,$$
$$A^T\big(X^{-1} - \alpha HH^T\big)^{-1}A + \alpha^{-1}E^T E + U < 0.$$

Lemma 2.4 ([167]) *Assume that the matrices A, H, E, and F with compatible dimensions such that $FF^T \le I$ are given. Let X be a symmetric positive definite matrix, and $\alpha > 0$ be an arbitrary positive constant such that $\alpha^{-1}I - EXE^T > 0$. Then, the following inequality holds*:

$$(A + HFE)X(A + HFE)^T \le A\big(X^{-1} - \alpha E^T E\big)^{-1}A^T + \alpha^{-1}HH^T.$$

Lemma 2.5 *Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and $\varepsilon > 0$. Then we have $2x^T y \le \varepsilon x^T x + \varepsilon^{-1} y^T y$.*

The following theorem provides a sufficient condition that is independent of the quantization effects Δ_k and $\bar{\Delta}_k$ but still guarantees the H_∞ performance and the stochastic stability of the closed-loop system (2.12) for the given two quantizers $q(\cdot)$ and $\bar{q}(\cdot)$.

Theorem 2.2 *Consider system (2.1). For a given disturbance attenuation level $\gamma > 0$ and two quantizers $q(\cdot)$ and $\bar{q}(\cdot)$, suppose that there exist two positive definite matrices $P^T = P > 0$, $Q^T = Q > 0$ and two positive scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ satisfying*

$$\gamma^2 I - \varepsilon_2 \mathscr{T}_2^T(\eta)\mathscr{T}_2(\eta) > 0, \tag{2.35}$$

$$R^{-1} - \Omega(\eta) - \varepsilon_1 \tilde{\mathscr{S}}_1(\eta)\tilde{\mathscr{S}}_1^T(\eta) > 0 \tag{2.36}$$

for all $\eta \in \mathbb{R}^{2n}$, *and*

$$\begin{aligned}
\widetilde{\mathfrak{H}}(\eta,\eta_d) := {} & \tilde{\mathscr{A}}^T(\eta,\eta_d)\big(R^{-1} - \Omega(\eta) - \varepsilon_1\tilde{\mathscr{S}}_1(\eta)\tilde{\mathscr{S}}_1^T(\eta)\big)^{-1}\tilde{\mathscr{A}}(\eta,\eta_d) \\
& + \mathscr{C}_{1c}^T(\eta)\big(P_p^{-1} - \varepsilon_1\mathscr{S}_{1p}(\eta)\mathscr{S}_{1p}^T(\eta)\big)^{-1}\mathscr{C}_{1c}(\eta) \\
& + \mathscr{C}_{2c}^T(\eta)\big(P_m^{-1} - \varepsilon_1\mathscr{S}_{2m}(\eta)\mathscr{S}_{2m}^T(\eta)\big)^{-1}\mathscr{C}_{2c}(\eta) \\
& + \mathscr{C}_{3c}^T(\eta)\big(I - \varepsilon_1\mathscr{S}_{3p}(\eta)\mathscr{S}_{3p}^T(\eta)\big)^{-1}\mathscr{C}_{3c}(\eta) \\
& + \varepsilon_1^{-1}\mathscr{T}_1^T(\eta)\mathscr{T}_1(\eta) + \varepsilon_1^{-1}\mathscr{T}_3^T(\eta)\mathscr{T}_3(\eta) \\
& + 2\varepsilon_1^{-1}\mathscr{T}_{1c}^T(\eta)\mathscr{T}_{1c}(\eta) + \varepsilon_1^{-1}\mathscr{T}_{2c}^T(\eta)\mathscr{T}_{2c}(\eta) + \mathscr{U}(\eta,\eta_d) \\
< {} & 0
\end{aligned} \tag{2.37}$$

for all nonzero $\eta, \eta_d \in \mathbb{R}^{2n}$, *where*

$$\begin{aligned}
& \tilde{\mathscr{A}}(\eta,\eta_d) = \big[\mathscr{A}_1^T(\eta,\eta_d) \quad \mathscr{A}_2^T(\eta,\eta_d)\big]^T, \qquad \tilde{\mathscr{B}}(\eta) = \big[\mathscr{B}_1^T(\eta) \quad \mathscr{H}_2^T(\eta)\big]^T, \\
& \tilde{\mathscr{S}}_2(\eta) = \big[\mathscr{S}_2^T(\eta) \quad 0\big]^T, \qquad \tilde{\mathscr{S}}_1(\eta) = \operatorname{diag}\big\{\mathscr{S}_1(\eta) + \mathscr{S}_2(\eta), \mathscr{S}_3(\eta)\big\}, \\
& \mathscr{U}(\eta,\eta_d) = \theta\mathscr{F}_w^T(\eta,\eta_d)P\mathscr{F}_w(\eta,\eta_d) + \eta^T(Q-P)\eta - \eta_d^T Q\eta_d, \\
& \mathscr{C}_{1c}(\eta) = \Big[\alpha_1\mathscr{C}_1^{1T}(\eta) \quad \alpha_2\mathscr{C}_1^{2T}(\eta) \quad \cdots \quad \alpha_p\mathscr{C}_1^{pT}(\eta)\Big]^T, \\
& \mathscr{T}_{1c}(\eta) = \Big[\alpha_1\mathscr{T}_4^{1T}(\eta) \quad \alpha_2\mathscr{T}_4^{2T}(\eta) \quad \cdots \quad \alpha_p\mathscr{T}_4^{pT}(\eta)\Big]^T, \\
& \mathscr{C}_{2c}(\eta) = \big[\beta_1\mathscr{C}_2^{1T}(\eta) \quad \beta_2\mathscr{C}_2^{2T}(\eta) \quad \cdots \quad \beta_m\mathscr{C}_2^{mT}(\eta)\big]^T, \\
& \mathscr{T}_{2c}(\eta) = \big[\beta_1\mathscr{T}_5^{1T}(\eta) \quad \beta_2\mathscr{T}_5^{2T}(\eta) \quad \cdots \quad \beta_m\mathscr{T}_5^{mT}(\eta)\big]^T, \\
& \mathscr{C}_{3c}(\eta) = \Big[\alpha_1\mathscr{C}_3^{1T}(\eta) \quad \alpha_2\mathscr{C}_3^{2T}(\eta) \quad \cdots \quad \alpha_p\mathscr{C}_3^{pT}(\eta)\Big]^T, \\
& P_p = \operatorname{diag}_p\{P\}, \qquad P_m = \operatorname{diag}_m\{P\}, \\
& \Omega(\eta) = \tilde{\mathscr{B}}(\eta)\big(\gamma^2 I - \varepsilon_2\mathscr{T}_2^T(\eta)\mathscr{T}_2(\eta)\big)^{-1}\tilde{\mathscr{B}}^T(\eta) + \varepsilon_2^{-1}\tilde{\mathscr{S}}_2(\eta)\tilde{\mathscr{S}}_2^T(\eta), \\
& R = \operatorname{diag}\{P, I\}, \qquad \mathscr{S}_{1p}(\eta) = \operatorname{diag}_p\big\{\mathscr{S}_1(\eta)\big\}, \\
& \mathscr{S}_{2m}(\eta) = \operatorname{diag}_m\big\{\mathscr{S}_2(\eta)\big\}, \qquad \mathscr{S}_{3p}(\eta) = \operatorname{diag}_p\big\{\mathscr{S}_3(\eta)\big\},
\end{aligned} \tag{2.38}$$

for some nonlinear parameter-functions f_c, g_c, *and* u_c. *Then the quantized nonlinear stochastic* H_∞ *control problem for system* (2.1) *is solved by the controller* (2.5).

Proof For presentation convenience, we first define

$$\tilde{\mathscr{F}}(\eta,\eta_d) = \big[\mathscr{F}_1^T(\eta,\eta_d) \quad \mathscr{F}_2^T(\eta,\eta_d)\big]^T, \qquad \tilde{\mathscr{H}}(\eta) = \big[\mathscr{H}_1^T(\eta) \quad \mathscr{H}_2^T(\eta)\big]^T.$$

After some manipulations, we have

$$\tilde{\mathscr{F}}(\eta,\eta_d)=\tilde{\mathscr{A}}(\eta,\eta_d)+\tilde{\mathscr{S}}_1(\eta)\tilde{F}\tilde{\mathscr{T}}(\eta), \tag{2.39}$$

$$\tilde{\mathscr{H}}(\eta)=\tilde{\mathscr{B}}(\eta)+\tilde{\mathscr{S}}_2(\eta)F\mathscr{T}_2(\eta), \tag{2.40}$$

where $\tilde{\mathscr{T}}(\eta)=[\mathscr{T}_1^T(\eta)\ \ \mathscr{T}_3^T(\eta)]^T$, $\tilde{F}=\mathrm{diag}\{F,F\}$, and $\tilde{\mathscr{A}}(\eta,\eta_d)$, $\tilde{\mathscr{B}}(\eta)$, $\tilde{\mathscr{S}}_1(\eta)$, $\tilde{\mathscr{S}}_2(\eta)$ are defined in (2.38).

By applying Schur complement, it is known that condition (2.35) is equivalent to

$$\varepsilon_2^{-1}I-\gamma^{-2}\mathscr{T}_2(\eta)\mathscr{T}_2^T(\eta)>0.$$

Hence, it follows from Lemma 2.4 that

$$\gamma^{-2}\tilde{\mathscr{H}}(\eta)\tilde{\mathscr{H}}^T(\eta)\le\Omega(\eta), \tag{2.41}$$

where $\Omega(\eta)$ is defined in (2.38). In addition, it can be easily seen from (2.36) that

$$R^{-1}-\Omega(\eta)>0. \tag{2.42}$$

Consequently, from (2.41)–(2.42) we get

$$R^{-1}-\gamma^{-2}\tilde{\mathscr{H}}(\eta)\tilde{\mathscr{H}}^T(\eta)>0,$$

which is obviously equivalent to (2.30) in Corollary 2.1.

On the other hand, we rewrite $\mathcal{H}(\eta,\eta_d)$ in the following compact form:

$$\begin{aligned}\mathcal{H}(\eta,\eta_d)=&\ \tilde{\mathscr{F}}^T(\eta,\eta_d)R\tilde{\mathscr{H}}(\eta)\big(\gamma^2I-\tilde{\mathscr{H}}^T(\eta)R\tilde{\mathscr{H}}(\eta)\big)^{-1}\tilde{\mathscr{H}}^T(\eta)R\tilde{\mathscr{F}}(\eta,\eta_d)\\&+\tilde{\mathscr{F}}^T(\eta,\eta_d)R\tilde{\mathscr{F}}(\eta,\eta_d)+\mathscr{G}_{1c}^T(\eta)P_p\mathscr{G}_{1c}(\eta)+\mathscr{G}_{2c}^T(\eta)P_m\mathscr{G}_{2c}(\eta)\\&+\mathscr{G}_{3c}^T(\eta)\mathscr{G}_{3c}(\eta)+\mathscr{U}(\eta,\eta_d),\end{aligned}$$

where

$$\mathscr{G}_{1c}(\eta)=\mathscr{C}_{1c}(\eta)+\mathscr{S}_{1p}(\eta)F_p\mathscr{T}_{1c}(\eta), \tag{2.43}$$

$$\mathscr{G}_{2c}(\eta)=\mathscr{C}_{2c}(\eta)+\mathscr{S}_{2m}(\eta)F_m\mathscr{T}_{2c}(\eta), \tag{2.44}$$

$$\mathscr{G}_{3c}(\eta)=\mathscr{C}_{3c}(\eta)+\mathscr{S}_{3p}(\eta)F_p\mathscr{T}_{1c}(\eta) \tag{2.45}$$

with $F_p=\mathrm{diag}_p\{F\}$, $F_m=\mathrm{diag}_m\{F\}$, and $\mathscr{U}(\eta,\eta_d)$, $\mathscr{C}_{1c}(\eta)$, $\mathscr{C}_{2c}(\eta)$, $\mathscr{C}_{3c}(\eta)$, $\mathscr{S}_{1p}(\eta)$, $\mathscr{S}_{2m}(\eta)$, $\mathscr{S}_{3p}(\eta)$, $\mathscr{T}_{1c}(\eta)$, $\mathscr{T}_{2c}(\eta)$ are defined in (2.38). Then, in virtue of Lemma 2.2 (Matrix Inverse Lemma), we obtain

$$\begin{aligned}\mathcal{H}(\eta,\eta_d)=&\ \tilde{\mathscr{F}}^T(\eta,\eta_d)\big(R^{-1}-\gamma^{-2}\tilde{\mathscr{H}}(\eta)\tilde{\mathscr{H}}^T(\eta)\big)^{-1}\tilde{\mathscr{F}}(\eta,\eta_d)+\mathscr{G}_{1c}^T(\eta)P_p\mathscr{G}_{1c}(\eta)\\&+\mathscr{G}_{2c}^T(\eta)P_m\mathscr{G}_{2c}(\eta)+\mathscr{G}_{3c}^T(\eta)\mathscr{G}_{3c}(\eta)+\mathscr{U}(\eta,\eta_d).\end{aligned} \tag{2.46}$$

Noting (2.41) and (2.42), it follows from (2.46) that

$$\begin{aligned}\overline{\mathcal{H}}(\eta,\eta_d) &:= \tilde{\mathscr{F}}^T(\eta,\eta_d)\big(R^{-1}-\Omega(\eta)\big)^{-1}\tilde{\mathscr{F}}(\eta,\eta_d)+\mathscr{G}_{1c}^T(\eta)P_p\mathscr{G}_{1c}(\eta)\\ &\quad+\mathscr{G}_{2c}^T(\eta)P_m\mathscr{G}_{2c}(\eta)+\mathscr{G}_{3c}^T(\eta)\mathscr{G}_{3c}(\eta)+\mathscr{U}(\eta,\eta_d)\\ &\geq \mathcal{H}(\eta,\eta_d).\end{aligned}\tag{2.47}$$

Next, let us "eliminate" the uncertainties in (2.47) by using Lemma 2.3. From (2.36) we have

$$\varepsilon_1^{-1}I-\tilde{\mathscr{S}}_1^T(\eta)\big(R^{-1}-\Omega(\eta)\big)^{-1}\tilde{\mathscr{S}}_1(\eta)>0.\tag{2.48}$$

Considering $\Omega(\eta)\geq 0$, it can also be obtained from (2.36) that $R^{-1}-\varepsilon_1\times\tilde{\mathscr{S}}_1(\eta)\tilde{\mathscr{S}}_1^T(\eta)>0$, which results in

$$I-\varepsilon_1\mathscr{S}_3(\eta)\mathscr{S}_3^T(\eta)>0,\tag{2.49}$$

$$P^{-1}-\varepsilon_1\big(\mathscr{S}_1(\eta)+\mathscr{S}_2(\eta)\big)\big(\mathscr{S}_1(\eta)+\mathscr{S}_2(\eta)\big)^T>0.\tag{2.50}$$

Noting that $\mathscr{S}_1(\eta)\mathscr{S}_2^T(\eta)=0$, we know that (2.50) implies

$$P^{-1}-\varepsilon_1\mathscr{S}_1(\eta)\mathscr{S}_1^T(\eta)>0,\tag{2.51}$$

$$P^{-1}-\varepsilon_1\mathscr{S}_2(\eta)\mathscr{S}_2^T(\eta)>0.\tag{2.52}$$

After using Schur complement again and conducting the augmented manipulation, it can be seen that (2.49), (2.51), and (2.52) are equivalent to

$$\varepsilon_1^{-1}I-\mathscr{S}_{3p}^T(\eta)\mathscr{S}_{3p}(\eta)>0,\tag{2.53}$$

$$\varepsilon_1^{-1}I-\mathscr{S}_{1p}^T(\eta)P_p\mathscr{S}_{1p}(\eta)>0,\tag{2.54}$$

$$\varepsilon_1^{-1}I-\mathscr{S}_{2m}^T(\eta)P_m\mathscr{S}_{2m}(\eta)>0,\tag{2.55}$$

respectively. Subsequently, by Lemma 2.3, we know that under conditions (2.48) and (2.53)–(2.55) together with (2.37), the inequality $\overline{\mathcal{H}}(\eta,\eta_d)<0$ is true, which implies $\mathcal{H}(\eta,\eta_d)<0$ from (2.47). So far, (2.30) and (2.31) in Corollary 2.1 have been shown to hold. Therefore, the rest of the proof can be directly obtained from Corollary 2.1, which is omitted here. □

Before giving further results, we make the following assumption on the plant (2.1) for the purpose of simplicity.

Assumption 2.1 *The system matrices $h_1(x)$, $h_2(x)$, and $k(x)$ are assumed to satisfy*

$$h_1(x)h_2^T(x)=0,\tag{2.56}$$

$$h_1(x)k^T(x)=0,\tag{2.57}$$

$$h_2(x)k^T(x)=0.\tag{2.58}$$

Assumption 2.1 means that the measurement noise, the output noise, and the system noise are mutually independent. Similar assumptions can be found in [2, 36].

Theorem 2.3 *Let the disturbance attenuation level* $\gamma > 0$, *the two quantizers* $q(\cdot)$ *and* $\bar{q}(\cdot)$, *and the controller parameter-functions* f_c, g_c, u_c *be given. The quantized nonlinear stochastic* H_∞ *control problem for system* (2.1) *is solved by the controller* (2.5) *if there exist positive definite matrices* $P_1^T = P_1 > 0$, $P_2^T = P_2 > 0$, $Q_1^T = Q_1 > 0$, $Q_2^T = Q_2 > 0$ *and positive scalars* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\lambda > 0$ *satisfying the following inequalities*:

$$\gamma^2 I - \varepsilon_2 k^T(x)\Lambda^2 k(x) \geq \lambda I, \tag{2.59}$$

$$\Phi_1(x) := P_1^{-1} - \lambda^{-1} h_1(x) h_1^T(x) - \varepsilon_1 g_1(x) g_1^T(x) > 0, \tag{2.60}$$

$$\Phi_2(x,\hat{x}) := P_2^{-1} - \lambda^{-1} g_c(\hat{x}) k(x) k^T(x) g_c^T(\hat{x}) - \left(\varepsilon_1 + \varepsilon_2^{-1}\right) g_c(\hat{x}) g_c^T(\hat{x}) > 0, \tag{2.61}$$

$$\Phi_3(x) := I - \lambda^{-1} h_2(x) h_2^T(x) - \varepsilon_1 g_2(x) g_2^T(x) > 0 \tag{2.62}$$

for all $x, \hat{x} \in \mathbb{R}^n$, *and*

$$\widehat{\mathcal{H}}(x, x_d, \hat{x}, \hat{x}_d) := \mathscr{W}_1(x, x_d, \hat{x}) + \mathscr{W}_2(x, \hat{x}) + 2\mathscr{W}_3(x, x_d, \hat{x}) + \mathscr{U}(\eta, \eta_d) < 0$$

for all nonzero $x, \hat{x}, x_d, \hat{x}_d \in \mathbb{R}^n$, *where*

$$\begin{aligned}
\mathscr{W}_1(x, x_d, \hat{x}) = {} & f_1^T(x, x_d)\Phi_1^{-1}(x) f_1(x, x_d) + f_c^T(\hat{x})\Phi_2^{-1}(x,\hat{x}) f_c(\hat{x}) \\
& + f_2^T(x, x_d)\Phi_3^{-1}(x) f_2(x, x_d) \\
& + u_c^T(\hat{x})\bar{\Xi} g_1^T(x)\Phi_1^{-1}(x) g_1(x)\bar{\Xi} u_c(\hat{x}) \\
& + l^T(x)\bar{\Gamma} g_c^T(\hat{x})\Phi_2^{-1}(x,\hat{x}) g_c(\hat{x})\bar{\Gamma} l(x) \\
& + u_c^T(\hat{x})\bar{\Xi} g_2^T(x)\Phi_3^{-1}(x) g_2(x)\bar{\Xi} u_c(\hat{x}) \\
& + 2\varepsilon_1^{-1}\left\|\bar{\Lambda}\bar{\Xi} u_c(\hat{x})\right\|^2 + \varepsilon_1^{-1}\left\|\Lambda\bar{\Gamma} l(x)\right\|^2, \\
\mathscr{W}_2(x,\hat{x}) = {} & \sum_{r=1}^{p} \alpha_r^2 u_c^T(\hat{x}) C_p^r g_1^T(x)\Psi_1^{-1}(x) g_1(x) C_p^r u_c(\hat{x}) \\
& + \sum_{r=1}^{m} \beta_r^2 l^T(x) C_m^r g_c^T(\hat{x})\Psi_2^{-1}(\hat{x}) g_c(\hat{x}) C_m^r l(x) \\
& + \sum_{r=1}^{p} \alpha_r^2 u_c^T(\hat{x}) C_p^r g_2^T(x)\Psi_3^{-1}(x) g_2(x) C_p^r u_c(\hat{x}) \\
& + 2\varepsilon_1^{-1}\sum_{r=1}^{p} \alpha_r^2\left\|\bar{\Lambda} C_p^r u_c(\hat{x})\right\|^2 + \varepsilon_1^{-1}\sum_{r=1}^{m} \beta_r^2\left\|\Lambda C_m^r l(x)\right\|^2,
\end{aligned} \tag{2.63}$$

$$\begin{aligned}\mathscr{W}_3(x, x_d, \hat{x}) &= f_1^T(x, x_d)\Phi_1^{-1}(x)g_1(x)\bar{\Xi}u_c(\hat{x}) \\ &\quad + f_c^T(\hat{x})\Phi_2^{-1}(x, \hat{x})g_c(\hat{x})\bar{\Gamma}l(x) \\ &\quad + f_2^T(x, x_d)\Phi_3^{-1}(x)g_2(x)\bar{\Xi}u_c(\hat{x}), \\ \mathscr{U}(\eta, \eta_d) &= \theta f_w^T(x, x_d)P_1 f_w(x, x_d) + x^T(Q_1 - P_1)x \\ &\quad + \hat{x}^T(Q_2 - P_2)\hat{x} - x_d^T Q_1 x_d - \hat{x}_d^T Q_2 \hat{x}_d,\end{aligned}$$

$$\begin{aligned}\Psi_1(x) &= P_1^{-1} - \varepsilon_1 g_1(x)g_1^T(x), \\ \Psi_2(\hat{x}) &= P_2^{-1} - \varepsilon_1 g_c(\hat{x})g_c^T(\hat{x}), \\ \Psi_3(x) &= I - \varepsilon_1 g_2(x)g_2^T(x).\end{aligned} \tag{2.64}$$

Proof Let $P = \text{diag}\{P_1, P_2\}$ and $Q = \text{diag}\{Q_1, Q_2\}$. It follows from (2.34) that (2.59) is equivalent to

$$\gamma^2 I - \varepsilon_2 \mathscr{T}_2^T(\eta)\mathscr{T}_2(\eta) \geq \lambda I,$$

which means that (2.35) is guaranteed by (2.59).

Under Assumption 2.1 and by a series of computations, it can be obtained from (2.59) that

$$R^{-1} - \Omega(\eta) - \varepsilon_1 \tilde{\mathscr{S}}_1(\eta)\tilde{\mathscr{S}}_1^T(\eta) \geq \begin{bmatrix} \Phi_1(x) & 0 & 0 \\ 0 & \Phi_2(x, \hat{x}) & 0 \\ 0 & 0 & \Phi_3(x) \end{bmatrix}. \tag{2.65}$$

Hence, (2.36) is obtained from (2.60)–(2.62).

Now, it remains to show that $\widetilde{\mathcal{H}}(\eta, \eta_d) < 0$. Considering (2.34) and (2.38), it follows from (2.65) that

$$\begin{aligned}&\tilde{\mathscr{A}}^T(\eta, \eta_d)\big(R^{-1} - \Omega(\eta) - \varepsilon_1 \tilde{\mathscr{S}}_1(\eta)\tilde{\mathscr{S}}_1^T(\eta)\big)^{-1}\tilde{\mathscr{A}}(\eta, \eta_d) \\ &\quad \leq f_1^T(x, x_d)\Phi_1^{-1}(x)f_1(x, x_d) + f_c^T(\hat{x})\Phi_2^{-1}(x, \hat{x})f_c(\hat{x}) \\ &\qquad + f_2^T(x, x_d)\Phi_3^{-1}(x)f_2(x, x_d) + u_c^T(\hat{x})\bar{\Xi}g_1^T(x)\Phi_1^{-1}(x)g_1(x)\bar{\Xi}u_c(\hat{x}) \\ &\qquad + l^T(x)\bar{\Gamma}g_c^T(\hat{x})\Phi_2^{-1}(x, \hat{x})g_c(\hat{x})\bar{\Gamma}l(x) \\ &\qquad + u_c^T(\hat{x})\bar{\Xi}g_2^T(x)\Phi_3^{-1}(x)g_2(x)\bar{\Xi}u_c(\hat{x}) \\ &\qquad + 2\big(f_1^T(x, x_d)\Phi_1^{-1}(x)g_1(x)\bar{\Xi}u_c(\hat{x}) + f_c^T(\hat{x})\Phi_2^{-1}(x, \hat{x})g_c(\hat{x})\bar{\Gamma}l(x) \\ &\qquad + f_2^T(x, x_d)\Phi_3^{-1}(x)g_2(x)\bar{\Xi}u_c(\hat{x})\big).\end{aligned} \tag{2.66}$$

By some straightforward manipulations and noting that $\Psi_1(x) > 0$, $\Psi_2(\hat{x}) > 0$ and $\Psi_3(x) > 0$ from (2.60)–(2.62), one can get

$$\begin{aligned}
&\mathscr{C}_{1c}^T(\eta)\big(P_p^{-1} - \varepsilon_1 \mathscr{S}_{1p}(\eta)\mathscr{S}_{1p}^T(\eta)\big)^{-1}\mathscr{C}_{1c}(\eta) \\
&\quad = \sum_{r=1}^{p} \alpha_r^2 u_c^T(\hat{x}) C_p^r g_1^T(x)\Psi_1^{-1}(x) g_1(x) C_p^r u_c(\hat{x}), \\
&\mathscr{C}_{2c}^T(\eta)\big(P_m^{-1} - \varepsilon_1 \mathscr{S}_{2m}(\eta)\mathscr{S}_{2m}^T(\eta)\big)^{-1}\mathscr{C}_{2c}(\eta) \\
&\quad = \sum_{r=1}^{m} \beta_r^2 l^T(x) C_m^r g_c^T(\hat{x})\Psi_2^{-1}(\hat{x}) g_c(\hat{x}) C_m^r l(x), \\
&\mathscr{C}_{3c}^T(\eta)\big(I - \varepsilon_1 \mathscr{S}_{3p}(\eta)\mathscr{S}_{3p}^T(\eta)\big)^{-1}\mathscr{C}_{3c}(\eta) \\
&\quad = \sum_{r=1}^{p} \alpha_r^2 u_c^T(\hat{x}) C_p^r g_2^T(x)\Psi_3^{-1}(x) g_2(x) C_p^r u_c(\hat{x}), \\
&\mathscr{T}_1^T(\eta)\mathscr{T}_1(\eta) = \big\|\bar{\Lambda}\bar{\Xi} u_c(\hat{x})\big\|^2 + \big\|\Lambda\bar{\Gamma} l(x)\big\|^2, \\
&\mathscr{T}_3^T(\eta)\mathscr{T}_3(\eta) = \big\|\bar{\Lambda}\bar{\Xi} u_c(\hat{x})\big\|^2, \\
&\mathscr{T}_{1c}^T(\eta)\mathscr{T}_{1c}(\eta) = \sum_{r=1}^{p} \alpha_r^2 \big\|\bar{\Lambda} C_p^r u_c(\hat{x})\big\|^2, \\
&\mathscr{T}_{2c}^T(\eta)\mathscr{T}_{2c}(\eta) = \sum_{r=1}^{m} \beta_r^2 \big\|\Lambda C_m^r l(x)\big\|^2, \\
&\mathscr{U}(\eta, \eta_d) = \theta f_w^T(x, x_d) P_1 f_w(x, x_d) + x^T(Q_1 - P_1)x \\
&\qquad\qquad + \hat{x}^T(Q_2 - P_2)\hat{x} - x_d^T Q_1 x_d - \hat{x}_d^T Q_2 \hat{x}_d.
\end{aligned} \tag{2.67}$$

It can be obtained from (2.66) and (2.67) that $\widetilde{\mathcal{H}}(\eta, \eta_d) \le \widehat{\mathcal{H}}(x, x_d, \hat{x}, \hat{x}_d) < 0$. Therefore, the proof of this theorem follows immediately from that of Theorem 2.2. □

In practice, the matrix functions $h_1(x)$, $h_2(x)$, $g_1(x)$, $g_2(x)$, and $k(x)$ are usually taken as constant matrices as follows:

$$\begin{aligned}
&h_1(x) = H_1, \qquad h_2(x) = H_2, \qquad g_1(x) = G_1, \\
&G_2(x) = G_2, \qquad k(x) = K,
\end{aligned} \tag{2.68}$$

and it is assumed that

$$H_1 H_2^T = 0, \qquad H_1 K^T = 0, \quad H_2 K^T = 0. \tag{2.69}$$

Furthermore, consider the issue of easy implementation, *linear time-invariant controller* is often designed in practical engineering. In view of this, we are going to show that the main results obtained so far can be directly specialized to the system with linear controller. We adopt the following linear observer-based controller:

$$\begin{cases} \hat{x}_{k+1} = F_c \hat{x}_k + G_c \bar{y}_k, \\ \bar{u}_k = U_c \hat{x}_k, \qquad \hat{x}_0 = 0, \end{cases} \tag{2.70}$$

where F_c, G_c, and U_c are the parameter matrices to be determined.

The following corollary is easily accessible from Theorem 2.3.

Corollary 2.2 *Let the disturbance attenuation level* $\gamma > 0$, *two quantizers* $q(\cdot)$ *and* $\bar{q}(\cdot)$, *and the controller parameter matrices* F_c, G_c, U_c *be given. Suppose that there exist positive definite matrices* $P_1^T = P_1 > 0$, $P_2^T = P_2 > 0$, $Q_1^T = Q_1 > 0$, $Q_2^T = Q_2 > 0$ *and positive scalars* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\lambda > 0$, $\mu > 0$ *satisfying the following inequalities*:

$$\gamma^2 I - \varepsilon_2 K^T \Lambda^2 K \geq \lambda I, \tag{2.71}$$

$$\Phi_1 := P_1^{-1} - \lambda^{-1} H_1 H_1^T - \varepsilon_1 G_1 G_1^T \geq \mu I, \tag{2.72}$$

$$\Phi_2 := P_2^{-1} - \lambda^{-1} G_c K K^T G_c^T - \left(\varepsilon_1 + \varepsilon_2^{-1}\right) G_c G_c^T \geq \mu I, \tag{2.73}$$

$$\Phi_3 := I - \lambda^{-1} H_2 H_2^T - \varepsilon_1 G_2 G_2^T \geq \mu I, \tag{2.74}$$

$$\begin{aligned} \mathcal{H}_1 :=\ & \mu^{-1}\left(1 + \varepsilon_3^{-1}\right) U_c^T \bar{\Xi} \left(G_1^T G_1 + G_2^T G_2\right) \bar{\Xi} U_c + \mu^{-1}(1 + \varepsilon_3) F_c^T F_c \\ & + \mu^{-1} \sum_{r=1}^{p} \alpha_r^2 U_c^T C_p^r \left(G_1^T G_1 + G_2^T G_2\right) C_p^r U_c \\ & + 2\varepsilon_1^{-1} \sum_{r=1}^{p} \alpha_r^2 U_c^T C_p^r \bar{\Lambda}^2 C_p^r U_c \\ & + 2\varepsilon_1^{-1} U_c^T \bar{\Xi} \bar{\Lambda}^2 \bar{\Xi} U_c + Q_2 - P_2 < 0, \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} \mathcal{H}_2(x, x_d) :=\ & \mu^{-1}(1 + \varepsilon_3)\left(\left\| f_1(x, x_d)\right\|^2 + \left\| f_2(x, x_d)\right\|^2\right) \\ & + \mu^{-1}\left(1 + \varepsilon_3^{-1}\right)\left\| G_c \bar{\Gamma} l(x)\right\|^2 + \varepsilon_1^{-1}\left\| \Lambda \bar{\Gamma} l(x)\right\|^2 \\ & + \mu^{-1} \sum_{r=1}^{m} \beta_r^2 \left\| G_c C_m^r l(x)\right\|^2 + \varepsilon_1^{-1} \sum_{r=1}^{m} \beta_r^2 \left\| \Lambda C_m^r l(x)\right\|^2 \\ & + \theta f_w^T(x, x_d) P_1 f_w(x, x_d) + x^T (Q_1 - P_1) x - x_d^T Q_1 x_d < 0 \end{aligned} \tag{2.76}$$

for all nonzero $x, x_d \in \mathbb{R}^n$. *Then the quantized nonlinear stochastic* H_∞ *control problem for system* (2.1) *is solved by the controller* (2.70).

Proof Under assumption (2.69), inequalities (2.59)–(2.62) follow from (2.71)–(2.74) by replacing H_1, H_2, G_1, G_2, K, and G_c with $h_1(x)$, $h_2(x)$, $g_1(x)$, $g_2(x)$, $k(x)$, and $g_c(\hat{x})$, respectively. Also, it follows from (2.72)–(2.74) that

$$\begin{aligned}
&\mathscr{W}_1(x, x_d, \hat{x}) \\
&\quad \leq \mu^{-1}\big(\big\|f_1(x, x_d)\big\|^2 + \|F_c\hat{x}\|^2 + \big\|f_2(x, x_d)\big\|^2 + \|G_1\bar{\Xi}U_c\hat{x}\|^2 \\
&\qquad + \big\|G_c\bar{\Gamma}l(x)\big\|^2 + \|G_2\bar{\Xi}U_c\hat{x}\|^2\big) + 2\varepsilon_1^{-1}\|\bar{\Lambda}\bar{\Xi}U_c\hat{x}\|^2 + \varepsilon_1^{-1}\big\|\Lambda\bar{\Gamma}l(x)\big\|^2.
\end{aligned} \tag{2.77}$$

Noting that (2.72)–(2.74) imply $\Psi_1(x) \geq \mu I$, $\Psi_2(\hat{x}) \geq \mu I$, and $\Psi_3(x) \geq \mu I$, respectively, one has

$$\begin{aligned}
&\mathscr{W}_2(x, \hat{x}) \\
&\quad \leq \mu^{-1}\left(\sum_{r=1}^{p}\alpha_r^2\big\|G_1C_p^rU_c\hat{x}\big\|^2 + \sum_{r=1}^{m}\beta_r^2\big\|G_cC_m^rl(x)\big\|^2 + \sum_{r=1}^{p}\alpha_r^2\big\|G_2C_p^rU_c\hat{x}\big\|^2\right) \\
&\qquad + 2\varepsilon_1^{-1}\sum_{r=1}^{p}\alpha_r^2\big\|\bar{\Lambda}C_p^rU_c\hat{x}\big\|^2 + \varepsilon_1^{-1}\sum_{r=1}^{m}\beta_r^2\big\|\Lambda C_m^rl(x)\big\|^2.
\end{aligned} \tag{2.78}$$

By Lemma 2.5, it follows from (2.72)–(2.74) that

$$\begin{aligned}
&\mathscr{W}_3(x, x_d, \hat{x}) \\
&\quad \leq \frac{1}{2}\mu^{-1}\big(\varepsilon_3\big(\big\|f_1(x, x_d)\big\|^2 + \|F_c\hat{x}\|^2 + \big\|f_2(x, x_d)\big\|^2\big) + \varepsilon_3^{-1}\big(\|G_1\bar{\Xi}U_c\hat{x}\|^2 \\
&\qquad + \big\|G_c\bar{\Gamma}l(x)\big\|^2 + \|G_2\bar{\Xi}U_c\hat{x}\|^2\big)\big).
\end{aligned} \tag{2.79}$$

Consequently, it can be obtained from (2.77)–(2.79) together with (2.63) that

$$\widehat{\mathcal{H}}(x, x_d, \hat{x}, \hat{x}_d) \leq \hat{x}^T\mathcal{H}_1\hat{x} + \mathcal{H}_2(x, x_d) - \hat{x}_d^TQ_2\hat{x}_d.$$

In view of (2.75)–(2.76) and noticing that $Q_2 > 0$, we have $\widehat{\mathcal{H}}(x, x_d, \hat{x}, \hat{x}_d) < 0$ for all nonzero $x, \hat{x}, x_d, \hat{x}_d \in \mathbb{R}^n$. Therefore, the rest of the proof immediately follows from that of Theorem 2.3. □

It is well known that the H_∞ controllers are difficult to be designed for *nonlinear* stochastic systems of *a very general form*. Therefore, it has been more and more common in the literature to assume that the nonlinearities are bounded by a linearity-like form (e.g., Lipschitz and sector conditions), and this makes it possible to deal with the problems by using an LMI approach. Such an approach, however, will inevitably lead to some conservatism due to the assumption on the nonlinearities. In this chapter, Theorem 2.3 is proved mainly by the "completing the square" technique, which results in very little conservatism.

2.3 Some Special Cases

To demonstrate that Theorem 2.3 serves as a theoretic basis for the H_∞ control problems of nonlinear stochastic systems, in this section, we aim to show that Theorem 2.3 can be specialized to the following three kinds of stochastic systems that have been extensively studied in the literature: (1) systems with Lipschitz-type nonlinearities, (2) systems with sector-bounded nonlinearities, and (3) linear systems. The specialized results are described in terms of the LMIs, which can be solved by the efficient Matlab LMI toolbox.

Case 1 We first consider a special class of nonlinear stochastic systems with nonlinearities described by Lipschitz condition. For this purpose, we assume that

$$f_1(x, x_d) = A_1 x + A_{1d} x_d + E\psi(x) + E_d \psi_d(x_d), \tag{2.80}$$

$$f_2(x, x_d) = A_2 x + A_{2d} x_d, \qquad l(x) = Lx, \tag{2.81}$$

$$f_w(x, x_d) = A_w x + A_{wd} x_d, \tag{2.82}$$

where A_i, A_{id} $(i = 1, 2)$, E, E_d, A_w, A_{wd}, and L are known real matrices. The nonlinear terms $\psi(x)$ and $\psi_d(x_d)$ satisfy the following Lipschitz condition:

$$\left\|\psi(x)\right\| \le \|Mx\|, \tag{2.83}$$

$$\left\|\psi_d(x_d)\right\| \le \|M_d x_d\|, \tag{2.84}$$

where M and M_d are given real matrices. The reason why we include the nonlinearity in $f_1(x, x_d)$ only is to avoid unnecessarily complicated notation and keep the mathematics exposition concise. It is not difficult to consider the Lipschitz-like nonlinearities in $f_2(x, x_d)$ and $f_w(x, x_d)$ and obtain the corresponding results.

The following corollary, which can be easily obtained from Corollary 2.2, shows that the quantized H_∞ control problem for stochastic time-delay systems with Lipschitz-like nonlinearities and missing measurements can be solved by the numerically appealing LMI approach.

Corollary 2.3 *Let the disturbance attenuation level $\gamma > 0$ be given. The quantized nonlinear stochastic H_∞ control problem for system (2.1) with the nonlinearities bounded by Lipschitz condition (2.83) and (2.84) is solved by the linear observer-based controller (2.70) if there exist positive definite matrices $P_1^T = P_1 > 0$, $R_2^T = R_2 > 0$, $Q_1^T = Q_1 > 0$, $\tilde{Q}_2^T = \tilde{Q}_2 > 0$, real matrices X, G_c, Y, and positive scalars $\kappa_1 > 0$, $\kappa_2 > 0$, $\varepsilon_2 > 0$, $\lambda > 0$ such that the following LMIs hold for given positive scalars $\varepsilon_1 > 0$, $\varepsilon_3 > 0$, and $\mu > 0$:*

$$\gamma^2 I - \varepsilon_2 K^T \Lambda^2 K \ge \lambda I, \tag{2.85}$$

$$\begin{bmatrix} -P_1 & P_1 H_1 & P_1 G_1 & P_1 \\ * & -\lambda I & 0 & 0 \\ * & * & -\varepsilon_1^{-1} I & 0 \\ * & * & * & -\mu^{-1} I \end{bmatrix} < 0, \tag{2.86}$$

$$\begin{bmatrix} -R_2 & G_c K & G_c & G_c & I \\ * & -\lambda I & 0 & 0 & 0 \\ * & * & -\varepsilon_1^{-1} I & 0 & 0 \\ * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & -\mu^{-1} I \end{bmatrix} < 0, \tag{2.87}$$

$$\begin{bmatrix} -I & H_2 & G_2 & I \\ * & -\lambda I & 0 & 0 \\ * & * & -\varepsilon_1^{-1} I & 0 \\ * & * & * & -\mu^{-1} I \end{bmatrix} < 0, \tag{2.88}$$

$$\begin{bmatrix} \tilde{Q}_2 - R_2 & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix} < 0, \tag{2.89}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \theta A_w^T P_1 & \Pi_{15} \\ * & \Pi_{22} & \Pi_{23} & \theta A_{wd}^T P_1 & 0 \\ * & * & \Pi_{33} & 0 & 0 \\ * & * & * & -\theta P_1 & 0 \\ * & * & * & * & \Pi_{55} \end{bmatrix} < 0, \tag{2.90}$$

where

$$\Theta_{12} = \begin{bmatrix} X^T \bar{\Xi} G_1^T & X^T \bar{\Xi} G_2^T & X^T C_{pc}^T G_{1p}^T & X^T C_{pc}^T G_{2p}^T & Y^T & X^T C_{pc}^T \bar{\Lambda}_p & X^T \bar{\Xi} \bar{\Lambda} \end{bmatrix},$$

$$\Theta_{22} = \mathrm{diag}\left\{ -\frac{\mu I}{1+\varepsilon_3^{-1}}, -\frac{\mu I}{1+\varepsilon_3^{-1}}, -\mu I, -\mu I, -\frac{\mu I}{1+\varepsilon_3}, -\frac{\varepsilon_1}{2}, -\frac{\varepsilon_1}{2} \right\},$$

$$\Pi_{11} = \mu^{-1}(1+\varepsilon_3)\big(A_1^T A_1 + A_2^T A_2\big) + Q_1 - P_1 + \varepsilon_1^{-1} L^T \bar{\Gamma} \Lambda^2 \bar{\Gamma} L \\ + \varepsilon_1^{-1} L^T C_{mc}^T \Lambda_m^2 C_{mc} L + \kappa_1 M^T M,$$

$$\Pi_{12} = \mu^{-1}(1+\varepsilon_3)\big(A_1^T A_{1d} + A_2^T A_{2d}\big), \qquad \Pi_{13} = \mu^{-1}(1+\varepsilon_3)\big[A_1^T E \quad A_1^T E_d\big],$$

$$\Pi_{15} = \big[L^T \bar{\Gamma} G_c^T \quad L^T C_{mc}^T G_{cm}^T\big], \qquad \Pi_{23} = \mu^{-1}(1+\varepsilon_3)\big[A_{1d}^T E \quad A_{1d}^T E_d\big],$$

$$\Pi_{22} = \mu^{-1}(1+\varepsilon_3)\big(A_{1d}^T A_{1d} + A_{2d}^T A_{2d}\big) - Q_1 + \kappa_2 M_d^T M_d,$$

$$\Pi_{33} = \begin{bmatrix} \mu^{-1}(1+\varepsilon_3) E^T E - \kappa_1 I & \mu^{-1}(1+\varepsilon_3) E^T E_d \\ * & \mu^{-1}(1+\varepsilon_3) E_d^T E_d - \kappa_2 I \end{bmatrix},$$

$$\Pi_{55} = \mathrm{diag}\left\{ -\frac{\mu I}{1+\varepsilon_3^{-1}}, -\mu I \right\}, \qquad \bar{\Lambda}_p = \mathrm{diag}_p\{\bar{\Lambda}\}, \qquad \Lambda_m = \mathrm{diag}_m\{\Lambda\},$$

$$G_{1p} = \mathrm{diag}_p\{G_1\}, \qquad G_{2p} = \mathrm{diag}_p\{G_2\}, \qquad G_{cm} = \mathrm{diag}_m\{G_c\},$$
$$C_{pc} = \begin{bmatrix}\alpha_1 C_p^1 & \alpha_2 C_p^2 & \cdots & \alpha_p C_p^p\end{bmatrix}^T,$$
$$C_{mc} = \begin{bmatrix}\beta_1 C_m^1 & \beta_2 C_m^2 & \cdots & \beta_m C_m^m\end{bmatrix}^T. \tag{2.91}$$

Moreover, if the LMIs (2.85)–(2.90) *are feasible, the desired controller parameters are given by* $F_c = Y R_2^{-1}$, G_c, *and* $U_c = X R_2^{-1}$.

Proof Setting $R_2 = P_2^{-1}$, $\tilde{Q}_2 = P_2^{-1} Q_2 P_2^{-1}$, $X = U_c R_2$, $Y = F_c R_2$ and applying Schur complement together with some algebraic manipulations, (2.72)–(2.75) follow directly from (2.86)–(2.89), respectively.

Letting

$$\vartheta = \begin{bmatrix}x^T & x_d^T & \psi^T(x) & \psi_d^T(x_d)\end{bmatrix}^T$$

and noting (2.80)–(2.82), (2.76) can be rewritten as

$$\mathcal{H}_2(x, x_d) = \vartheta^T \Upsilon_1 \vartheta,$$

where

$$\Upsilon_1 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \mu^{-1}(1+\varepsilon_3)A_1^T E & \mu^{-1}(1+\varepsilon_3)A_1^T E_d \\ * & \Sigma_{22} & \mu^{-1}(1+\varepsilon_3)A_{1d}^T E & \mu^{-1}(1+\varepsilon_3)A_{1d}^T E_d \\ * & * & \mu^{-1}(1+\varepsilon_3)E^T E & \mu^{-1}(1+\varepsilon_3)E^T E_d \\ * & * & * & \mu^{-1}(1+\varepsilon_3)E_d^T E_d \end{bmatrix},$$

$$\begin{aligned}
\Sigma_{11} &= \mu^{-1}(1+\varepsilon_3)\big(A_1^T A_1 + A_2^T A_2\big) + \theta A_w^T P_1 A_w + Q_1 - P_1 \\
&\quad + \mu^{-1}\big(1+\varepsilon_3^{-1}\big)L^T \bar{\Gamma} G_c^T G_c \bar{\Gamma} L + \varepsilon_1^{-1} L^T \bar{\Gamma} \Lambda^2 \bar{\Gamma} L \\
&\quad + \mu^{-1} L^T C_{mc}^T G_{cm}^T G_{cm} C_{mc} L + \varepsilon_1^{-1} L^T C_{mc}^T \Lambda_m^2 C_{mc} L, \\
\Sigma_{12} &= \mu^{-1}(1+\varepsilon_3)\big(A_1^T A_{1d} + A_2^T A_{2d}\big) + \theta A_w^T P_1 A_{wd}, \\
\Sigma_{22} &= \mu^{-1}(1+\varepsilon_3)\big(A_{1d}^T A_{1d} + A_{2d}^T A_{2d}\big) + \theta A_{wd}^T P_1 A_{wd} - Q_1.
\end{aligned}$$

From (2.83) and (2.84) it can be easily seen that

$$\begin{aligned}
\mathcal{H}_2(x, x_d) &\le \vartheta^T \Upsilon_1 \vartheta + \kappa_1 \begin{bmatrix} x \\ \psi(x) \end{bmatrix}^T \begin{bmatrix} M^T M & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ \psi(x) \end{bmatrix} \\
&\quad + \kappa_2 \begin{bmatrix} x_d \\ \psi_d(x_d) \end{bmatrix}^T \begin{bmatrix} M_d^T M_d & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x_d \\ \psi_d(x_d) \end{bmatrix} \\
&= \vartheta^T \Upsilon_2 \vartheta,
\end{aligned}$$

where

$$\Upsilon_2 = \begin{bmatrix} \Sigma_{11} + \kappa_1 M^T M & \Sigma_{12} & \mu^{-1}(1+\varepsilon_3)A_1^T E & \mu^{-1}(1+\varepsilon_3)A_1^T E_d \\ * & \Sigma_{22} + \kappa_2 M_d^T M_d & \mu^{-1}(1+\varepsilon_3)A_{1d}^T E & \mu^{-1}(1+\varepsilon_3)A_{1d}^T E_d \\ * & * & \mu^{-1}(1+\varepsilon_3)E^T E - \kappa_1 I & \mu^{-1}(1+\varepsilon_3)E^T E_d \\ * & * & * & \mu^{-1}(1+\varepsilon_3)E_d^T E_d - \kappa_2 I \end{bmatrix}.$$

By Schur complement, (2.90) is equivalent to $\Upsilon_2 < 0$, which implies $\mathcal{H}_2(x, x_d) < 0$. Therefore, the proof of this corollary is accomplished in virtue of Corollary 2.2. □

Case 2 Let us now deal with the nonlinear terms $\psi(x)$ and $\psi_d(x_d)$ described by the following sector-bounded conditions that are more general than the Lipschitz-like ones:

$$\big(\psi(x) - Ux\big)^T \big(\psi(x) - Vx\big) \le 0, \tag{2.92}$$

$$\big(\psi_d(x_d) - U_d x_d\big)^T \big(\psi_d(x_d) - V_d x_d\big) \le 0, \tag{2.93}$$

where U, V, U_d, V_d are known real constant matrices, and $U - V$, $U_d - V_d$ are symmetric positive definite matrices.

In order to obtain the corresponding results for Case 2, we decompose the sector-bounded nonlinear term $\psi(x)$ and $\psi_d(x_d)$ into a linear part and a nonlinear part as follows:

$$\psi(x) = \frac{1}{2}(U + V)x + \tilde{\psi}(x), \tag{2.94}$$

$$\psi_d(x_d) = \frac{1}{2}(U_d + V_d)x_d + \tilde{\psi}_d(x_d), \tag{2.95}$$

where

$$\big\|\tilde{\psi}(x)\big\| \le \left\|\frac{1}{2}(U - V)x\right\|, \tag{2.96}$$

$$\big\|\tilde{\psi}_d(x_d)\big\| \le \left\|\frac{1}{2}(U_d - V_d)x_d\right\|. \tag{2.97}$$

Letting

$$\begin{aligned} \tilde{A}_1 &= A_1 + \frac{1}{2}E(U + V), & \tilde{M} &= \frac{1}{2}(U - V), \\ \tilde{A}_{1d} &= A_{1d} + \frac{1}{2}E_d(U_d + V_d), & \tilde{M}_d &= \frac{1}{2}(U_d - V_d), \end{aligned} \tag{2.98}$$

the nonlinear functions $f_1(x, x_d)$ can be rewritten as

$$f_1(x, x_d) = \tilde{A}_1 x + \tilde{A}_{1d} x_d + E\tilde{\psi}(x) + E_d\tilde{\psi}_d(x_d), \tag{2.99}$$

where

$$\left\|\tilde{\psi}(x)\right\| \le \|\tilde{M}x\|, \tag{2.100}$$

$$\left\|\tilde{\psi}_d(x_d)\right\| \le \|\tilde{M}_d x_d\|. \tag{2.101}$$

Subsequently, by replacing A_1, A_{1d}, M, and M_d with $\tilde{A}_1$, $\tilde{A}_{1d}$, $\tilde{M}$, and $\tilde{M}_d$, respectively, the following corollary can be immediately obtained from Corollary 2.3.

Corollary 2.4 *Let the disturbance attenuation level $\gamma > 0$ be given. The quantized nonlinear stochastic H_∞ control problem for system (2.1) with the nonlinearities bounded by sector-bounded conditions (2.92) and (2.93) is solved by the linear observer-based controller (2.70) if there exist positive definite matrices $P_1^T = P_1 > 0$, $R_2^T = R_2 > 0$, $Q_1^T = Q_1 > 0$, $\tilde{Q}_2^T = \tilde{Q}_2 > 0$, real matrices X, G_c, Y, and positive scalars $\kappa_1 > 0$, $\kappa_2 > 0$, $\varepsilon_2 > 0$, $\lambda > 0$ satisfying the LMIs (2.85)–(2.90) with*

$$\begin{aligned}
\Pi_{11} &= \mu^{-1}(1+\varepsilon_3)\big(\tilde{A}_1^T\tilde{A}_1 + A_2^T A_2\big) + Q_1 - P_1 + \varepsilon_1^{-1}L^T\bar{\Gamma}\Lambda^2\bar{\Gamma}L \\
&\quad + \varepsilon_1^{-1}L^T C_{mc}^T\Lambda_m^2 C_{mc}L + \kappa_1\tilde{M}^T\tilde{M}, \\
\Pi_{12} &= \mu^{-1}(1+\varepsilon_3)\big(\tilde{A}_1^T\tilde{A}_{1d} + A_2^T A_{2d}\big), \\
\Pi_{13} &= \mu^{-1}(1+\varepsilon_3)\big[\tilde{A}_1^T E \quad \tilde{A}_1^T E_d\big], \\
\Pi_{22} &= \mu^{-1}(1+\varepsilon_3)\big(\tilde{A}_{1d}^T\tilde{A}_{1d} + A_{2d}^T A_{2d}\big) - Q_1 + \kappa_2\tilde{M}_d^T\tilde{M}_d, \\
\Pi_{23} &= \mu^{-1}(1+\varepsilon_3)\big[\tilde{A}_{1d}^T E \quad \tilde{A}_{1d}^T E_d\big],
\end{aligned} \tag{2.102}$$

for given positive scalars $\varepsilon_1 > 0$, $\varepsilon_3 > 0$, and $\mu > 0$, where Θ_{12}, Θ_{22}, Π_{15}, Π_{33}, Π_{55}, $\bar{\Lambda}_p$, Λ_m, G_{1p}, G_{2p}, G_{cm}, C_{pc}, and C_{mc} are defined in (2.91), and $\tilde{A}_1$, $\tilde{A}_{1d}$, $\tilde{M}$, $\tilde{M}_d$ are defined in (2.98). Moreover, if the LMIs (2.85)–(2.90) with (2.102) are feasible, the desired controller parameters are given by $F_c = YR_2^{-1}$, G_c, and $U_c = XR_2^{-1}$.

Case 3 When the function $f_1(x, x_d)$ is taken as a linear form,

$$f_1(x, x_d) = A_1 x + A_{1d}x_d,$$

Corollary 2.3 further degenerates to the following result.

Corollary 2.5 *Let the disturbance attenuation level $\gamma > 0$ be given. The quantized stochastic H_∞ control problem for system (2.1) with a linear form is solved by linear observer-based controller (2.70) if there exist positive definite matrices $P_1^T = P_1 > 0$, $R_2^T = R_2 > 0$, $Q_1^T = Q_1 > 0$, $\tilde{Q}_2^T = \tilde{Q}_2 > 0$, real matrices X, G_c, Y, and* *positive scalars* $\kappa_1 > 0$, $\kappa_2 > 0$, $\varepsilon_2 > 0$, $\lambda > 0$ *satisfying the LMIs* (2.85)–(2.89) *and*

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \theta A_w^T P_1 & L^T \bar{\Gamma}^T G_c^T & L^T C_{mc}^T G_{cm}^T \\ * & \Pi_{22} & \theta A_{wd}^T P_1 & 0 & 0 \\ * & * & -\theta P_1 & 0 & 0 \\ * & * & * & -\frac{\mu I}{1+\varepsilon_3^{-1}} & 0 \\ * & * & * & * & -\mu I \end{bmatrix} < 0 \tag{2.103}$$

for given positive scalars $\varepsilon_1 > 0$, $\varepsilon_3 > 0$, *and* $\mu > 0$, *where*

$$\begin{aligned} \Pi_{11} &= \mu^{-1}(1+\varepsilon_3)\big(A_1^T A_1 + A_2^T A_2\big) + Q_1 - P_1 \\ &\quad + \varepsilon_1^{-1} L^T \bar{\Gamma} \Lambda^2 \bar{\Gamma} L + \varepsilon_1^{-1} L^T C_{mc}^T \Lambda_m^2 C_{mc} L, \\ \Pi_{22} &= \mu^{-1}(1+\varepsilon_3)\big(A_{1d}^T A_{1d} + A_{2d}^T A_{2d}\big) - Q_1, \end{aligned} \tag{2.104}$$

and Θ_{12}, Θ_{22}, Π_{12}, C_{pc}, *and* C_{mc} *are defined in* (2.91). *Moreover, if the LMIs* (2.85)–(2.89), (2.103) *with* (2.104) *are feasible, the desired controller parameters are given by* $F_c = Y R_2^{-1}$, G_c, *and* $U_c = X R_2^{-1}$.

In this chapter, the quantized H_∞ control problem is dealt with for nonlinear stochastic time-delay systems with missing measurements. We first consider a very general stochastic system (2.1) where *all the system parameters and controller parameters* are nonlinear functions or functionals. In this case, sufficient conditions are given in Corollary 2.1 which make sure that system (2.12) is stochastically stable and H_∞ criterion in (2.15) is satisfied. Note that, at this stage, the nonlinear parameters are very general since there are no assumptions posed on them. Therefore, as expected, the sufficient conditions established in Corollary 2.1 serve as a theoretical basis for *general* nonlinear stochastic systems. It is shown in subsequent analysis that the fundamental results given in Corollary 2.1 can be specialized to numerically tractable ones in practical cases where the nonlinear parameters take certain commonly used forms. Based on Corollary 2.1, the aim of Theorem 2.2 is to provide a particular condition that eliminates the quantization effects Δ_k and $\bar{\Delta}_k$ but still guarantees the H_∞ performance and the stochastic stability. Next, we take some practically justifiable forms, in a gradual way, for the nonlinear parameters with hope to obtain easy-to-verify conditions for the addressed design problem. Under the assumption that the measurement noise, the output noise, and the system noise are mutually independent, Theorem 2.3 offers a more specific condition that ensures both the stability and the H_∞ performance, and such a condition is further simplified in Corollary 2.2. Furthermore, in this section, three special cases are considered, respectively, for stochastic systems with Lipschitz-like nonlinearities, sector-bounded nonlinearities, and a linear nominal part, and LMI-based results are obtained that can be easily checked using standard numerical software such as Matlab toolbox.

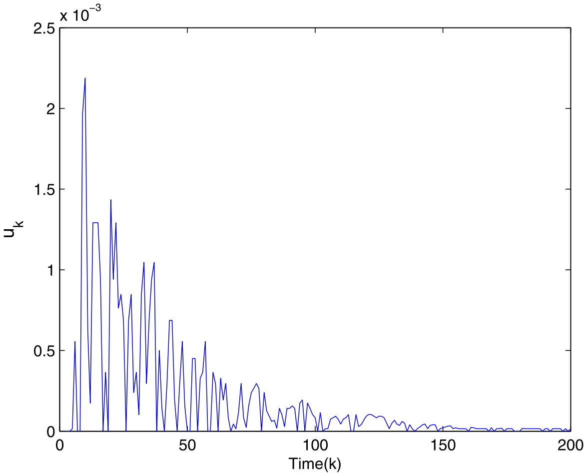


Fig. 2.2 The control input with quantization by $\bar{q}(\cdot)$

2.4 Illustrative Examples

In this section, three examples are employed to demonstrate the theory presented in this chapter.

Example 1 Nonlinear H_∞ control design.

Consider the following nonlinear discrete-time stochastic system:

$$\begin{cases} x_{k+1} = \dfrac{1}{3}x_k + \dfrac{1}{6}x_{k-1}\sin x_k + \dfrac{1}{4}v_k + \dfrac{1}{3}u_k + \dfrac{1}{50}x_k \cos x_{k-1} w_k, \\ z_k = \dfrac{1}{3}x_k \sin x_k - \dfrac{1}{6}x_{k-1} + \dfrac{1}{\sqrt{2}}u_k, \end{cases} \tag{2.105}$$

with the initial conditions $\varphi_{-1} = \varphi_0 = 0$. The measurement with sensors data missing is described as

$$y_k = \frac{1}{3}\gamma_k x_k \cos x_k. \tag{2.106}$$

We choose the dynamic observer-based controller parameters as $F_c = \frac{2}{7}$, $G_c = 1$, $U_c = \frac{1}{5}$ and obtain the following dynamic observer-based controller:

$$\begin{cases} \hat{x}_{k+1} = \dfrac{2}{7}\hat{x}_k + \bar{y}_k, \\ \mathbf{u}_k = \dfrac{1}{5}\xi_k \hat{x}_k. \end{cases} \tag{2.107}$$

In this example, let the probability $\bar{\gamma} = \bar{\xi} = 0.8$, the variance $\theta = 0.25$, the disturbance attenuation level $\gamma = 0.85$, and the disturbance input $v_k = \exp(-k/35) \times n_k$,

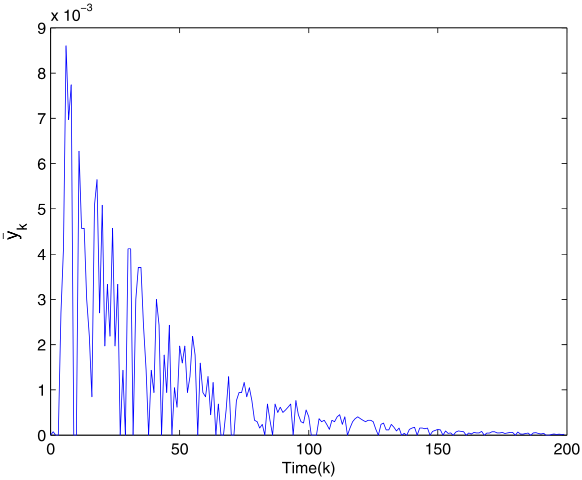


Fig. 2.3 The measurement with quantization by $q(\cdot)$

where n_k is uniformly distributed over $[0, 0.1]$. The parameters of the two logarithmic quantizers $q(\cdot)$ and $\bar{q}(\cdot)$ are set as $\chi_0 = \bar{\chi}_0 = 0.003$ and $\rho = \bar{\rho} = 0.9$. According to Corollary 2.2, it can be seen that the controller of the form (2.107) is a desired controller for system (2.105) with parameters $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, $\lambda = 0.7155$, $\mu = 0.5$, $P_1 = 1.4317$, $P_2 = 0.4$, $Q_1 = 0.2223$, and $Q_2 = 0.002$.

Simulation results are shown in Figs. 2.2–2.4. Specifically, the control input after quantization by quantizers $\bar{q}(\cdot)$ is given in Fig. 2.2, and the measurement after quantization by quantizers $q(\cdot)$ is shown in Fig. 2.3, which correspond to the controlled system and the dynamic controller, respectively. Figure 2.4 depicts the simulation result of the state response of the closed-loop system. In this example, we can calculate the H_∞ performance constraint is 0.0469, which is less than the given disturbance attenuation level $\gamma = 0.85$. Therefore, this example has verified the theories obtained in this chapter.

Example 2 H_∞ control design for special nonlinear systems.

Consider a class of sector-bounded nonlinear systems with the following parameters:

$$A_1 = \begin{bmatrix} -0.1 & -0.2 \\ 0 & -0.1 \end{bmatrix}, \qquad A_{1d} = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \qquad E = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix},$$

$$E_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & 0.2 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} 0.1 & 0.1 \\ -0.1 & -0.1 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} -0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix},$$

$$A_w = \begin{bmatrix} -0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix}, \qquad A_{wd} = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix},$$

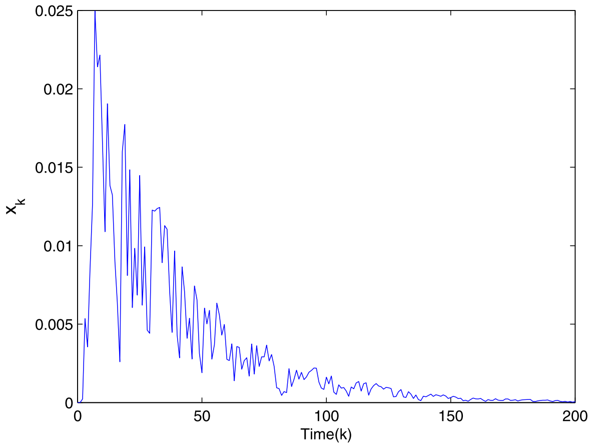


Fig. 2.4 The state response of the closed-loop system

$$A_{2d} = \begin{bmatrix} -0.1 & 0.1 \\ 0 & -0.1 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix}, \qquad U = U_d = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix},$$

$$V = V_d = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, \qquad K = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}.$$

In this example, let $\theta = 0.25$, $\rho_1 = \rho_2 = \bar{\rho}_1 = \bar{\rho}_2 = 0.9$, $\bar{\gamma}^1 = \bar{\xi}^2 = 0.8$, and $\bar{\gamma}^2 = \bar{\xi}^1 = 0.9$. The H_∞ performance level is taken as $\gamma = 0.5$. In order to design output feedback controller, we first choose $\varepsilon_1 = \varepsilon_3 = 1$, $\mu = 0.1$. With the above parameters and by using the Matlab LMI toolbox, we solve the LMIs (2.85)–(2.90) with (2.102) and obtain the parameters of the desired output feedback controller as follows:

$$F_c = YR_2^{-1} = \begin{bmatrix} 0.0577 & 0.0027 \\ 0.0046 & 0.0251 \end{bmatrix}, \qquad G_c = \begin{bmatrix} 0.8413 & 0.0051 \\ 0.0715 & 1.1657 \end{bmatrix},$$

$$U_c = XR_2^{-1} = \begin{bmatrix} 0.9339 & -0.0999 \\ -0.1013 & 1.2420 \end{bmatrix}.$$

Example 3 H_∞ control design for F-404 aircraft engine system.

To further demonstrate the applicability of the proposed design techniques, in this example, we consider the F-404 aircraft engine system in [41]. By linearizing the model of an F-404 aircraft engine system, the nominal system matrix A_c and the measurement output matrix L_c are obtained as follows:

$$A_c = \begin{bmatrix} -1.4600 & 0 & 2.4280 \\ 0.1643 & -0.4000 & -0.3788 \\ 0.3107 & 0 & -2.2300 \end{bmatrix}, \qquad L_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

By assuming the sampling time to be $T = 0.05s$, the discretized nominal system matrix A_1 and the measurement output matrix L are given by

$$A_1 = \begin{bmatrix} 0.9270 & 0 & 0.1214 \\ 0.0082 & 0.9800 & -0.0189 \\ 0.0155 & 0 & 0.8885 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

According to the expatiation in [153], the disturbances produced by external circumstance such as wind gusts, gravity gradients, and sensor and actuator noise, may enter the aircraft engine systems in many different ways. It is, therefore, reasonable to take linearization errors (nonlinear disturbances), time delays, and Itô-type stochastic perturbations into account when the aircraft engine system is modeled. In addition to the main system parameters A_1 and L, we set other parameters as follows:

$$A_{1d} = \begin{bmatrix} 0.0030 & -0.0030 & 0.0040 \\ 0.0020 & -0.0008 & 0.0030 \\ -0.0035 & -0.0006 & -0.0020 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} -0.0020 & 0.0030 \\ -0.0040 & 0.0060 \\ 0.0030 & -0.0045 \end{bmatrix},$$

$$A_w = \begin{bmatrix} -0.0025 & 0.0040 & 0.0030 \\ -0.0025 & 0.0055 & 0.0035 \\ 0.0030 & -0.0040 & 0.0060 \end{bmatrix},$$

$$A_{wd} = \begin{bmatrix} -0.0025 & 0.0035 & 0.0025 \\ -0.0020 & 0.0065 & 0.0033 \\ 0.0030 & -0.0035 & 0.0055 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.042 & 0.035 & 0.028 \\ 0.028 & 0.049 & 0.014 \end{bmatrix},$$

$$A_{2d} = \begin{bmatrix} -0.01 & 0.01 & 0 \\ 0 & -0.01 & 0 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.0100 & 0.0050 & 0 \\ 0 & -0.0050 & 0.0050 \\ 0 & 0 & 0.0050 \end{bmatrix}, \qquad E_d = \begin{bmatrix} -0.0050 & 0 & 0.0050 \\ -0.0050 & 0.0100 & 0 \\ 0 & 0 & 0.0050 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0.1 & 0 \end{bmatrix}, \qquad M_d = \begin{bmatrix} 0 & 0 & 0.1 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} -0.0100 & 0 \\ 0.0050 & 0.0050 \\ 0 & 0.0050 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{bmatrix}, \qquad K = \begin{bmatrix} -0.06 & 0.03 \\ -0.04 & 0.02 \end{bmatrix}.$$

In order to achieve the desired performance of the F-404 aircraft engine system, the information needs to be transmitted between the control flat on the ground and the aircraft in air via wireless communication channels. In this case, both the phenomena of data missing and quantization effect should be considered. In practice, the probabilities $\bar{\gamma}^1$, $\bar{\gamma}^2$, $\bar{\xi}^1$, and $\bar{\xi}^2$ can be derived beforehand, and the quantization densities ρ_1, ρ_2, $\bar{\rho}_1$, and $\bar{\rho}_2$ can be chosen according to the desired requirement of accuracy. In this example, we set $\bar{\gamma}^1 = 0.85$, $\bar{\gamma}^2 = 0.95$, $\bar{\xi}^1 = 0.7$, $\bar{\xi}^2 = 0.9$,

$\rho_1 = \rho_2 = 0.9$, and $\bar{\rho}_1 = \bar{\rho}_2 = 0.95$. The H_∞ performance level is taken as $\gamma = 0.4$, and the variance of Gaussian white noise is given by $\theta = 0.5$. By choosing $\varepsilon_1 = 1$, $\varepsilon_3 = 0.01$, and $\mu = 0.3$, the LMIs (2.85)–(2.90) with (2.91) can be solved by using the Matlab LMI toolbox, and the following controller parameters are obtained:

$$F_c = Y R_2^{-1} = \begin{bmatrix} 0.6909 & -0.1266 & -0.1767 \\ -0.1267 & 0.8179 & 0.0493 \\ -0.1767 & 0.0492 & 0.7684 \end{bmatrix},$$

$$G_c = \begin{bmatrix} -0.0027 & 0.0076 \\ 0.0166 & 0.0048 \\ -0.0028 & 0.0076 \end{bmatrix},$$

$$U_c = X R_2^{-1} = \begin{bmatrix} 0.5186 & -0.0788 & 0.5975 \\ 0.3464 & 0.4252 & -0.0789 \end{bmatrix}.$$

2.5 Summary

In this chapter, the quantized H_∞ control problem has been addressed for a class of nonlinear stochastic time-delay network-based systems with data missing. Two logarithmic quantizers have been employed to quantize both the measured output and the input signals in the NCSs, and one diagonal matrix whose leading diagonal elements are Bernoulli-distributed stochastic variables has been used to model the data missing phenomena. Then, we have derived a sufficient condition under which the closed-loop system is stochastically stable and the controlled output satisfies H_∞ performance constraint for all nonzero exogenous disturbances under the zero initial condition by applying the method of sector bound uncertainties. For the purpose of easy checking, the sufficient condition has been decoupled into some inequalities. Based on that, quantized H_∞ controllers have been designed successfully for some special classes of nonlinear stochastic time-delay systems. Finally, three examples have been provided to show the effectiveness and applicability of the proposed methods.