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Solutions of Problems 001–500

In this chapter we present solutions of problems 001–500 of the main text. We will need to refer both to the main text problems and to their solutions; to distinguish a problem from its solution, we use the prefix "S." to the number of the solved problem. For example, the solution of Problem 349 is numbered as S.349. We tried to uniformize and clarify all possible references by formulating useful statements as independent facts. We have always avoided statements like "it is clear that the proof of Fact 3 of S.351 gives a stronger property". If such a property was needed after the Fact was proved, we preferred to return to that Fact, reformulate it and redo its proof if necessary.

In quite a few cases the author had a strong temptation to refer to future results to make a solution short and elegant. He managed to resist it so that this is never done in this book; we only refer to *previous* problems and solutions. The short and elegant proofs are given after the respective methods are introduced and studied systematically. This sounds like a merit but has its price: sticking to this principle implied that sometimes a solution occupies ten or more pages because the author had to develop the basics of a theory in the same solution. The author fully understands that it is ridiculous to expect that someone develops a theory to solve an exercise; however, the merit seems to make up for this drawback. Besides, I don't think this is going to bother a specialist; a beginner student could be annoyed of course, but my experience shows that a beginner gets much more annoyed by a reference to nowhere, like, for example, the phrase "Gul'ko proved that this space can be condensed into a Σ_*-product of real lines". Another justification is that, in the author's biased opinion, there will be no more than 1% of the readers who will *really* try to solve the problems without opening the solutions. The rest of the people will use this composition as a reference book or as a textbook so, if a result is needed, one needs the basics of the relevant theory anyway.

Another principle the author tried to implement was to get the solutions to be as independent as reasonably possible. This will be of help for specialists wishing to look up the solution of any problem without browsing through the previous ones. There are references, of course, but only to the results and never to definitions or constructions. We also use some minimal number of concepts and notation without explanations but they are always standard and can be found in introductory parts or

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in the index. In most cases even the fundamental concepts are introduced again in the respective solution to make it as self-contained as possible.

The last thing to keep in mind is that any solution is a continuation of the respective problem. Thus, if the problem says: "Let X be a metrizable compact space. Prove that X ...", we do not repeat this assumption about X considering that it was already made in the formulation of the problem. Thus our solution could start with the phrase: "Since X is metrizable and compact...".

The material of this volume is an introductory one. An evident consequence is an abundance of simple facts proved in great detail. This could be a nuisance for a specialist but of help for the beginner. Five hundred solutions are presented here; some of them are quite difficult to understand and much more difficult to figure out on one's own without consulting this chapter.

The base of the theory presented in this chapter is given by the problems of the main text. However, there are quite a few (more than 200) auxiliary statements; they are presented as facts or observations inside solutions. Some of them are well-known theorems, other are just simple lemmas formulated to avoid repetitions of the same proof.

The reader will notice that we expect him/her to have a higher level of understanding as the theory builds up. Eventually, there are more phrases like "it is easy to see" or "it is an easy exercise"; the reader should trust the author's word and experience that the statements like that are *really* easy to prove as soon as one has the necessary background. The author is convinced that if a student attacks a problem after understanding the solutions of all previous ones, his/her preparation is more than sufficient for being able to grasp the current solution.

In fact, much less is necessary than all previous problems. The main text is really a bulk of many topics which are developed independently or with little dependency. I am afraid, a beginner will not be able to trace the minimal set of topics to master, needed for understanding current solution. However this can be easily done by a specialist so consult one if you want to minimize your efforts.

A beginner should also remember that we use without explanations the most well-known properties of the real numbers and some simple facts of the set theory. A one-year calculus course is more than sufficient to cover all formal prerequisites. The informal prerequisite is to be able to understand logical implications and to be persistent enough not to give up even if the solution is not understood after ten readings. In the worst case, try to prove that the problem or the solution is false. It normally shouldn't be, but no big work is free of errors (and believe me this is a huge one!) so try to find them and communicate them to me to correct the respective parts.

S.001. *Let X be a topological space. Given an arbitrary set $A \subset X$, prove that $x \in \overline{A}$ if and only if $U \cap A \neq \emptyset$ for any $U \in \tau(X)$ such that $x \in U$.*

Solution. Suppose that $x \in \overline{A}$ and $x \in U \in \tau(X)$. If $U \cap A = \emptyset$ then $A \subset F = X\backslash U$ and the set $X\backslash U$ is closed in X. Since the closure of A is the intersection of all closed sets which contain A, we have $\overline{A} \subset F$. Thus $x \in \overline{A} \subset F$ and $x \in U = X\backslash F$, a contradiction.

Assume that $U \cap A \neq \emptyset$ for all open $U \ni x$. If $x \notin \overline{A}$ then the set $V = X\backslash\overline{A}$ is open and $x \in V$. However, $V \cap A = \emptyset$, a contradiction.

S.002. *Given a topological space X and a family $\mathcal{B} \subset \tau(X)$, prove that $\mathcal{B}$ is a base of X if and only if for any $U \in \tau(X)$ and $x \in U$ there exists $V \in \mathcal{B}$ such that $x \in V \subset U$.*

Solution. Suppose that $\mathcal{B} \subset \tau(X)$ is a base. If $x \in U \in \tau(X)$ then there is $\mathcal{B}' \subset \mathcal{B}$ such that $U = \bigcup\mathcal{B}'$. Therefore, there is $V \in \mathcal{B}'$ with $x \in V \subset \bigcup\mathcal{B}' = U$. Then $V \in \mathcal{B}$ and $x \in V \subset U$.

Now, suppose that $\mathcal{B} \subset \tau(X)$ and, for any $x \in U \in \tau(X)$, there exists $V \in \mathcal{B}$ such that $x \in V \subset U$. If $U = \emptyset$ then, letting $\mathcal{B}' = \emptyset$, we have $\mathcal{B}' \subset \mathcal{B}$ and $U = \bigcup\mathcal{B}'$. If $U \in \tau(X)$ is non-empty find for each $x \in U$ a set $V_x \in \mathcal{B}$ such that $x \in V_x \subset U$ and consider the family $\mathcal{B}' = \{V_x : x \in U\}$. It is immediate that $\mathcal{B}' \subset \mathcal{B}$ and $\bigcup\mathcal{B}' = U$ which proves that $\mathcal{B}$ is a base of the topology $\tau(X)$.

S.003. *Let X be a topological space. Prove that the family $\mathcal{F}$ of all closed subsets of X has the following properties:*

(F1) $X \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$.
(F2) If $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$.
(F3) If $\gamma \subset \mathcal{F}$ then $\bigcap\gamma \in \mathcal{F}$.

Now suppose that X is a set and $\mathcal{F} \subset \exp(X)$ has the properties (F1)–(F3). Prove that there exists a unique topology τ on X such that $\mathcal{F}$ is the family of closed subsets of (X, τ).

Solution. Clearly, $F \in \mathcal{F}$ iff $X\backslash F \in \tau(X)$. Thus $\emptyset = X\backslash X \in \mathcal{F}$ and $X = X\backslash\emptyset \in \mathcal{F}$ (we applied (TS1) twice). Therefore (F1) holds for $\mathcal{F}$. To check (F2), note that if $A, B \in \mathcal{F}$ then $X\backslash A \in \tau(X)$ and $X\backslash B \in \tau(X)$ and applying (TS2) we obtain $(X\backslash A) \cap (X\backslash B) = X\backslash(A \cup B) \in \tau(X)$, whence $A \cup B \in$ F. Finally, to see that (F3) holds, let $\mu = \{X\backslash F : F \in \gamma\}$. Then $\mu \subset \tau(X)$ and hence $\bigcup \mu \in \tau(X)$ by (TS3). Since $X\backslash\bigcap\gamma = \bigcup \mu$, the property (F3) is also proved.

Now, suppose that $\mathcal{F}$ is a family of subsets of X with properties (F1)–(F3). Let $\tau = \{X\backslash F : F \in \mathcal{F}\}$. The laws of De Morgan together with (F1)–(F3) imply that the axioms (TS1)–(TS3) hold for τ. The definition of τ shows that $\mathcal{F}$ is precisely the family of all closed sets for τ. Now, if τ' is another topology for which $\mathcal{F}$ consists of closed sets for τ', then $\tau' = \{X\backslash F : F \in \mathcal{F}\} = \tau$ and this proves the uniqueness.

S.004. *Let X be a topological space. Show that the operator of the closure has the following properties:*

(C1) $\overline{\emptyset} = \emptyset$.
(C2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ *for any* $A, B \subset X$.
(C3) $A \subset \overline{A}$ *for any* $A \subset X$.
(C4) $\overline{\overline{A}} = \overline{A}$ *for any* $A \subset X$.

Now, suppose that X is a set and $[\cdot]$ *is an operator on* $\exp(X)$ *with (C1)–(C4) (i.e.,* $[\emptyset] = \emptyset$, $[A \cup B] = [A] \cup [B]$, $A \subset [A]$ *and* $[[A]] = [A]$ *for all* $A, B \subset X$*). Prove that there exists a unique topology* τ *on X such that* $[A] = \overline{cl}_\tau(A)$ *for any* $A \subset X$*. We will say that* τ *is generated by the closure operator* $[\cdot]$.

Solution. Since $\emptyset$ is a closed set and $\emptyset \subset \emptyset$, we have $\overline{\emptyset} \subset \emptyset$, so (C1) holds. The set $F = \overline{A} \cup \overline{B}$ is closed and $A \subset \overline{A} \subset F$ as well as $B \subset \overline{B} \subset F$ whence $A \cup B \subset F$. Since the closure is the intersection of all closed supersets, we have $\overline{A \cup B} \subset F$. Now, $A \subset A \cup B \subset \overline{A \cup B}$ and the set $G = \overline{A \cup B}$ is closed. Since the closure is the intersection of all closed supersets, we have $\overline{A} \subset G$. Reasoning in the same way we obtain $\overline{B} \subset G$ and hence $F = \overline{A} \cup \overline{B} \subset G$. This shows that $F = G$ so (C2) is settled. The property (C3) is immediate from the definition of the closure. Since $\overline{A}$ is closed and $\overline{A} \subset \overline{A}$, we have $\overline{\overline{A}} \subset \overline{A}$. The reverse inclusion is a consequence of (C3) and therefore we proved (C4).

Now, let X be an arbitrary set without topology. Assume that $[\cdot]$ is an operator on $\exp(X)$ with the properties (C1)–(C4) i.e., $[\emptyset] = \emptyset$, $[A \cup B] = [A] \cup [B]$, $A \subset [A]$ and $[[A]] = [A]$ for all $A, B \subset X$. Observe first that the condition $A \subset B$ implies $[B = A \cup (B \backslash A)] = [A] \cup [B \backslash A] \supset [A]$ and hence $[A] \subset [B]$. Now, let $\mathcal{F} = \{A \subset X : [A] = A\}$. We are going to check that $\mathcal{F}$ satisfies (F1)–(F3) from 003.

The property (C1) shows that $\emptyset \in \mathcal{F}$ and (C3) implies $X \subset \overline{X}$ whence $X = \overline{X}$ and $X \in \mathcal{F}$ which settles (F1). If $A, B \in \mathcal{F}$ then $A = [A]$ and $B = [B]$. Applying (C2) we conclude that $[A \cup B] = [A] \cup [B] = A \cup B$ and hence $A \cup B \in \mathcal{F}$. This proves (F2). Now, take any $\gamma \subset \mathcal{F}$ and denote by G the set $\bigcap \gamma$. For any $A \in \gamma$, we have $G \subset A$ and therefore $[G] \subset [A] = A$. Thus $[G] \subset \bigcap\{A : A \in \gamma\} = \bigcap \gamma = G$. By property (C3) we have $[G] = G$ and hence (F3) is checked. Note that we did not use (C4) yet.

By Problem 003 there exists a unique topology τ for which $\mathcal{F}$ consists of closed sets for τ and hence $\tau = \{X \backslash A : A \in \mathcal{F}\}$. Let us prove that $\overline{A} = [A]$ for any $A \subset X$. Since $\overline{A}$ is closed, we have $[\overline{A}] = \overline{A}$ and hence $[A] \subset [\overline{A}] = \overline{A}$. The set $[A] \supset A$ is closed by (C4) and hence $\overline{A} \subset [A]$ which proves the promised equality.

Suppose finally that τ' is a topology such that $[A] = \mathrm{cl}_{\tau'}(A)$ for any $A \subset X$. Thus F is closed in τ if and only if $F = \overline{F} = [F] = \mathrm{cl}_{\tau'}(F)$ which happens iff F is closed in τ'. This shows that $\mathcal{F}$ is precisely the family of all closed subsets of τ' so by the uniqueness part of Problem 003, we have $\tau = \tau'$.

S.005. *Let X be a topological space. Show that the operator of the interior has the following properties:*

(I1) $\mathrm{Int}(X) = X$.
(I2) $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$ *for any* $A, B \subset X$.
(I3) $\mathrm{Int}(A) \subset A$ *for any* $A \subset X$.
(I4) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$ *for any* $A \subset X$.

Now, suppose that X is a set and $\langle\cdot\rangle$ is an operator on on $\exp(X)$ *with (I1)–(I4) (i.e., $\langle\emptyset\rangle = \emptyset$, $\langle A \cap B\rangle = \langle A\rangle \cap \langle B\rangle$, $\langle A\rangle \subset A$ and $\langle\langle A\rangle\rangle = \langle A\rangle$ for all $A, B \subset X$). Prove that there exists a unique topology τ on X such that $\langle A\rangle = \mathrm{Int}_\tau(A)$ for any $A \subset X$. We will say that τ is generated by the interior operator $\langle\cdot\rangle$.*

Solution. To see that (I1) holds, observe that there is nothing contained in an empty set so $\mathrm{Int}(\emptyset) = \emptyset$. Since $\mathrm{Int}(A \cap B)$ is an open set contained in $A \cap B$, it is contained in A. The set $\mathrm{Int}(A)$ is the union of all open sets contained in A, so $\mathrm{Int}(A \cap B) \subset \mathrm{Int}(A)$. The same reasoning shows that $\mathrm{Int}(A \cap B) \subset \mathrm{Int}(B)$ and therefore $\mathrm{Int}(A \cap B) \subset \mathrm{Int}(A) \cap \mathrm{Int}(B)$. On the other hand, $\mathrm{Int}(A) \cap \mathrm{Int}(B) \subset A \cap B$ is an open set, so we can again use the fact that $\mathrm{Int}(A \cap B)$ is the union of all open sets contained in $A \cap B$ and hence $\mathrm{Int}(A) \cap \mathrm{Int}(B) \subset \mathrm{Int}(A \cap B)$. This settles (I2). The property (I3) is an immediate consequence of the definition of the interior. To check (I4), observe that $\mathrm{Int}(A) \subset \mathrm{Int}(A)$ is an open set and hence $\mathrm{Int}(A) \subset \mathrm{Int}(\mathrm{Int}(A))$. Since the reverse inclusion is stated in (I3), we have proved (I4).

Now, let X be an arbitrary set without topology. Suppose that $\langle\cdot\rangle$ is an operator on $\exp(X)$ with the properties (I1)–(I4), i.e., $\langle\emptyset\rangle = \emptyset$, $\langle A \cap B\rangle = \langle A\rangle \cap \langle B\rangle$, $\langle A\rangle \subset A$ and $\langle\langle A\rangle\rangle = \langle A\rangle$ for all $A, B \subset X$). Observe first that, if $A \subset B$ then (I2) implies $\langle A\rangle = \langle A \cap B\rangle = \langle A\rangle \cap \langle B\rangle \subset \langle B\rangle$. Now, let $\tau = \{A \subset X : \langle A\rangle = A\}$ and check that τ is a topology on X. The property (I1) implies $X \in \tau$ and $\emptyset \in \tau$ by (I3). If $U, V \in \tau$ then $U = \langle U\rangle$ and $V = \langle V\rangle$ and therefore $U \cap V = \langle U\rangle \cap \langle V\rangle = \langle U \cap V\rangle$ which shows that $U \cap V \in \tau$. Now, take any $\gamma \subset \tau$ and denote by W the set $\bigcup\gamma$. For any $U \in \gamma$, we have $U \subset W$ and therefore $U = \langle U\rangle \subset \langle W\rangle$. This shows that $W = \bigcup\gamma \subset \langle W\rangle$. Since the reverse inclusion is a consequence of (I3), we have $\langle W\rangle = W$ and (TS3) is proved together with the fact that τ is a topology. Note that we did not need the property (I4) yet.

Let us show that $\langle A\rangle = \mathrm{Int}(A)$ for any $A \subset X$. Since $\mathrm{Int}(A)$ is an open set, we have $\langle\mathrm{Int}(A)\rangle = \mathrm{Int}(A)$ which shows that $\mathrm{Int}(A) = \langle\mathrm{Int}(A)\rangle \subset \langle A\rangle$. The property (I4) shows that $\langle A\rangle$ is an open set and hence $\langle A\rangle \subset \mathrm{Int}(A)$ which proves that $\langle A\rangle = \mathrm{Int}(A)$. Finally, let τ' be a topology on X such that $\langle A\rangle = \mathrm{Int}_{\tau'}(A)$ for any $A \subset X$. Then U belongs to τ if and only if $U = \mathrm{Int}(U) = \langle U\rangle = \mathrm{Int}_{\tau'}(U)$ and this happens iff $U \in \tau'$. Hence $\tau = \tau'$ and the uniqueness is also proved.

S.006. *Suppose that X is a topological space and $\mathcal{B}$ is a base of X. Prove that $\mathcal{B}$ has the following properties:*

(B1) $\bigcup\mathcal{B} = X$.
(B2) If $U, V \in \mathcal{B}$ and $x \in U \cap V$ then there is $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$.

Now, let X be a set without topology. Prove that, for any family $\mathcal{B} \subset$ $\exp(X)$ *with the properties (B1) and (B2), there exists a unique topology τ on the set X such that $\mathcal{B}$ is a base for* (X, τ). *We will call τ the topology generated by $\mathcal{B}$ as a base.*

Solution. The property (B1) holds because X is an open subset of X. Suppose that $U, V \in \mathcal{B}$ and $x \in U \cap V$. Since the set $U \cap V$ is open, there exists $\mathcal{B}' \subset \mathcal{B}$ such that $U \cap V = \cup\mathcal{B}'$. This means that there is $W \in \mathcal{B}'$ with $x \in W$. It is clear that $W \in \mathcal{B}$ and $x \in W \subset U \cap V$ so that (B2) is also fulfilled.

Now, suppose that X is an arbitrary set without topology in which we have a family $\mathcal{B} \subset \exp(X)$ with the properties (B1) and (B2). Let us prove that the family $\tau = \{\bigcup\gamma : \gamma \subset \mathcal{B}\}$ is a topology on X. If $\gamma = \emptyset$ then $\gamma \subset \mathcal{B}$ and $\bigcup\gamma = \emptyset$ which shows that $\emptyset \in \tau$. The property (B1) implies $X \in \tau$ (just take $\gamma = \mathcal{B}$) so the axiom (TS1) is checked. If $U, V \in \tau$ then there are $\gamma, \mu \subset \mathcal{B}$ such that $U = \bigcup\gamma$ and $V = \bigcup\mu$. For an arbitrary $x \in U \cap V$ there is $U_x \in \gamma$ and $V_x \in \mu$ such that $x \in U_x \cap V_x$ and it is immediate from the definition of γ and μ that $U_x \subset U$ and $V_x \subset V$. The property (B2) guarantees the existence of $W_x \in \mathcal{B}$ such that $x \in W_x \subset U_x \cap V_x \subset U \cap V$. It is evident that, for the family $\mathcal{B}' = \{W_x : x \in U \cap V\}$, we have $\mathcal{B}' \subset \mathcal{B}$ and $\cup\mathcal{B}' = U \cap V$ which proves that $U \cap V \in \tau$ settling (TS2). Given a family $\gamma \subset \tau$, for any $U \in \gamma$, fix a family $\gamma_U \subset \mathcal{B}$ such that $\bigcup\gamma_U = U$. Then $\mathcal{B}' = \bigcup\{\gamma_U : U \in \gamma\} \subset \mathcal{B}$ and $\bigcup\mathcal{B}' = \bigcup\gamma$ which proves that $\bigcup\gamma \in \tau$ and hence (TS3) holds as well.

By the definition of τ, the unions of all subfamilies of $\mathcal{B}$ represent all elements of τ which shows that $\mathcal{B}$ is a base for τ. Finally, if τ' is another topology on X for which $\mathcal{B}$ is a base then $\mathcal{B} \subset \tau'$ and $\tau \subset \tau'$ because any union of elements of τ' has to belong to τ'. On the other hand, any element of the topology τ' is a union of some subfamily of $\mathcal{B}$ and hence belongs to τ by the definition of τ. This proves $\tau = \tau'$ and the uniqueness is also established.

S.007. *Suppose that X is a topological space and, for each $x \in X$ we have a fixed local base $\mathcal{B}_x$ at the point x. Show that the family $\{\mathcal{B}_x : x \in X\}$ has the following properties:*

(LB1) $\mathcal{B}_x \neq \emptyset$ and $\bigcap\mathcal{B}_x \ni x$ for every $x \in X$.
(LB2) If $x \in X$ and $U, V \in \mathcal{B}_x$ then there is $W \in \mathcal{B}_x$ such that $W \subset U \cap V$.
(LB3) If $x \in U \in \mathcal{B}_y$ then there is $V \in \mathcal{B}_x$ such that $V \subset U$.

Now, suppose that X is an arbitrary set without topology and $\mathcal{B}_x$ is a family of subsets of X for any $x \in X$ such that the collection $\{\mathcal{B}_x : x \in X\}$ has the properties (LB1)–(LB3). Show that there exists a unique topology τ on the set X such that $\mathcal{B}_x$ is a local base of (X, τ) at x for any $x \in X$. We will call τ the topology generated by the families $\{\mathcal{B}_x : x \in X\}$ as local bases.

Solution. The second part of (LB1) is a part of the definition of a local base. Since $x \in X \in \tau(X)$, there is $U \in \mathcal{B}_x$ with $x \in U \subset X$ and hence $\mathcal{B} \neq \emptyset$ so that (LB1) is true. The properties (LB2) and (LB3) follow immediately from the fact that U and $U \cap V$ are both open sets.

Now, let X be an arbitrary set without topology. Suppose that $\mathcal{B}_x \subset \exp(X)$ for any $x \in X$ and the family $\{\mathcal{B}_x : x \in X\}$ has the properties (LB1)–(LB3). Consider the family $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$. Since $x \in \bigcup\mathcal{B}_x \subset \bigcup\mathcal{B}$ for each $x \in X$, we have $\bigcup\mathcal{B} = X$ so the axiom (B1) holds for $\mathcal{B}$. Assume that $U, V \in \mathcal{B}$ and $x \in U \cap V$. There are $y, z \in X$ with $U \in \mathcal{B}_y$ and $V \in \mathcal{B}_z$. Applying (LB3) we obtain $U', V' \in \mathcal{B}_x$ such that $U' \subset U$ and $V' \subset V$. The property (LB2) shows that there exists a set $W \in \mathcal{B}_x$ for which $x \in W \subset U' \cap V' \subset U \cap V$. As a consequence the family $\mathcal{B}$ satisfies (B2).

By Problem 006 there is a unique topology τ on X such that $\mathcal{B}$ is a base for τ. In particular, $\mathcal{B}_x \subset \tau$ for all $x \in X$. If $x \in U \in \tau$ then there is $V \in \mathcal{B}$ such that $x \in V \subset U$.

By the definition of $\mathcal{B}$ we have $V \in \mathcal{B}_y$ for some $y \in X$. Applying (LB3) we can obtain $W \in \mathcal{B}_x$ such that $x \in W \subset V \subset U$. This proves that $\mathcal{B}_x$ is a local base for τ at x for each $x \in X$.

Suppose finally, that τ' is a topology on X such that $\mathcal{B}_x$ is a local base for τ' at x for every $x \in X$. It is immediate from the definition of a local base that $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ is a base for τ'. By the uniqueness part of Problem 006, we have $\tau' = \tau$ and our proof is complete.

S.008. *Prove that* $\bigcup \mathcal{S} = X$ *for any subbase* $\mathcal{S}$ *of a topological space* X. *Now, let* X *be an arbitrary set without topology. Prove that, for any family* $\mathcal{S} \subset \exp(X)$ *with* $\bigcup \mathcal{S} = X$, *there exists a unique topology* τ *on* X *such that* $\mathcal{S}$ *is a subbase for* (X, τ). *We will call* τ *the topology generated by* $\mathcal{S}$ *as a subbase.*

Solution. Since the finite intersections of the elements of $\mathcal{S}$ form a base of X, the set X is a union of those finite intersections. As a consequence, for any $x \in X$, there are $U_1, \ldots, U_n \in \mathcal{S}$ for which $x \in U_1 \cap \cdots \cap U_n \subset \bigcup \mathcal{S}$. This proves that $\bigcup \mathcal{S} = X$.

Now, let X be an arbitrary set without topology for which we have a family $\mathcal{S} \subset \exp(X)$ with $\bigcup \mathcal{S} = X$. Denote by $\mathcal{B}$ the family of all finite intersections of the elements of $\mathcal{S}$. The property (B1) holds for $\mathcal{B}$ because $\mathcal{S} \in \mathcal{B}$. It is immediate that, for any $U, V \in \mathcal{B}$, we have $U \cap V \in \mathcal{B}$. Therefore, taking $W = U \cap V$ in the hypothesis of (B2) shows that (B2) holds for $\mathcal{B}$ as well.

Take the unique topology τ on X for which $\mathcal{B}$ is a base (see Problem 006). Since $\mathcal{B}$ coincides with all finite intersections of the elements of S, the family $\mathcal{S}$ is a subbase for τ. Finally, if τ' is another topology with $\mathcal{S}$ a subbase of τ' then $\mathcal{B}$ is a base for τ' by the definition of subbase. Hence $\tau' = \tau$ by the uniqueness part of Problem 006.

S.009. *Suppose that* X *and* Y *are topological spaces and* $f : X \to Y$. *Prove that the following conditions are equivalent:*

(*i*) *f is a continuous map.*
(*ii*) *There is a base* $\mathcal{B}$ *in* Y *such that* $f^{-1}(U)$ *is open in* X *for every* $U \in \mathcal{B}$.
(*iii*) *There is a subbase* $\mathcal{S}$ *in* Y *such that* $f^{-1}(U)$ *is open in* X *for every* $U \in \mathcal{S}$.
(*iv*) *f is continuous at every point* $x \in X$.
(*v*) $f^{-1}(F)$ *is closed in* X *whenever* F *is closed in* Y.
(*vi*) $f(\mathrm{cl}_X(A)) \subset \mathrm{cl}_Y(f(A))$ *for any* $A \subset X$.
(*vii*) $\mathrm{cl}_X(f^{-1}(B)) \subset f^{-1}(\mathrm{cl}_Y(B))$ *for any* $B \subset Y$.
(*viii*) $f^{-1}(\mathrm{Int}_Y(B)) \subset \mathrm{Int}_X(f^{-1}(B))$ *for any* $B \subset Y$.

Solution. To see that (i)$\Longrightarrow$(ii) let $\mathcal{B} = \tau(Y)$. The implication (ii)$\Longrightarrow$(iii) is obtained looking at the subbase $\mathcal{S} = \mathcal{B}$.

To establish the implication (iii)$\Longrightarrow$(iv) assume that $f(x) \in U \in \tau(Y)$. There exist sets $U_1, \ldots, U_n \in \mathcal{S}$ such that $f(x) \in U_1 \cap \cdots \cap U_n \subset U$. Now observe that the set $V = f^{-1}(U_1) \cap \cdots \cap f^{-1}(U_n)$ is open in X, contains x and $f(V) \subset U$ which proves continuity of f at the point x.

To show that (iv)$\Longrightarrow$(v), take any closed $F \subset Y$. Given $x \in X \backslash f^{-1}(F)$, we have $f(x) \in U = Y \backslash F \in \tau(Y)$. By continuity of the function f at the point x, there is a set

$V_x \in \tau(X)$ such that $f(V_x) \subset U$ and, as a consequence, $V_x \subset X\backslash f^{-1}(F)$. Therefore $X\backslash f^{-1}(F) = \bigcup\{V_x : x \in X\backslash f^{-1}(F)\}$ is an open set and hence $f^{-1}(F)$ is closed.

Let us prove the implication (v)$\Longrightarrow$(vi). Take any $A \subset X$ and any point $y \in F = \mathrm{cl}_Y(f(A))$. The set $G = f^{-1}(F)$ is closed in X and contains the set $f^{-1}(f(A)) \supset A$. Therefore $\mathrm{cl}_X(A) \subset G$ and hence $f(\mathrm{cl}_X(A)) \subset F = \mathrm{cl}_Y(f(A))$.

To show that (vi)$\Longrightarrow$(vii) take an arbitrary set $B \subset Y$ and observe that $f(\mathrm{cl}_X(f^{-1}(B))) \subset \mathrm{cl}_Y(f(f^{-1}(B))) \subset \mathrm{cl}_Y(B)$ and, as a consequence, we have $\mathrm{cl}_X(f^{-1}(B)) \subset f^{-1}(\mathrm{cl}_Y(B))$.

To prove the implication (vii)$\Longrightarrow$(viii) take any $B \subset Y$ and note that the set $F = Y\backslash \mathrm{Int}_Y(B)$ is closed in Y. Thus $\mathrm{cl}_X(f^{-1}(F)) \subset f^{-1}(\mathrm{cl}_Y(F)) = f^{-1}(F)$, which shows that $f^{-1}(F)$ is closed in X. Hence the set $f^{-1}(\mathrm{Int}_Y(B)) = X\backslash f^{-1}(F)$ is open in X and $f^{-1}(\mathrm{Int}_Y(B)) \subset f^{-1}(B)$. Since the interior of $f^{-1}(B)$ is the union of all open sets contained in $f^{-1}(B)$, we have $f^{-1}(\mathrm{Int}_Y(B)) \subset \mathrm{Int}_X(f^{-1}(B))$.

To establish the implication (viii)$\Longrightarrow$(i) take any open $U \subset Y$ and observe that $\mathrm{Int}_X(f^{-1}(U)) \supset f^{-1}(\mathrm{Int}_Y(U)) = f^{-1}(U)$ which shows that $f^{-1}(U)$ is open in X.

S.010. *Show that any T_1-space is a T_0-space. Give an example of a T_0-space which is not a T_1-space.*

Solution. Suppose that X is a T_1-space. Given distinct points $x, y \in X$, the set $U = X\backslash\{x\}$ is open and $U \cap \{x, y\} = \{y\}$ which shows that X is a T_0-space. To see that the classes T_0 and T_1 do not coincide, please, check that the space $X = \{0, 1\}$ with $\tau(X) = \{\emptyset, X, \{0\}\}$ is a T_0-space which is not a T_1-space.

S.011. *Show that any T_2-space is a T_1-space. Give an example of a T_1-space which is not a T_2-space.*

Solution. Suppose that X is a Hausdorff space and $x \in X$. For any $y \in X\backslash\{x\}$ fix open disjoint sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. We have $y \in V_y \subset X\backslash \{x\}$ and therefore $X\backslash\{x\} = \cup\{V_y : y \in X\backslash\{x\}\}$ is an open set. Hence $\{x\}$ is closed.

Now, let $X = \mathbb{R}$ and $\tau(X) = \{\emptyset\} \cup \{\mathbb{R}\backslash A : A$ is a finite subset of $\mathbb{R}\}$. Since $\mathbb{R}\backslash\{x\} \in \tau(X)$ for any $x \in X$, the space X is a T_1-space. However, there are no disjoint open sets U and V such that $0 \in U$ and $1 \in V$ and therefore X is not a Hausdorff space.

S.012. *Show that any T_3-space is a T_2-space. Give an example of a T_2-space which is not a T_3-space.*

Solution. Let X be a T_3-space. Given distinct $x, y \in X$, the set $F = \{y\}$ is closed and does not contain x. Therefore there are open $U, V \subset X$ such that $x \in U$, $\{y\} \subset V$ and $U \cap V = \emptyset$. It is clear that the sets U and V separate the points x and y.

To get an example of a Hausdorff non-regular space, denote the set $\{\frac{1}{n} : n \in \mathbb{N}\}$ by S and let $\mathcal{U} = \mathcal{N}_R \cup \{\mathbb{R} \setminus S\}$. Since $\bigcup \mathcal{U} \supset \bigcup \mathcal{N}_R = \mathbb{R}$, the family $\mathcal{U}$ generates a topology τ on $\mathbb{R}$ as a subbase (Problem 008). Denote the space $(\mathbb{R}, \tau)$ by X. Given distinct $p, q \in X$, let $r = |p - q|$ (remember that p and q are real numbers!) and consider the intervals $U = (x - \frac{r}{2}, x + \frac{r}{2})$ and $V = (y - \frac{r}{2}, y + \frac{r}{2})$ which are open and disjoint.

Observe that any open interval $(a, b) \subset \mathbb{R}$ is open in the natural topology $\mathcal{N}_R$ on $\mathbb{R}$ because, if $x \in (a, b)$ then, for $\varepsilon = \min\{x - a, b - x\}$, we have $(x - \varepsilon, x + \varepsilon) \subset (a, b)$. Since $p \in U$, $q \in V$, $U, V \in \mathcal{N}_R \subset \tau$ and $U \cap V = \emptyset$, we proved that X is Hausdorff.

To see that X is not regular, let $x = 0$ and $F = S$. It is clear that F is a closed subset of X which does not contain x. Suppose that $U, V \in \tau$, $x \in U$, $F \subset V$ and $U \cap V = \emptyset$. As the family $\mathcal{U}$ is a subbase for τ, there are $U_1, \dots, U_n \in \mathcal{U}$ with $x \in W = U_1 \cap \dots \cap U_n \subset U$. Then $W' = W \cap (\mathbb{R} \backslash S) \in \tau$ and if we take the intersection G of all U_i's which belong to $\mathcal{N}_R$ then $W' = G \cap (\mathbb{R}\backslash S)$. Since $G \in \mathcal{N}_R$, there is $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset G$. It is clear that the set $W'' = (-\varepsilon, \varepsilon)\backslash S$ belongs to τ and $W'' \cap V = \emptyset$. The sequence $\left\{\frac{1}{n}\right\}$ converges to zero and hence $\frac{1}{m-1} < \varepsilon$ for some $m \in \mathbb{N}$. Since $\frac{1}{m} \in V \in \tau$, there are $V_1, \dots, V_k \in \mathcal{U}$ such that $\frac{1}{m} \in V_1 \cap \dots \cap V_k \subset V$. Observe that $V_i \neq (\mathbb{R}\backslash S)$ for all i because $\frac{1}{m} \notin (\mathbb{R}\backslash S)$. Thus, there exists $\delta > 0$ such that $\delta < \min\left\{\frac{1}{m-1} - \frac{1}{m}, \frac{1}{m} - \frac{1}{m+1}\right\}$ and $H = \left(\frac{1}{m} - \delta, \frac{1}{m} + \delta\right) \subset V_i$ for all $i \leqslant k$. Now, the point $z = \frac{1}{m} + \frac{\delta}{2}$ belongs to $H \cap (-\varepsilon, \varepsilon) \cap (\mathbb{R}\backslash S) \subset U \cap V = \emptyset$ which is a contradiction.

S.013. *Show that any Tychonoff space is a T_3-space.*

Solution. Any Tychonoff space is T_1 by definition. Suppose that X is Tychonoff, $x \in X$ and F is a closed subset of X with $x \notin F$. There exists a continuous function $f : X \to \mathbb{R}$ with $f(x) = 1$ and $f(y) = 0$ for all $y \in F$. In the previous problem, we proved that any interval $(a, b) \subset \mathbb{R}$ is open in $\mathbb{R}$. By continuity of f the sets $U = f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ and $V = f^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ are open in X and $U \cap V = \emptyset$. It is immediate that $x \in U$ and $F \subset V$ so X is regular.

S.014. *Let $Y = \{(x, y) \in \mathbb{R}^2 : y \geqslant 0\}$. Denote by L the set $\{(x, y) \in Y : y = 0\}$. For each $z = (x, 0) \in L$ let $N_z = \{(x, t) : 0 < t \leqslant 2\} \cup \{(t + x, t) : 0 < t \leqslant 2\}$. If $z \in Y\backslash L$ we put $\mathcal{B}_z = \{\{z\}\}$. Given $z \in L$, let $\mathcal{B}_z = \{\{z\} \cup (N_z\backslash A) : A$ is a finite subset of $N_z\}$. Let $p = (0, -1)$ and $X = Y \cup \{p\}$. Denote by $\mathcal{B}_p$ the family $\{\{p\} \cup O_n : n \in \omega\}$, where $O_n = \{z = (x, y) \in Y : x > n\}$ for any $n \in \omega$.*

(*i*) *Show that the families $\{\mathcal{B}_z : z \in X\}$ satisfy the conditions (LB1)–(LB3) of the Problem 007 and hence they generate a topology τ on X as local bases. Denote by μ_Y the topology of subspace of X on Y.*

(*ii*) *Prove that any $U \in \mathcal{B}_z$ is closed in X for any $z \in Y$. Deduce from this fact that the space (Y, μ_Y) is a Tychonoff one.*

(*iii*) *Let $f : Y \to \mathbb{R}$ be a continuous function. Assume that $f(z) = 0$ for some $z \in L$. Prove that there exists a countable set $N(f, z) \subset N_z$ such that $f(u) = 0$ for any $u \in N_z \backslash N(f, z)$.*

(*iv*) *Suppose that $f : Y \to \mathbb{R}$ is a continuous function such that $f|A \equiv 0$ for an infinite $A \subset N_z$ for some $z \in L$. Prove that $f(z) = 0$.*

(*v*) *Given $r \in \mathbb{R}$, assume that $f : Y \to \mathbb{R}$ is a continuous function such that $f|B \equiv 0$ for an infinite set $B \subset [r, r + 1] \times \{0\} \subset L$. Show that there is an infinite set $B' \subset [r + 1, r + 2] \times \{0\}$ for which $f|B' \equiv 0$.*

(*vi*) *Denote by W_n the set $\{p\} \cup O_n$. Prove that $\overline{W}_{n+2} \subset W_n$ for any $n \in \omega$ (the closure is taken in X). Deduce from this fact that X is a T_3-space.*

(*vii*) *Let* $F = \{(t, 0) : t \in (-\infty, 0]\}$. *Prove that* F *is closed in* X *and* $f(p) = 0$ *for any function* $f \in C(X)$ *such that* $f(x) = 0$ *for all* $x \in F$. *Conclude that* X *is an example of a* T_3*-space which is not completely regular.*

Solution. (i) The property (LB1) is clear. It is also immediate that $U, V \in \mathcal{B}_z$ implies $U \cap V \in \mathcal{B}_z$ for all $z \in X$. Thus (LB2) also holds. The property (LB3) is evident for any $z \in Y \backslash L$ because $\mathcal{B}_z = \{\{z\}\}$ and $z \in U \in \mathcal{B}_t$ implies $\{z\} \subset U$. Now, if $z \in L$ and $z \in U \in \mathcal{B}_y$ for some $y \neq z$ then $y = p$ and $U = O_n$ for some $n \in \omega$. Therefore $M_z = \{z\} \cup N_z \subset O_n = U$ and (LB3) is checked for z. Now, if $z = p$ then $z \in U \in \mathcal{B}_y$ is possible only if $y = p$ so this case is trivial.

(ii) If $z \in Y \backslash L$ then $U = \{z\}$ and, for any $y \in Y \backslash \{z\}$, the set $V \backslash \{z\}$ is open for every $V \in \mathcal{B}_y$. Indeed, if $y \in Y \backslash L$ then $V = \{y\}$ and $V \backslash \{z\} = \{y\} \backslash \{z\} = V$. If $y \in L$ then $V = M_y \backslash A$ for some finite $A \subset N_y$ and $V \backslash \{z\} = M_y \backslash (A \cup \{z\}) \in \mathcal{B}_y$. This shows that, for every point $y \in Y \backslash \{z\}$, there is an open set W_y such that $y \in W_y \subset Y \backslash \{z\}$. Thus $Y \backslash \{z\} = \bigcup\{W_y : y \in Y \backslash \{z\}\}$ is an open set and hence $U = \{z\}$ is closed. In particular, $\{z\}$ is closed for any $z \in Y \backslash L$. If $z \in L$ then the set $Y \backslash \{z\} = \bigcup\{N_t : t \in L \backslash \{z\}\} \cup (\bigcup\{\{z\} : z \in Y \backslash L\})$ is open and hence $\{z\}$ is also closed in Y. Thus, Y is a T_1-space.

Suppose now that $U \in \mathcal{B}_z$ for some point $z \in L$. Then $U = M_z \backslash A$ for some finite $A \subset N_z$. If $y \in (Y \backslash L) \backslash U$ then we let $W_y = \{y\} \in \mathcal{B}_y$. It is clear that $y \in W_y \subset Y \backslash U$. Now, if $y \in L \backslash \{z\}$ then the sets N_y and N_z can have at most two points (say a, b) in their intersection. If we let $W_y = M_y \backslash \{a, b\}$ then again $W_y \in \mathcal{B}_y$ and $W_y \subset Y \backslash U$. Therefore $Y \backslash U = \bigcup\{W_y : y \in Y \backslash U\}$ is an open set and hence U is closed.

Let us prove that (Y, μ_Y) is completely regular (and hence Tychonoff) space. Given a point $y \in Y$ and a closed set $F \subset Y$ such that $y \notin F$, there is $U \in \mathcal{B}_y$ such that $U \cap F = \emptyset$ because $Y \backslash F$ is an open neighbourhood of y and $\mathcal{B}_y$ is a local base of Y at y. Now, let $f(x) = 1$ if $x \in U$ and $f(x) = 0$ if $x \in Y \backslash U$. For the function $f : Y \to [0, 1]$ we have $f(y) = 1$ and $f(x) = 0$ for any $x \in F$ so we only must prove that f is continuous. We will use the condition 009(vi) which is equivalent to continuity of f. For any $A \subset Y$ the set $f(A)$ consists of one or two points of $[0, 1]$ so $\overline{f(A)} = f(A)$ (the bar denotes the closure in $[0, 1]$). If $f(A) = \{0, 1\}$ then $f(\mathrm{cl}_Y(A)) \subset f(Y) = \{0, 1\} = f(A) = \overline{f(A)}$. If $f(A)$ is a one-point set, say $f(A) = \{1\}$ then $A \subset U$ and, the set U being closed, we have $\mathrm{cl}_Y(A) \subset U$ whence $f(\mathrm{cl}_Y(A)) \subset f(U) = \{1\} = \overline{f(A)}$. If $f(A) = \{0\}$ then $A \subset Y \backslash U$ and $\mathrm{cl}_Y(A) \subset Y \backslash U$ because $Y \backslash U$ is also closed. Thus $f(\mathrm{cl}_Y(A)) \subset f(Y \backslash U) = \{0\} = \overline{f(A)}$.

(iii) We use the condition 009(iv) which is equivalent to continuity of f. Recall that $M_z = N_z \cup \{z\}$ for any $z \in L$. For any $n \in \mathbb{N}$, the interval $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is an open set which contains 0 and therefore there is an open $U \ni z$ for which $f(U) \subset \left(-\frac{1}{n}, \frac{1}{n}\right)$. Since $\mathcal{B}_z$ is a local base at z, there is a finite $A_n \subset N_z$ such that $M_z \backslash A_n \subset U$ and hence $f(M_z \backslash A_n) \subset f(U) \subset \left(-\frac{1}{n}, \frac{1}{n}\right)$. We claim that the countable set $N(f, z) = \bigcup\{A_n : n \in \mathbb{N}\}$ is as promised. Indeed, if $u \in N_z \backslash N(f, z)$ then $u \in \bigcap\{N_z \backslash A_n : n \in \mathbb{N}\}$ and hence $|f(u)| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore $f(u) = 0$.

(iv) Note that $z \in \mathrm{cl}_Y(A)$ so we can apply the condition 009(vi) to conclude that $f(z) \in \overline{f(A)} = \overline{\{0\}} = \{0\}$ (the closure is taken in $\mathbb{R}$) and hence $f(z) = 0$.

(v) Given an arbitrary point $z = (x, 0) \in L$, let $N_z^0 = \{(x, t) : 0 < t < 2\}$ and $N_z^1 = \{(t + x, t) : 0 < t \leqslant 2\}$. We can assume that B is countable because every infinite set contains an infinite countable subset. Let $N_B = \bigcup\{N(f, z) : z \in B\}$. Denote by $\overline{pr}$ the projection of the set Y onto L, i.e., $\overline{pr}((x, y)) = (x, 0)$ for any $(x, y) \in Y$. Note that, for any $z \in B$, we have $\overline{pr}(N_z^1) \supset [r+1, r+2] \times \{0\}$ and hence $N_z^1 \cap N_y^0 \neq \emptyset$ for any $y \in [r+1, r+2]$. The set $P = \overline{pr}(N_B)$ is countable and therefore there is an infinite set $B' \subset ([r+1, r+2] \times \{0\}) \backslash P$. Let us prove that B' is as promised. If $z \in B'$ and $y \in B$, denote by t_y the unique intersection point of the sets N_z^0 and N_y^1. Since $N_z^0 \cap N_B = \emptyset$, we have $t_y \notin N(f, y)$ (see (iii)) and hence $f(t_y) = 0$. It is evident that $y \neq y'$ implies $t_y \neq t_{y'}$ and therefore $\{t_y : y \in B\}$ is an infinite subset of N_z on which the function f is equal to zero. Now the statement (iv) shows that $f(z) = 0$.

(vi) Since $W_{n+2} \subset W_n$, we only have to prove that $\overline{W}_{n+2} \backslash W_{n+2} \subset W_n$. All points of $\overline{W}_{n+2} \backslash W_{n+2}$ belong to L so let $z = (x, 0) \in L$. If $x \leqslant n$ then $N_z \cap W_{n+2} = \emptyset$ and hence $x \notin \overline{W}_{n+2}$. Thus $\overline{W}_{n+2} \subset W_n$.

To prove that X is a T_3-space, note that we proved that all points of Y are closed in Y. Observe that, for any $z \in Y$, there is $n \in \mathbb{N}$ such that $z \notin W_n$. This shows that p cannot be in the closure of $\{z\}$ and hence $\{z\}$ is also closed in X. It is immediate that Y is open in X and therefore $\{p\}$ is closed. This proves that X is a T_1-space so we only need to establish regularity of X. Observe first that, for any $z \in Y$, the point p is not in the closure of any $U \in \mathcal{B}_z$ and hence U is also closed and open in X. If $z \in Y$ and F is a closed subset of X with $z \notin F$ then there is $U \in \mathcal{B}_z$ with $U \cap F = \emptyset$. Then for the open sets U and $V = X \backslash U$, we have $F \subset V$, $x \in U$ and $U \cap V = \emptyset$. Finally, suppose that F is a closed subset of X with $p \notin F$. Since the family $\{W_n : n \in \mathbb{N}\}$ is a local base at p, there is $n \in \mathbb{N}$ such that $W_n \cap F = \emptyset$. Now let $U = W_{n+2}$ and $V = X \backslash \overline{W}_{n+2}$. It is easy to see that U and V are open sets such that $p \in U$, $F \subset V$ and $U \cap V = \emptyset$.

(vii) The set F must be closed in the space X because its complement $X \backslash F = \bigcup\{\{z\} : z \in Y \backslash L\} \cup \{W_z : z \in L \backslash F\} \cup W_1$ is open being a union of open sets. Now suppose that $f : X \to \mathbb{R}$ is a continuous function with $f|F \equiv 0$. Apply (v) for $r = -1$. The function f has infinitely many zeros on $[-1, 0] \times \{0\}$ and hence it has infinitely many zeros on $[0, 1] \times \{0\}$ and the same on $[1, 2] \times \{0\}$, on $[2, 3] \times \{0\}$, etc. As a consequence, for any $n \in \mathbb{N}$, there is $x > n$ such that $f(z) = 0$ for $z = (x, 0)$. This implies $p \in \overline{A}$ where $A = f^{-1}(0)$. Since $\{0\}$ is closed in $\mathbb{R}$, the set A is closed in X and hence $p \in A$. This proves that $f(p) = 0$, so the space X cannot be completely regular.

S.015. *(Urysohn's lemma). Let X be a normal space. Suppose that F and G are non-empty closed subsets of X with* $F \cap G = \emptyset$. *Prove that for any rational number* $q \in [0, 1]$ *one can choose an open set* U_q *in such a way that the following properties will hold:*

(*i*) $F \subset U_0$ *and* $U_1 = X \backslash G$.
(*ii*) $\overline{U}_r \subset U_s$ *if* $r < s$.

Show that the function $f : X \to \mathbb{R}$ *defined by the formula*

$$f(x) = \begin{cases} \inf\{r : x \in U_r\}, & \text{if } x \in X \backslash G; \\ 1, & \text{if } x \in G \end{cases}$$

is continuous and $f(F) \subset \{0\}$, $f(G) \subset \{1\}$. *Deduce from this fact that any* T_4*-space is a Tychonoff space.*

Solution. Apply normality of the space X to find open sets U_0 and W such that $F \subset U_0$, $G \subset W$ and $U_0 \cap W = \emptyset$. The set $X \backslash W$ is closed and $U_0 \subset X \backslash W$ which implies $\overline{U}_0 \subset X \backslash W \subset U_1$. Take any faithful enumeration $\{q_n : n \in \omega\}$ of the set $\mathbb{Q} \cap [0, 1]$ such that $q_0 = 0$ and $q_1 = 1$. We already have open sets U_0 and U_1 such that $\overline{U}_0 \subset U_1$. Suppose that $k \geqslant 1$ and, for any $n \leqslant k$, we have an open set U_{q_n} such that $q_n < q_m$ implies $\overline{U}_{q_n} \subset U_{q_m}$ for all $m, n \leqslant k$. Consider the numbers $r = \max\{q_i : i \leqslant k$ and $q_i < q_{k+1}\}$ and $s = \min\{q_i : i \leqslant k$ and $q_{k+1} < q_i\}$. It is clear that $r < q_{k+1} < s$ and $\overline{U}_r \subset U_s$ by the inductive hypothesis. By normality of X there exist open sets $U_{q_{k+1}}$ and W such that $\overline{U}_r \subset U_{q_{k+1}}$, $X \backslash U_s \subset W$ and $U_{q_{k+1}} \cap W = \emptyset$. Since the set $X \backslash W$ is closed and $U_{q_{k+1}} \subset X \backslash W$, we have $\overline{U}_{q_{k+1}} \subset X \backslash W \subset U_s$. We claim that the sets $\{U_{q_n} : n \leqslant k + 1\}$ satisfy the inductive hypothesis $\overline{U}_{q_n} \subset U_{q_m}$ for all $m, n \leqslant k + 1$ such that $q_n < q_m$.

Indeed, if both numbers q_n and q_m are distinct from q_{n+1} then the inclusion takes place by the inductive hypothesis. If $m = k + 1$ then $\overline{U}_{q_n} \subset \overline{U}_r \subset U_{q_{k+1}}$ by our construction and the inductive hypothesis. If $n = k + 1$ then $\overline{U}_{q_{k+1}} \subset U_s \subset U_{q_m}$ by our construction and the inductive hypothesis. Thus, our inductive construction can be fulfilled for all $n \in \omega$ and hence we will have open sets $\{U_q : q \in \mathbb{Q} \cap [0, 1]\}$ with the properties (i) and (ii).

Note that $x \in U_0$ for any $x \in F$ and hence $f(x) = 0$. By the definition of f we have $f(x) = 1$ for all $x \in G$. Let us prove that f is continuous. Since the family $\mathcal{S} = \{(r, +\infty) : r \in \mathbb{R}\} \cup \{(-\infty, r) : r \in \mathbb{R}\}$ is a subbase of $\mathbb{R}$, it is sufficient to prove that $f^{-1}(U)$ is open in X for any $U \in \mathcal{S}$ (009(iii)).

Now, if $r \leqslant 0$ then $f^{-1}((-\infty, r)) = \emptyset$ and $f^{-1}((-\infty, r)) = X$ if $r > 1$. If $0 < r \leqslant 1$ then the set $f^{-1}((-\infty, r)) = \bigcup\{U_s : s \in \mathbb{Q} \cap [0, 1]$ and $s < r\}$ is also open as a union of open sets. If $r < 0$ then $f^{-1}((r, +\infty)) = X$ and $f^{-1}((r, +\infty)) = \emptyset$ if $r \geqslant 1$. Finally, if $0 \leqslant r < 1$ we have $f^{-1}((r, +\infty)) = \bigcup\{X \backslash \overline{U}_s : s \in \mathbb{Q} \cap [0, 1]$ and $s > r\}$ which proves that $f^{-1}((r, +\infty))$ is an open set.

Suppose finally that X is a T_4-space. Given $x \in X$ and a closed set $F \subset X$ with $x \notin F$, the sets $\{x\}$ and F are closed and disjoint so by the Urysohn's lemma there is a continuous function $f : X \to [0, 1]$ such that $f(\{x\}) = \{1\}$ and $f(y) = 0$ for all $y \in F$. Therefore $f(x) = 1$, $f(F) \subset \{0\}$ and this proves that X is a Tychonoff space.

S.016. *Prove that the space* (Y, μ_Y), *constructed in Problem 014 is Tychonoff but not normal.*

Solution. It was proved in 014 that (Y, μ_Y) is a Tychonoff space. To see that it is not normal, consider the disjoint closed sets $F = \{(t, 0) : t \in (-\infty, 0]\}$ and $G = \{(t, 0) : t \in (0, +\infty)\}$ of the space Y. Suppose that Y is normal. By Problem 015 there is a continuous function $f : Y \to \mathbb{R}$ such that $f(F) = \{0\}$ and $f(G) = \{1\}$. Now apply 014(v) for $r = -1$ to conclude that there are infinitely many points in $[0, 1] \times \{0\} \subset G$ in which f is equal to zero. However, this is a contradiction because $f(x) = 1$ for every $x \in G$.

S.017. *Let* X *be a* T_i*-space for* $i \leqslant 3\frac{1}{2}$. *Prove that any subspace of* X *is a* T_i*-space.*

Solution. Let X be a T_0-space. If $Y \subset X$ and x, y are distinct points of Y then there is $U \in \tau(X)$ such that $U \cap \{x, y\}$ consists of exactly one point. Then $U' = U \cap Y \in \tau(Y)$ and the set $U' \cap \{x, y\} = U \cap \{x, y\}$ also consists of exactly one point.

Now, if X is a T_1-space and $Y \subset X$ then, for any $y \in Y$, the set $\{y\}$ is closed in X. Therefore $X \backslash \{y\} \in \tau(X)$ and $Y \backslash \{y\} = (X \backslash \{y\}) \cap Y$ is open in Y. As a consequence, $\{y\}$ is closed in Y.

Assume that X is a Hausdorff space and $Y \subset X$. If x and y are distinct points of Y, then there are $U, V \in \tau(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Now, if $U' = U \cap Y$ and $V' = V \cap Y$ then $U', V' \in \tau(Y)$, $x \in U'$, $y \in V'$ and $U' \cap V' = \emptyset$, which proves that Y is Hausdorff.

If X is a T_3-space and $Y \subset X$, then we already established that Y must be a T_1-space so we must only check that Y is regular. Take a point $y \in Y$ and a set $F \notin y$ which is closed in Y. The set $Y \backslash F$ is open in Y so there is $U \in \tau(X)$ with $U \cap Y = Y \backslash F$. The set $P = X \backslash U$ is closed and $y \notin P$. By regularity of X there are $V, W \in \tau(X)$ such that $y \in V, P \subset W$ and $V \cap W = \emptyset$. If $V' = V \cap Y$ and $W' = W \cap Y$ then $V', W' \in \tau(Y)$, $y \in V'$, $F \subset W'$ and $V' \cap W' = \emptyset$.

Suppose finally that X is Tychonoff and $Y \subset X$. We already established that Y must be a T_1-space so we must only check that Y is completely regular. Take a point $y \in Y$ and a set $F \notin y$ which is closed in Y. The set $Y \backslash F$ is open in Y so there is $U \in \tau(X)$ with $U \cap Y = Y \backslash F$. The set $P = X \backslash U$ is closed and $y \notin P$. By complete regularity of X there exists a continuous $f : X \to [0, 1]$ such that $f(y) = 1$ and $f(x) = 0$ for all $x \in P$. Let $g(x) = f(x)$ for all $x \in Y$. Then $g : Y \to [0, 1]$, $g(y) = 1$, $g(F) \subset \{0\}$ so the only thing left is to prove that g is continuous. But this follows immediately from the equality $g^{-1}(U) = f^{-1}(U) \cap Y$ for any $U \in \tau([0, 1])$.

S.018. *Show that a closed subspace of a T_4-space is a T_4-space. Give an example of a T_4-space X such that some $Y \subset X$ is not normal.*

Solution. Suppose that Z is a normal T_1-space and P is a closed subspace of Z. By 017, the space P is T_1 so it is sufficient to prove that P is normal. If F and G are closed disjoint subsets of P then F and G are closed in Z (it is an easy exercise) so there are $U, V \in \tau(Z)$ such that $F \subset U$, $G \subset V$ and $U \cap V = \emptyset$. Let $U' = U \cap P$ and $V' = V \cap P$. For the sets U' and V' we have $U', V' \in \tau(P)$, $F \subset U'$, $G \subset V'$ and $U' \cap V' = \emptyset$. Hence the space P is normal.

To construct the promised space X, we must develop some technique for proving normality of spaces. Given a space Z, say that $U \subset Z$ is *clopen* if $U \in \tau(Z)$ and U is closed in Z.

Claim 1. Suppose that a space Z is a finite union of its clopen normal subspaces. Then Z is normal.

Proof of the claim. Suppose that $Z = C_1 \cup \cdots C_n$, where C_i is a clopen normal subspace of Z for all $i \leqslant n$. We can assume that the sets C_i are disjoint, because if not, then we can consider the sets $C'_1 = C_1$ and $C'_i = C_i \backslash (C_1 \cup \cdots \cup C_{i-1})$ for all $i \in \{2, \ldots, n\}$. Every subspace C'_i is closed in C_i and hence normal. Besides, each C'_i is clopen in Z and $Z = C'_1 \cup \cdots \cup C'_n$. So, from now on we assume that the sets C_i

are disjoint. Suppose that F and G are closed disjoint subspaces of Z. Then $F_i = F \cap C_i$ and $G_i = G \cap C_i$ are closed disjoint subsets of C_i for each $i \leqslant n$. By normality of C_i, there are $U_i, V_i \in \tau(C_i)$ such that $F_i \subset U_i, G_i \subset V_i$ and $U_i \cap V_i = \emptyset$. For the sets $U = U_1 \cup \cdots \cup U_n$ and $V = V_1 \cup \cdots \cup V_n$, we have $U, V \in \tau(X), F \subset U, G \subset V$ and $U \cap V = \emptyset$. Therefore Z is normal.

Claim 2. Let Z be any space with a unique non-isolated point z_0. Then Z is normal.

Proof of the claim. Suppose that F and G are closed disjoint subspaces of Z. The point z_0 cannot belong to both of them; suppose, for example, that $z_0 \notin F$. Then F consists of isolated points of Z and hence F is open in Z. Let $U = F$ and $V = Z \backslash F$. Then $U, V \in \tau(Z), F \subset U, G \subset V$ and $U \cap V = \emptyset$ so the space Z is normal.

For each $n \in \mathbb{N}$, let $P_n = [0, 1] \times \left\{\frac{1}{n}\right\} \subset \mathbb{R}^2$ and $P_0 = [0, 1] \times \{0\} \subset \mathbb{R}^2$. Our space X will be the set $P = \bigcup\{P_n : n \in \omega\}$ with some topology we are going to describe. Let $p = (0,0), D = \{(x, y) \in P : x > 0 \text{ and } y > 0\}$ and $Q = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$. If $z \in D$ then let $\mathcal{B}_z = \{\{z\}\}$. For any $z = (x, 0) \in P_0 \backslash \{p\}$, we let $\mathcal{B}_z = \{O(x, k) : k \in \mathbb{N}\}$ where $O(x, k) = \{(x, 0)\} \cup \{(x, \frac{1}{n}) : n \in \mathbb{N} \text{ and } n \geqslant k\}$ for each $k \in \mathbb{N}$. Now, if $z = (0, \frac{1}{n}) \in Q$ then $\mathcal{B}_z = \{V(A, n) : A \text{ is a finite subset of } [0, 1]\}$, where $V(A, n) = \{z\} \cup (([0, 1]\backslash A) \times \frac{1}{n})$. Denote by N_k the set $\{0\} \cup \{\frac{1}{n} : n \geqslant k\}$ for each $k \in \mathbb{N}$ and let $\mathcal{B}_p = \{W(A, k) = ([0, 1]\backslash A) \times N_k : A \text{ is a finite subset of } (0, 1] \text{ and } k \in \mathbb{N}\}$. We leave to the reader the evident verification of the properties (LB1)–(LB3) (Problem 007) for the families $\{\mathcal{B}_z : z \in P\}$. Let $X = (P, \tau)$, where τ is the topology generated by the families $\{\mathcal{B}_z : z \in P\}$ as local bases. Observe also that every $V \in \mathcal{B}_z$ is clopen for any $z \in P$.

The space X is normal. Indeed, suppose that F and G are closed disjoint subspaces of X. The point p cannot belong to both of them; assume, for example, that $p \notin F$. The set $X \backslash F$ is an open neighbourhood of p and therefore there is a finite $A \subset (0, 1]$ and $k \in \mathbb{N}$ such that $W(A, k) \cap F = \emptyset$. Note that $X \backslash W(A, k) = P_1 \cup \cdots \cup P_{k-1} \cup (\bigcup\{\{a\} \times N_1 : a \in A\})$. Each P_i is clopen in X as well as $\{a\} \times N_1$ for each $a \in A$. The subspaces P_i and $\{a\} \times N_1$ are normal (Claim 2) for all $i \in \mathbb{N}$ and $a \in A$ being spaces with a unique non-isolated point. They are also clopen subspaces of X so Claim 1 is applicable to conclude that $X \backslash W(A, k)$ is a normal space.

The sets F and $G' = G \cap (X \backslash W(A, k))$ are closed and disjoint in $X \backslash W(A, k)$ so it is possible to choose $U, V' \in \tau(X \backslash W(A, k))$ with $F \subset U, G' \subset V'$ and $U \cap V' = \emptyset$. Now, if $V = V' \cup W(A, k)$ then $U, V \in \tau(X), F \subset U, G \subset V$ and $U \cap V = \emptyset$ whence the space X is normal.

The last thing we have to prove is to find a non-normal subspace of X. Let $Y = X \backslash \{p\}$. To see that Y is not normal, consider the sets $F = (\{0\} \times N_1) \backslash \{p\}$ and $G = P_0 \backslash \{p\}$. It is easy to verify that F and G are closed subspaces of Y and $F \cap G = \emptyset$. Suppose that $U, V \in \tau(Y)$ are disjoint and $F \subset U, G \subset V$. Then U and V are also open in X and hence, for each $z_k = (0, \frac{1}{k}) \in F$ there exists a finite set $A_k \subset [0, 1]$ such that $V(A_k, k) \subset U$. The set $A = \bigcup\{A_k : k \in \mathbb{N}\}$ is countable and hence there is $x \in (0, 1] \backslash A$. We have $(x, \frac{1}{n}) \in U$ for any $n \in \mathbb{N}$ and therefore $O(x, n) \cap U \neq \emptyset$ for all $n \in \mathbb{N}$. However, $\{O(x, n) : n \in \mathbb{N}\}$ is a local base at $(x, 0)$ which implies $O(x, n) \subset V$ for some $n \in \mathbb{N}$. As a consequence, $U \cap V \supset U \cap O(x, n) \neq \emptyset$ which is a contradiction. Thus Y cannot be normal.

S.019. *Prove that* $\mathbb{R}$ *is a* T_4*-space.*

Solution. Observe that any open interval $(a, b) \subset \mathbb{R}$ is open in the natural topology $\mathcal{N}_R$ on $\mathbb{R}$ because, if $x \in (a, b)$ then, for the number $\varepsilon = \min\{x - a, b - x\}$, we have $(x - \varepsilon, x + \varepsilon) \subset (a, b)$. For any $x \in \mathbb{R}$, the set $\mathbb{R}\backslash\{x\}$ is open because it is easy to represent it as a union of open intervals. Hence $\{x\}$ is closed and $\mathbb{R}$ is a T_1-space. To prove the normality of $\mathbb{R}$, we will need the following auxiliary function. Given $A \subset \mathbb{R}$, let $d_A(x) = \inf\{|x - y| : y \in A\}$.

Claim. The function d_A is continuous for any $A \subset \mathbb{R}$.

Proof of the claim. We use the condition 009(iv) which is equivalent to continuity. Take any $x_0 \in \mathbb{R}$ and suppose that $r_0 = d_A(x_0) \in U \in \mathcal{N}_R$. There is $\varepsilon > 0$ such that $(r_0 - \varepsilon, r_0 + \varepsilon) \subset U$. The proof of the claim will be finished if we show that $d_A(V) \subset (r_0 - \varepsilon, r_0 + \varepsilon) \subset U$, where $V = \left(x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}\right)$. Pick any $x \in V$. The infimum condition in the definition of d_A implies the existence of $y \in A$ such that $|x_0 - y| < d_A(x_0) + \frac{\varepsilon}{2}$. Then $|x - y| \leqslant |x - x_0| + |x_0 - y| < \frac{\varepsilon}{2} + d_A(x_0) + \frac{\varepsilon}{2} = d_A(x_0) + \varepsilon$. Therefore $d_A(x) \leqslant |x - y| < r_0 + \varepsilon$. To prove that $d_A(x) > r_0 - \varepsilon$ suppose not. Then $d_A(x) < r_0 - \frac{\varepsilon}{2}$ and hence we can find $z \in A$ such that $|x - z| < r_0 - \frac{\varepsilon}{2}$. Now, $|x_0 - z| \leqslant |x_0 - x| + |x - z| < \frac{\varepsilon}{2} + r_0 - \frac{\varepsilon}{2} = r_0$ and, as a consequence, $d_A(x_0) < |x - z| < r_0$ which is a contradiction. Thus $d_A(x) \in (r_0 - \varepsilon, r_0 + \varepsilon)$ and our claim is proved.

To prove that $\mathbb{R}$ is normal, take disjoint closed sets $F, G \subset \mathbb{R}$. The function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = d_F(x) - d_G(x)$ (see the claim) is continuous (it is a good exercise that the difference of two continuous functions is a continuous function; we leave its easy proof to the reader because a more general statement will be established in subsequent material). Observe that the sets $U = \varphi^{-1}((-\infty, 0))$ and $V = \varphi^{-1}((0, +\infty))$ are open in $\mathbb{R}$ and disjoint being inverse images of disjoint open sets under the continuous function φ. If $x \in F$ then $d_F(x) = 0$ and $d_G(x) > 0$ and hence $\varphi(x) < 0$ which implies $x \in U$. This shows that $F \subset U$. Analogously, if $x \in G$ then $d_G(x) = 0$ and $d_F(x) > 0$. Therefore, $\varphi(x) > 0$ and $x \in V$. Thus $G \subset V$ and we proved the normality of $\mathbb{R}$.

S.020. *Let* U *be a subspace of* $\mathbb{R}$. *Given a function* $f : U \to \mathbb{R}$, *prove that* f *is continuous in the sense of Calculus (that is, for any* $x \in U$ *and* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $y \in U$ *and* $|y - x| < \delta$ *implies* $|f(y) - f(x)| < \varepsilon$*) if and only if it is continuous as a map between the spaces* U *and* $\mathbb{R}$.

Solution. Suppose that f is continuous in the sense of calculus and fix $x \in U$. If $W \ni f(x)$ is an open set then there is $\varepsilon > 0$ such that $(f(x) - \varepsilon, f(x) + \varepsilon) \subset W$. By the calculus definition of continuity at the point x, there is $\delta > 0$ such that $|y - x| < \delta$, $y \in U$ implies $|f(y) - f(x)| < \varepsilon$. But what the last statement says is that $f(V) \subset (f(x) - \varepsilon, f(x) + \varepsilon) \subset W$ for the open set $V = (x - \delta, x + \delta) \cap U \ni x$. This shows that f is continuous in the topological sense.

Now, suppose that $f : U \to \mathbb{R}$ is continuous and fix $x \in U$ and $\varepsilon > 0$. The set $W = (f(x) - \varepsilon, f(x) + \varepsilon)$ is open in $\mathbb{R}$ and contains the point $f(x)$. By 009(iv) there is $V \in \tau(U)$ with $x \in V$ such that $f(V) \subset W$. The set V is also open in $\mathbb{R}$ and therefore there is a $\delta > 0$ such that $(x - \delta, x + \delta) \subset V$. Now we have $f((x - \delta, x + \delta)) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$ and this means precisely that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$.

S.021. *Let X, Y and Z be topological spaces. Suppose that $f \in C(X, Y)$ and $g \in C(Y, Z)$. Prove that $h = g \circ f \in C(X, Z)$. In other words, the composition of continuous maps is a continuous map.*

Solution. Let U be an open subset of Z. Then $g^{-1}(U)$ is open in Y by continuity of g. The set $W = f^{-1}(g^{-1}(U))$ is open in X by continuity of f. Now observe that $h^{-1}(U) = W$ is also open in X and hence h is continuous.

S.022. *Suppose that X and Z are topological spaces and $f : X \to Z$. Given a subspace $Y \subset X$, let $(f|Y)(x) = f(x)$ for any point $x \in Y$; this defines a map $f|Y : Y \to Z$. Prove that, if f is a continuous map, then $f|Y$ is also continuous. In other words, the restriction of a continuous map to a subspace is a continuous map.*

Solution. To prove continuity of the function $g = f|_Y$ observe that we have $g^{-1}(U) = f^{-1}(U) \cap Y$ for any $U \subset Z$. As a consequence, the set $g^{-1}(U)$ is open in Y if U is open in Z.

S.023. *Let X and Z be topological spaces. Suppose that, for a map $f : X \to Z$, we have $f(X) \subset T \subset Z$. Prove that f is a continuous map if and only if f is continuous considered as a mapping of X to T.*

Solution. Denote by f_0 the function f considered to map X to T. Suppose that f is continuous. If $U \in \tau(T)$ then there is $V \in \tau(X)$ such that $V \cap T = U$. Now, $f_0^{-1}(U) = f^{-1}(U) \in \tau(X)$ and hence f_0 is continuous.

On the other hand, if the map f_0 is continuous and $U \in \tau(Z)$ then we have $f^{-1}(U) = f_0^{-1}(U \cap T) \in \tau(X)$ and hence f is continuous.

S.024. *Prove that a composition of homeomorphisms is a homeomorphism.*

Solution. Suppose that $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms. The map $h = g \circ f$ is clearly a bijection which is continuous by Problem 021. Now the map $h^{-1} = f^{-1} \circ g^{-1}$ is also continuous by Problem 021 applied to the continuous maps f^{-1} and g^{-1}.

S.025. *Prove that, for any $a, b \in \mathbb{R}$ with $a < b$, the interval (a, b) is homeomorphic to $\mathbb{R}$.*

Solution. Define a function $f : (a, b) \to (-1, 1)$ by $f(t) = \frac{2}{b-a}(t - a) - 1$ for any $t \in (a, b)$. Then f is a homeomorphism as well as the map $g : (-1, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ defined by $g(t) = \frac{\pi}{2}t$. Finally, the map $h : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ for which $h(t) = \arctan(t)$ is also a homeomorphism and therefore, $h \circ g \circ f$ is a homeomorphism between (a, b) and $\mathbb{R}$ by Problem 024.

S.026. *Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{I}$.*

Solution. If $f : \mathbb{R} \to \mathbb{I}$ is a homeomorphism then $g = f^{-1} : [-1, 1] \to \mathbb{R}$ is a continuous surjective function which is also continuous in the sense of calculus by

Problem 020. However, a standard theorem of any basic course of calculus says that every continuous real-valued function on $\mathbb{I}$ is bounded and hence g cannot be surjective, a contradiction.

S.027. *Let X be a topological space. Given two mappings $f, g : X \to \mathbb{R}$ and $\lambda \in \mathbb{R}$, show that*

(*i*) *If $f, g \in C(X)$ then $f + g \in C(X)$ and $f \cdot g \in C(X)$.*
(*ii*) *If $f, g \in C(X)$ and $g(x) \neq 0$ for any $x \in X$ then $\frac{f}{g} \in C(X)$.*
(*iii*) *If $f \in C(X)$ then $\lambda f \in C(X)$ for any $\lambda \in \mathbb{R}$.*

Solution. To prove (i), fix a point $x_0 \in X$ and $\varepsilon > 0$. By continuity of f, there is $U \in \tau(X)$ such that $x_0 \in U$ and $f(U) \subset \left(f(x_0) - \frac{\varepsilon}{2}, f(x_0) + \frac{\varepsilon}{2}\right)$. Since g is also continuous, there is $V \in \tau(X)$ such that $x_0 \in V$ and $g(V) \subset (g(x_0) - \frac{\varepsilon}{2}, g(x_0) + \frac{\varepsilon}{2})$. Now, if $x \in W = U \cap V$ then $|(f+g)(x) - (f+g)(x_0)| = |f(x)+g(x) - f(x_0) - g(x_0)| \leqslant |f(x) - f(x_0)| + |g(x) - g(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ which shows that $(f + g)(W) \subset ((f + g)(x_0) - \varepsilon, (f + g)(x_0) + \varepsilon)$ and hence the function $f + g$ is continuous.

To prove continuity of $f \cdot g$ at the point x_0, fix $U, V \in \tau(X)$ such that $x_0 \in U \cap V$ and $|f(x) - f(x_0)| < 1$ for any $x \in U$ as well as $|g(x) - g(x_0)| < 1$ for any $x \in V$. For the number $K = |f(x_0)| + |g(x_0)| + 1$, find $W, W' \in \tau(X)$ for which $x \in W \cap W'$, $|f(x) - f(x_0)| < \frac{\varepsilon}{2K}$ if $x \in W$ and $|g(x) - g(x_0)| < \frac{\varepsilon}{2K}$ for all $x \in W'$. Observe that if $x \in U \cap V$ then $|f(x_0)| < |f(x_0)|+1 \leqslant K$ and $|g(x)| < |g(x_0)| + 1 \leqslant K$. The set $O = U \cap V \cap W \cap W'$ is open in X and contains the point x_0. Given $x \in O$, we have

$$\begin{aligned} |(f \cdot g)(x) - (f \cdot g)(x_0)| &= |f(x)g(x) - f(x_0)g(x_0)| \\ &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\leqslant |f(x)||g(x) - g(x_0)| + |g(x_0)||f(x) - f(x_0)| \\ &< K \cdot \tfrac{\varepsilon}{2K} + K \cdot \tfrac{\varepsilon}{2K} = \varepsilon. \end{aligned}$$

Therefore $(f \cdot g)(O) \subset ((f \cdot g)(x_0) - \varepsilon, ((f \cdot g)(x_0) + \varepsilon)$ and continuity of $f \cdot g$ at the point x_0 is proved.

(ii) Let us prove first that $\frac{1}{g}$ is continuous at the point x_0. By continuity of g, there is an open set $U \ni x_0$ such that $|g(x) - g(x_0) < \frac{|g(x_0)|}{2}$ for all $x \in U$. As a consequence, $|g(x)| > \frac{|g(x_0)|}{2}$ and hence $\frac{1}{|g(x)|} < \frac{2}{|g(x_0)|}$ for all $x \in U$. There is an open $V \ni x_0$ such that $|g(x) - g(x_0)| < \varepsilon \cdot \frac{(g(x_0))^2}{2}$ for all $x \in V$. The set $W = U \cap V$ is open in X and contains x_0. If $x \in W$ then $|(\frac{1}{g})(x) - (\frac{1}{g})(x_0)| = \frac{1}{|g(x)||g(x_0)|} \cdot |g(x) - g(x_0)| < \frac{2}{(g(x_0))^2} \cdot |g(x) - g(x_0)| < \varepsilon \cdot \frac{(g(x_0))^2}{2} \cdot \frac{2}{(g(x_0))^2} = \varepsilon$. This shows that the function $\frac{1}{g}$ is continuous at the point x_0 and hence the function $\frac{f}{g} = f \cdot \frac{1}{g}$ is also continuous by (i).

(iii) To prove continuity of λf at the point x_0, suppose first that $\lambda = 0$. Then the function λf is equal to zero at all points and its continuity is evident. If $\lambda \neq 0$ let $U \ni x_0$ be an open set with $|f(x) - f(x_0)| < \frac{\varepsilon}{|\lambda|}$ for any $x \in U$. Then, for any $x \in U$, we have $|\lambda f(x) - \lambda f(x_0)| = |\lambda| \cdot |f(x) - f(x_0)| < |\lambda| \frac{\varepsilon}{|\lambda|} = \varepsilon$ which proves continuity of λf at the point x_0.

S.028. *Let X be a topological space. Given two mappings $f, g : X \to \mathbb{R}$, show that, if $f, g \in C(X)$, then max$(f, g) \in C(X)$ and* $\min(f, g) \in C(X)$.

Solution. Observe that we have $\max(f, g)(x) = \frac{1}{2}(f(x) + g(x) + |f(x) + g(x)|)$ and $\min(f, g)(x) = \frac{1}{2}(f(x) + g(x) - |f(x) - g(x)|)$ for any $x \in X$. The functions $f + g$ and $f - g$ are continuous by Problem 027. The function $|f - g|$ is also continuous being a composition of $\varphi(t) = |t|$ with the continuous function $f - g$ (Problem 021). Applying problem 027 once more, we can conclude that $\max(f, g)$ and $\min(f, g)$ are continuous.

S.029. *Suppose that X is a topological space and $f_n : X \to \mathbb{R}$ for all $n \in \omega$. Prove that, if $\{f_n : n \in \omega\} \subset C(X)$ and $f_n \rightrightarrows f$, then $f \in C(X)$. In other words, the limit of a uniformly convergent sequence of continuous functions is a continuous function.*

Solution. Fix $x_0 \in X$ and $\varepsilon \geqslant 0$. There exists $m \in \omega$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n \geqslant m$ and $x \in X$. The function f_m is continuous at the point x_0 so there is an open $U \ni x_0$ such that $|f_m(x) - f_m(x_0)| < \frac{\varepsilon}{3}$ for all $x \in U$. If $x \in U$ then $|f(x) - f(x_0)| = |f(x) - f_m(x) + f_m(x) - f_m(x_0) + f_m(x_0) - f(x_0)| \leqslant |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, which proves continuity of f at the point x_0.

S.030. *Suppose that X is a set and we are given a function $f_n : X \to \mathbb{R}$ for each $n \in \omega$. Let $g_n = f_0 + \cdots + f_n$ and assume that $|f_n(x)| \leqslant c_n$ for every $x \in X$ and $n \in \omega$. Assume additionally that the series $\sum_{n=0}^{\infty} c_n$ converges, i.e., there is $c \in \mathbb{R}$ such that for every $\varepsilon > 0$ we can find a number $m \in \omega$ such that $|\sum_{k=0}^{n} c_k - c| < \varepsilon$ for all $n \geqslant m$. Prove that the sequence $\{g_n : n \in \omega\}$ converges uniformly on X, i.e., $g_n \rightrightarrows g$ for some $g : X \to \mathbb{R}$.*

Solution. It is clear that $c_n \geqslant 0$ for all $n \in \omega$. Fix $x \in X$ and consider the numeric sequence $\{g_n(x)\}$. Given $\varepsilon > 0$ there exists $m \in \omega$ such that $|\sum_{k=0}^{n} c_n - c| < \frac{\varepsilon}{2}$ for all $n \geqslant m$. As a consequence, for any natural $p, q \geqslant m$ with $p \leqslant q$, we have

$$\begin{aligned} |g_p(x) - g_q(x)| &= |f_{p+1}(x) + \cdots + f_q(x)| \leqslant c_{p+1} + \cdots + c_q \\ &= \left(\sum\nolimits_{i=1}^{q} c_i - c\right) - \left(\sum\nolimits_{i=1}^{p} c_i - c\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

which proves that $\{g_n(x)\}$ is a Cauchy sequence. Denote its limit by $g(x)$. Since $g(x)$ exists for every $x \in X$, we obtain a function $g : X \to \mathbb{R}$. To see that $g_n \rightrightarrows g$, let $\varepsilon > 0$. There exists a natural s such that, for all $n \geqslant s$ we have $|\sum_{k=0}^{n} c_n - c| < \frac{\varepsilon}{3}$. Now, if $p, q \geqslant s$ then

$$\begin{aligned} |g_p(x) - g_q(x)| &= |f_{p+1}(x) + \cdots + f_q(x)| \leqslant c_{p+1} + \cdots + c_q \\ &= \left(\sum\nolimits_{i=1}^{q} c_i - c\right) - \left(\sum\nolimits_{i=1}^{p} c_i - c\right) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \tfrac{2\varepsilon}{3} \end{aligned}$$

for all $x \in X$.

If $q \to \infty$ in the inequality $|g_p(x) - g_q(x)| \leqslant \frac{2\varepsilon}{3}$, we obtain $|g_p(x) - g(x)| \leqslant \frac{2\varepsilon}{3} < \varepsilon$ for all $x \in X$ and $p \geqslant s$ and this proves that $g_n \rightrightarrows g$.

S.031. *(The Tietze-Urysohn theorem) Let X be a normal space. Suppose that A is a closed subspace of X and* $f : A \to [a, b] \subset \mathbb{R}$ *is a continuous function. Prove that there exists a continuous function* $F : X \to [a, b]$ *such that* $F|A = f$*, i.e.,* $F(x) = f(x)$ *for all* $x \in A$.

Solution. Observe that, if $a = b$, our statement is trivially true, so from now on we assume that $a < b$. Suppose first that $a = -1$ and $b = 1$. We will need the following lemma.

Lemma. Let $h : A \to \mathbb{R}$ be a continuous function with $|h(x)| \leqslant c$ for all $x \in A$. Then there is a continuous function $g : X \to \mathbb{R}$ such that $|g(x)| \leqslant \frac{c}{3}$ for all $x \in X$ and $|g(x) - h(x)| \leqslant \frac{2c}{3}$ for all $x \in A$.

Proof of the Lemma. The disjoint sets $P = h^{-1}\left(\left[-c, -\frac{c}{3}\right]\right)$ and $Q = h^{-1}\left(\left[\frac{c}{3}, c\right]\right)$ are closed in A and hence in X. Apply the Urysohn's lemma (015) to find a continuous function $k : X \to [0, 1]$ such that $k(P) \subset \{0\}$ and $k(Q) \subset \{1\}$. The function $g(x) = \frac{2}{3}c\left(k(x) - \frac{1}{2}\right)$ is as promised. Observe first that g is continuous being obtained by arithmetical operations from the function k (Problem 027). Now, $k(x) \in [0, 1]$ implies $k(x) - \frac{1}{2} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ for all $x \in X$. Thus $g(x) \in \left[-\frac{2}{3} \cdot c \cdot \frac{1}{2}, \frac{2}{3} \cdot c \cdot \frac{1}{2}\right] = \left[-\frac{c}{3}, \frac{c}{3}\right]$ for all $x \in X$. To finish the proof of the lemma, we must show that $|g(x) - h(x)| \leqslant \frac{2c}{3}$ for all $x \in A$. If $x \in P$ then $h(x) \in \left[-c, -\frac{c}{3}\right]$ and $g(x) = -\frac{c}{3}$. As a consequence, $h(x) - g(x) \in \left[-\frac{2}{3}c, 0\right]$ and $|h(x) - g(x)| \leqslant \frac{2}{3}c$. If $x \in Q$ then we must have $h(x) \in \left[\frac{c}{3}, c\right]$ and $g(x) = \frac{c}{3}$. Thus $h(x) - g(x) \in \left[0, \frac{2}{3}c\right]$ and $|h(x) - g(x)| \leqslant \frac{2}{3}c$. Now, if $x \in A \backslash (P \cup Q)$ then $|h(x)| < \frac{c}{3}$. It follows from the inequality $|g(x)| \leqslant \frac{c}{3}$ that $|h(x) - g(x)| \leqslant |h(x) + g(x)| \leqslant \frac{c}{3} + \frac{c}{3} = \frac{2}{3}c$ and our lemma is proved.

Remembering that $|f(x)| \leqslant 1$ for all $x \in A$, we can apply the lemma for $c = 1$ and $h = f$. This gives us a continuous function $g_1 : X \to \mathbb{R}$ such that $|g_1(x)| \leqslant \frac{1}{3}$ for all $x \in X$ and $|f(x) - g_1(x)| \leqslant \frac{2}{3}$ for all $x \in A$. Proceeding by induction, suppose that we have continuous functions $g_1, \ldots g_n$ on the space X such that, for all $i \leqslant n$, we have

$(*)$ $|g_i(x)| \leqslant \frac{1}{3}\left(\frac{2}{3}\right)^{i-1}$ for all $x \in X$ and
$(**)$ $|f(x) - (g_1(x) + \cdots + g_i(x))| < \left(\frac{2}{3}\right)^i$ for any $x \in A$.

Apply the lemma to the function $h(x) = (f(x) - (g_1(x) + \cdots + g_i(x)))|_A$ and $c = \left(\frac{2}{3}\right)^n$ obtaining thus a continuous $g : X \to \mathbb{R}$ such that $|g(x)| \leqslant \frac{1}{3}\left(\frac{2}{3}\right)^n$ for each $x \in X$ and $|h(x) - g(x)| \leqslant \frac{2}{3}\left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^{n+1}$ for any $x \in A$. It is clear that the function $g_{n+1} = g$ satisfies $(*)$ and $(**)$ for all $i \leqslant n + 1$ which means that our construction is fulfilled for each $n \in \mathbb{N}$. Since the series $\sum_{i=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{i-1}$ is convergent, the property $(*)$ implies that the series $\sum_{i=1}^{\infty} g_i(x)$ converges uniformly to a function F (030). The function F is continuous by Problem 029 so it suffices to show that $F : X \to \mathbb{I}$ and $F(x) = f(x)$ for all $x \in A$. Given $x \in X$, apply $(*)$ to see that $|F(x)| \leqslant \sum_{i=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{i-1} = 1$ and hence $F(x) \in [-1, 1]$ for all $x \in X$. Taking any

$x \in A$ and considering the sum $G_n(x) = \sum_{i=1}^n g_i(x)$, we obtain from $(**)$ that $|f(x) - G_n(x)| \leqslant \left(\frac{2}{3}\right)^n$ for each $n \in \mathbb{N}$. Now, if $n \to \infty$, we obtain $|f(x) - F(x)| = 0$ because $G_n(x) \to F(x)$ when $n \to \infty$. Hence $f(x) = F(x)$ for all $x \in A$ and our statement is proved for $a = -1$ and $b = 1$.

Let us prove it for an arbitrary closed interval $[a, b] \subset \mathbb{R}$ with $a < b$. Observe that the function $r(t) = \frac{2}{b-a}(t - a) - 1$ is a homeomorphism between $[a, b]$ and $[-1, 1]$ with the inverse function $s(t) = \frac{1}{2}((b - a)t + a + b)$. Let $h = r \circ f$. Then h is a continuous function on A and $h : A \to \mathbb{I}$. Since we proved that for such functions the relevant extension exists, we can take a continuous $H : X \to [-1, 1]$ such that $H|_A = h$. The function $F = s \circ H$ is as promised because it is continuous, maps X into $[a, b]$ and $F(x) = s(H(x)) = s(h(x)) = s(r(f(x))) = f(x)$ for all $x \in A$.

S.032. *Let X be a normal space. Suppose that A is a closed subspace of X and $f : A \to \mathbb{R}$ is a continuous function. Prove that there exists a continuous function $F : X \to \mathbb{R}$ such that $F|A = f$, i.e., $F(x) = f(x)$ for all $x \in A$.*

Solution. Let $h(x) = \frac{2}{\pi}\arctan(f(x))$ for any $x \in A$. Then $h : A \to \mathbb{R}$ is a continuous function and $h(A) \subset (-1, 1) \subset \mathbb{I}$. Thus we can apply Problem 031 to conclude that there exists a continuous function $H : X \to \mathbb{I}$ such that $H|_A = h$. The set $B = H^{-1}(-1) \cup H^{-1}(1)$ is closed and disjoint from A because $|H(x)| = |h(x)| < 1$ for all $x \in A$. The space X is normal so we can apply the Urysohn's lemma (Problem 015) to find a continuous function $k : X \to [0, 1]$ such that $k(x) = 1$ for all $x \in A$ and $k(y) = 0$ for any $y \in B$. The function $H_1 = k \cdot H$ is continuous by Problem 027(i) and, for all $x \in A$, we have $H_1(x) = k(x) \cdot H(x) = H(x) \cdot 1 = h(x)$. Now, if $x \in B$ then $H_1(x) = H(x) \cdot k(x) = 0$ because $k(x) = 0$ for any $x \in B$. If $x \notin B$ then $|H_1(x)| = |H(x)| \cdot |k(x)| < 1$ because $|k(x)| \leqslant 1$ and $|H(x)| < 1$. It turns out that $H_1(x) \subset (-1, 1)$ for all $x \in X$ and the function $F(x) = \tan\left(\frac{\pi}{2} \cdot H_1(x)\right)$ is well defined and continuous on X. If $x \in A$ then $F(x) = \tan\left(\frac{\pi}{2} \cdot H_1(x)\right) = \tan\left(\frac{\pi}{2} \cdot H(x) \cdot k(x)\right) = \tan\left(\frac{\pi}{2} \cdot H(x) \cdot 1\right) = \tan\left(\frac{\pi}{2} \cdot \frac{2}{\pi} \cdot \arctan(f(x))\right) = \tan(\arctan(f(x))) = f(x)$ and therefore $F(x) = f(x)$ for all $x \in A$.

S.033. *Prove that X is a normal space if and only if for any closed $F, G \subset X$ with $F \cap G = \emptyset$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f(F) \subset \{0\}$ and $f(G) \subset \{1\}$.*

Solution. Necessity is precisely the Urysohn's lemma proved in Problem 015. To establish sufficiency, suppose that F and G are disjoint closed subsets of X. Take a continuous function $f : X \to \mathbb{R}$ with $f(F) \subset \{0\}$ and $f(G) \subset \{1\}$. The sets $U = f^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ and $V = f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ are open, disjoint and $F \subset U$, $G \subset V$ which proves the normality of X.

S.034. *Prove that, for any Tychonoff space X, if we are given distinct points $x_1, \ldots, x_n \in X$ and (not necessarily distinct) $r_1, \ldots, r_n \in \mathbb{R}$, then there exists a function $f \in C(X)$ such that $f(x_i) = r_i$ for all $i = 1, \ldots, n$.*

Solution. There is nothing to prove if $n = 1$, so suppose that $n \in \mathbb{N}$ and $n \geqslant 2$. If $Y = \{x_1, \ldots, x_n\}$ then, for every $i \leqslant n$, the set $F_i = Y \setminus \{x_i\}$ is closed in X and does not contain x_i. The Tychonoff property of X guarantees the existence of a continuous function $f_i : X \to \mathbb{R}$ such that $f_i(x_i) = 1$ and $f_i(F_i) \subset \{0\}$. It is easy to check that the function $f = r_1 \cdot f_1 + \cdots + r_n \cdot f_n$ is as promised.

S.035. *Let X be an arbitrary set. Suppose that $f_n, g_n : X \to \mathbb{R}$ for each $n \in \omega$. Prove that, if $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$, then $f_n + g_n \rightrightarrows f + g$.*

Solution. Let $\varepsilon > 0$. As $f_n \rightrightarrows f$, there is $m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $n \geqslant m$ and $x \in X$. Since $g_n \rightrightarrows g$, there is $k \in \mathbb{N}$ such that $|g_n(x) - g(x)| < \frac{\varepsilon}{2}$ for all $n \geqslant k$ and $x \in X$. Now, if $l = k + m$ then, for any $n \geqslant l$ and any $x \in X$, we have $|f_n(x) + g_n(x) - f(x) - g(x)| \leqslant |f_n(x) - f(x)| + g_n(x) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ which shows that $f_n + g_n \rightrightarrows f + g$.

S.036. *Let $f_n, g_n \in C(X)$ for some topological space X. Assume that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$. Is it always true that*

(i) $\max(f_n, g_n) \rightrightarrows \max(f, g)$*?*
(ii) $f_n \cdot g_n \rightrightarrows f \cdot g$*?*

Solution. (i) This is true. To prove it, we will need the (easy to prove) inequality $||a| - |b|| \leqslant |a - b|$ for any $a, b \in \mathbb{R}$. Observe first that if $f_n \rightrightarrows f$ then $|f_n| \rightrightarrows |f|$. Indeed, let $\varepsilon > 0$. Find $m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geqslant m$ and $x \in X$. Then $||f_n(x)| - |f(x)|| \leqslant |f_n(x) - f(x)| < \varepsilon$ for all $n \geqslant m$ and $x \in X$. As a consequence, $|f_n| \rightrightarrows |f|$. Another easy observation is that $(-f_n) \rightrightarrows (-f)$. Applying Problem 035 we can conclude that $\max(f_n, g_n) = \frac{1}{2}(f_n + g_n + |f_n - g_n|) \rightrightarrows \frac{1}{2}(f + g + |f - g|) = \max(f, g)$.

(ii) This is not necessarily true. Let $f_n = x + \frac{1}{n} \in C(\mathbb{R})$. It is trivial that $f_n \rightrightarrows f$, where $f(x) = x$ for all $x \in \mathbb{R}$. However, it is not true that $f_n^2 \rightrightarrows f^2$. To see this, let $n \in \mathbb{N}$. Then $f_n^2(x) - f(x) = \left(x + \frac{1}{n}\right)^2 - x^2 = 2x \cdot \frac{1}{n} + \frac{1}{n^2}$. Therefore $\left|f_n^2(n) - f(n)\right| = 2 + \frac{1}{n^2} > 2$ which shows that the condition of the uniform convergence is not fulfilled for $\varepsilon = 2$.

S.037. *Call a function $f : \mathbb{R} \to \mathbb{R}$ increasing (decreasing) if $x \leqslant y$ implies $f(x) \leqslant f(y)$ ($f(x) \geqslant f(y)$ respectively), for all $x, y \in \mathbb{R}$. Prove that the set of all increasing functions as well as the set of all decreasing functions is closed in $C_p(\mathbb{R})$.*

Solution. Let $I \subset C_p(\mathbb{R})$ be the set of all increasing functions. If $f \notin I$ then there are $x, y \in \mathbb{R}$ such that $x < y$ and $f(x) > f(y)$. If $r = f(x) - f(y)$ then the set $W = \left[x, y; \left(f(x) - \frac{r}{3}, f(x) + \frac{r}{3}\right), \left(f(y) - \frac{r}{3}, f(y) + \frac{r}{3}\right)\right]$ is open in $C_p(X)$ and contains the function f. If $g \in W$ then $g(y) < f(y) + \frac{r}{3} < f(x) - \frac{r}{3} < g(x)$ and hence $g(x) > g(y)$ whence g is not increasing. This shows that $W \cap I = \emptyset$. Since each $f \in C_p(X) \backslash I$ has a neighbourhood contained in $C_p(X) \backslash I$, the set $C_p(X) \backslash I$ is open.

Denote by $D \subset C_p(\mathbb{R})$ be the set of all decreasing functions. If $f \notin I$ then there are $x, y \in \mathbb{R}$ such that $x < y$ and $f(x) < f(y)$. If $r = f(y) - f(x)$ then the set $W = \left[x, y; \left(f(x) - \frac{r}{3}, f(x) + \frac{r}{3}\right), \left(f(y) - \frac{r}{3}, f(y) + \frac{r}{3}\right)\right]$ is open in $C_p(X)$ and contains the function f. If $g \in W$ then $g(x) < f(x) + \frac{r}{3} < f(y) - \frac{r}{3} < g(y)$ and hence $g(x) < g(y)$ whence g is not decreasing. This shows that $W \cap I = \emptyset$. Since each

$f \in C_p(X)\backslash D$ has a neighbourhood contained in $C_p(X)\backslash D$, the set $C_p(X)\backslash D$ is also open.

S.038. *Prove that there is a subspace of* $C_p(\mathbb{R})$ *which is homeomorphic to* $C_p(\mathbb{I})$.

Solution. Define a function $\varphi : \mathbb{R} \to I$ as follows: $\varphi(x) = x$ if $|x| < 1$; if $x > 1$ then $\varphi(x) = 1$ and $\varphi(x) = -1$ for all $x < -1$. For any $f \in C_p(\mathbb{I})$, let $\varphi^*(f) = f \circ \varphi$. Thus we have a map $\varphi^* : C_p(\mathbb{I}) \to C_p(\mathbb{R})$. It is sufficient to prove that φ^* is an embedding. Observe first that φ is an injection. Indeed, if $f \neq g$ then $f(x) \neq g(x)$ for some $x \in \mathbb{I}$. Then $\varphi^*(f)(x) = f(x) \neq g(x) = \varphi^*(g)(x)$ and therefore $\varphi^*(f) \neq \varphi^*(g)$. To see that φ is continuous, take any $f_0 \in C_p(\mathbb{I})$ and any $U \ni \varphi^*(f_0)$ with $U \in \tau(C_p(\mathbb{R}))$. There are $x_1, \dots, x_n \in \mathbb{R}$ and $O_1, \dots, O_n \in \tau(\mathbb{R})$ such that $\varphi^*(f_0) \in O \subset U$, where $O = [x_1, \dots, x_n; O_1, \dots, O_n]$ (the standard open set is taken in $C_p(\mathbb{R})$). If we let $O' = [\varphi(x_1), \dots, \varphi(x_n); O_1, \dots, O_n]$ (the standard open set is taken in $C_p(\mathbb{I})$) then $f_0 \in O'$ and $\varphi^*(O') \subset O$ which proves continuity of φ^* at the point f_0. To see that $(\varphi^*)^{-1}$ is also continuous, consider the map $\pi : C_p(\mathbb{R}) \to C_p(\mathbb{I})$ defined by $\pi(f) = f_{\mathbb{I}}$. It is immediate that π restricted to $\varphi^*(C_p(\mathbb{I}))$ is the inverse map for φ^* so it is sufficient to prove that π is continuous.

Fix any $f_0 \in C_p(\mathbb{R})$ and let $g_0 = \pi(f_0)$. If $g_0 \in U \in \tau(C_p(\mathbb{I}))$ then there is a standard open set $O = [x_1, \dots, x_n; O_1, \dots, O_n]$ such that $f \in O \subset U$. Now, if $O' = \{f \in C_p(\mathbb{R}) : f(x_i) \in O_i$ for all $i \leqslant n\}$ then O' is a standard open set in $C_p(\mathbb{R})$ and $f_0 \in \pi(O') \subset O \subset U$ which proves continuity of π at the point f_0.

S.039. *Prove that* $\overline{C^*(\mathbb{R})} = C_p(\mathbb{R})$ *and* $\mathrm{Int}(C^*(\mathbb{R})) = \emptyset$.

Solution. Given a topological space Z, it is an easy exercise to prove that $\overline{A} = Z$ for some $A \subset Z$ if and only if there is a base $\mathcal{B}$ of Z such that $A \cap U \neq \emptyset$ for all $U \in \mathcal{B}$. Therefore to prove that $\overline{C^*(\mathbb{R})} = C_p(\mathbb{R})$, it suffices to show that $C^*(\mathbb{R}) \cap O \neq \emptyset$ for any standard open $O \subset C_p(\mathbb{R})$. Let $O = [x_1, \dots, x_n; O_1, \dots, O_n]$. Since $\mathbb{R}$ is a Tychonoff space (Problem 019), there exists a function $f \in C_p(\mathbb{R})$ such that $f(x_i) = r_i \in O_i$ for all $i \leqslant n$. Let $r = |x_1| + \cdots + |x_n| + 1$. Define a function $g \in C(\mathbb{R})$ in the following way: $g(x) = f(x)$ if $|x| \leqslant r$; if $x > r$ then $g(x) = f(r)$ and $g(x) = f(-r)$ for all $x < -r$. The easy verification of the fact that $g \in C^*(\mathbb{R}) \cap O$, is left to the reader.

S.040. *Given* $n \in \omega$, *denote by* $P_n \subset C_p(\mathbb{R})$ *the set of all polynomials of degree* $\leqslant n$. *Prove that* P_n *is a closed subset of* $C_p(\mathbb{R})$.

Solution. We will use the following well-known properties of polynomials (the proofs can be found in any textbook of algebra):

(∗) If $x_0, \dots, x_n$ are distinct points of $\mathbb{R}$ and $r_0, \dots, r_n \in \mathbb{R}$ then there exists a polynomial p of degree $\leqslant n$ such that $p(x_i) = r_i$ for all $i \leqslant n$.

(∗∗) If $x_0, \dots, x_n$ are distinct points of $\mathbb{R}$ and p is a polynomial with $p(x_i) = 0$ for all $i \leqslant n$, then $p(x) = 0$ for all $x \in \mathbb{R}$.

Claim. Suppose that $f \in \overline{P}_n \backslash P_n$. Then $f + p \in \overline{P}_n \backslash P_n$ for any $p \in P_n$ and $\lambda f \in \overline{P}_n \backslash P_n$ for all $\lambda \in \mathbb{R}\backslash\{0\}$.

Proof of the claim. If $f + p = q \in P_n$ then $f = q - p \in P_n$ which is a contradiction. Thus $f + p \notin P_n$. Suppose that $f + p \notin \overline{P}_n$. Then there is a standard open set $W = [x_1, \ldots, x_k; O_1, \ldots, O_k]$ such that $f + p \in W \subset C_p(\mathbb{R})\backslash P_n$. Making the sets O_i smaller if necessary, we can consider that $O_i = (a_i, b_i)$ for all $i \leqslant k$. Let $O_i' = (a_i - p(x_i), b_i - p(x_i))$ for all $i \leqslant k$. The standard set $W' = \left[x_1, \ldots, x_k; O_1', \ldots, O_k'\right]$ contains the function f because $f(x_i)+p(x_i) \in (a_i, b_i)$ implies $f(x_i) \in (a_i - p(x_i), b_i - p(x_i)) = O_i'$ for all $i \leqslant k$. Since $f \in \overline{P}_n$, there is $q \in P_n$ with $q \in W'$. Therefore $a_i - p(x_i) < q(x_i) < b_i - p(x_i)$ and hence $a_i < p(x_i) + q(x_i) < b_i$ for all $i \leqslant k$ which implies $r = p + q \in W \cap P_n$, a contradiction showing that $f + p \in \overline{P}_n\backslash P_n$.

The proof of $\lambda f \in \overline{P}_n\backslash P_n$ is much the same. If $\lambda f = q \in P_n$ then $f = \frac{1}{\lambda} q \in P_n$ which is false. Hence $\lambda f \notin P_n$. Suppose that $\lambda f \notin \overline{P}_n$. Then there is a standard open set $W = [x_1, \ldots, x_k; O_1, \ldots, O_k]$ such that $\lambda f \in W \subset C_p(\mathbb{R})\backslash P_n$. Making the sets O_i smaller if necessary, we can consider that $O_i = (a_i, b_i)$ for all $i \leqslant k$. Let $O_i' = \left(\frac{1}{\lambda}a_i, \frac{1}{\lambda}b_i\right)$ for all $i \leqslant k$ if $\lambda > 0$ and $O_i' = \left(\frac{1}{\lambda}b_i, \frac{1}{\lambda}a_i\right)$ for all $i \leqslant k$ if $\lambda < 0$.

The standard set $W' = \left[x_1, \ldots, x_k; O_1', \ldots, O_k'\right]$ contains the function f because $\lambda f(x_i) \in O_i$ implies $f(x_i) \in O_i'$ for all $i \leqslant k$. Since $f \in \overline{P}_n$, there is $q \in P_n$ with $q \in W'$. It is easy to see that $r = \lambda q \in W \cap P_n$ which is a contradiction.

Suppose that P_n is not closed and hence there exists $f \in \overline{P}_n\backslash P_n$. Let $x_i = i$ for $i = 0, \ldots, n$. Apply $(*)$ to find $p \in P_n$ such that $p(x_i) = f(x_i)$ for all $i \leqslant n$. The claim says that $g = f - p \in \overline{P}_n\backslash P_n$. The function g is not identically zero because $g \notin P_n$. Hence there is $y \in \mathbb{R}\backslash\{x_0, \ldots, x_n\}$ such that $g(y) \neq 0$. The point y is not necessarily smaller than all x_i's but we can assume this performing the following trick. Take $y_0 = \min\{y, x_0, \ldots, x_n\}$. If $y = y_0$ then there is nothing to do. If not, then there is $r \in P_n$ such that $g(x) = r(x)$ for all $x \in K = \{y, x_0, \ldots, x_n\}\backslash\{y_0\}$. Again we have $h = g - r \in \overline{P}_n\backslash P_n$ and $h(x) = 0$ for all $x \in K$. Note that it is impossible that $h(y_0) = 0$ because then $r(x) = 0$ for all x from the set $(K \cup \{y_0\})\backslash\{y\}$ which has $n + 1$ elements. The property $(**)$ implies that $r \equiv 0$ which contradicts $r(y) = g(y) \neq 0$. As a consequence, $h(y_0) \neq 0$ and $h(K) \subset \{0\}$. Multiplying the function h by an appropriate number and applying the claim once more, we can assume that $h(y_0) = 1$.

Enumerate K as $y_1 < \cdots < y_{n+1}$ and use $(*)$ once more to find $s \in P_n$ such that $s(y_i) = (-1)^i$ for all $i = 1, \ldots, n + 1$. The function s is continuous and hence bounded on the closed interval $[y_0, y_{n+1}]$ (we are applying here the relevant well-known theorem of Calculus: any continuous $s : [a, b] \to \mathbb{R}$ is bounded; continuity of any polynomial is an easy exercise). Take any $M \in \mathbb{R}$ such that $|s(x)| < M$ for any $x \in [y_0, y_{n+1}]$ Then $t = \frac{1}{2M}s$ is also a polynomial from P_n for which $|t(x)| < \frac{1}{2}$ for all $x \in [y_0, y_{n+1}]$ and $t(y_i) = w_i = (-1)^i \frac{1}{2M}$ for all $i = 1, \ldots, n + 1$. Finally, let $u = h - t$. Still $u \in \overline{P}_n\backslash P_n$ and the number $u_i = u(y_i)$ is positive if i is even and negative otherwise. Consider the open sets $O_i = \left(\frac{w_i}{2}, \frac{3w_i}{2}\right)$ if i is even and $O_i = \left(\frac{3w_i}{2}, \frac{w_i}{2}\right)$ otherwise. The standard open set $O = [y_0, \ldots, y_{n+1}; O_0, \ldots, O_{n+1}]$ contains u and does not intersect P_n because any $w \in P_n \cap O$ must have at least $n + 1$ sign changes on $[y_0, y_{n+1}]$ and hence at least $n + 1$ roots which implies $w \equiv 0$, a contradiction which shows that $u \notin \overline{P}_n$ and this last contradiction finishes our proof. Note that we used another well-known theorem of Calculus which says that, if $w : [a, b] \to \mathbb{R}$ is a continuous function such that $w(a) \cdot w(b) < 0$ then $w(x) = 0$ for some $x \in (a, b)$.

S.041. *Denote by* $P \subset C_p(\mathbb{R})$ *the set of all polynomials. Prove that* $\overline{P} = C_p(\mathbb{R})$ *and* $\mathrm{Int}(P) = \emptyset$.

Solution. Let $O = [x_1, \ldots, x_n; O_1, \ldots, O_n]$ be a standard non-empty open set. Choose $r_i \in O_i$ for all $i \leqslant n$ and find a polynomial p such that $p(x_i) = r_i$ (see the solution of Problem 040). It is clear that $p \in O \cap P$ and therefore $\overline{P} = C_p(\mathbb{R})$. Now, if $\mathrm{Int}(P) \neq \emptyset$ then $O = [x_1, \ldots, x_n; O_1, \ldots, O_n] \subset P$ for some standard non-empty open $O \subset C_p(\mathbb{R})$. Take any interval $[a, b] \supset \{x_1, \ldots x_n\}$ and choose $r_i \in O_i \backslash \{0\}$ for all $i \leqslant n$. It is easy to construct a function $f \in C_p(\mathbb{R})$ such that $f(x_i) = r_i$ for all $i \leqslant n$ and $f(x) = 0$ for any $x \in \mathbb{R} \backslash [a, b]$. It is clear that $f \in O$ and f is not identically zero having infinitely many zeros in $\mathbb{R}$. Thus f cannot be a polynomial which is a contradiction with the fact that $O \subset P$.

S.042. *Let* $H(\mathbb{R}) \subset C_p(\mathbb{R})$ *be the set of all homeomorphisms of* $\mathbb{R}$ *onto* $\mathbb{R}$. *Is it true that* $\overline{H(\mathbb{R})} = C_p(\mathbb{R})$?

Solution. No, this is not true. Let $O = [1, 2, 3; (-2, -1), (1, 2), (-2, -1)]$. Then O is a non-empty open subset of $C_p(\mathbb{R})$. If $f \in O$ then f has two sign changes and hence it has at least two zeros: one on $(1, 2)$ and another on $(2, 3)$. As a consequence, the function f cannot be injective. This proves that $H(\mathbb{R}) \cap O = \emptyset$.

S.043. *Let* U *be the set of all uniformly continuous functions from* $C(\mathbb{R})$ *(that is,* $f \in U$ *if and only if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $|f(x) - f(y)| < \varepsilon$ *whenever* $|x - y| < \delta$*). Is it true that* $\overline{U} = C_p(\mathbb{R})$?

Solution. We will prove that this is true. Let $O = [x_1, \ldots, x_k; O_1, \ldots, O_k]$ be any non-empty standard open set. Choose $a, b \in \mathbb{R}$ in such a way that $a < b$ and $\{x_1, \ldots, x_k\} \subset [a, b]$. Let $p(x)$ be a polynomial such that $p(x_i) \in O_i$ for each $i \leqslant k$. The polynomial p is not necessarily uniformly continuous. To correct it, define a function f as follows: $f(x) = p(x)$ for all $x \in [a, b]$, $f(x) = p(b)$ for all $x > b$ and $f(x) = p(a)$ for all $x < a$. It is easy to see that f is continuous. It is a well-known fact of Calculus that f has to be uniformly continuous on $[a, b]$, i.e., for any $\varepsilon > 0$ there is $\delta > 0$ such that, for any $x, y \in [a, b]$ we have $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. It turns out that the same δ proves the uniform continuity of f on the whole $\mathbb{R}$. Indeed, if $|x - y| < \delta$ and $x, y \in [a, b]$ then $|f(x) - f(y)| < \varepsilon$ because f is uniformly continuous on $[a, b]$. If $x < a$ then $|a - y| < \delta$ and therefore $|f(x) - f(y)| = |f(a) - f(y)| < \varepsilon$. Analogously, if $x > b$ then $|b - y| < \delta$ and hence $|f(x) - f(y)| = |f(b) - f(y)| < \varepsilon$. The cases when $y < a$ or $y > b$ are considered identically. Thus f is a uniformly continuous function and $f \in O$. This proves that $\overline{U} = C_p(\mathbb{R})$.

S.044. *Prove that, for every* $f \in C_p(\mathbb{R})$, *there exist open sets* $\{U_n : n \in \omega\}$ *such that* $\{f\} = \bigcap\{U_n : n \in \omega\}$.

Solution. Let $\mathbb{Q} = \{q_i : i \in \omega\}$ be some enumeration of the rationals. For each number $n \in \omega$, consider the standard open set $U_n = \left[r_0, \ldots, r_n; O_2^n, \ldots, O_n^n\right]$, where $r_i = f(q_i)$ and $O_i^n = \left(r_i - \frac{1}{n}, r_i + \frac{1}{n}\right)$ for all $i \in \omega$. It is clear that $f \in U_n$ for all $n \in \omega$. To prove that $\{f\} = \bigcap\{U_n : n \in \omega\}$, suppose that $g \in \bigcap\{U_n : n \in \omega\}$. Then $g(q_i) = f(q_i)$ for all $i \in \omega$. The function $h = g - f$ is continuous and $h(q) = 0$ for all $q \in \mathbb{Q}$.

Since $\overline{\mathbb{Q}} = \mathbb{R}$, for any $x \in \mathbb{R}$, we have $h(x) \in \overline{h(\mathbb{Q})} = \overline{\{0\}} = \{0\}$ (Problem 009(vi)). Thus $h(x) = 0$ for all $x \in \mathbb{R}$ and $f = g$.

S.045. *Prove that each of the spaces $C_p(\mathbb{Q})$ and $C_p(\mathbb{N})$ has a countable base.*

Solution. Let $\mathcal{U}$ be the family of all open intervals in $\mathbb{R}$ with rational endpoints. The family $\mathcal{B}$ of all sets $[x_1, \ldots, x_n; O_1, \ldots, O_n]$ where $n \in \mathbb{N}$, $x_i \in \mathbb{Q}$ (or $x \in \mathbb{N}$) and $O_i \in \mathcal{U}$ for all $i \leqslant n$, is a countable base in $C_p(\mathbb{Q})$ (or $C_p(\mathbb{N})$, respectively).

S.046. *Is there a countable local base at some $f \in C_p(\mathbb{R})$?*

Solution. We will prove that there is no countable local base at any $f \in C_p(\mathbb{R})$. Suppose that $\{U_n : n \in \omega\}$ is such a base at f. For each $n \in \omega$ fix a standard open set W_n such that $f \in W_n \subset U_n$. It is evident that $\mathcal{B} = \{W_n : n \in \omega\}$ is also a local countable base at f. Let $W_n = \left[x_1^n, \ldots, x_{k_n}^n; O_1^n, \ldots, O_{k_n}^n\right]$ for all $n \in \omega$. The set $P = \{x_j^i : i \in \omega, j \in \{1, \ldots k_i\}\}$ is countable and hence there exists $x \in \mathbb{R}\backslash P$. The set $W = [x, (f(x) - 1, f(x)+1)]$ is open in $C_p(\mathbb{R})$ and $f \in W$. Since $\mathcal{B}$ is a local base at f, there is $n \in \omega$ such that $W_n \subset W$. Apply Problem 034 to find a function $g \in C_p(\mathbb{R})$ such that $g(x) = f(x) + 2$ and $g(x_i^n) = f(x_i^n)$ for all $i \leqslant k_n$. It is immediate that $g \in W_n \backslash W$ which is a contradiction.

S.047. *Prove that there is a countable set $A \subset C_p(\mathbb{R})$ such that $\overline{A} = C_p(\mathbb{R})$.*

Solution. Let $\mathcal{B} = \{\{O_1, \ldots, O_n\} : n \in \mathbb{N},\ O_i = (a_i, b_i)$ is a rational open interval for all $i \leqslant n$ and $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ if $i \neq j\}$. Given $O = \{O_1, \ldots, O_n\} \in \mathcal{B}$, let $m(O) = n$. If $q = (q_1, \ldots, q_n)$ is an n-tuple of rational numbers, fix $f_{O,\,q} \in C_p(\mathbb{R})$ such that $f_{O,\,q}(O_i) = \{q_i\}$ for all $i \leqslant n$ (the existence of such a function is an easy exercise). The set $A = \{f_{O,\,q} : O \in \mathcal{B}$ and q is an $m(O)$-tuple of rationals$\}$ is countable. Let us prove that we have $\overline{A} = C_p(\mathbb{R})$. Given a standard open set $W = [x_1, \ldots, x_n; U_1, \ldots, U_n]$, choose $q_i \in U_i \cap \mathbb{Q}$ for all $i \leqslant n$ and let $q = (q_1, \ldots q_n)$. There exists $O = \{O_1, \ldots, O_n\} \in \mathcal{B}$ such that $x_i \in O_i$ for all $i \leqslant n$ (note that we are not losing generality assuming that $\{x_1, \ldots, x_n\}$ are distinct points of $\mathbb{R}$). Then $f_{O,\,q} \in W \cap A$ and hence $\overline{A} = C_p(\mathbb{R})$.

S.048. *Let $h_n : \mathbb{R} \to \mathbb{R}$ be a homeomorphism for every $n \in \omega$ and suppose that $h_n \rightrightarrows h$. Is it always true that h is a homeomorphism?*

Solution. No, it is not always true. To see this, let us define a homeomorphism $h_n : \mathbb{R} \to \mathbb{R}$ as follows: $h_n(x) = x$ if $x \notin [0, 1]$; if $x \in \left[0, \frac{1}{2}\right]$ then $h_n(x) = \frac{2}{n+1}x$ and $h_n(x) = 2\left(1 - \frac{1}{n+1}\right)\left(x - \frac{1}{2}\right) + \frac{1}{n+1}$ for $x \in \left[\frac{1}{2}, 1\right]$. Now, let $h(x) = x$ for all $x \notin [0, 1]$, $h(x) = 0$ for $x \in \left[0, \frac{1}{2}\right]$ and $h(x) = 2x - 1$ if $x \in \left[\frac{1}{2}, 1\right]$. We leave to the reader the routine verification of the fact that h_n is a homeomorphism for each n. It is also easy to see that $h_n \rightrightarrows h$ and h is not an injective map.

S.049. *Let $u_n : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function for all $n \in \omega$ and suppose that $u_n \rightrightarrows u$. Is it always true that u is a uniformly continuous function?*

Solution. Yes, it is always true. Given $\varepsilon > 0$, there is a number $n \in \mathbb{N}$ such that $|u_n(x) - u(x)| < \frac{\varepsilon}{3}$ for all $x \in \mathbb{R}$. Since the function u_n is uniformly continuous, there exists $\delta > 0$ such that $|u_n(x) - u_n(y)| < \frac{\varepsilon}{3}$ whenever $|x - y| < \delta$. For such x and y, we have $|u(x) - u(y)| \leqslant |u(x) - u_n(x)| + |u_n(x) - u_n(y)| + |u_n(y) - u(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ and hence u is uniformly continuous.

S.050. *Let $\{f_n : n \in \omega\} \subset A \subset C_p(\mathbb{R})$. Suppose that $f_n \rightrightarrows f$. Prove that $f \in \overline{A}$. Is it true that, if $A \subset C_p(\mathbb{R}), f \in C_p(\mathbb{R})$ and $f \in \overline{A}$, then $f_n \rightrightarrows f$ for some $\{f_n : n \in \omega\} \subset A$?*

Solution. Suppose that $W = [x_1, \dots, x_n; O_1, \dots, O_n]$ is a standard open set with $f \in W$. We have $f(x_i) \in O_i$ for all $i \leqslant n$ and hence there exists $\varepsilon > 0$ such that $(f(x_i) - \varepsilon, f(x_i) + \varepsilon) \subset O_i$ for all $i \leqslant n$. There is $n \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$. It is immediate that $f_n \in W$ and hence $f \in \overline{\{f_n : n \in \omega\}} \subset \overline{A}$.

The second statement is not true. Define $f_n \in C_p(\mathbb{R})$ as follows: $f_n(x) = 0$ if $x \notin \left[0, \frac{2}{n}\right]$; if $x \in \left[0, \frac{1}{n}\right]$ then $f_n(x) = nx$ and $f_n(x) = n(\frac{1}{n} - x) + 1$ whenever $x \in \left[\frac{1}{n}, \frac{2}{n}\right]$. If $A = \{f_n : n \in \mathbb{N}\}$ then $f \in \overline{A}$ if $f(x) = 0$ for all $x \in \mathbb{R}$. However, no sequence of elements of A can converge uniformly to $f \equiv 0$ because every $g \in A$ is equal to 1 at some point, namely, if $g = f_n$ then $g\left(\frac{1}{n}\right) = 1$.

S.051. *Denote by D the set of all continuously differentiable functions $f \in C_p(\mathbb{R})$ (that is, D consists of the functions $f : \mathbb{R} \to \mathbb{R}$ such that the derivative f' of f exists and is continuous). Give D the topology inherited from $C_p(\mathbb{R})$ and consider the map $d : D \to C_p(\mathbb{R})$ defined by the formula $d(f) = f'$. Is the map d continuous?*

Solution. The map d is discontinuous. To see this, let $f_n = \frac{\sin nx}{n}$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$. It is clear that $A = \{f_n : n \in \mathbb{N}\}$ is contained in D. It is easy to prove that the function $f_0 \equiv 0$ is in the closure of A. Observe that $d(A) = \{\cos nx : n \in \mathbb{N}\}$ and $f(0) = 1$ for any $f \in d(A)$. Thus $f_0 \in W = \left[0; \left(-\frac{1}{2}, \frac{1}{2}\right)\right]$ while $W \cap d(A) = \emptyset$ and therefore $d(f_0) = f_0 \notin \overline{d(A)}$. Now apply Problem 009(vi) to see that d is not continuous.

S.052. *Let P be the set of all polynomials in $C_p(\mathbb{R})$. Give P the topology inherited from $C_p(\mathbb{R})$ and consider the map $d : P \to P$ defined by the formula $d(p) = p'$ (i.e., a polynomial is mapped to its derivative). Is the map $d : P \to P$ continuous?*

Solution. No, the map $d : P \to P$ is not continuous. To establish this, we will need the theorem of Stone–Weierstrass proved in Calculus: suppose that $[a, b] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is a continuous function. Then, for any $\varepsilon > 0$, there is a polynomial p such that $|p(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$. Anyway, we will prove this theorem later (see Problem 193) as a consequence of a general topological result (191).

For each $n \in \mathbb{N}$, define a function $\varphi_n : [-n.n] \to \mathbb{R}$ as follows: $\varphi_n(t) = \frac{1}{t}$ if $t \in \left[-n, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, n\right]$ and $\varphi_n(t) = n^2 t$ if $t \in \left[-\frac{1}{n}, \frac{1}{n}\right]$. Since φ_n is continuous, the Stone–Weierstrass theorem is applicable and we can find a polynomial p_n such that $|p_n(t) - \varphi_n(t)| < \frac{1}{n^3}$ for all $t \in [-n.n]$. In particular, $\left|p_n(t) - \frac{1}{t}\right| < \frac{1}{n^3}$ for all $t \in \left[-n, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, n\right]$. Let $q_n(t) = t - t^2 p_n(t)$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Then $A = \{q_n : n \in \mathbb{N}\} \subset P$ and $f_0 \in \overline{A}$, where $f_0 \equiv 0$. To see this, take any $\varepsilon > 0$ and $x_1, \dots, x_k \in \mathbb{R}$. There exists a number $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and

$\{x_2,\dots,x_k\}\backslash\{0\} \subset \left[-n,-\frac{1}{n}\right] \cup \left[\frac{1}{n},n\right]$. We will check that $|q_n(x_i)| < \varepsilon$ for all $i \leqslant k$ and therefore $q_n \in [x_1,\dots,x_n; (-\varepsilon,\varepsilon),\dots,(-\varepsilon,\varepsilon)]$. Since $q_n(0) = 0$, it suffices to prove that $|q_n(x_i)| < \varepsilon$ for all $x_i \neq 0$. All such x_i's belong to $\left[-n,-\frac{1}{n}\right] \cup \left[\frac{1}{n},n\right]$ so it is sufficient to show that $|q_n(t)| < \varepsilon$ for all $t \in \left[-n,-\frac{1}{n}\right] \cup \left[\frac{1}{n},n\right]$. We have $\left|p_n(t) - \frac{1}{t}\right| < \frac{1}{n^3}$ and hence $|q_n(t)| = t^2 \cdot \left|p_n(t) - \frac{1}{t}\right| < n^2 \cdot \frac{1}{n^3} < \frac{1}{n} < \varepsilon$. As a consequence, $f_0 \in \overline{A}$. However, $f_0 = d(f_0) \notin \overline{d(A)}$ because $f(0) = 1$ for all $f \in d(A)$ and hence $W = \left[0, \left(-\frac{1}{2},\frac{1}{2}\right)\right]$ is a neighbourhood of f_0 such that $W \cap d(A) = \emptyset$. Now apply Problem 009(vi) to see that d is not continuous.

S.053. *Assume that $a, b \in \mathbb{R}$ and $a < b$; give the set $[a, b] \subset \mathbb{R}$ the topology inherited from the space $\mathbb{R}$ and define the map* int : $C_p([a,b]) \to \mathbb{R}$ *by the formula* $\mathrm{int}(f) = \int_a^b f(t)dt$ *for each $f \in C_p([a,b])$. Is the map* int *continuous?*

Solution. The map int : $C_p([0,3]) \to \mathbb{R}$ is not continuous. Given $n \in \mathbb{N}$, let $f_n(t) = 0$ for all $t \in \left[\frac{2}{n},3\right]$. If $t \in \left[0,\frac{1}{n}\right]$ we let $f_n(t) = n^2t$ and $f_n(t) = -n^2(t-\frac{2}{n})$ for all $t \in \left[\frac{1}{n},\frac{2}{n}\right]$. Then $A = \{f_n : n \in \mathbb{N}\} \subset C_p([0,3])$ and $h \equiv 0$ belongs to the closure of A. However, $\mathrm{int}(f_n) = \int_0^3 f_n(t)\mathrm{d}t = 1$ for all $n \in \mathbb{N}$ and hence $0 = \mathrm{int}(h) \notin \overline{\mathrm{int}(A)} = \overline{\{1\}} = \{1\}$. Now apply Problem 009(vi) to see that int is not continuous.

S.054. *Assume that $a, b \in \mathbb{R}$ and $a < b$; give the set $[a, b] \subset \mathbb{R}$ the topology inherited from $\mathbb{R}$. Let $P \subset C_p([a,b])$ be the set of all polynomials on $[a,b]$. Define the map* int*: $P \to \mathbb{R}$ by the formula* $\mathrm{int}(p) = \int_a^b p(t)\mathrm{d}t$ *for every polinomial $p \in P$. Is the map* int *continuous?*

Solution. We will show that, for $a = 0$ and $b = 3$ the map int : $P \to \mathbb{R}$ is not continuous. To establish this, we will need the theorem of Stone–Weierstrass proved in Calculus: suppose that $[a,b] \subset \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is a continuous function. Then, for any $\varepsilon > 0$, there is a polynomial p such that $|p(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$. Anyway, we will prove this theorem later (see Problem 193) as a consequence of a general topological result (Problem 191). Given $n \in \mathbb{N}$, let $f_n(t) = 0$ for all $t \in \left[\frac{2}{n},3\right]$. If $t \in \left[0,\frac{1}{n}\right]$ we let $f_n(t) = n^2t$ and $f_n(t) = -n^2(t-\frac{2}{n})$ for all $t \in \left[\frac{1}{n},\frac{2}{n}\right]$. Apply the theorem of Stone–Weierstrass to obtain a polynomial $p_n(t)$ such that $|p_n(t) - f_n(t)| < \frac{1}{6n}$ for all $n \in \mathbb{N}$ and $t \in [0,3]$. Then

$$\begin{aligned}\mathrm{int}(p_n) &= \int_0^3 p_n(t)\mathrm{d}t \\ &= \int_0^3 f_n(t)\mathrm{d}t + \int_0^3 (p_n(t) - f_n(t))\mathrm{d}t \geqslant 1 - \int_0^3 \tfrac{1}{6n}\mathrm{d}t = 1 - \tfrac{1}{2n} \geqslant \tfrac{1}{2}.\end{aligned}$$

If $f_0 \equiv 0$ then $f_0 \in \overline{A}$ for $a = \{p_n : n \in \mathbb{N}\}$. To see this, let $x_1,\dots,x_k \in [0,3]$ and $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $f_n(x_i) = 0$ for all $i < k$ and $\frac{1}{6n} < \varepsilon$. Then $|p_n(x_i)| = |p_n(x_i) - f_n(x_i)| < \frac{1}{6n} < \varepsilon$. Hence $p_n \in [x_1,\dots,x_k; O_1,\dots,O_k]$, where $O_i = (-\varepsilon,\varepsilon)$ for all $i < k$. This proves that $f_0 \in \overline{A}$. However, $\mathrm{int}(f_0) = 0 \notin \overline{\mathrm{int}(A)} \subset \left[\frac{1}{2},+\infty\right]$. Now apply Problem 009(vi) to see that int is not continuous.

S.055. *Assume that* $a, b \in \mathbb{R}$ *and* $a < b$*; give the set* $[a, b] \subset \mathbb{R}$ *the topology inherited from* $\mathbb{R}$. *Define the map* $\mathrm{prm} : C_p([a, b]) \to C_p([a, b])$ *by the formula* $\mathrm{prm}(f)(x) = \int_a^x f(t)\mathrm{d}t$ *for every* $f \in C_p([a, b])$. *Is the map* prm *continuous?*

Solution. Let us prove that the function prm is not continuous for $a = 0$ and $b = 3$. Given $n \in \mathbb{N}$, let $f_n(t) = 0$ for all $t \in \left[\frac{2}{n}, 3\right]$. If $t \in \left[0, \frac{1}{n}\right]$ we let $f_n(t) = n^2 t$ and $f_n(t) = -n^2(t - \frac{2}{n})$ for all $t \in \left[\frac{1}{n}, \frac{2}{n}\right]$. Then $A = \{f_n : n \in \mathbb{N}\} \subset C_p([0, 3])$ and $h \equiv 0$ is in the closure of A. Since $\int_0^3 f_n(t)\mathrm{d}t = 1$ for all $n \in \mathbb{N}$, we have $\mathrm{prm}(f_n)(3) = 1$ for all $n \in \mathbb{N}$. Hence $h = \mathrm{prm}(h) \notin \overline{\mathrm{prm}(A)}$ because the standard open set $W = \left[3; \left(-\frac{1}{2}, \frac{1}{2}\right)\right]$ contains h and $W \cap \mathrm{prm}(A) = \emptyset$. Now apply Problem 009(vi) to see that prm is not continuous.

S.056. *Given a space* X, *show that the family* $\{[x_1, \dots, x_n; O_1, \dots, O_n] : n \in \mathbb{N}, x_1, \dots, x_n \in X$ *and* O_i *is a rational open interval for any* $i \leqslant n\}$ *is a base of the space* $C_p(X)$.

Solution. Suppose that $f \in U \in \tau(C_p(X))$. Then there exists a standard open set $V = [x_1, \dots, x_n; V_1, \dots, V_n]$ such that $f \in V \subset U$. Since every set V_i is open in $\mathbb{R}$ and $f(x_i) \in V_i$, there is $\varepsilon_i > 0$ such that $(f(x_i) - \varepsilon_i, f(x_i) + \varepsilon_i) \subset V_i$ for each $i \leqslant n$. Since any non-empty interval must contain a rational number, we can choose a point $p_i \in \mathbb{Q} \cap (f(x_i) - \varepsilon_i, f(x_i))$ and $q_i \in \mathbb{Q} \cap (f(x_i), f(x_i) + \varepsilon_i)$ for all $i \leqslant n$. As a consequence, we have $f(x_i) \in (p_i, q_i) = W_i \subset V_i$ for all $i \leqslant n$. It is immediate that we have $f \in W = [x_1, \dots, x_n; W_1, \dots, W_n] \subset V \subset U$. Now apply Problem 002 to finish the proof.

S.057. *Prove that, for any space* X, *the family* $\{[x; O] : x \in X$ *and* O *is a rational open interval*$\}$ *is a subbase of the space* $C_p(X)$.

Solution. Just note that $\cap\{[x_i; V_i] : i \leqslant n\} = [x_1, \dots, x_n; V_1, \dots, V_n]$ and apply Problem 056.

S.058. *Let* X *be a topological space. Given* $f \in C_p(X)$, *points* $x_1, \dots, x_n \in X$ *and* $\varepsilon > 0$, *let* $O(f, x_1, \dots, x_n, \varepsilon) = \{g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon$ *for all* $i \leqslant n\}$. *Prove that the family* $\{O(f, x_1, \dots, x_n, \varepsilon) : n \in \mathbb{N}, x_1, \dots, x_n \in X, \varepsilon > 0\}$ *is a local base of* $C_p(X)$ *at* f.

Solution. Suppose that $f \in U \in \tau(C_p(X))$. Then there exists a standard open set $V = [x_1, \dots, x_n; V_1, \dots, V_n]$ such that $f \in V \subset U$. Since each V_i is open in $\mathbb{R}$ and $f(x_i) \in V_i$, there is $\varepsilon_i > 0$ such that $(f(x_i) - \varepsilon_i, f(x_i) + \varepsilon_i) \subset V_i$ for each $i \leqslant n$. Now, if $\varepsilon = \min\{\varepsilon_i : i \leqslant n\}$ then $f \in O(f, x_1, \dots, x_n, \varepsilon) \subset V \subset U$. The last thing to observe is that $O(f, x_1, \dots, x_n, \varepsilon)$ is an open set in $C_p(X)$ because $O(f, x_1, \dots, x_n, \varepsilon) = [x_1, \dots, x_n; O_1, \dots, O_n]$, where $O_i = (f(x_i) - \varepsilon, f(x_i) + \varepsilon)$ for all $i \leqslant n$.

S.059. *For any sets* $A, B \subset C_p(X)$, *let* $A + B = \{a + b : a \in A$ *and* $b \in B\}$. *Prove that, if* A *is an open set and* B *is an arbitrary subset of* $C_p(X)$, *then* $A + B$ *is an open set.*

Solution. We will first establish that $A + b = \{a + b : a \in A\}$ has to be an open set for any $b \in B$. Say that an open set W is *standard of second type* if $W = O(f, x_1, \dots, x_n, \varepsilon)$ for some function $f \in C_p(X)$, points $x_1, \dots, x_n \in X$ and $\varepsilon > 0$. Observe that we have $O(f, x_1, \dots, x_n, \varepsilon) + g = O(f + g, x_1, \dots, x_n, \varepsilon)$ for all $n \in \mathbb{N}, f, g \in C_p(X)$,

$x_1 \ldots, x_n \in X$ and $\varepsilon > 0$. An immediate consequence is that $W + g$ is open for any open W which is standard of second type. It was proved in Problem 058 that the standard open sets of second type form a local base at every element of $C_p(X)$. Thus, for any $a \in A$ there is W_a such that W_a is standard open of second type and $a \in W_a \subset A$. Therefore $A = \bigcup\{W_a : a \in A\}$. Then $A + b = \bigcup\{W_a + b : a \in A\}$ for each $b \in B$ and hence $A + b$ is open. Note, finally, that $A + B = \bigcup\{A + b : b \in B\}$ is open being a union of open sets.

S.060. *For any $A, B \subset C_p(X)$, prove that $\overline{A} + \overline{B} \subset \overline{A + B}$.*

Solution. Observe first that, for any $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ we have $O(f, x_1, \ldots, x_n, \varepsilon) + O(g, x_1, \ldots, x_n, \varepsilon) \subset O(f + g, x_1, \ldots, x_n, 2\varepsilon)$ for arbitrary functions $f, g \in C_p(X)$. Here, as before, $O(f, x_1, \ldots, x_n, \varepsilon) = \{g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon$ for all $i \leqslant n\}$. It was proved in Problem 058 that, for any $f \in C_p(X)$, the sets $O(f, x_1, \ldots, x_n, \varepsilon)$ form a local base at f. Suppose that $f \in \overline{A} + \overline{B}$ and f belongs to a set $U \in \tau(C_p(X))$. Then $O(f, x_1, \ldots, x_n, \varepsilon) \subset U$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$. Now, $f = a + b$ for some $a \in \overline{A}$ and $b \in \overline{B}$. Take $a' \in O(a, x_1, \ldots, x_n, \frac{\varepsilon}{2}) \cap A$ and $b' \in O(b, x_1, \ldots, x_n, \frac{\varepsilon}{2}) \cap B$. Then we have $a' + b' \in (A + B) \cap O(f, x_1, \ldots, x_n, \varepsilon) \subset U \cap (A + B)$. It turned out that, for an arbitrary open $U \ni f$, we have $U \cap (A + B) \neq$. Now apply Problem 001 to conclude that $f \in \overline{A + B}$.

S.061. *Let U be a non-empty open subset of $C_p(X)$. Prove that there exists a countable $A \subset C_p(X)$ such that $A + U = C_p(X)$.*

Solution. Let $f_0(x) = 0$ for all $x \in X$. It is easy to see that we have the equality $O(f, x_1, \ldots, x_n, \varepsilon) = O(f_0, x_1, \ldots, x_n, \varepsilon) + f$ for any $f \in C_p(X)$. Here $O(f, x_1, \ldots, x_n, \varepsilon) = \{g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon$ for all $i \leqslant n\}$. It was proved in Problem 058 that, for any $f \in C_p(X)$, the sets $O(f, x_1, \ldots, x_n, \varepsilon)$ form a local base at f. Thus, given $f \in U$, there exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $O(f, x_1, \ldots, x_n, \varepsilon) \subset U$. For any $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$, fix a function $f_q \in C_p(X)$ such that $f_q(x_i) = q_i$ for all $i \leqslant n$ (see Problem 034). The set $A = \{f_q - f : q \in \mathbb{Q}^n\}$ is countable. Let us prove that $A + U = C_p(X)$. Take an arbitrary $g \in C_p(X)$ and let $r_i = g(x_i)$ for all $i \leqslant n$. There exists $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$ such that $|q_i - r_i| < \varepsilon$ for all $i \leqslant n$. Then $g - f_q \in O(f_0, x_1, \ldots, x_n, \varepsilon)$ and hence $g - f_q + f \in O(f, x_1, \ldots, x_n, \varepsilon) \subset U$. Therefore $g \in U + (f_q - f) \subset U + A$ and our proof is complete.

S.062. *Let $f_0 \in C_p(X)$ be equal to zero at all points of X. Suppose that $\mathcal{B}$ is a local base at f_0. Prove that, for any $f \in C_p(X)$, the family $\mathcal{B} + f = \{U + f : U \in \mathcal{B}\}$ is a local base at f. Here $U + f = \{u + f : u \in U\}$.*

Solution. All elements of the family $\mathcal{B} + f$ are open by Problem 059. Suppose that $f \in V \in \tau(C_p(X))$. The set $V - f = \{g - f : g \in V\}$ is open (Problem 059) and contains f_0. Thus, there is $U \in \mathcal{B}$ such that $U \subset V - f$. It is immediate that $U + f \subset V$ and $U + f \in \mathcal{B} + f$ whence $\mathcal{B} + f$ is a local base at f.

S.063. *Let $f_0 \in C_p(X)$ be equal to zero at all points of X. Suppose that U is an open set of $C_p(X)$ which contains f_0. Prove that there exists an open set V of $C_p(X)$ such that $f_0 \in V$ and $V + V \subset U$.*

Solution. The sets $O(f_0, x_1, \dots, x_n, \varepsilon)$ constitute a local base at f_0 (Problem 058). As a consequence, we can find a number $n \in \mathbb{N}$, points $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that $O(f_0, x_1, \dots, x_n, \varepsilon) \subset U$. Now, let $V = O(f_0, x_1, \dots, x_n, \frac{\varepsilon}{2})$. It is straightforward that $V + V \subset O(f_0, x_1, \dots, x_n, \varepsilon) \subset U$.

S.064. *Define $f_0 \in C_p(X)$ to be equal to zero at all points of X. Let U be an open set of $C_p(X)$ which contains f_0. Suppose that V is an open set of $C_p(X)$ such that $f_0 \in V$ and $V + V \subset U$. Prove that $\overline{V} \subset U$.*

Solution. The sets $O(f_0, x_1, \dots, x_n, \varepsilon)$ constitute a local base at f_0 (Problem 058). As a consequence, it is possible to find $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that $W = O(f_0, x_1, \dots, x_n, \varepsilon) \subset V$. For any $f \in \overline{V}$, the set $W + f$ is open (Problem 059) and contains f. Therefore there is $g \in (W + f) \cap V$. This implies that $g \in V$ and $g = f + w$ for some $w \in W$. Note that $(-w) \in W$ and hence $f = g - w \in V + W \subset V + V \subset U$. The function $f \in \overline{V}$ being arbitrary, we proved that $\overline{V} \subset U$.

S.065. *Let $f_0 \in C_p(X)$ be equal to zero at all points of X. Take any local base $\mathcal{B}$ of $C_p(X)$ at the point f_0. Prove that, for any set $A \subset C_p(X)$, we have $\overline{A} = \bigcap\{A + U : U \in \mathcal{B}\}$.*

Solution. Take any $f \in \overline{A}$ and $U \in \mathcal{B}$. It is possible to find $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that $O(f_0, x_1, \dots, x_n, \varepsilon) \subset U$. For the set $V = O(f_0, x_1, \dots, x_n, \frac{\varepsilon}{2})$ we have $V + V \subset O(f_0, x_1, \dots, x_n, \varepsilon) \subset U$. Since $V + f$ is open (Problem 059) and contains f, we have $(V + f) \cap A \neq \emptyset$. Take any $g \in (V + f) \cap A$ and observe that there is $v \in V$ such that $g = v + f$. Since $(-v) \in V$, we have $f = g - v \in g + V \subset A + V \subset A + U$. The function $f \in \overline{A}$ and $U \in \mathcal{B}$ being arbitrary, we proved that $\overline{A} \subset \bigcap\{A + U : U \in \mathcal{B}\}$.

Now take any $f \notin \overline{A}$. By Problem 062 there exist $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that $U + f \subset C_p(X) \setminus \overline{A}$ where $U = O(f_0, x_1, \dots, x_n, \varepsilon)$. To finish the proof, it is sufficient to establish that $f \notin A + U$. Suppose not. Then $f = a + u$ for some $a \in A$ and $u \in U$. As a consequence, we have $(-u) + f = a \in A$ and $-u + f \in U + f \subset C_p(X) \setminus \overline{A}$ which is a contradiction.

S.066. *Given a set $A \subset C_p(X)$, let $A^n = \{f_1 + \dots + f_n : f_i \in A, i = 1, \dots, n\}$ for any $n \in \mathbb{N}$. Denote by f_0 the function which is equal to zero at all points of X. Prove that, for any open set $U \ni f_0$, we have $\bigcup\{U^n : n \in \mathbb{N}\} = C_p(X)$.*

Solution. Take any $f \in C_p(X)$. There exist $k \in \mathbb{N}$, $x_1, \dots, x_k \in X$ and $\varepsilon > 0$ such that $V = O(f_0, x_1, \dots, x_k, \varepsilon) \subset U$. Take $n \in \mathbb{N}$ such that $\frac{1}{n}|f(x_i)| < \varepsilon$ for all $i \leqslant k$. Now, if $f_i = \frac{1}{n} \cdot f$ for all $i \leqslant n$, then $f = f_1 + \dots + f_n$ and $f_i \in V \subset U$ for any $i \leqslant n$. Thus $f \in U^n$.

S.067. *Let X be a topological space. For an arbitrary $x \in X$, define the function $e_x : C_p(X) \to \mathbb{R}$ as follows: $e_x(f) = f(x)$ for any $f \in C_p(X)$. Show that the map e_x is continuous for any $x \in X$.*

Solution. Let $f \in C_p(X)$ and $\varepsilon > 0$. Then $U = O(f, x, \varepsilon) = \{g \in C_p(X) : |g(x) - f(x)| < \varepsilon\}$ is an open set such that $f \in U$ and $e_x(U) \subset (e_x(f) - \varepsilon, e_x(f) + \varepsilon)$ which proves that e_x is continuous at f.

S.068. *Prove that $C_p(X)$ is a Tychonoff space for any topological space X.*

Solution. If $f, g \in C_p(X)$ and $f \neq g$ then there is $x \in X$ such that $f(x) \neq g(x)$. For $\varepsilon = \frac{1}{2}|f(x) - g(x)|$ we have $O(f,x,\varepsilon) \cap O(g,x,\varepsilon) = \emptyset$ which proves that $C_p(X)$ is Hausdorff and hence T_1.

To prove that $C_p(X)$ is completely regular, take any $f \in C_p(X)$ and any closed $F \subset C_p(X)$ with $f \notin F$. Given $r \in \mathbb{R}$ and $\varepsilon > 0$, consider the function $\varphi_{r,\varepsilon}$ defined as follows: $\varphi_{r,\varepsilon}(t) = 0$ if $t \notin [r - \varepsilon, r + \varepsilon]$; if $t \in [r - \varepsilon, r]$ then $\varphi_{r,\varepsilon}(t) = \frac{1}{\varepsilon}(t - r + \varepsilon)$ and $\varphi_{r,\varepsilon}(t) = -\frac{1}{\varepsilon}(t - r - \varepsilon)$ for all $t \in [r, r + \varepsilon]$. There are $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $O(f, x_1, \ldots, x_n, \varepsilon) \subset C_p(X)\backslash F$. Letting $r_i = f(x_i)$ for all $i \leqslant n$, consider the function $\varphi = (\varphi_{r_2,\varepsilon} \circ e_{x_2}) \cdot \ldots \cdot (\varphi_{r_n,\varepsilon} \circ e_{x_n})$. It is straightforward that $\varphi(f) = 1$. If $g \in F$ then $g \notin O(f, x_1, \ldots, x_n, \varepsilon)$ and hence $g(x_i) \notin (r_i - \varepsilon_i, r_i + \varepsilon)$ for some $i \leqslant n$. Thus $\varphi_{r_i,\varepsilon}(g(x_i)) = \varphi_{r_i,\varepsilon}(e_{x_i}(g)) = 0$. As a consequence, $\varphi(g) = 0$ and the function φ witnesses the fact that $C_p(X)$ is completely regular.

S.069. *Call a subset $C \subset C_p(X)$ convex if, for any $f, g \in C$ and $t \in [0, 1]$, we have $tf + (1 - t)g \in C$. Prove that, for an arbitrary X, the space $C_p(X)$ has a base consisting of convex sets.*

Solution. Recall that we have established in Problem 056 that the family $\mathcal{B} = \{[x_1, \ldots, x_n; O_1, \ldots, O_n] : n \in \mathbb{N}, x_1, \ldots, x_n \in X$ and O_i is a rational open interval for any $i \leqslant n\}$ forms a base of the space $C_p(X)$. Let us show that every $U = [x_1, \ldots, x_n; O_1, \ldots, O_n] \in \mathcal{B}$ is convex. Suppose that $f, g \in U$, $t \in [0, 1]$ and $h = tf + (1 - t)g$. Write O_i as (a_i, b_i) where $a_i, b_i \in \mathbb{Q}$ for all $i \leqslant n$. We have $r_i = f(x_i) \in (a_i, b_i)$ and $s_i = g(x_i) \in (a_i, b_i)$ for all $i \leqslant n$. Now, $h(x_i) = tr_i + (1 - t)s_i \in (ta_i + (1 - t)a_i, tb_i + (1 - t)b_i) = (a_i, b_i)$ for all $i \leqslant n$. Thus $h \in U$ and hence U is convex.

S.070. *Prove that the intersection of any family of convex subsets of $C_p(X)$ is a convex set. Show that the union of two convex sets is not necessarily a convex set.*

Solution. If U_α is convex for each $\alpha \in A$ then, for any $f, g \in U = \bigcap\{U_\alpha : \alpha \in A\}$ and $t \in [0, 1]$ we have $f, g \in U_\alpha$ for each α and hence $tf + (1 - t)g \in U_\alpha$. Therefore, $tf + (1 - t)g \in U$ and U is convex.

Given $x \in X$, the standard open sets $U = [x; (-1, 1)]$ and $V = [x; (2, 4)]$ are convex (see the solution of Problem 069 for the proof). However, $U \cup V$ is not convex because $f \equiv 0 \in U, f \equiv 3 \in V$ while $\frac{1}{2}f + \frac{1}{2}g \equiv \frac{3}{2} \notin U \cup V$.

S.071. *Prove that, if $A, B \subset C_p(X)$ are convex subsets of $C_p(X)$ then the set $A + B = \{a + b : a \in A$ and $b \in B\}$ is convex.*

Solution. Suppose that $f, g \in A + B$ and $t \in [0, 1]$. There exist $a, a' \in A$ and $b, b' \in B$ such that $f = a + b$ and $g = a' + b'$. Letting $a'' = ta + (1 - t)a' \in A$ and $b'' = tb + (1 - t)b' \in B$, we have $tf + (1 - t)g = t(a + b) + (1-t)(a'+b') = a'' + b'' \in A + B$ and hence $A + B$ is convex.

S.072. *Given a space X and a set $A \subset C_p(X)$, let $conv(A) = \{t_1 f_1 + \cdots + t_n f_n : n \in \mathbb{N}, f_i \in A, t_i \in [0, 1]$ for all $i \leqslant n$ and $\sum_{i=1}^{n} t_i = 1\}$. The set* conv(*A*) *is called the* *convex hull of A. Prove that the convex hull of A is a convex set and coincides with the intersection of all convex subsets of $C_p(X)$ which contain A.*

Solution. Let $f, g \in \text{conv}(A)$. Then there are $t_1, \ldots, t_n, s_1, \ldots, l_k \in [0, 1]$ such that $\sum_{i=1}^n t_i = \sum_{i=1}^k s_i = 1$ and $f = t_1 f_1 + \cdots t_n f_n$, $g = s_1 g_1 + \cdots s_k g_k$ for some $f_1, \ldots, f_n$, $g_1, \ldots, g_k \in A$. Then $tf + (1-t)g = \sum_{i=1}^n t t_i f_i + \sum_{i=1}^k (1-t) s_i g_i$, where $\sum_{i=1}^n t t_i + \sum_{i=1}^k (1-t)s_i = t \cdot 1 + (1-t) \cdot 1 = 1$ which shows that $tf + (1-t)g \in \text{conv}(A)$ and hence $\text{conv}(A)$ is convex.

Now, let γ be the family of all convex subsets of $C_p(X)$ which contain A. Since $A \subset \text{conv}(A)$ and $\text{conv}(A)$ is convex, we have $\text{conv}(A) \in \gamma$ and $\bigcap \gamma \subset \text{conv}(A)$. To prove that $\bigcap \gamma = \text{conv}(A)$ we must take an arbitrary convex $U \supset A$ and prove that $\text{conv}(A) \subset U$. We will prove by induction on n the following statement $I(n)$: if $f_1, \ldots, f_n \in U$ then $t_1 f_1 + \cdots + t_n f_n \in U$ whenever $t_1, \ldots, t_n \in [0, 1]$ and $\sum_{i=1}^n t_i = 1$. Since $I(1)$ is evident and $I(2)$ is true because U is convex, let us fix $n \geqslant 3$ such that $I(k)$ holds for all $k < n$. For each $i \in \{2, \ldots, n\}$, let $s_i = \frac{t_i}{t_2 + \cdots + t_n}$. Then $s_i \in [0, 1]$ for all i and $\sum_{i=2}^n = 1$. Since $I\,(n-1)$ holds, we have $g_1 = s_2 f_2 + \cdots + s_n f_n \in U$. It is easy to check that $t_1 f_1 + \cdots + t_n f_n = t_1 f_1 + (1 - t_1) g_1 \in U$ because U is convex and $I(n)$ is proved.

S.073. *Show that, for any open $U \subset C_p(X)$, its convex hull conv(U) is also an open set.*

Solution. It follows from Problem 059 that, for any sets $U_1, \ldots, U_n \in \tau(C_p(X))$, the set $U_1 + \cdots + U_n = \{f_1 + \cdots + f_n : f_i \in U_i \text{ for all } i \leqslant n\}$ is open in $C_p(X)$. Let us prove first that $tU = \{tf : f \in U\}$ is an open set for any $t \neq 0$ and $U \in \tau(C_p(X))$. If U is empty, everything is clear so assume that $U \neq \emptyset$. Given $f \in tU$, there is $g \in U$ such that $f = tg$. There are $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $O(g, x_1, \ldots, x_n, \varepsilon) \subset U$. Then $O(f, x_1, \ldots, x_n, |t|\,\varepsilon) \subset tU$. Indeed, suppose that $|h(x_i) - f(x_i)| < |t|\,\varepsilon$ for all $i \leqslant n$. Then $|\frac{1}{t} h(x_i) - g(x_i)| < \varepsilon$ for all $i \leqslant n$ and therefore $\frac{1}{t} h \in O(g, x_1, \ldots, x_n, \varepsilon) \subset U$ which implies $h \in tU$. This proves that tU is also open.

Finally, if $f = t_1 f_1 + \cdots t_n f_n \in \text{conv}(U)$ then the set $U_f = t_1 U + \cdots + t_n U$ is open by the previous remarks and $f \in U_f \subset \text{conv}(U)$. As an immediate consequence, $\text{conv}(U) = \bigcup \{U_f : f \in \text{conv}(U)\}$ is an open set.

S.074. *Call a function φ: $C_p(X) \to \mathbb{R}$ a linear functional, if we have the equality $\varphi(\alpha f + \beta g) = \alpha\varphi(f) + \beta\varphi(g)$ for any $f, g \in C_p(X)$ and $\alpha, \beta \in \mathbb{R}$. Prove that a linear functional $\varphi : C_p(X) \to \mathbb{R}$ is continuous if and only if $\varphi^{-1}(0)$ is closed in $C_p(X)$.*

Solution. Since $\{0\}$ is closed in $\mathbb{R}$, the set $\varphi^{-1}(0)$ is closed in $C_p(X)$ if φ is continuous (Problem 009(v)). Now, suppose that $\varphi^{-1}(0)$ is closed. If $\varphi^{-1}(0) = C_p(X)$ then $\varphi \equiv 0$ is continuous. If not, fix $f \in C_p(X) \setminus \varphi^{-1}(0)$. If $f_0 \equiv 0$ then $f \neq f_0$ because $\varphi(f_0) = 0$. Letting $r = \varphi(f) \neq 0$, observe that $\varphi^{-1}(r) = \varphi^{-1}(0) + f$. Indeed, if $g \in \varphi^{-1}(r)$ then $\varphi(g - f) = \varphi(g) - \varphi(f) = r - r = 0$ and hence $h = g - f \in \varphi^{-1}(0)$ whence $g = f + h \in \varphi^{-1}(0) + f$. This proves the inclusion $\varphi^{-1}(r) \subset \varphi^{-1}(0) + f$. Now, if $g \in \varphi^{-1}(0) + f$ then $g = h + f$ for some $h \in \varphi^{-1}(0)$ and hence

$\varphi(g) = \varphi(h) + \varphi(f) = 0 + r = r$. Another important observation is that $C_p(X)\backslash\varphi^{-1}(r) = C_p(X)\backslash(\varphi^{-1}(0) + f) = (C_p(X)\backslash\varphi^{-1}(0)) + f$ is an open set by Problem 059. Thus $\varphi^{-1}(r)$ is closed and $f_0 \notin \varphi^{-1}(r)$. There exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $U = O(f_0, x_1, \ldots, x_n, \varepsilon) \subset C_p(X)\backslash\varphi^{-1}(r)$. If there is some function $g \in U$ with $s = \varphi(g) > r$ then $h = \frac{r}{s} g \in U$ and $\varphi(h) = r$, a contradiction. This proves that $\varphi(U) \subset (-r, r)$. Now, given $g \in C_p(X)$ and $\delta > 0$, the set $W = \frac{\delta}{r} \cdot U + g$ is an open neighbourhood of g (see Problem 059 and the solution of Problem 073). It is easy to check that $\varphi(W) \subset (\varphi(g) - \delta, \varphi(g) + \delta)$ and hence φ is continuous at g.

S.075. *Let $\varphi : C_p(X) \to \mathbb{R}$ be a discontinuous linear functional. Prove that $\overline{\varphi^{-1}(0)} = C_p(X)$.*

Solution. Assume the contrary and fix an open non-empty $U \subset C_p(X)\backslash\varphi^{-1}(0)$. It is possible to find $f \in U$, $n \in \mathbb{N}$, $\varepsilon > 0$ and points $x_1, \ldots, x_n \in X$ such that $O(f, x_1, \ldots, x_n, \varepsilon) \subset U$. If $f_0 \equiv 0$ and $r = \varphi(f)$ then $W \cap \varphi^{-1}(r) = \emptyset$ where $W = O(f_0, x_1, \ldots, x_n, \varepsilon)$. Indeed, if $\varphi(h) = r$ and $|h(x_i)| < \varepsilon$ for all $i \leqslant n$ then $f - h \in U \cap \varphi^{-1}(0)$ which is a contradiction. Now, if there is some $g \in W$ with $s = \varphi(g) > r$ then $h = \frac{r}{s} g \in W$ and $\varphi(h) = r$, a contradiction. This proves that $\varphi(W) \subset (-r, r)$. Now, given a function $g \in C_p(X)$ and $\delta > 0$, the set $V = \frac{\delta}{r} \cdot W + g$ is an open neighbourhood of g (see Problem 059) It is easy to check that $\varphi(V) \subset (\varphi(g) - \delta, \varphi(g) + \delta)$ and hence φ is continuous at g. The point g being arbitrary, we proved continuity of φ which is again a contradiction.

S.076. *Let X be an arbitrary space. Suppose that $\varphi : C_p(X) \to \mathbb{R}$ is a continuous linear functional such that $\varphi(f) \neq 0$ for some $f \in C_p(X)$. Prove that $\varphi(U)$ is open in $\mathbb{R}$ for any $U \in \tau(C_p(X))$.*

Solution. If the set U is empty there is nothing to prove. If not, take any number $r \in f(U)$ and any $g \in U$ with $\varphi(g) = r$. The set $U + (-g)$ is open (see Problem 059 and the solution of Problem 073) and contains the function $f_0 \equiv 0$. Therefore there exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $f_0 \in V = O\ (f_0, x_1, \ldots, x_n, \varepsilon) \subset U + (-g)$. For each $h \in C_p(X)\backslash V$ and $M_h = \max\{|h(x_i)| : i \leqslant n\} \neq 0$, note that $h' = \frac{\varepsilon}{M_h} \cdot h \in V$. As a consequence, we have $\varphi(h) = \frac{M_h}{\varepsilon} \varphi(h')$ which shows that $\varphi \equiv 0$ if $\varphi(V) = \{0\}$. Since φ is not identically zero, there is $f' \in V$ such that $\delta = |\varphi(f')| > 0$. Observe that $t \cdot V = \{tv : v \in V\} \subset V$ for any $t \in [-1, 1]$. Thus, for any $\mu \in (-\delta, \delta)$, we have $g' = \frac{\mu}{\delta} f' \in V$ and $\varphi(g') = \mu$. This proves that $\varphi(V) \supset (-\delta, \delta)$. For the set $W = V + g$ we have $W \subset U$ and $\varphi(U) \supset \varphi(W) \supset (r - \delta, r + \delta)$ whence $\varphi(U)$ is an open set in $\mathbb{R}$.

S.077. *Let X be an arbitrary space. Suppose that $f, g \in C_p(X)$ and $f \neq g$. Prove that there exists a continuous linear functional $\varphi : C_p(X) \to \mathbb{R}$ such that $\varphi(f) \neq \varphi(g)$.*

Solution. Since f and g are distinct, there is $x \in X$ with $f(x) \neq g(x)$. The map $e_x : C_p(X) \to \mathbb{R}$, defined by $e_x(h) = h(x)$ for all $h \in C_p(X)$, is continuous (Problem 067) and it is easy to verify that e_x is linear. Hence the linear functional $\varphi = e_x$ is what we are looking for, because $\varphi(f) = f(x) \neq g(x) = \varphi(g)$.

S.078. *Let X be an arbitrary space. Denote by $L_p(X)$ the set of continuous linear functionals on $C_p(X)$. Prove that $L_p(X)$ is closed in $C_p(C_p(X))$.*

Solution. Take any function $\varphi \in \overline{L_p(X)}$ and any $f,\ g \in C_p(X)$, $\alpha,\ \beta \in \mathbb{R}$. For the numbers $A = \varphi(\alpha f + \beta g), B = \alpha\varphi(f)$ and $C = \beta\varphi(g)$, we must prove that $A = B + C$. Fix an arbitrary $\varepsilon > 0$. For $\varepsilon' = \frac{\varepsilon}{\alpha+\beta+1}$, there exists $\varphi' \in L_p(X)$ such that $|\varphi(f) - \varphi'(f)| < \varepsilon'$, $|\varphi(g) - \varphi'(g)| < \varepsilon'$ and $|\varphi(\alpha f + \beta g) - \varphi'(\alpha f + \beta g)| < \varepsilon'$. Observe that, if $A' = \varphi'(\alpha f + \beta g), B' = \alpha\varphi'(f)$ and $C' = \beta\varphi'(g)$ then $A' = B' + C'$ and $|A - B - C| \leqslant |A - A'| + |B - B'| + |C - C'| + |A' - B' - C'| < \varepsilon' + \alpha\varepsilon' + \beta\varepsilon' = \varepsilon$. This proves that, for any $\varepsilon > 0$, we have $|A - B - C| < \varepsilon$ and hence $A - B - C = 0$.

S.079. *For a topological space X and a function $f \in C_p(X)$, consider the map $T_f : C_p(X) \to C_p(X)$ defined by the formula $T_f(g) = f + g$ for every $g \in C_p(X)$. Prove that T_f is a homeomorphism for any $f \in C_p(X)$.*

Solution. It is evident that T_{-f} is the inverse map for T_f and hence T_f is a bijection. If we prove that, for any $f \in C_p(X)$, the map T_f is continuous then, in particular, the map T_{-f} is continuous and hence T_f is a homeomorphism. Now, if $U \in \tau(C_p(X))$ then $T_f^{-1}(U) = U + (-f)$ is an open set by Problem 059 and hence T_f is continuous.

S.080. *For a topological space X and a function $f \in C_p(X)$, consider the map $M_f : C_p(X) \to C_p(X)$ defined by the formula $M_f(g) = f \cdot g$ for every $g \in C_p(X)$. Prove that M_f is a continuous map for any $f \in C_p(X)$.*

Solution. Let us fix $g \in C_p(X)$ and an open $U \ni M_f(g) = g \cdot f$. There exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $O(f \cdot g, x_1, \ldots, x_n, \varepsilon) \subset U$. Take any $\delta > 0$ such that $\delta|f(x_i)| < \varepsilon$ for all $i \leqslant n$. Is suffices to prove that, for the set $V = O(g, x_1, \ldots, x_n, \delta)$, we have $M_f(V) \subset U$. For any $h \in V$ and any $i \leqslant n$, we have $|M_f(h)(x_i) - M_f(g)(x_i)| = |f(x_i)||g(x_i) - h(x_i)| < \delta|f(x_i)| < \varepsilon$ which shows that $M_f(h) \in O(f \cdot g, x_1, \ldots, x_n, \varepsilon) \subset U$.

S.081. *Let X be a topological space. For an arbitrary $f \in C_p(X)$, consider the map $M_f : C_p(X) \to C_p(X)$ defined by the formula $M_f(g) = f \cdot g$. Prove that the map $M_f : C_p(X) \to C_p(X)$ is a homeomorphism iff $f(x) \neq 0$ for all $x \in X$.*

Solution. For $g = \frac{1}{f}$, apply Problem 080 to see that M_f and M_g are continuous. Is it immediate that the maps are mutually inverse and hence M_f is a homeomorphism.

S.082. *Given a topological space X and a function $f \in C_p(X)$, consider the maps $U_f, D_f : C_p(X) \to C_p(X)$ defined by $U_f(g) = \max(f,g)$ and $D_f(g) = \min(f,g)$ for any $g \in C_p(X)$. Prove that the maps U_f and D_f are continuous for any $f \in C_p(X)$.*

Solution. Let us prove that the map U_f is continuous at any point $g \in C_p(X)$. Given any set $U \in \tau(C_p(X))$ with $U_f(g) \in U$, we can find $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $V = O(U_f(g), x_1, \ldots, x_n, \varepsilon) \subset U$. We are going to prove that $U_f(W) \subset U$ where $W = O(g, x_1, \ldots, x_n, \varepsilon)$. Indeed, take an arbitrary $h \in W$. Then $|h(x_i) - g(x_i)| < \varepsilon$ for each $i \leqslant n$. Let us see the possibilities for the value of each one of the numbers $U_f(h)(x_i)$ and $U_f(g)(x_i)$. If $f(x_i) \leqslant g(x_i)$ and $f(x_i) \leqslant h(x_i)$ then $|U_f(h)(x_i) - U_f(g)(x_i)| = |h(x_i) - g(x_i)|$. If $g(x_i) < f(x_i) \leqslant h(x_i)$ then $|U_f(h)(x_i) - U_f(g)(x_i)| = |h(x_i) - f(x_i)| \leqslant |h(x_i) - g(x_i)|$.

Analogously, if we have $h(x_i) < f(x_i) \leqslant g(x_i)$ then $|U_f(h)(x_i) - U_f(g)(x_i)| = |f(x_i) - g(x_i)| \leqslant |h(x_i) - g(x_i)|$. Finally, if $f(x_i) \geqslant g(x_i)$ and $f(x_i) \geqslant h(x_i)$ then $|U_f(h)(x_i) - U_f(g)(x_i)| = |f(x_i) - f(x_i)| = 0 \leqslant |g(x_i) - h(x_i)|$. This proves that, for any $h \in W$, we have $|U_f(h)(x_i) - U_f(g)(x_i)| \leqslant |h(x_i) - g(x_i)| < \varepsilon$. Thus $U_f(W) \subset V \subset U$ and U_f is continuous at g. The proof of continuity of D_f is identical.

S.083. *Given a topological space X, let $CI(X) = \{f \in C(X) : f(x) \neq 0$ for any $x \in X\}$. Considering that the set $CI(X)$ carries the topology inherited from $C_p(X)$, define the map $i : CI(X) \to CI(X)$ by the formula $i(f) = \frac{1}{f}$. Is the map i continuous?*

Solution. Let us prove that i is continuous at any $f \in CI(X)$. Given an arbitrary open set $U \ni \frac{1}{f}$ there are $n \in \mathbb{N}, x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $V = O(\frac{1}{f}, x_1, \ldots, x_n, \varepsilon) \cap CI(X) \subset U$. Making ε smaller if necessary, we can assume that $|f(x_i)| > 2\varepsilon$ for any $i \leqslant n$. Take any $\delta > 0$ such that $\delta < \varepsilon$ and $\delta < 2\varepsilon^3$. It suffices to prove that $i(W) \subset U$ for $W = O(f, x_1, \ldots, x_n, \delta) \cap CI(X)$. Take any $h \in W$. Then $|i(h)(x_i) - i(f)(x_i)| = |\frac{1}{f(x_i)} - \frac{1}{h(x_i)}| = |\frac{1}{h(x_i)f(x_i)}| \cdot |f(x_i) - h(x_i)|$. Since $|f(x_i)| > 2\varepsilon$ and $|h(x_i) - f(x_i)| < \delta < \varepsilon$, we have $|h(x_i)| > \varepsilon$ and hence $|i(h)(x_i) - i(f)(x_i)| < \frac{1}{\varepsilon} \cdot \frac{1}{2\varepsilon} \cdot \delta < \frac{2\varepsilon^3}{2\varepsilon^2} = \varepsilon$ which shows that $i(h) \in V \subset U$.

S.084. *For an arbitrary space X and any $A \subset C(X)$ denote by $\overline{A}^u$ the set $\{f \in C(X)$: there exists a sequence $\{f_n : n \in \omega\} \subset A$ such that $f_n \rightrightarrows f\}$. Prove that the operator $A \to \overline{A}^u$ has the properties (C1)–(C4) listed in Problem 004. Therefore there exists a topology τ_u called the uniform convergence topology on $C(X)$ such that $\overline{A}^u = cl_{\tau_u}(A)$ for every $A \subset C(X)$. The space $(C(X), \tau_u)$ will be denoted $C_u(X)$.*

Solution. The property (C1) is evident. Given $f \in C_p(X)$ we have $f_n \rightrightarrows f$ if $f_n = f$ for all n. This shows that (C3) also holds. Another evident observation is that $f_n \rightrightarrows f$ implies $g_k = f_{n_k} \rightrightarrows f$ for any subsequence $\{f_{n_k}\} \subset \{f_n\}$. If $f \in \overline{A \cup B}^u$ then $f_n \rightrightarrows f$ for some sequence $\{f_n\} \subset A \cup B$. Clearly, there is a subsequence $\{f_{n_k}\} \subset \{f_n\}$ which is contained in one of the sets A, B. Since $f_{n_k} \rightrightarrows f$, we have $f \in \overline{A}^u \cup \overline{B}^u$ which proves the inclusion $\overline{A \cup B}^u \subset \overline{A}^u \cup \overline{B}^u$. The reverse inclusion is obvious because any sequence in A or in B is also a sequence in $A \cup B$.

Finally, to see that (C4) is also true, take any $f \in \overline{\overline{A}^u}^u$ and fix a sequence $\{f_n\} \subset \overline{A}^u$ with $f_n \rightrightarrows f$. For each $n \in \omega$, there is a sequence $\{f_{nk}\} \subset A$ such that $f_{nk} \rightrightarrows f_n$. Thus, for each $n \in \omega$, there is $k_n \in \omega$ such that $|f_{nk_n}(x) - f_n(x)| < \frac{1}{n+1}$ for all $x \in X$. We have $\{g_n\} \subset A$, where $g_n = f_{nk_n}$ for all $n \in \omega$. Let us prove that $g_n \rightrightarrows f$. Given $\varepsilon > 0$, there exists $m \in \omega$ such that $\frac{1}{m+1} < \frac{\varepsilon}{2}$ and $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $x \in X$ and $n \geqslant m$. Then

$$|g_n(x) - f(x)| \leqslant |f_{nk_n}(x) - f_n(x)| + |f_n(x) - f(x)| < \frac{1}{n+1} + \frac{\varepsilon}{2} < \varepsilon$$

for all $x \in X$ and hence $f \in \overline{A}^u$.

S.085. *Prove that, for any space X, the space $C_u(X)$ has a countable local base at any point.*

Solution. Fix any function $f \in C_u(X)$ and define, for every number $n \in \mathbb{N}$, the set $U_n = \{g \in C_u(X) : |g(x) - f(x)| < \frac{1}{n}$ for all $x \in X\}$. Observe that $f \notin \overline{F}_n$ where $F_n = C_u(X)\backslash U_n$. Indeed, if not, then there is a sequence $\{f_k\} \subset F_n$ such that $f_k \rightrightarrows f$. As a consequence, there is $k \in \omega$ such that $|f(x) - f_k(x)| < \frac{1}{n}$ for all $x \in X$. Then $f_k \in U_n = C_u(X)\backslash F_n$ which is a contradiction. This shows that $W_n = \text{Int}(U_n)$ is an open set which contains f. To see that $\mathcal{B} = \{W_n : n \in \mathbb{N}\}$ is a local base at f, take any open $U \subset C_u(X)$ with $f \in U$. If $W_n \backslash U \neq \emptyset$ for all $n \in \mathbb{N}$, pick $f_n \in W_n \backslash U$ for each $n \in \mathbb{N}$. Since $|f_n(x) - f(x)| < \frac{1}{n}$ for all $n \in \mathbb{N}$ and $x \in X$, it is evident that $f_n \rightrightarrows f$. Thus, $f \in \overline{C_u(X)\backslash U} = C_u(X)\backslash U$ which is a contradiction. Therefore, $W_n \subset U$ for some n and $\mathcal{B}$ is a countable local base at f.

S.086. *Define the identity map* id : $C(X) \to C(X)$ *by* id$(f) = f$ *for all* $f \in C(X)$. *Prove that* id : $C_u(X) \to C_p(X)$ *is continuous for any space* X.

Solution. Since $\text{id}^{-1}(U) = U$ for any $U \subset C_p(X)$, it is sufficient to prove that there is a subbase $\mathcal{S}$ of $C_p(X)$ such that every $U \in \mathcal{S}$ is open in $C_u(X)$ (Problem 009(iii)). By Problem 057 it suffices to prove that the set $W = [x, O]$ is open in $C_u(X)$ for any $x \in X$ and any rational interval $O \subset \mathbb{R}$. Assume that $O = (a, b)$ and take any $f \in W$. Take $\delta = \min\{f(x) - a, b - f(x)\} > 0$ and consider the set $U_\delta = \{g \in C_u(X) : |g(y) - f(y)| < \delta$ for any $y \in X\}$. It is clear that $f \in U_\delta \subset W$. If $f \in \overline{C_u(X)\backslash U_\delta}^u$ then there is a sequence $\{f_n : n \in \omega\} \subset C_u(X)\backslash U_\delta$ with $f_n \rightrightarrows f$. This means that, for some $k \in \omega$ we have $|f_k(y) - f(y)| < \delta$ for all $y \in X$ and hence $f_k \in U_\delta$ which is a contradiction. As a consequence, there is $V_f \in \tau_u$ such that $f \in V_f \subset U_\delta \subset W$. Therefore $W = \bigcup\{V_f : f \in W\}$ is open in $C_u(X)$ and our proof is complete.

S.087. *For any* $X \in T_{3\frac{1}{2}}$ *prove that the identity map* id : $C_u(X) \to C_p(X)$ *is a homeomorphism if and only if the space* X *is finite.*

Solution. Take any finite space $X = \{x_1, \dots, x_k\}$. By Problem 086 the map id is continuous, so it suffices to show that $i = \text{id}^{-1}$ is also continuous. Since $i^{-1}(F) = F$ for any $F \subset C_u(X)$, it suffices to establish that F is closed in $C_p(X)$ for every closed $F \subset C_u(X)$ (Problem 009)(v)). Suppose that F is closed in $C_u(X)$ and non-closed in $C_p(X)$. Take any $f \in \overline{F}\backslash F$ (the bar will denote the closure in $C_p(X)$). Then, for every $n \in \mathbb{N}$ there exists $f_n \in F \cap W_n$ where $W_n = [x_1, \dots, x_n; O_1, \dots, O_n]$ with $O_m = (f(x_m) - \frac{1}{n}, f(x_m) + \frac{1}{n})$ for each $m \leqslant k$. It is immediate that $f_n \rightrightarrows f$ and hence $f \in \overline{F}^u = F$, a contradiction. This shows that id is a homeomorphism if X is finite. Note that we did not use the Tychonoff property of X.

Assume now that X is an infinite Tychonoff space and id is a homeomorphism. Take $f_0 \equiv 0$ and consider the set $W = \{f \in C_u(X) : |f(x)| < 1$ for all $x \in X\}$. In S.085, we proved that f_0 belongs to the set $U = \text{Int}(W)$ (the interior is taken in $C_u(X)$). Since U is open in $C_u(X)$ and $(\text{id}^{-1})^{-1}(U) = U$, the set U must be open in $C_p(X)$. Therefore, there exist $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that $V = O(f_0, x_1, \dots, x_n, \varepsilon) \subset U$ (058). Since the space X is Tychonoff and the set $F = \{x_1, \dots, x_n\}$ is closed and finite, we can take $x \in X\backslash F$ and $h \in C_p(X)$ such that $h|_F \equiv 0$ and $h(x) = 1$. It is evident that $h \in V\backslash W$ which contradicts the fact that $V \subset U \subset W$. Thus X has to be finite.

S.088. *Prove that the space $C_u(\mathbb{N})$ does not have countable base.*

Solution. We proved in S.085 that, for any space X and any $f \in C_u(X)$, the sets $\{W_n(f) : n \in \mathbb{N}\}$ form a local base of $C_u(X)$ at f. Here $W_n(f)$ is the interior of the set $\{g \in C_u(X)$: for any $x \in X$, we have $|g(x) - f(x)| < \frac{1}{n}\}$ for each $n \in \mathbb{N}$. As a consequence, the family $\{W_n(f) : f \in C(X), n \in \mathbb{N}\}$ is a base in $C_u(X)$.

Claim. Let Z be a space which has a countable base $\mathcal{B}$. If $\mathcal{C}$ is any other base of Z then there is a countable $\mathcal{C}' \subset \mathcal{C}$ such that $\mathcal{C}'$ is a base in Z. In other words, if there is a countable base in a space then any base of this space contains a countable base.

Proof of the claim. Let $\mathcal{B} = \{U_n : n \in \omega\}$. Call a pair $\mu = (m, n)$ of elements of ω *admissible*, if there is $C \in \mathcal{C}$ such that $U_m \subset C \subset U_n$. If a pair $\mu = (m, n)$ is admissible then fix some $C = C(\mu) \in \mathcal{C}$ such that $U_m \subset C \subset U_n$. The family $\mathcal{C}' = \{C(\mu) : \mu$ is an admissible pair$\} \subset \mathcal{C}$ is countable. Let us prove that $\mathcal{C}'$ is a base in Z. Indeed, if $x \in U \in \tau(Z)$ then there is $m \in \omega$ such that $x \in U_n \subset U$ because $\mathcal{B}$ is a base in Z (Problem 002). By the same reason, there is $C' \in \mathcal{C}$ with $x \in C' \subset U_n$. Applying Problem 002 once more, we can find $U_m \in \mathcal{B}$ such that $x \in U_m \subset C'$. Since $U_m \subset C' \subset U_n$, the pair $\mu = (m, n)$ is admissible and therefore $C = C(\mu) \in \mathcal{C}'$ and $x \in C \subset U_n \subset U$. Applying Problem 002 once again, we see that $\mathcal{C}'$ is a base in Z and our claim is proved.

Applying our claim we can observe that, if there is a countable base in the space $C_u(\mathbb{N})$ then there exists a countable set $A = \{f_n : n \in \mathbb{N}\} \subset C_u(X)$ such that the family $\mathcal{B} = \{W_n(f) : f \in A, n \in \mathbb{N}\}$ is a base in $C_u(\mathbb{N})$. To bring this to a contradiction, let $g(n) = f_n(n) + 2$ for each $n \in \mathbb{N}$. Since $\mathcal{B}$ is a base, we have $g \in W_k(f_n)$ for some $k, n \in \mathbb{N}$. As a consequence, $|f_n(x) - g(x)| < \frac{1}{k} \leqslant 1$ for any $x \in \mathbb{N}$. However, $|f_n(n) - g(n)| = 2$ which is a contradiction. Hence the space $C_u(\mathbb{N})$ cannot have a countable base.

S.089. *Let X and Y be topological spaces. Given $Z \subset Y$, each $f \in C(X, Z)$ can also be considered a function from X to Y. Thus, $C(X,Z) \subset C(X,Y)$. Prove that the topology of $C_p(X, Z)$ coincides with the topology on $C(X, Z)$ induced from $C_p(X, Y)$ and hence $C_p(X, Z) \subset C_p(X,Y)$.*

Solution. For an arbitrary standard open set

$H = [x_1, \dots, x_n; O_1, \dots, O_n] = \{f \in C_p(X, Y) : f(x_i) \in O_i$ for all $i \leqslant n\}$ in the space $C_p(X, Y)$, it is immediate that

$H \cap C_p(X, Z) = \{f \in C_p(X, Z) : f(x_i) \in O_i \cap Z$ for all $i \leqslant n\}$ is a standard open set in $C_p(X, Z)$. Now, if a set U is open in the space $C_p(X,Y)$ then $U = \bigcup\gamma$ where all elements of the family γ are standard open sets. Since $U \cap C_p(X, Z) = \bigcup\{H \cap C_p(X, Z) : H \in \gamma\}$ is a union of standard open sets of $C_p(X, Z)$, it is open in $C_p(X, Z)$. This shows that the subspace topology of $C_p(X, Z)$ is contained in $\tau(C_p(X, Z))$.

Now let

$W = [x_1, \dots, x_n; U_1, \dots, U_n] = \{f \in C_p(X, Z) : f(x_i) \in U_i$ for all $i \leqslant n\}$ be a standard open set in the space $C_p(X, Z)$. For each $i \leqslant n$, take a set $O_i \in \tau(Y)$ such that $U_i = O_i \cap Z$. Then $W' = W \cap C_p(X, Z)$ where

$W' = [x_1, \dots, x_n; O_1, \dots, O_n] = \{f \in C_p(X, Y) : f(x_i) \in O_i$ for all $i \leqslant n\}$ is a standard open set in the space $C_p(X, Y)$. Finally, if $U \in \tau(C_p(X, Z))$ then $U = \bigcup\gamma$

where γ consists of standard open sets in $C_p(X, Z)$. We proved that each $W \in \gamma$ is the intersection of a standard open set in $C_p(X, Y)$ with $C_p(X, Z)$ and hence every $W \in \gamma$ belongs to the subspace topology on $C_p(X, Z)$. As a consequence, $U = \bigcup\gamma$ also belongs to the subspace topology on $C_p(X, Z)$. This proves that the subspace topology of $C_p(X, Z)$ coincides with $\tau(C_p(X, Z))$.

S.090. *Let X and Y be topological spaces. Given a closed $Z \subset Y$, prove that $C_p(X, Z)$ is a closed subspace of $C_p(X, Y)$.*

Solution. If $f \in H = C_p(X, Y)\backslash C_p(X, Z)$ then there is a point $x \in X$ such that $f(x) \in O = Y \setminus Z$. Hence $U_f = \{g \in C_p(X, Y) : f(x) \in O\}$ is an open set in $C_p(X, Y)$ such that $f \in U_f \subset H$. Therefore $H = \bigcup\{U_f : f \in H\}$ is an open set.

S.091. *Let X and Y be topological spaces. If $w : Y \to Z$ is a continuous map, define a map $h_w : C_p(X, Y) \to C_p(X, Z)$ in the following way: for any $f \in C_p(X, Y)$ let $h_w(f) = w \circ f$. Show that the map h_w is continuous.*

Solution. Take any function $f \in C_p(X, Y)$ and any set $U \in \tau(C_p(X, Z))$ such that $g = h_w(f) \in U$. There is a standard open set $V = [x_1, \ldots, x_n; O_1, \ldots, O_n]$ in $C_p(X, Z)$ such that $g \in V \subset U$. Then $W = \{h \in C_p(X, Y) : h(x_i) \in w^{-1}(O_i) \text{ for all } i \leqslant n\}$ is a standard open set in $C_p(X, Y)$ such that $f \in W$ and $h_w(W) \subset V \subset U$. Therefore, the map h_w is continuous at f.

S.092. *Show that, for an arbitrary space X, there exists a continuous mapping $\rho : C_p(X) \to C_p(X, \mathbb{I})$ such that $\rho(f) = f$ whenever $f \in C_p(X, \mathbb{I})$.*

Solution. Define a function $w : \mathbb{R} \to \mathbb{I}$ as follows: $w(t) = -1$ for all $t < -1$; if $t \in \mathbb{I}$ then $w(t) = t$ and if $t > 1$ then $w(t) = 1$. It is clear that w is continuous and hence $\rho = h_w : C_p(X) \to C_p(X,\mathbb{I})$ defined by $\rho(f) = w \circ f$, is a continuous map (Problem 091). It is also immediate that $\rho(f) = f$ for any $f \in C_p(X, \mathbb{I})$ so our proof is complete.

S.093. *Let $\varphi : C_p(X) \to C_p(Y)$ be an isomorphism. Prove that*

(i) If $f, g \in C_p(X)$ and $f(x) \leqslant g(x)$ for any $x \in X$ then $\varphi(f)(y) \leqslant \varphi(g)(y)$ for any $y \in Y$.
(ii) If $f, g \in C_p(X)$ then $\varphi(\max(f, g)) = \max(\varphi(f), \varphi(g))$.
(iii) If $f, g \in C_p(X)$ then $\varphi(\min(f, g)) = \min(\varphi(f), \varphi(g))$.

Solution. (i) Suppose that $h \in C_p(X)$ and $h(x) \geqslant 0$ for all $x \in X$. Then, we have $\varphi(h)(y) \geqslant 0$ for all $y \in Y$. Indeed, the function $h_1 = \sqrt{h}$ for which $h_1(x) = \sqrt{h(x)}$ for all $x \in X$, is well defined and continuous and therefore $\varphi(h) = \varphi(h_1^2) = \varphi(h_1))^2$ and hence $\varphi(h)(y) \geqslant 0$ for all $y \in Y$. Now, if $f(x) \leqslant g(x)$ for all $x \in X$ then $h(x) \geqslant 0$ for all x, where $h = g - f$. As a consequence, $\varphi(h)(y) = \varphi(g - f)(y) = \varphi(g)(y) - \varphi(f)(y) \geqslant 0$ for all $y \in Y$.

(ii) Given any $h \in C_p(X)$, define $|h|(x) = |h(x)|$ for all $x \in X$. We have $(\varphi(|h|))^2 = \varphi(|h|^2) = \varphi(h^2) = (\varphi(h))^2$ and hence $\varphi(|h|) = |\varphi(h)|$ for any function $h \in C_p(X)$. Observing that $\max(f, g) = \frac{1}{2}(f + g + |f - g|)$, we conclude that $\varphi(\max(f,g)) = \frac{1}{2}(\varphi(f) + \varphi(g) + |\varphi(f) - \varphi(g)|) = \max(\varphi(f), \varphi(g))$.

(iii) Observing that $\min(f, g) = \frac{1}{2}(f + g - |f - g|)$, we conclude that we have the equalities $\varphi(\min(f,g)) = \frac{1}{2}(\varphi(f) + \varphi(g) - |\varphi(f) - \varphi(g)|) = \min(\varphi(f), \varphi(g))$.

S.094. *Prove that, if $\varphi : C_p(X) \to C_p(Y)$ is an isomorphism, then we have the equality $\varphi(C^*(X)) = C^*(Y)$.*

Solution. Given $r \in \mathbb{R}$, let $p_r(x) = r$ for all $x \in X$. Now, $p_1 = p_1^2$ and hence $\varphi(p_1) = (\varphi(p_1))^2$ whence $\varphi(p_1)(y)$ can only be equal to 0 or 1. Therefore, for any $n \in \mathbb{N}$, we have $\varphi(p_n) = n\varphi(p_1)$ so p_n can only take values 0 or n. If $f \in C^*(X)$ then $|f| \leqslant p_n$ for some $n \in \mathbb{N}$ which implies $|\varphi(f)| = \varphi(|f|) \leqslant \varphi(p_n)$ (see Problem 093 and it solution). Since $|\varphi(p_n)(y)| \leqslant n$ for any $y \in Y$, the function $\varphi(f)$ is bounded. This proves that $\varphi(C^*(X)) \subset C^*(Y)$. The same proof gives $\varphi^{-1}(C^*(Y)) \subset C^*(X)$ and hence $\varphi(C^*(X)) = C^*(Y)$.

S.095. *Let $\varphi : C_p(X) \to C_p(Y)$ be an isomorphism. Suppose that $\boldsymbol{c} \in \mathbb{R}$ and $f(x) = \boldsymbol{c}$ for every $x \in X$. Prove that $\varphi(f)(y) = \boldsymbol{c}$ for any $y \in Y$.*

Solution. Given $r \in \mathbb{R}$, let $p_r(x) = r$ for all $x \in X$ and $q_r(y) = r$ for all $y \in Y$. We must prove that $\varphi(p_c) = q_c$ for all $c \in \mathbb{R}$. Note first that $\varphi(p_0) + \varphi(p_0) = \varphi(p_0 + p_0) = \varphi(p_0)$ and hence $\varphi(p_0) = q_0$. Since φ is an isomorphism, there is some function f with $\varphi(f) = q_1$. Then $\varphi(p_1) = \varphi(p_1) \cdot \varphi(f) = \varphi(p_1 \cdot f) = \varphi(f) = q_1$. Since $q_n = q_1 + \cdots + q_1$ (the sum is of n summands), we obtain the equality $\varphi(p_n) = \varphi(p_1 + \cdots + p_1) = q_1 + \cdots + q_1 = q_n$ (all sums have n summands) for each $n \in \mathbb{N}$. Now, $\varphi(-f) + \varphi(f) = \varphi(p_0) = q_0$ which implies $\varphi(-f) = -\varphi(f)$ for any $f \in C(X)$. Therefore $\varphi(p_n) = q_n$ for any integer n. If r is a rational number then $r = \frac{a}{b}$ for some integers a and b. Therefore, $b\varphi(p_r) = \varphi(bp_r) = \varphi(p_a) = q_a$ and we have $\varphi(p_r) = \frac{1}{b}q_a = q_n$.

Finally, take any $c \in \mathbb{R}$ and any $\varepsilon > 0$. Pick any rational numbers r and s with $r \in (c - \frac{\varepsilon}{2}, c)$ and $s \in (c, c + \frac{\varepsilon}{2})$. Then $|\varphi(p_c) - q_c| \leqslant |\varphi(p_s) - \varphi(p_r)| = |q_s - q_r| < \varepsilon$ (see Problem 093(i)). Since $\varepsilon > 0$ was taken arbitrarily, we proved that $\varphi(p_c) = q_c$.

S.096. *Prove that there is no isomorphism between $C_p(\mathbb{R})$ and $C_p(\mathbb{I})$.*

Solution. All continuous functions on $\mathbb{I}$ are bounded so if $\varphi : C(\mathbb{I}) \to C(\mathbb{R})$ is an isomorphism then $C^*(\mathbb{R}) = \varphi(C^*(\mathbb{I})) = \varphi(C(\mathbb{I})) = C(\mathbb{R})$ by Problem 094, which is a contradiction because the identity function is not bounded on $\mathbb{R}$.

S.097. *Suppose that X is a set and (Y, τ) is a topological space. Given a map $f : X \to Y$, denote the family $\{f^{-1}(U) : U \in \tau\}$ by $f^{-1}(\tau)$. Prove that*

(i) *$\mu = f^{-1}(\tau)$ is a topology on X such that f is continuous considered as a map from (X, μ) to (Y, τ);*
(ii) If v is any topology on X, such that the map f : $(X, v) \to (Y, \tau)$ *is continuous, then $\mu \subset v$.*

Solution. (i) Since $\emptyset = f^{-1}(\emptyset)$ and $X = f^{-1}(Y)$, the axiom (TS1) holds for the family μ. If $U, V \in \mu$ then $U = f^{-1}(U')$ and $V = f^{-1}(V')$ for some $U', V' \in \tau$. Then $U \cap V = f^{-1}(U' \cap V') \in \mu$ so (TS2) also holds. Finally, if $\gamma \subset \mu$ then, for any $U \in \gamma$ fix $U' \in \tau$ with $U = f^{-1}(U')$. Then $\bigcup\gamma = f^{-1}(\bigcup\{U' : U \in \gamma\}) \in \mu$ and (TS3) is also proved. The map $f : (X, \mu) \to (Y, \tau)$ is continuous because $f^{-1}(U) \in \mu$ for each $U \in \tau$.

(ii) Let ν be a topology on X such that $f: (X, \nu) \to (Y, \tau)$ is continuous. For any $U \in \mu$ we have $U = f^{-1}(U') \in \nu$ for some $U' \in \tau$. The map f is continuous and therefore $U = f^{-1}(U') \in \nu$ whence $\mu \subset \nu$.

S.098. *Let X be a set and let $f: X \to \mathbb{R}$ be a map. Prove that the space $(X, f^{-1}(\mathcal{N}_R))$ is completely regular.*

Solution. Denote the topology $f^{-1}(\mathcal{N}_R)$ by τ and take any $x \in X$. If F is τ-closed and $x \notin F$ then $X \backslash F = f^{-1}(U)$ for some open $U \subset \mathbb{R}$. Since $y = f(x) \in U$ and $\mathbb{R}$ is completely regular, there is a continuous function $h : \mathbb{R} \to [0, 1]$ such that $h(y) = 1$ and $h|(\mathbb{R}\backslash U) \equiv 0$. The function $g = h \circ f : X \to [0, 1]$ is continuous and $f(x) = 1$, $f(F) \subset \{0\}$ so X is completely regular.

S.099. *Let $\mathcal{T}$ be a non-empty family of topologies on a set X. Show that the family $\bigcup\mathcal{T}$ satisfies the condition of Problem 008 for generating a topology as a subbase. The topology thus generated is called the supremum or the least upper bound of the family of topologies from $\mathcal{T}$. Prove that the least upper bound of T_i-topologies is always a T_i-topology for $i \leqslant 3\frac{1}{2}$.*

Solution. Fix any topology $\nu_0 \in \mathcal{T}$ and note that $\bigcup(\bigcup\mathcal{T}) \supset \bigcup \nu_0 = X$ so $\bigcup\mathcal{T}$ generates a topology τ as a subbase. Suppose that any $\nu \in \mathcal{T}$ is T_0 and take any distinct $x, y \in X$. There exists $U \in \nu_0$ such that $U \cap \{x, y\}$ has exactly one element. Since $U \in \tau$, this proves that τ is T_0. Suppose that any $\nu \in \mathcal{T}$ is T_1 and take any point $x \in X$. Since $\{x\}$ is closed in (X, ν_0), we have $X \setminus \{x\} \in \nu_0 \subset \tau$. Thus, $X \setminus \{x\}$ is open in (X, τ) and hence $\{x\}$ is closed in (X, τ).

If all elements of $\mathcal{T}$ are Hausdorff then, for any distinct $x, y \in X$, there exist sets $U, V \in \nu_0$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since $U, V \in \tau$, we proved that the space (X, τ) is also Hausdorff. Assume that all elements of $\mathcal{T}$ are T_3. Then τ is T_1 so we only have to show that τ is regular. Given a τ-closed $F \subset X$ and $x \in X \setminus F$, the set $V = X \setminus F$ is τ-open and $x \in V$. Since $\mathcal{T}$ is a subbase of τ, there are $\nu_1, \ldots, \nu_n \in \mathcal{T}$ such that $x \in U = U_1 \cap \cdots \cap U_n \subset V$ for some $U_1 \in \nu_1, \ldots, U_n \in \nu_n$. As ν_i is regular, there are disjoint $W_i, H_i \in \nu_i$ such that $x \in W_i$ and $X \setminus U_i \subset H_i$ for all $i \leqslant n$.

The sets $W = W_1 \cap \cdots \cap W_n$ and $H = H_1 \cup \cdots \cup H_n$ are τ-open, disjoint and $x \in W$, $F \subset H$ which proves that X is regular. Suppose, finally, that each $\nu \in \mathcal{T}$ is Tychonoff. Then (X, τ) is T_1 so we must only prove complete regularity of (X, τ).

Given a τ-closed set $F \subset X$ and a point $x \in X \setminus F$, observe that the set $V = X \setminus F$ is τ-open and $x \in V$. Since $\mathcal{T}$ is a subbase of τ, there exist sets $\nu_1, \ldots, \nu_n \in \mathcal{T}$ such that $x \in U = U_1 \cap \cdots \cap U_n \subset V$ for some $U_1 \in \nu_1, \ldots, U_n \in \nu_n$. As ν_i is completely regular, there are ν_i-continuous functions $f_i : X \to [0, 1]$ such that $f_i(x) = 1$ and $f_i(X \setminus U_i) \subset \{0\}$ for all $i \leqslant n$. Each f_i is also τ-continuous because $f_i^{-1}(H) \in \nu_i \subset \tau$ for each open $H \subset [0, 1]$. Therefore $f = f_1 \cdots f_n$ is τ-continuous, $f: X \to [0, 1]$ and $f(x) = 1$, $f(F) \subset \{0\}$ whence (X, τ) is a Tychonoff space.

S.100. (*Tychonoff spaces capture all of C_p-theory*). *Let X be a topological space. Given $x, y \in X$, define $x \equiv y$ to mean that $f(x) = f(y)$ for every $f \in C(X)$. Observe that this is an equivalence relation. Let X_c be the set of all equivalence classes. For each*

$f \in C(X)$, let us define $\varphi_f : X_c \to \mathbb{R}$ in the following way: $\varphi_f(y) = f(x)$, where x is an arbitrary point of y (remember that $y \subset X$ is an equivalence class). Given $x \in X$, let $\pi(x) = y$ where y is the equivalence class containing x.

(*i*) *Observe that the map φ_f is well defined for each $f \in C(X)$.*
(*ii*) *Denote by τ_f the topology $\varphi_f^{-1}(\mathcal{N}_R)$. Prove that the least upper bound τ of the topologies $\{\tau_f : f \in C(X)\}$ is a Tychonoff topology on X_c.*
(*iii*) *Show that the mapping $\pi : X \to Y = (X_c, \tau)$ is continuous.*
(*iv*) *Prove that the spaces $C_p(X)$ and $C_p(Y)$ are topologically isomorphic.*

Solution. Clearly, $x \equiv x$ and $x \equiv y$ implies $y \equiv x$ for all $x, y \in X$. If $x \equiv y$ and $y \equiv z$ then, for any $f \in C(X)$, we have $f(x) = f(y)$ and $f(y) = f(z)$ whence $f(x) = f(z)$. Thus, $\equiv$ is an equivalence relation. Now, $X_c = \{[x] : x \in X\}$, where $[x] = \{y \in X : y \equiv x\}$ for all $x \in X$. Thus $\pi(x) = [x]$ for any $x \in X$.

(i) The map φ_f is well defined if the value of $\varphi_f(y)$ is well defined for every $y \in X_c$. This happens if and only if $f(z) = f(x)$ for any $x, z \in y$. But $x, z \in y$ implies $x \equiv z$ and hence $f(x) = f(z)$ so the value of $\varphi_f(y)$ is well defined.
(ii) The topology τ_f is completely regular for each $f \in C(X)$ (Problem 098) and hence τ is completely regular by Problem 099. Observe also that φ_f is τ-continuous on X_c for each $f \in C(X)$ (Problem 097). Now take any $y \in X_c$. Then $y = [x]$ for any $x \in y$ so fix such x. If $y' \in X_c \setminus \{y\}$ then $y' = [x']$ for some $x' \in X$ such that x' ò x. This implies that there is $f \in C(X)$ such that $f(x) \neq f(x')$. As a consequence, $\varphi_f(y) \neq \varphi_f(y')$. Take any open $U, U' \subset \mathbb{R}$ such that $f(x) \in U$, $f(x') \in U'$ and $U \cap U' = \emptyset$ (they exist because $\mathbb{R}$ is Hausdorff (Problem 019)). Then $y \in \varphi_f^{-1}(U)$, $y' \in \varphi_f^{-1}(U')$, the sets $\varphi_f^{-1}(U)$, $\varphi_f^{-1}(U')$ are open in (X_c, τ) and disjoint. This shows that (X_c, τ) is Hausdorff and hence Tychonoff.
(iii) Let us prove continuity of π at an arbitrary point $x \in X$. Suppose that $y = [x] = \pi(x) \in U \in \tau$. Since $\bigcup\{\tau_f : f \in C(X)\}$ is a subbase of τ, there are $f_1, \dots, f_n \in C(X)$ and $O_1, \dots, O_n \in \mathcal{N}_R$ such that $y \in V \subset U$ where $V = \bigcap\{\varphi_{f_i}^{-1}(O_i) : i \leqslant n\}$. It suffices to prove that $x \in W$ and $\pi(W) \subset U$ for the open set $W = \bigcap\{f_i^{-1}(O_i) : i \leqslant n\}$. Note first that $f_i(x) = \varphi_{f_i}(y) \in O_i$ for all $i \leqslant n$ and hence $x \in W$. Now, if $x' \in W$ and $y' = \pi(x')$ then $\varphi_{f_i}(y') = f_i(x') \in O_i$. As a consequence $y' \in \varphi_{f_i}^{-1}(O_i)$ for each $i \leqslant n$ and hence $\pi(x') = y' \in V \subset U$.
(iv) Given $f \in C_p(X)$, let $\varphi(f) = \varphi_f \in C_p(X_c)$. It suffices to show that φ is a topological isomorphism. If $f \neq g$ then take any $x \in X$ with $f(x) \neq g(x)$. Then $\varphi_f([x]) = f(x) \neq g(x) = \varphi_g([x])$ and hence $\varphi(f) \neq \varphi(g)$. Therefore, φ is an injection. Given any $\xi \in C(X_c)$, let $f = \xi \circ \pi$. Then $f \in C(X)$ and $\varphi(f) = \xi$ so φ is a bijection. Assume that $f, g \in C_p(X)$ and $h = f + g$. Then $\varphi(h)([x]) = h(x) = f(x) + g(x) = \varphi(f)([x]) + \varphi(g)([x])$ for all $x \in X$ which proves that $\varphi(f + g) = \varphi(f) + \varphi(g)$. The proof of the equality $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$ is identical. Let us finally prove that φ is a homeomorphism. The family $\mathcal{B} = \{[y, O] : y \in X_c$ and O is a rational open interval$\}$ is a subbase in $C_p(X_c)$ (Problem 057) so, to prove continuity of φ, it suffices to establish that $\varphi^{-1}(U)$ is open in $C_p(X)$ for any $U = [y, O] \in \mathcal{B}$ (Problem 009(iii)). Pick any $x \in y$ and note that

$\varphi^{-1}(U) = V = \{f \in C_p(X) : f(x) \in O\}$. Indeed, $f \in U$ iff $\varphi_f(y) = f(x) \in O$ which happens if and only if $f \in V$. Thus φ is continuous. The family $\mathcal{C} = \{[x, O] : x \in X$ and O is a rational open interval$\}$ is a subbase in $C_p(X)$ (057) so, to prove continuity of φ^{-1}, it is sufficient to show that $(\varphi^{-1})^{-1}(U) = \varphi(U)$ is open in $C_p(X_c)$ for any $U = [x, O] \in \mathcal{C}$. This will follow from the fact that $\varphi(U) = V = \{\xi \in C_p(X_c) : \xi([x]) \in O\}$ because V is a standard open subset of $C_p(X_c)$. Now, $\xi = \varphi(f) \in V$ iff $\xi([x]) = f(x) \in O$ which happens iff $f \in U$ and our proof is complete.

S.101. *Suppose that X_t is a space for each index $t \in T$. Prove that the family $\mathcal{B} = \{\prod_{t\in T} U_t : U_t \in \tau(X_t)$ for all t, and the set $\{t \in T : U_t \neq X_t\}$ is finite$\}$ is a base for the space $X = \prod_{t\in T} X_t$. It is called the canonical (or standard) base of the product $\prod_{t\in T} X_t$.*

Solution. Let $\tau_t = p_t^{-1}(\tau(X_t))$ for each $t \in T$. Since $\nu = \cup\{\tau_t : t \in T\}$ generates the topology of X as a subbase, the family $\mathcal{B}' = \{U_1 \cap \cdots \cap U_n : n \in \mathbb{N}, U_i \in \nu$ for all $i \leqslant n\}$ is a base of X. If $U = U_1 \cap \cdots \cap U_n \in \mathcal{B}'$ then $U_i \in \tau_{t_i}$ for each $i \leqslant n$. Choose some faithful ($\equiv$ without repetitions) enumeration $\{s_1, \ldots, s_k\}$ of the set $A = \{t_1, \ldots, t_n\}$ and, for each $i \leqslant k$, denote by $\{t^i_1, \ldots, t^i_{m_i}\}$ the set of all $t \in A$ such that $t = s_i$. If $V_i = U_{t^i_2} \cap \cdots \cap U_{t^i_{m_i}}$ for all $i \leqslant k$ then $V_i \in \tau_{s_i}$ and $U = V_1 \cap \cdots \cap V_k$, where $s_i \neq s_j$ if $i \neq j$. This shows that we may assume without loss of generality that $\mathcal{B}' = \{U_1 \cap \cdots \cap U_n : n \in \mathbb{N}, U_i \in \tau_{t_i}$ for all $i \leqslant n$ and $t_i \neq t_j$ if $i \neq j\}$. In what follows, this is assumed for all elements of $\mathcal{B}'$.

For any $U = U_1 \cap \cdots \cap U_n \in \mathcal{B}'$ pick $V_i \in \tau(X_{t_i})$ such that $p_{t_i}^{-1}(V_i)\nu_i$ and observe that $U = \prod_{t\in T} U_t$ where $U_t = X_t$ if $t \notin \{t_1, \ldots, t_n\}$ and $U_t = V_i$ if $t = t_i$. As a consequence, every element of $\mathcal{B}'$ belongs to $\mathcal{B}$. Given a set $U = \prod_{t\in T} U_t \in \mathcal{B}$, let $\text{supp}(U) = \{t \in T : U_t \neq X_t\}$. The set $\text{supp}(U)$ can be empty; in this case $U = X$ and the set U is trivially open. If $\{t_1, \ldots, t_n\}$ is a faithful enumeration of supp $(U) \neq \emptyset$ then $U = U_1 \cap \cdots \cap U_n$ where $U_i = p_{t_i}^{-1}(U_{t_i})$ for all $i \leqslant n$. Therefore, $U \in \mathcal{B}'$ and hence $\mathcal{B} = \mathcal{B}'$ is a base of X.

S.102. *Suppose X_t is a space for any $t \in T$ and we are given a space Y together with a map $f : Y \to \prod_{t\in T} X_t$. Prove that f is continuous if and only if $p_t \circ f$ is continuous for any $t \in T$.*

Solution. If the map f is continuous then so is $p_t \circ f$ for each $t \in T$ because p_t is continuous. Now suppose that $p_t \circ f$ is continuous for all $t \in T$. To show that f is continuous, it suffices to find a subbase $\mathcal{S}$ in X such that $f^{-1}(U) \in \tau(Y)$ for any set $U \in \mathcal{S}$. By definition of the product topology, the family $\mathcal{S} = \cup\{p_t^{-1}(\tau(X_t)) : t \in T\}$ is a subbase of X. If $U \in \mathcal{S}$ then $U = p_t^{-1}(V)$ for some $t \in T$ and $V \in \tau(X_t)$. Thus $f^{-1}(U) = (p_t \circ f)^{-1}(V)$ is open in Y because $p_t \circ f$ is continuous.

S.103. *Let $\{X_t : t \in T\}$ be a family of topological spaces and suppose that $T = \bigcup\{T_s : s \in S\}$, where $T_s \neq \emptyset$ for all $s \in S$, and $T_s \cap T_{s'} = \emptyset$ if $s \neq s'$. Prove that the spaces $\prod_{t\in T} X_t$ and $\prod_{s\in S} (\prod_{t\in T_s} X_t)$ are homeomorphic, i.e., the topological product is associative.*

Solution. Let $X = \prod_{t\in T} X_t$ and $Y = \prod_{s\in S} (\prod_{t\in T_s} X_t)$. Given $x \in X$, define $\varphi(x)(s) = y_s(x) = x|T_s$ for each $s \in S$. It is immediate that $y_s(x) \in \prod_{t\in T_s} X_t$ for every $s \in S$ and therefore $\varphi(x) \in Y$. Let us prove that the map $\varphi : X \to Y$ is a homeomorphism. If $x \neq x'$ then there is $t \in T$ such that $x(t) \neq x'(t)$. Now, there exists $s \in S$ with $t \in T_s$ because $T = \bigcup\{T_s : s \in S\}$.

As a consequence, $\varphi(x)(s) = y_s(x) \neq y_s(x') = \varphi(x')(s)$ and $\varphi(x) \neq \varphi(x')$ whence φ is an injection. Given $y \in Y$, define $x \in X$ as follows: if $t \in T$, find the unique $s \in S$ with $t \in T_s$ (such s is unique because the sets T_s are disjoint) and let $x(t) = y(s)(t)$. It is immediate that $\varphi(x) = y$ and hence φ is a bijection. The last thing we must prove is continuity of φ and φ^{-1}. Let $p_t : X \to X_t$ be the natural projection. Denote by q_s the natural projection of Y onto $Y_s = \prod_{t\in T_s} X_t$ for each $s \in S$ and let $r_{st} : Y_s \to X_t$ also be the natural projection for all $s \in S$ and $t \in T_s$.

The map φ is continuous if and only if the composition $q_s \circ \varphi : X \to Y_s$ is continuous for all $s \in S$ (Problem 102). Since Y_s is also a product, the map $q_s \circ \varphi$ is continuous if and only if $r_{st} \circ q_s \circ \varphi$ is continuous for each $t \in T_s$. Note that $r_{st} \circ q_s \circ \varphi(x) = r_{st}(x|T_s) = (x|T_s)(t) = x(t)$ for any $x \in X$ and hence the map $r_{st} \circ q_s \circ \varphi$ is continuous because it coincides with the continuous map p_t. As a consequence, the map φ is continuous. Given $t \in T$, there is a unique $s \in S$ such that $t \in T_s$. It is immediate that $p_t \circ \varphi^{-1} : Y \to X_t$ coincides with the continuous map $r_{st} \circ q_s$ and hence φ^{-1} is also continuous.

S.104. *Let $\{X_t : t \in T\}$ be a family of spaces. Suppose that $\varphi : T \to T$ is a bijection. Prove that the spaces $\prod_{t\in T} X_t$ and $\prod_{t\in T} X_{\varphi(t)}$ are homeomorphic, i.e., the topological product is commutative.*

Solution. Let $Y_t = X_{\varphi(t)}$ for each $t \in T$. Our aim is to prove that the spaces $X = \prod_{t\in T} X_t$ and $Y = \prod_{t\in T} Y_t$ are homeomorphic. Denote by $p_t : X \to X_t$ and $q_t : Y \to Y_t$ the respective natural projections. Given $x \in X$, for the function $\delta(x) = x \circ \varphi$ we have $\delta(x)(t) = x(\varphi(t)) \in Y_t$ for each $t \in T$ and therefore $\delta(x) \in Y$, i.e., $\delta : X \to Y$. Given any $y \in Y$, for the function $\mu(y) = y \circ \varphi^{-1}$ we have $\mu(y)(t) = y(\varphi^{-1}(t)) \in Y_{\varphi^{-1}(t)} = X_t$. This shows that $\mu : Y \to X$ and it is clear that μ is the inverse function for δ and vice versa. To see that δ is continuous, note that $q_t \circ \delta$ is continuous for each $t \in T$ because $q_t \circ \delta = p_{\varphi(t)}$. Analogously, $\mu = \delta^{-1}$ is continuous because $p_t \circ \mu = q_{\varphi^{-1}(t)}$ is a continuous map for each $t \in T$. Therefore $\delta : X \to Y$ is a homeomorphism.

S.105. *Show that, for every $i \in \{0, 1, 2, 3, 3\frac{1}{2}\}$, the Tychonoff product of any family of T_i-spaces is a T_i-space.*

Solution. Let $X = \prod_{t\in T} X_t$. (i) Suppose that X_t is a T_0-space for each $t \in T$. Given distinct $x, y \in X$, fix any $t \in T$ with $x(t) \neq y(t)$. There exists $U \in \tau(X_t)$ such that $U \cap \{(x(t), y(t)\}$ consists of exactly one point. Then $V = p_t^{-1}(U)$ is open in X and $V \cap \{x, y\}$ consists of exactly one point. Therefore X is a T_0-space.

(ii) Assume that X_t is a T_1-space for all $t \in T$. Take any point $x \in X$. For any point $y \in X \setminus \{x\}$ find $t \in T$ with $x(t) \neq y(t)$. Since the space X_t is T_1, the set $U = X_t \setminus \{x(t)\}$ is open and $y(t) \in U$. Therefore $V_y = p_t^{-1}(U)$ is an open set in X and $y \in V_y \subset X$

$\backslash\{x\}$. Thus $X \backslash \{x\} = \cup\{V_y : y \in X\backslash\{x\}\}$ is an open set. As a consequence, $\{x\}$ is closed in X.

(iii) If all X_t's are Hausdorff, take distinct $x, y \in X$. Pick any $t \in T$ with $x(t) \neq y(t)$. There are $U', V' \in \tau(X_t)$ such that $x(t) \in U'$, $y(t) \in V'$ and $U' \cap V' = \emptyset$. If $U = p_t^{-1}(U')$ and $V = p_t^{-1}(V')$ then $U, V \in \tau(X)$, $x \in U$, $y \in V$ and $U \cap V = \emptyset$. This means X is a Hausdorff space.

(iv) Let X_t be a T_3-space for each $t \in T$. By (ii), the space X is T_1 so we only must prove regularity of X. Take any $x \in X$ and any closed $F \subset X$ such that $x \notin F$. There exist $t_1, \ldots, t_n \in T$ and $O_i \in \tau(X_{t_i})$ such that $x \in U \subset X\backslash F$ where $U = \bigcap\{p_{t_i}^{-1}(O_i) : i \leqslant n\}$. Let $x_i = p_{t_i}(x)$ for each $i \leqslant n$. Then $x_i \in O_i$ and by regularity of X_{t_i} there are $V_i, W_i \in \tau(X_{t_i})$ such that $x_i \in V_i$, $X_{t_i}\backslash O_i \subset W_i$ and $V_i \cap W_i = \emptyset$ for all $i \leqslant n$. The sets $G = \bigcap\{p_{t_i}^{-1}(V_i) : i \leqslant n\}$ and $H = \cup\{p_{t_i}^{-1}(W_i) : i \leqslant n\}$ are open in X and $G \cap H = \emptyset$. Since $x_i \in V_i$ for all $i \leqslant n$, we have $x \in G$. If $y \in F$ then $y \notin U$ and therefore $y \notin p_{t_i}^{-1}$ for some $i \leqslant n$. This implies $p_{t_i}(y) \in W_i$ and hence $y \in H$. This proves that $F \subset H$ and X is regular.

(v) Let us tackle the case when X_t is a Tychonoff space for each $t \in T$. By (ii) the space X is T_1 so we must only prove complete regularity of X. Take any $x \in X$ and any closed $F \subset X$ such that $x \notin F$. There exist $t_1, \ldots, t_n \in T$ and $O_i \in \tau(X_{t_i})$ such that $x \in U \subset X\backslash F$ where $U = \cap\{p_{t_i}^{-1}(O_i) : i \leqslant n\}$. Let $x_i = p_{t_i}(x)$ for each $i \leqslant n$. Then $x_i \in O_i$ and by complete regularity of X_{t_i} there are continuous functions $f_i : X_{t_i} \to [0, 1]$ such that $f_i(x_i) = 1$ and $f_i(X_{t_i}\backslash O_i) \subset \{0\}$ for all $i \leqslant n$. Now, let $g_i = f_i \circ p_{t_i}$ for all $i \leqslant n$. Then $g_i : X \to [0, 1]$ is a continuous function and $g_i(x) = 1$ for each $i \leqslant n$. As a consequence, $f = g_1 \cdot \ldots \cdot g_n : X \to [0, 1]$ is a continuous function on X and $f(x) = 1$. If $y \in F$ then $y \notin U$ which implies $p_{t_i}(y) \in X_{t_i}\backslash O_i$ for some $i \leqslant n$. By the choice of f_i, we have $g_i(y) = f_i(p_{t_i}(y)) = 0$ and therefore $f(y) = 0$. This proves that $f(F) \subset \{0\}$ and hence the space X is Tychonoff.

S.106. *Show that, for any non-empty topological product $X = \prod_{t\in T} X_t$ and any $s \in T$, the space X has a closed subspace homeomorphic to X_s.*

Solution. Pick a point $a_t \in X_t$ for any $t \in T$. Given $x \in X_s$, let $\varphi(x)(t) = a_t$ if $t \neq s$ and $\varphi(x)(s) = x$. It is clear that $\varphi : X_s \to X$. Denote the set $\varphi(X_s)$ by Y. Let us prove that the map $\varphi : X_s \to Y$ is a homeomorphism. If $x \neq y$ then $\varphi(x)(s) = x \neq y = \varphi(y)(s)$ which shows that $\varphi : X_s \to Y$ is a bijection. If $t \neq s$ then $p_t \circ \varphi(x) = a_t$ for each $x \in X_s$. As a consequence $p_t \circ \varphi$ is continuous. Now $p_s \circ \varphi$ is also continuous because $p_s \circ \varphi(x) = x$ for any $x \in X_s$. This shows that φ is continuous (Problem 102). It is easy to see that φ^{-1} is also continuous because it coincides with the function $p_s|Y$. This proves that φ is a homeomorphism. To prove that Y is closed in X, take any $f \in X\backslash Y$. If $f(t) \neq a_t$ for some $t \in T\backslash\{s\}$ then $U_f = p_t^{-1}(X_t\backslash\{a_t\})$ is an open set in X with $f \in U_f \subset X\backslash Y$. If $f(t) = a_t$ for all $t \in T\backslash\{s\}$ then $f(s) = x \in X_s$ and hence $f = \varphi(x) \in Y$, a contradiction. Thus, for every $f \in X \backslash Y$, there is an open U_f such that $f \in U_f \subset X\backslash Y$. Therefore $X\backslash Y = \cup\{U_f : f \in X \backslash Y\}$ is an open subset of X.

S.107. *Given a topological product $X = \prod\{X_t : t \in T\}$ and $S \subset T$, define the S-face of X to be the product $X_S = \prod\{X_t : t \in S\}$. Let $p_S : X \to X_S$ be defined by the formula $p_S(x) = x|S$ for every $x \in X$. The map p_S is called a natural projection of X onto its* *face X_S. Prove that the natural projection onto any face is a continuous open map. In particular, the projections to the factors of any product are continuous open maps.*

Solution. We will use the following easy observation. If $f : T \to Z$ is a continuous onto map then f is open if and only if T has a base $\mathcal{T}$ such that $f(U)$ is open for each $U \in \mathcal{T}$. It was proved in Problem 101 that the family $\mathcal{B} = \{\prod_{t\in T} U_t : U_t \in \tau(X_t)$ for all t, and the set $\{t \in T : U_t \neq X_t\}$ is finite$\}$ is a base for the space X so it suffices to show that $p_S(U)$ is open in X_S for any $U \in \mathcal{B}$. Given $U = \prod_{t\in T} U_t \in \mathcal{B}$, let $\mathrm{supp}(U) = \{t \in T : U_t \neq X_t\}$. Observe that $p_S(U) = \prod_{t\in S} U_t$ where $U_t \in \tau(X_t)$ for all $t \in S$ and $U_t \neq X_t$ only for $t \in \mathrm{supp}(U) \cap S$ which is a finite set. Therefore $p_S(U)$ is open in X_S (101) and we established that the map p_S is open.

Let $q_t : X_S \to X_t$ be the natural projection for any $t \in S$. The map $p_t : X \to X_t$ is the respective natural projection in X. To see that p_S is continuous observe that $q_t \circ p_S = p_t$ is a continuous map for every $t \in S$ and hence p_S is continuous (Problem 102). It is evident that p_S is onto so our solution is complete.

S.108. *(The Hewitt–Marczewski–Pondiczery theorem) Given an infinite cardinal κ, suppose that $|T| \leqslant 2^\kappa$ and X_t is a space such that $d(X_t) \leqslant \kappa$ for all $t \in T$. Prove that $d(\prod_{t\in T} X_t) \leqslant \kappa$. In particular, the product of at most 2^ω-many separable spaces is separable.*

Solution. Observe that if Y is a continuous image of X then $d(Y) \leqslant d(X)$. If Z is a dense subspace of X then $d(X) \leqslant d(Z)$; this follows from the fact that any dense subspace of Z is also dense in X. Fix a set $A_t \subset X_t = \overline{A_t}$ with $|A_t| \leqslant \kappa$ for each $t \in T$. Take a surjection $f_t : D(\kappa) \to A_t$ for each $t \in T$. Since any map defined on a discrete space is continuous, f_t is continuous for all $t \in T$. Given $x \in D(\kappa)^T$, let $\varphi(x)(t) = f_t(x(t))$ for all $t \in T$. Then $\varphi : D(\kappa)^T \to X = \prod_{t\in T} X_t$.

We leave to the reader the boring verification of the fact that φ is a continuous map and $\varphi(D(\kappa)^T) = A = \prod_{t\in T} A_t$. It is another easy exercise for the reader to check that A is dense in X and therefore $d(X) \leqslant d(A) \leqslant d(D(\kappa)^T)$. As a consequence, it suffices to prove that $D(\kappa)^T$ has density $\leqslant \kappa$. The space $D = D(2)$ is Tychonoff and hence D^κ is also Tychonoff by Problem 105. The family $\mathcal{B} = \{\prod_{t\in\kappa} U_t : U_t \in \tau(D)$ for all t, and the set $\{t \in T : U_t \neq D\}$ is finite$\}$ is a base for the space D^κ. Since there are only κ-many finite subsets of κ and $\tau(D)$ is finite, we have $|\mathcal{B}| = \kappa$ and hence $w(D^\kappa) \leqslant \kappa$. It is trivial that $w(Y) \leqslant w(D^\kappa)$ for any $Y \subset D^\kappa$. Now, $|D^\kappa| = 2^\kappa$ and hence there exists an injection $i : T \to D^\kappa$.

We have a topology $\mu = i^{-1}(\tau(D^\kappa))$ on the set T which is Tychonoff and has weight $\leqslant \kappa$ (see Problem 097). Fix any base $\mathcal{C}$ of (T, μ) of cardinality $\leqslant \kappa$. Consider the set $B = \{x \in D(\kappa)^T$: there exists a finite disjoint family $\gamma \subset \mathcal{C}$ such that $x(U)$ has at most one element for all $U \in \gamma$ and the same is true for $x(T\backslash\bigcup\gamma)\}$. In other words, B consists of functions on T which are constant on the elements of some disjoint finite $\gamma \subset \mathcal{C}$ as well as on the set $T \setminus \bigcup\gamma$. It takes some routine cardinal arithmetic to prove that $|B| \leqslant \kappa$. Let us show that B is dense in $D(\kappa)^T$.

Take any non-empty basic open set $U = \prod_{t\in T} U_t$ where $\mathrm{supp}(U) = \{t \in T : U_t \neq D(\kappa)\}$ is a finite set. If $\mathrm{supp}(U) = \emptyset$ then, evidently, $U \cap B \neq \emptyset$, so suppose that

$\text{supp}(U) = \{t_1, \ldots, t_n\}$. For each $i \leqslant n$, choose $u_i \in U_{t_i}$ and fix some $a \in D(\kappa)$. Since (T, μ) is a Tychonoff space, there is a disjoint $\gamma = \{C_1, \ldots, C_n\} \subset \mathcal{C}$ such that $t_i \in C_i$ for each $i \leqslant n$. Given $t \in T$, let $x(t) = u_i$ if $t \in C_i$ for some $i \leqslant n$ and $x(t) = a$ for all indices $t \in D(\kappa) \setminus \cup\gamma$. It is clear that $x \in B \cap U$ and hence $U \cap B \neq \emptyset$ for every basic open set U. Therefore, B is dense in $D(\kappa)^T$ and hence $d(D(\kappa)^T) \leqslant |B| \leqslant \kappa$.

S.109. *Prove that any product of separable spaces has the Souslin property. In particular, the space $\mathbb{R}^A$ has the Souslin property for any set A.*

Solution. Let X_t be a separable space for each $t \in T$. Assume that the space $X = \prod_{t\in T} X_t$ does not have the Souslin property and fix a family $\{U_\alpha : \alpha < \omega_1\}$ of non-empty disjoint open subsets of X. We may assume (taking smaller sets if necessary) that every U_α is an element of the standard open base of X, namely $U_\alpha = \prod\{U_t^\alpha : t \in T\}$ where $\text{supp}(U_\alpha) = \{t \in T : U_t^\alpha \neq X_t\}$ is finite for all $\alpha < \omega_1$. The set $S = \bigcup\{\text{supp}(U_\alpha) : \alpha < \omega_1\}$ has cardinality $\leqslant \omega_1 \leqslant 2^\omega$. The set $V_\alpha = p_S(U_\alpha)$ is non-empty and open in $X_S = \prod\{X_t : t \in S\}$ (Problem 107) and the family $\{V_\alpha : \alpha < \omega_1\}$ is disjoint.

Indeed, if $\alpha \neq \beta$ and $x \in V_\alpha \cap V_\beta$ then take any $x_t \in X_t$ for all $t \in T \setminus S$ and consider a function $y \in X$ defined by $y(t) = x(t)$ if $t \in S$ and $y(t) = x_t$ for all $t \in T \setminus S$. Note that $\text{supp}(U_\alpha) \cup \text{supp}(U_\beta) \subset S$ and we have $y(t) = x(t) \in U_t^\alpha$ and $y(t) = x(t) \in U_t^\beta$ for all $t \in S$. But we also have $y(t) \in U_t^\alpha$ and $y(t) \in U_t^\beta$ for all $t \in T \setminus S$ because $U_t^\alpha = U_t^\beta = X_t$ for all $t \in T\setminus S$. This shows that $y \in U_\alpha \cap U_\beta$ which is a contradiction.

Now apply Problem 108 to see that the space X_S is separable. If A is a countable dense subset of X_S then, for each $\alpha < \omega_1$, choose $a_\alpha \in V_\alpha \cap A$. If $\alpha \neq \beta$ then $V_\alpha \cap V_\beta = \emptyset$ which implies $a_\alpha \neq a_\beta$. Thus $\{a_\alpha : \alpha < \omega_1\}$ is an uncountable subset of a countable set A which is a contradiction. Finally, to see that $\mathbb{R}^A$ has the Souslin property, note that $\mathbb{R}$ is separable because the countable set $\mathbb{Q}$ is dense in $\mathbb{R}$.

S.110. *Suppose that X is a space and Y is a dense subspace X. Prove that $c(X) = c(Y)$.*

Solution. Suppose that γ is a disjoint family of non-empty open subsets of X. Then $\mu = \{U \cap Y : U \in \gamma\}$ is also a disjoint family of non-empty open subsets of Y with $|\mu| = |\gamma|$. Thus $c(X) \leqslant c(Y)$. On the other hand, if μ a disjoint family of non-empty open subsets of Y then, for each $U \in \mu$, take $W(U) \in \tau(X)$ with $W(U) \cap Y = U$. Observe that any non-empty open set of X has to meet Y because Y is dense in X. Now, if $W(U) \cap W(U') \neq \emptyset$ then the non-empty open set $W(U) \cap W(U')$ has to intersect Y and therefore $\emptyset \neq W(U) \cap W(U') \cap Y = U \cap U'$ which is a contradiction which proves that the family $\gamma = \{W(U) : U \in \mu\}$ is disjoint. Since $|\mu| = |\gamma|$, this shows that $c(Y) \leqslant c(X)$.

S.111. *Given spaces X and Y, note that $C(X, Y) \subset Y^X$. Prove that the topology of $C_p(X, Y)$ coincides with the topology induced in $C(X, Y)$ from the Tychonoff product Y^X. In particular, $C_p(X)$ is a subspace of $\mathbb{R}^X$. Prove that $C_p(X)$ dense in $\mathbb{R}^X$. Hence, for any space X, the space $C_p(X)$ has the Souslin property.*

Solution. By the definition of the product, the set Y^X consists of all mappings from X to Y. Thus $C(X, Y) \subset Y^X$. To see that the topology induced on $C(X, Y)$ from Y^X is precisely the topology of $C_p(X, Y)$, we will use the following evident fact. Suppose that Z is a space and $T \subset Z$ has a topology τ. Then τ is the subspace topology on T if and only if there exists a base $\mathcal{C}$ of Z such that the family $\mathcal{C}|T = \{U \cap T : U \in \mathcal{C}\}$ is a base for (T, τ). We leave to the reader the simple proof of this fact. Now if we take an element $U = \prod_{x \in X} U_x$ of the standard base $\mathcal{B}$ in Y^X (Problem 101) then $U \cap C(X, Y) = [x_1, \ldots, x_n; U_1, \ldots, U_n]$ where $\{x_1, \ldots, x_n\}$ is some enumeration of the finite set $\{x \in X : U_x \neq Y\}$ and $U_i = U_{x_i}$ for all $i \leqslant n$. If we denote by $\mathcal{B}'$ the standard base of $C_p(X, Y)$ then $\mathcal{B}|C(X, Y) \subset \mathcal{B}'$. On the other hand, if $V = [x_1, \ldots, x_n; O_1, \ldots, O_n] \in \mathcal{B}'$ then $U = \prod_{x \in X} U_x \in \mathcal{B}$ and $U \cap C(X, Y) = V$ if we take $U_{x_i} = O_i$ for all $i \leqslant n$ and $U_x = Y$ otherwise. Therefore $\mathcal{B}|C(X, Y) = \mathcal{B}'$ and we proved that the topology of $C_p(X, Y)$ is the topology of subspace of Y^X and, in particular, $C_p(X)$ is a subspace of $\mathbb{R}^X$.

To prove that $C_p(X)$ is dense in $\mathbb{R}^X$, take any non-empty standard open subset $U = \prod_{x \in X} U_x$ of the space $\mathbb{R}^X$. The set $A = \{x \in X : U_x \neq \mathbb{R}\}$ is finite and it is possible to choose $r_x \in U_x$ for each $x \in A$. Apply Problem 034 to find a function $f \in C_p(X)$ such that $f(x) = r_x$ for any $x \in A$. Then $f \in U \cap C_p(X)$ and therefore $C_p(X) \cap U \neq \emptyset$ for any standard open $U \subset \mathbb{R}^X$ which shows that $C_p(X)$ is dense in $\mathbb{R}^X$. The last part of the assertion of this problem follows from Problems 109 and 110.

S.112. *Let Y_t be a space for every $t \in T$. Show that, for any space X, the space $C_p(X, \prod_{t \in T} Y_t)$ is homeomorphic to $\prod\{C_p(X, Y_t) : t \in T\}$.*

Solution. Let us establish first the following general fact $(*)$. Given spaces Z and W, take any $z \in Z$ and define $\pi_z : W^Z \to W$ to be the natural projection, i.e., $\pi_z(f) = f(z)$ for all $f \in W^Z$. Since $C_p(Z, W) \subset W^Z$ (Problem 111), the map $\pi_z|C_p(Z, W)$ is continuous. For the spaces $Y = \prod_{t \in T} Y_t$ and $C = \prod\{C_p(X, Y_t) : t \in T\}$, let $p_t : Y \to Y_t$ and $q_t : C \to C_p(X, Y_t)$ be the respective natural projections. Given $x \in X$, we will also need the natural projection $r_x : Y^X \to Y$ defined by $r_x(f) = f(x)$ for all $f \in Y^X$.

For any $f \in C_p(X, Y)$, define $\varphi(f) \in C$ letting $\varphi(f)(t) = p_t \circ f$ for each $t \in T$. We are going to prove that the map $\varphi : C_p(X, Y) \to C$ is a homeomorphism. Fix any $t \in T$ and let $(p_t)_*(f) = p_t \circ f$. The mapping $(p_t)_* : C_p(X, Y) \to C_p(X, Y_t)$ is continuous by Problem 091. Since $q_t \circ \varphi = (p_t)_*$ for each t, the mapping φ is also continuous (see Problem 102). If $f, g \in C_p(X, Y)$ and $f \neq g$ then $f(x) \neq g(x)$ for some point $x \in X$. Furthermore, there is an index $t \in T$ such that $f(x)(t) \neq g(x)(t)$. Therefore, $\varphi(f)(t)(x) = p_t(f(x)) = f(x)(t) \neq g(x)(t) = p_t(g(x)) = \varphi(g)(t)(x)$ and hence $p_t \circ f \neq p_t \circ g$ which implies $\varphi(f) \neq \varphi(g)$.

To see that φ is surjective, take any $g \in C$. Then $g(t) \in C_p(X, Y_t)$ for each t so we can let $f(x)(t) = g(t)(x)$ for each $x \in X$. We have $f \in C_p(X, Y)$ by Problem 102 and $\varphi(f) = g$ whence φ is a continuous bijection. To establish that φ^{-1} is continuous, observe that φ^{-1} is also a map from C in Y^X (Problem 111) so it suffices to prove that $\varphi^{-1} : C \to Y^X$ is continuous (Problem 023). Continuity of φ^{-1} is equivalent to continuity of all compositions $r_x \circ \varphi^{-1}$ for $x \in X$ (Problem 102). Observe that $r_x \circ \varphi^{-1} : C \to Y$

and Y is also a product so continuity of $r_x \circ \varphi^{-1}$ is equivalent to continuity of $p_t \circ r_x \circ \varphi^{-1}$ for each $t \in T$. For any $f \in C$, we have $p_t \circ r_x \circ \varphi^{-1}(f) = f(t)(x)$ and therefore the map $p_t \circ r_x \circ \varphi^{-1}$ coincides with $\pi_x \circ q_t : C \to Y_t$ where $\pi_x(h) = h(x)$ for any $h \in C_p(X, Y_t)$. The map q_t is continuous and so is π_t by (*) which proves continuity of φ^{-1}.

S.113. *Suppose that X_t is a space for each $t \in T$ and let $X = \bigcup\{X_t \times \{t\} : t \in T\}$. For every $t \in T$, define the map $q_t : X_t \times \{t\} \to X_t$ by the formula $q_t(x, t) = x$ for each $x \in X_t$. If $U \subset X$, let $U \in \tau$ if $q_t(U \cap (X_t \times \{t\}))$ is open in X_t for all $t \in T$. Prove that τ is a topology on X. The space (X, τ) is denoted by $\bigoplus \{X_t : t \in T\}$ and is called the discrete (or free) union of the spaces X_t. Show that*

(*i*) *If $X_t \times \{t\}$ is given the topology of subspace of $\bigoplus\{X_t : t \in T\}$ then the map q_t is a homeomorphism for each t. Thus $X_t \times \{t\}$ is a copy of X_t.*
(*ii*) *Each $X_t \times \{t\}$ is a clopen ($\equiv$ closed-and-open) subset of $\bigoplus\{X_t : t \in T\}$.*
(*iii*) *If a space X can be represented as a union of a family $\{X_t : t \in T\}$ of pairwise disjoint open subsets of X, then X is homeomorphic to $\bigoplus\{X_t : t \in T\}$.*

Solution. If $U = \emptyset$ then $q_t(U \cap (X_t \times \{t\})) = \emptyset \in \tau(X_t)$ for each $t \in T$ and hence $\emptyset \in \tau$. If $U = X$ then $q_t(U \cap (X_t \times \{t\})) = X_t \in \tau(X_t)$ for each $t \in T$ which shows that $X \in \tau$. Given $U, V \in \tau$, we have

$$q_t(U \cap V \cap (X_t \times \{t\})) = q_t(U \cap (X_t \times \{t\})) \cap q_t(V \cap (X_t \times \{t\})) \in \tau(X_t)$$

because each one of the sets of the intersection belongs to $\tau(X_t)$. As a consequence, $U \cap V \in \tau$. Finally, if $\gamma \subset \tau$ then

$$q_t((\bigcup\gamma) \cap (X_t \times \{t\})) = \bigcup\{q_t(U \cap (X_t \times \{t\})) : U \in \gamma\} \in \tau(X_t)$$

for each $t \in T$ because every element of the union belongs to τ. Therefore $\cup\gamma \in \tau$ and we proved that τ is a topology on X. Denote by Y_t the set $X_t \times \{t\}$ for each $t \in T$.

(i) It is clear that q_t is a bijection. If $U \in \tau(X_t)$ then $W = q_t^{-1}(U) \cup (X \setminus Y_t)$ is an open set in X with $W \cap Y_t = q_t^{-1}(U)$. Therefore $q_t^{-1}(U) \in \tau(Y_t)$ and q_t is continuous. If U is open in Y_t then there is $W \in \tau$ such that $W \cap Y_t = U$. By definition of τ, we have $(q_t^{-1})^{-1}(U) = q_t(U) = q_t(W \cap Y_t)$ is an open set in X_t and hence q_t is a homeomorphism.
(ii) Each $U = Y_t$ is open because, for any $s \neq t$, we have $q_s(U \cap Y_s) = \emptyset \in \tau(X_s)$ and $q_t(U \cap Y_t) = q_t(Y_t) = X_t \in \tau(X_t)$. The set U is also closed being a complement of an open set $X \setminus U = \bigcup\{Y_s : s \in T \setminus \{t\}\}$.
(iii) Define a map $q : Y = \bigoplus\{X_t : t \in T\} \to X$ by the equality $q(x) = q_t(x)$ if $x \in Y_t = X_t \times \{t\}$. It is evident that q is a well-defined bijection. If U is open in Y then $q(U) = \cup\{q_t(U \cap Y_t) : t \in T\}$. The set $q_t(U \cap Y_t)$ is open in X_t and hence in X whence $q(U)$ is open in X and therefore q^{-1} is continuous. Finally, if V is open in X then $q_t(q^{-1}(V) \cap Y_t) = V \cap X_t$ is open in X_t for each $t \in T$ and hence the set $q^{-1}(V)$ is open in Y. This proves that q is also continuous.

S.114. *Suppose that $X = \bigoplus\{X_t : t \in T\}$. Prove that, for any space Y, the space $C_p(X, Y)$ is homeomorphic to the space $\prod\{C_p(X_t, Y) : t \in T\}$.*

Solution. Let us consider the restriction map $\pi_t : C_p(X, Y) \to C_p(X_t, Y)$ defined by $\pi_t(f) = f|X_t$ for all $t \in T$ and $f \in C_p(X, Y)$. Given an element $U = [x_1, \ldots, x_n;$

$O_1, \ldots, O_n]$ of the standard base of the space $C_p(X_t, Y)$, it is easy to convince ourselves that $\pi_t^{-1}(U) = \{f \in C_p(X, Y) : f(x_i) \in O_i$, for all $i \leqslant n\} \in \tau(C_p(X, Y))$ and hence π_t is continuous. For any $f \in C_p(X, Y)$, define $\varphi(f) \in C = \prod\{C_p(X_t, Y) : t \in T\}$ by the equality $\varphi(f)(t) = \pi_t(f) = f|X_t$. If $q_t : C \to C_p(X_t, Y)$ is the respective natural projection then $q_t \circ \varphi = \pi_t$ is a continuous map for any $t \in T$ and therefore the map $\varphi : C_p(X, Y) \to C$ is continuous. If $f, g \in C_p(X, Y)$, $f \neq g$ then pick any point $x \in X$ with $f(x) \neq g(x)$ and $t \in T$ such that $x \in X_t$. Then $f|X_t \neq g|X_t$, i.e., $\pi_t(f) \neq \pi_t(g)$ which implies $\varphi(f) \neq \varphi(g)$ and hence φ is an injection. Given any $g \in C$ define a function $f \in C_p(X, Y)$ as follows: for any $x \in X$ find the unique $t \in T$ with $x \in X_t$ and let $f(x) = g(t)(x)$. We omit the simple verification of the fact that f is continuous and $\varphi(f) = g$ which shows that φ is a bijection. To prove that φ is a homeomorphism, we must show that φ^{-1} is continuous. Let us consider φ^{-1} as a map from C to Y^X (Problem 111). By Problem 102, continuity of φ^{-1} is equivalent to continuity of the map $r_x \circ \varphi^{-1}$ for all $x \in X$ where $r_x(h) = h(x)$ for all $h \in Y^X$. So take any $x \in X$ and $t \in T$ with $x \in X_t$. Then $r_x \circ \varphi^{-1}(f) = f(t)(x) = q_t(f)(x) = s_x \circ q_t(f)$ where $s_x : C_p(X_t, Y) \to Y$ is defined by $s_x(h) = h(x)$ for all $h \in C_p(X_t, Y)$. The map s_x is continuous being the restriction of the respective natural projection of Y^{X_t} to $C_p(X_t, Y)$. Therefore $r_x \circ \varphi^{-1} = s_x \circ q_t$ is also continuous which proves continuity of φ^{-1}.

S.115. *Given a space X, define the map* sm $: C_p(X) \times C_p(X) \to C_p(X)$ *by the equality* sm$(f, g) = f + g$ *for any* $f, g \in C_p(X)$. *Prove that the map* sm *is continuous.*

Solution. Take any functions $f_0, g_0 \in C_p(X)$. If $h_0 = f_0 + g_0 \in U \in \tau(C_p(X))$ then there exist $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $O(h_0, x_1, \ldots, x_n, \varepsilon) \subset U$. The sets $V = O(f_0, x_1, \ldots, x_n, \frac{\varepsilon}{2})$ and $W = O(g_0, x_1, \ldots, x_n, \frac{\varepsilon}{2})$ are open in $C_p(X)$ and therefore $(f_0, g_0) \in V \times W \in \tau(C_p(X) \times C_p(X))$. Given $f \in V$ and $g \in W$, we have $|f(x_i) + g(x_i) - h_0(x_i)| \leqslant |f(x_i) - f_0(x_i)| + |g(x_i) - g_0(x_i)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $i \leqslant n$. As a consequence, $f + g \in O(h_0, x_1, \ldots, x_n, \varepsilon)$ and we established that sm$(V \times W) = V + W \subset O(h_0, x_1, \ldots, x_n, \varepsilon) \subset U$ and hence the map sm is continuous at the point (f_0, g_0).

S.116. *Given a space X, define the map* pr $: C_p(X) \times C_p(X) \to C_p(X)$ *by the equality pr*$(f, g) = f \cdot g$ *for any* $f, g \in C_p(X)$. *Prove that the map* pr *is continuous.*

Solution. Take any functions $f_0, g_0 \in C_p(X)$. If $h_0 = f_0 \cdot g_0 \in U \in \tau(C_p(X))$ then there exist $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $O(h_0, x_1, \ldots, x_n, \varepsilon) \subset U$. Let $M = \sum_{i=1}^n |f_0(x_i)| + \sum_{i=1}^n |g_0(x_i)| + 2$ and $\delta = \min\{\frac{\varepsilon}{2M}, 1\}$. Observe that the sets $V = O(f_0, x_1, \ldots, x_n, \delta)$ and $W = O(g_0, x_1, \ldots, x_n, \delta)$ are open in $C_p(X)$ and therefore $(f_0, g_0) \in V \times W \in \tau(C_p(X) \times C_p(X))$. If we are given $f \in V$ and $g \in W$, we have $|g(x_i)| < 1 + |g_0(x_i)| < M$ for each $i = 1, \ldots, n$ and hence $|f(x_i) \cdot g(x_i) - h_0(x_i)| = |g(x_i) \cdot (f(x_i) - f_0(x_i)) + f_0(x_i) \cdot (g(x_i) - g_0(x_i))| \leqslant |g(x_i)||f(x_i) - f_0(x_i)| + |f_0(x_i)||g(x_i) - g_0(x_i)| < M \cdot \delta + M \cdot \delta \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $i \leqslant n$. As a consequence, $f \cdot g \in O(h_0, x_1, \ldots, x_n, \varepsilon)$ and we established that pr$(V \times W) = V \cdot W \subset O(h_0, x_1, \ldots, x_n, \varepsilon) \subset U$ and hence the map pr is continuous at the point (f_0, g_0).

S.117. *Let X be an arbitrary set. Given a family $\mathcal{F} \subset \exp(X)$ with a property $\mathcal{P}$, we say that $\mathcal{F}$ is a maximal family with the property $\mathcal{P}$, if $\mathcal{F}$ has $\mathcal{P}$ and for any $\gamma \subset \exp(X)$ with the property $\mathcal{P}$, we have $\gamma = \mathcal{F}$ whenever $\mathcal{F} \subset \gamma$. Prove that*

(*i*) *Any filter is a filter base and any filter base is a centered family.*
(*ii*) *For any centered family $\mathcal{C}$ on X, there is a filter $\mathcal{F}$ on X such that $\mathcal{C} \subset \mathcal{F}$.*
(*iii*) *If $\mathcal{F}$ is a filter on X then there is a maximal filter $\mathcal{U}$ on X such that $\mathcal{F} \subset \mathcal{U}$. A maximal filter is called ultrafilter, so applying (ii), this statement could be formulated as follows: every centered family on X is contained in an ultrafilter on X.*
(*iv*) *A family $\mathcal{U} \subset \exp(X)$ is an ultrafilter if and only if it is a maximal centered family. As a consequence, any centered family on X is contained in a maximal centered family on X.*
(*v*) *A family $\mathcal{U} \subset \exp(X)$ is an ultrafilter if and only if it is a centered family and, for any $A \subset X$, we have $A \in \mathcal{U}$ or $X \backslash A \in \mathcal{U}$.*
(*vi*) *If X is a topological space and $\gamma \subset \tau^*(X)$ is disjoint then there is a maximal disjoint $\mu \subset \tau^*(X)$ such that $\gamma \subset \mu$.*
(*vii*) *There are no maximal point-finite families of non-empty open subsets of $\mathbb{R}$.*

Solution. (i) To see that any filter $\mathcal{F}$ is a filter base observe that, for any sets $A, B \in \mathcal{F}$, the set $C = A \cap B \subset A \cap B$ belongs to $\mathcal{F}$. Now, if $\mathcal{B}$ is a filter base and $B_1, \ldots, B_n \in \mathcal{B}$ then it is easy to prove by induction on n that there is $C \in \mathcal{B}$ such that $C \subset B_1 \cap \cdots \cap B_n$. Since $C \neq \emptyset$, this proves that $\mathcal{B}$ is centered.

(ii) Consider the family $\mathcal{F} = \{F \subset X : \text{there exist } n \in \mathbb{N} \text{ and } C_1, \ldots, C_n \in \mathcal{C} \text{ with } C_1 \cap \cdots \cap C_n \subset F\}$. It is immediate that $\mathcal{C} \subset \mathcal{F}$ and $\mathcal{F}$ is a filter.

(iii) We will apply the Zorn's Lemma which says that any partially ordered set in which every chain has an upper bound, has a maximal element. Consider the family $\mathbf{U} = \{\gamma \subset \exp(X) : \gamma \text{ is a filter and } \mathcal{F} \subset \gamma\}$. The partial order on $\mathbf{U}$ is the inclusion. If $\mathcal{U} \in \mathbf{U}$ is a maximal element, then $\mathcal{F} \subset \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter, so by Zorn's Lemma, it suffices to prove that every chain in $\mathbf{U}$ has an upper bound. Take an arbitrary non-empty chain $\mathbf{C} \subset \mathbf{U}$. Then $\mu = \bigcup \mathbf{C} \subset \exp(X)$ and $\gamma \subset \mu$ for any $\gamma \in \mathbf{C}$. This shows that μ is an upper bound for $\mathbf{C}$ so the last thing we must prove is that $\mu \in \mathbf{U}$, i.e., that μ is a filter. Since $\mathbf{C} \neq \emptyset$, we have $\mu \neq \emptyset$. Given $A, B \in \mu$, there are $\gamma_1, \gamma_2 \in \mathbf{C}$ such that $A \in \gamma_1$ and $B \in \gamma_2$. The family $\mathbf{C}$ being a chain, we have $\gamma_1 \subset \gamma_2$ or $\gamma_2 \subset \gamma_1$. In both cases, $A, B \in \gamma_i$ for some $i \in \{1, 2\}$ and hence $A \cap B \in \gamma_i$ because γ_i is a filter. Therefore $A \cap B \in \mu$. Now, if $A \in \mu$ and $A \subset B$ then there is $\gamma \in \mathbf{C}$ such that $A \in \gamma$. Since γ is a filter, we have $B \in \gamma$ and hence $B \in \mu$. Therefore μ is a filter. To finish the proof observe that any centered family is contained in a filter by (ii) and any filter is contained in an ultrafilter. Therefore any centered family is contained in an ultrafilter.

(iv) Suppose that $\mathcal{U}$ is an ultrafilter on X. Then $\mathcal{U}$ is a centered family by (i). To see that $\mathcal{U}$ is maximal centered observe that if $\mathcal{U} \subset \mathcal{U}'$ and $\mathcal{U}'$ is centered then, by (ii), there is a filter $\mathcal{F} \supset \mathcal{U}' \supset \mathcal{U}$. By the maximality of $\mathcal{U}$ as a filter, we have $\mathcal{F} = \mathcal{U}$ and hence $\mathcal{U} = \mathcal{U}'$ which shows that $\mathcal{U}$ is a maximal centered family. Now, if $\mathcal{U}$ is maximal centered then apply (iii) to find an ultrafilter $\xi \supset \mathcal{U}$. Since ξ is also centered, by the maximality of $\mathcal{U}$ as a centered family, we have $\mathcal{U} = \xi$ and hence $\mathcal{U}$ is an ultrafilter.

(v) Suppose first that $\mathcal{U}$ is an ultrafilter. Then it is maximal centered and, given any $A \subset X$ with $A \notin \mathcal{U}$, the family $\mathcal{U} \cup \{A\}$ is not centered by maximality of $\mathcal{U}$ as a centered family. Therefore there are $U_1, \ldots, U_n \in \mathcal{U}$ with $U_1 \cap \cdots \cap U_n \cap A = \emptyset$. Then $U = U_1 \cap \cdots \cap U_n \in \mathcal{U}$ and $U \subset X \setminus A$. Now it is easy to see that $\mathcal{U}' = \mathcal{U} \cup \{X \setminus A\}$ is centered and hence $\mathcal{U}' = \mathcal{U}$ by the maximality of $\mathcal{U}$ as a centered family. As a consequence, $X \setminus A \in \mathcal{U}$. Assume finally that $\mathcal{U}$ is a centered family such that $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$ for any $A \subset X$. By (iv), it is sufficient to show that $\mathcal{U}$ is maximal centered. So take any $A \notin \mathcal{U}$. We have $B = X \setminus A \in \mathcal{U}$, so for $\mathcal{U}' = \mathcal{U} \cup \{A\}$ we have $A, B \in \mathcal{U}'$ and $A \cap B = \emptyset$ and therefore $\mathcal{U}'$ is not centered. This proves that $\mathcal{U}$ is maximal centered and hence an ultrafilter.

(vi) Let $\mathcal{U} = \{\delta \subset \tau^*(X) : \delta$ is disjoint and $\gamma \subset \delta\}$. The partial order on $\mathcal{U}$ is the inclusion. If $\mu \in \mathcal{U}$ is a maximal element, then $\gamma \subset \mu$ and μ is a maximal disjoint family, so by Zorn's Lemma, it suffices to prove that every chain in $\mathcal{U}$ has an upper bound. Take an arbitrary non-empty chain $\mathcal{C} \subset \mathcal{U}$. Then $\nu = \bigcup \mathcal{C} \subset \tau^*(X)$ and $\delta \subset \nu$ for any $\delta \in \mathcal{C}$. This shows that ν is an upper bound for $\mathcal{C}$ so the last thing we must prove is that $\nu \in \mathcal{U}$, i.e., that ν is a disjoint family. Given $A, B \in \nu$, there are $\delta_1, \delta_2 \in \mathcal{C}$ such that $A \in \delta_1$ and $B \in \delta_2$. The family $\mathcal{C}$ being a chain, we have $\delta_1 \subset \delta_2$ or $\delta_2 \subset \delta_1$. In both cases, $A, B \in \delta_i$ for some $i \in \{1, 2\}$ and hence $A \cap B = \emptyset$ because δ_i is a disjoint family. Therefore ν is a disjoint family.

(vii) Suppose that $\gamma \subset \tau^*(\mathbb{R})$ is a point-finite family. Observe that $\tau^*(\mathbb{R})$ is not point-finite so there is $U \in \tau^*(\mathbb{R}) \setminus \gamma$. It is clear that $\gamma \cup \{U\}$ is point-finite and hence γ is not maximal.

S.118. *Prove that the following properties are equivalent for any (not necessarily Tychonoff) space X:*

- (*i*) *X is compact.*
- (*ii*) *There is a base $\mathcal{B}$ in X such that every cover of X with the elements of $\mathcal{B}$ has a finite subcover.*
- (*iii*) *There is a subbase $\mathcal{S}$ in X such that every cover of X with the elements of $\mathcal{S}$ has a finite subcover.*
- (*iv*) *If $\mathcal{P}$ is a filter base in X then $\bigcap\{\overline{P} : P \in \mathcal{P}\} \neq \emptyset$.*
- (*v*) *If $\mathcal{F}$ is a filter on X then $\bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$.*
- (*vi*) *Given an ultrafilter $\mathcal{U}$ on the set X we have $\bigcap\{\overline{U} : U \in \mathcal{U}\} \neq \emptyset$.*
- (*vii*) *If $\mathcal{C}$ is a centered family of subsets of X then $\bigcap\{\overline{C} : C \in \mathcal{C}\} \neq \emptyset$.*
- (*viii*) *If $\mathcal{D}$ is a centered family of closed subsets of X then $\bigcap\{D : D \in \mathcal{D}\} \neq \emptyset$.*
- (*ix*) *If $\mathcal{G}$ is a filter base of closed subsets of X then $\bigcap\{G : G \in \mathcal{G}\} \neq \emptyset$.*
- (*x*) *For any infinite $A \subset X$ there is a point $x \in X$ such that $|U \cap A| = |A|$ for any neighbourhood U of the point x (such a point x is called a complete accumulation point of A). Thus, this criterion could be formulated as follows: a space X is compact iff any infinite subset of X has a complete accumulation point.*

Solution. (i) $\Rightarrow$ (ii) because $\mathcal{B} = \tau(X)$ is a base of X. Since every base of X is a subbase of X, taking $\mathcal{S} = \mathcal{B}$, we obtain (ii) $\Rightarrow$ (iii). Since any filter is a filterbase and any ultrafilter is a filter, we have (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi). Suppose that (vi) is fulfilled and take any centered family $\mathcal{C}$ of subsets of X. By Problem 117(iii) there exists an ultrafilter $\mathcal{U} \supset \mathcal{C}$. Then $\bigcap\{\overline{C} : C \in \mathcal{C}\} \supset \bigcap\{\overline{C} : C \in \mathcal{U}\} \neq \emptyset$ by (vi) and this proves that (vi) $\Rightarrow$ (vii). If (vii) holds and $\mathcal{C}$ is a centered family of closed subsets of X then by (vii) we have $\bigcap\{F : F \in \mathcal{C}\} = \bigcap\{\overline{F} : F \in \mathcal{B}\} \neq \emptyset$ which proves (vii) $\Rightarrow$ (viii). Since any filter base is centered, we have (viii) $\Rightarrow$ (ix). Now, suppose that (ix) holds and $\mathcal{P}$ is a filter base on X. It is easily checked that $\mathcal{G} = \{\overline{P} : P \in \mathcal{P}\}$ is also a filter base, so by (ix), we have $\bigcap\{\overline{P} : P \in \mathcal{P}\} = \bigcap \mathcal{G} \neq \emptyset$ which proves that (ix) $\Rightarrow$ (iv) and hence the conditions (iv)–(ix) are equivalent.

If the space X is compact and $A \subset X$ is an infinite set suppose that, for every point $x \in X$, there is $U_x \in \tau(x, X)$ such that $|U_x \cap A| < |A|$. There are $x_1, \dots, x_n \in X$ such that $X = \cup\{U_{x_i} : i \leqslant n\}$. Therefore $A = \cup\{A \cap U_{x_i} : i \leqslant n\}$ and $|A| = \Sigma\{|A \cap U_{x_i}| : i \leqslant n\} < |A|$ because every summand has cardinality less than $|A|$. The obtained contradiction shows that (i) $\Rightarrow$ (x).

Assume that (x) holds for the space X. If X is not compact choose a family $\gamma = \{U_\alpha : \alpha < \kappa\} \subset \tau(X)$ of minimal cardinality κ such that $\cup\gamma = X$ and γ has no finite subcover. For each $\alpha < \kappa$, let $V_\alpha = \bigcup\{U_\beta : \beta \leqslant \alpha\}$. Note first that $V_\alpha \neq X$ for any $\alpha < \kappa$ for otherwise the family $\gamma_\alpha = \{U_\beta : \beta \leqslant \alpha\}$ is an open cover of X of cardinality $< \kappa$ which has, by the choice of κ, a finite subcover, a contradiction. The second observation is that $\kappa = |\gamma|$ is a regular cardinal, i.e., κ has no cofinal subset of cardinality less than κ. To prove it, assume the contrary and take a cofinal $B \subset \kappa$ with $|B| < \kappa$. By cofinality of B the family $\mu = \{V_\alpha : \alpha \in B\}$ is an open cover of X. Since $|\mu| < \kappa$, there are $\alpha_1, \dots, \alpha_n \in B$ such that $\cup\{V_{\alpha_i} : i \leqslant n\} = X$. If α_j is the biggest one from the ordinals $\{\alpha_1, \dots, \alpha_n\}$ we have $X = V_{\alpha_j}$ which is a contradiction with the first observation. The third observation is that the cardinality of $P_\alpha = X \setminus V_\alpha$ cannot be less than κ. Indeed, if $|P_\alpha| < \kappa$ for some $\alpha < \kappa$ then for each $x \in P_\alpha$, fix $\alpha_x < \kappa$ such that $x \in U_{\alpha_x} \subset V_{\alpha_x}$. The set $\{\alpha_x : x \in P_\alpha\}$ cannot be cofinal in κ by the second observation so there is $\beta < \kappa$ such that $\beta > \alpha$ and $\alpha_x < \beta$ for all $x \in P_\alpha$. It is easy to see that this implies $X = V_\beta$ which is a contradiction with the first observation.

By the first observation, we can choose a point $x_0 \in X \setminus V_0$. Suppose that we have chosen $x_\alpha \in X \setminus V_\alpha$ for each $\alpha < \beta$. The set $X \setminus V_\beta$ has cardinality at least κ by the third observation so there exists $x_\beta \in (X \setminus V_\beta) \setminus \{x_\alpha : \alpha < \beta\}$ which shows that we can construct the set $A = \{x_\alpha : \alpha < \kappa\} \subset X$. Clearly, $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$ and hence $|A| = \kappa$. Now, for any $x \in X$ there is $\alpha < \kappa$ with $x \in U_\alpha \subset V_\alpha$. Since $x_\beta \notin V_\alpha$ for each $\beta > \alpha$, we have $|A \cap U_\alpha| < \kappa$ and therefore A has no complete accumulation point in X, a contradiction. This establishes that (i) $\Leftrightarrow$ (x).

Now, suppose that X is compact and $\mathcal{D}$ is a centered family of closed subsets of X such that $\bigcap \mathcal{D} = \emptyset$. The family $\gamma = \{X \setminus C : C \in \mathcal{C}\}$ is an open cover of X and, by compactness of X, there are $C_1, \dots, C_n \in \mathcal{C}$ such that $(X \setminus C_1) \cup \dots \cup (X \setminus C_n) = X$ whence $C_1 \cap \dots \cap C_n = \emptyset$ which is a contradiction with the fact that $\mathcal{C}$ is centered. This proves (i) $\Rightarrow$ (viii). Now, if (viii) holds, take any open cover γ of the space X. If γ has no finite subcover then the family $\mathcal{G} = \{X \setminus U : U \in \gamma\}$ is centered and consists of closed subsets of X. By (viii) we have $\bigcap \mathcal{G} \neq \emptyset$ and, as an immediate consequence, $\bigcup \gamma \neq X$ which is a contradiction which proves the implication (viii) $\Rightarrow$ (i).

The last implication left is (iii) ⇒(i). The following technical definition will be useful in this proof: an open cover of X will be called *marked* if it has no finite subcover. It is clear that any subcover of a marked cover is also marked.

Fix a subbase $\mathcal{S}$ like in (iii) and assume that there exists an open cover γ of X which does not have a finite subcover, i.e., γ is marked. A routine application of Zorn's Lemma shows that there exists a family $\mu \subset \tau(X)$ such that $\gamma \subset \mu$ and μ is a maximal marked cover of X. If $v = \mu \cap \mathcal{S}$ is a cover of X then it is marked which is a contradiction with the choice of $\mathcal{S}$. Therefore there is $x \in X \setminus (\bigcup v)$. Since γ is a cover of X, there exists $U \in \gamma$ with $x \in U$. The family $\mathcal{S}$ being a subbase, there are $U_1, \ldots, U_n \in \mathcal{S}$ such that $x \in U_1 \cap \cdots \cap U_n \subset U$. Now, $U_i \notin \mu$ for each $i \leqslant n$ for otherwise $U_i \in \mu \cap \mathcal{S} = v$ and $x \in \bigcup v$ which is a contradiction. Since μ is maximal marked, the family $\mu \cup \{U_i\}$ is not marked, i.e., there is a finite $\mu_i \subset \mu$ such that $(\bigcup \mu_i) \cup U_i = X$ for each $i \leqslant n$. Observe that $\delta = \mu_1 \cup \cdots \cup \mu_n \cup \{U\}$ is a finite subfamily of μ. Given $y \in X$, if $y \in \bigcap_{i \leqslant n} U_i$ then $y \in U \subset \bigcup \delta$. If $y \notin \bigcap_{i \leqslant n} U_i$ then $y \notin U_i$ for some $i \leqslant n$ and therefore $y \in \cup \mu_i \subset \bigcup \delta$. Thus $y \in \bigcup \delta$ and $\bigcup \delta = X$ which is contradiction with the fact that μ has no finite subcover. Hence X is compact and our proof is complete.

S.119. *Prove that a continuous image of a compact space is a compact space.*

Solution. Assume that X is a compact space and $f : X \to Y$ is a continuous onto map. If γ is an open cover of Y then $\mu = \{f^{-1}(U) : U \in \gamma\}$ is an open cover of X. By compactness of X there are $U_1, \ldots, U_n \in \gamma$ with $X = f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n)$. As a consequence, $Y = f(X) = f(f^{-1}(U_1)) \cup \cdots \cup f(f^{-1}(U_n)) = U_1 \cup \cdots \cup U_n$ and hence Y is compact.

S.120. *Prove that a closed subspace of a compact space is a compact space.*

Solution. If X is compact and Y is a closed subspace of X take any centered family $\mathcal{D}$ of closed subsets of Y. Then the elements of $\mathcal{D}$ are also closed in X and hence $\bigcap \mathcal{D} \neq \emptyset$. Apply Problem 118(viii) to conclude that Y is compact.

S.121. *Prove that, if X is a Hausdorff space and Y is a compact subspace of X, then Y is closed in X. Show that this is not true if X is not Hausdorff.*

Solution. Take any $x \in X \setminus Y$. For any $y \in Y$ fix open sets U_y, V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. The family $\{U_y \cap Y : y \in Y\}$ is an open cover of the compact space Y. Thus there are $y_1, \ldots, y_n \in Y$ with $Y \subset U_{y1} \cup \cdots \cup U_{y_n}$. The set $W_x = V_{y1} \cap \cdots \cap V_{y_n}$ is open in X and contains x. Now, if $z \in Y$ then $z \in U_{y_i}$ for some $i \leqslant n$ and hence $z \notin V_{y_i} \supset W_x$. This shows that $W_x \cap Y = \emptyset$ and hence $X \setminus Y = \bigcup \{W_x : x \in X \setminus Y\}$ is an open set.

To show that our statement may be false when X is not Hausdorff, define a topology v on $\mathbb{R}$ as follows: a set $A \subset \mathbb{R}$ belongs to v if $A = \emptyset$ or $\mathbb{R} \setminus A$ is finite. We leave to the reader the simple verification that $X = (\mathbb{R}, v)$ is compact as well as any $Y \subset X$. Thus the set $Y = [0, 1]$ is compact but not closed in X because any $x \in \mathbb{R} \setminus [0, 1]$ belongs to the closure of Y.

S.122. *Let X be a compact space. Show that, for any Hausdorff space Y, any continuous surjective mapping $f : X \to Y$ is closed.*

Solution. Let F be a closed subset of X. The subspace F is compact by Problem 120 and hence $f(F)$ is a compact subset of Y by 119. Now apply Problem 121 to conclude that $f(F)$ is closed in Y.

S.123. *Show that, if X is a compact space and $f : X \to Y$ is a condensation then f is a homeomorphism.*

Solution. We must only prove that f^{-1} is continuous. Given a closed $F \subset X$, the set $(f^{-1})^{-1}(F) = f(F)$ is closed in Y by Problem 122 and hence f^{-1} is continuous by 009(v).

S.124. *Show that any Lindelöf T_3-space is normal as well as any Hausdorff compact space.*

Solution. For possible future references we will prove a little more, namely, that

(i) Any Lindelöf T_3-space is normal and hence Tychonoff.
(ii) Any Hausdorff compact space is normal and hence Tychonoff.

Let X be a Lindelöf T_3-space. The first observation is that, for any closed $F \subset X$ and any $\gamma \subset \tau(X)$ with $F \subset \bigcup \gamma$, there is a countable $\gamma' \subset \gamma$ with $F \subset \bigcup \gamma'$. Indeed, the family $\Gamma = \gamma \cup \{X \backslash F\}$ is an open cover of the space X and therefore there is a countable $\gamma' \subset \gamma$ such that $X = (X \backslash F) \cup (\bigcup \gamma')$. It is immediate that $F \subset \bigcup \gamma'$. The second observation is that, given a closed $F \subset X$ and $x \in X \backslash F$, there is $U \in \tau(x, X)$ such that $\overline{U} \cap F = \emptyset$. This is true because, by regularity of X, there are $U, V \in \tau(X)$ such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$. It is clear that $\overline{U} \cap F = \emptyset$. Now, take any disjoint closed non-empty sets $F, G \subset X$. For any $x \in F$, apply the second observation to find an open set U_x with $x \in U_x$ and $\overline{U}_x \cap G = \emptyset$. Analogously, for any $y \in G$ there is $V_y \in \tau(y, X)$ such that $\overline{V}_y \cap F = \emptyset$. Now, $\mathcal{F} = \{U_x : x \in F\}$ is an open cover of the closed set F so by the first observation, we can find a family $\{U_i : i \in \omega\} \subset \mathcal{F}$ with $F \subset \cup\{U_i : i \in \omega\}$. Since $\mathcal{G} = \{V_y : y \in G\}$ is an open cover of the closed set G, by the first observation, there is a family $\{V_i : i \in \omega\} \subset \mathcal{G}$ such that $G \subset \bigcup\{V_i : i \in \omega\}$. Note that $\overline{U}_i \cap G = \emptyset$ and $\overline{V}_i \cap F = \emptyset$ for any $i \in \omega$. Consider the sets $U_i' = U_i \backslash (\overline{V}_0 \cup \dots \cup \overline{V}_i)$ and $V_i' = V_i \backslash (\overline{U}_0 \cup \dots \cup \overline{U}_i)$ for all $i \in \omega$ and let $U = \bigcup_{i \in \omega} U_i'$ and $V = \cup_{i \in \omega} V_i'$. It is evident that U and V are open sets. We claim that $F \subset U$, $G \subset V$ and $U \cap V = \emptyset$. For any $x \in F$ and $i \in \omega$ such that $x \in U_i$ we have $x \in U_i'$ because $\overline{V}_0 \cup \dots \cup \overline{V}_i$ does not meet F. Therefore $F \subset U$. Analogously, if $y \in G$ and $y \in V_i$ then $y \in V_i'$ and hence $G \subset V$. To see that U and V are disjoint, take any $x \in U \cap V$. Then $x \in U_i' \cap V_j'$ for some $i, j \in \omega$. If $i \leqslant j$ then $x \in V_j'$ implies $x \notin \overline{U}_k$ for any $k \leqslant j$ and, in particular, $x \notin U_i \supset U_i'$, a contradiction. The case $j < i$ gives us a contradiction in the same way and hence $U \cap V = \emptyset$. The proof of the normality of any Lindelöf T_3-space is complete.

Now, assume that X is compact Hausdorff space. Since any compact space is Lindelöf, by the previous statement it is sufficient to establish regularity of the space X. Take any closed $F \subset X$ and any $x \in X \setminus F$. For any $y \in F$ there are $U_y, V_y \in \tau(X)$

such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. The space F is compact, so the cover $\{U_y \cap F : y \in F\}$ has a finite subcover. Choose $y_1, \ldots, y_n \in F$ such that $F \subset V_{y1} \cup \cdots \cup V_{y_n}$. Consider the sets $U = \bigcap_{i \leqslant n} U_{y_i}$ and $V = \bigcup_{i \leqslant n} V_{y_i}$. It is clear that $U \in \tau(x, X)$ and $V \in \tau(F, X)$ so it is sufficient to show that $U \cap V = \emptyset$. Indeed, if $z \in U \cap V$ then $z \in U_{y_i}$ for all $i \leqslant n$ and $z \in V_{y_j}$ for some number $j \leqslant n$. This implies $z \in U_{y_j} \cap V_{y_j} = \emptyset$ which is a contradiction. We proved (i) and (ii) so our solution is complete.

S.125. *(The Tychonoff theorem) Show that any product of compact spaces is a compact space.*

Solution. Suppose that the space X_t is compact for any $t \in T$. We will prove that $X = \prod\{X_t : t \in T\}$ is compact using the criterion given in Problem 118(iii). Consider the family $\mathcal{S} = \bigcup_{t \in T} \mathcal{S}_t$, where $\mathcal{S}_t = \{p_t^{-1}(U) : U \in \tau(X_t)\}$ for each index $t \in T$. It is easy to see that the family of all finite intersections of the elements of $\mathcal{S}$ is precisely the standard base (see Problem 101) of the product $\prod\{X_t : t \in T\}$ which shows that $\mathcal{S}$ is a subbase of X. Let us prove that $\gamma \subset \mathcal{S}$ and $\bigcup \gamma = X$ implies that γ has a finite subcover. Note that $\gamma = \bigcup\{\gamma_t : t \in T\}$ where $\gamma_t = \gamma \cap \mathcal{S}_t$ for each $t \in T$. Fix a family $\mu_t \subset \tau(X_t)$ such that $\gamma_t = \{p_t^{-1}(U) : U \in \mu_t\}$. We claim that there is $t_0 \in T$ such that $\bigcup \gamma_{t_0} = X$. Indeed, if this were not the case, then, for each $t \in T$, there is $y_t \in X \backslash (\cup \gamma_t)$. Now let $y(t) = y_t(t)$ for each $t \in T$. It is evident that $y \in X$. If $t \in T$ then $y(t) = y_t(t) \notin \bigcup \mu_t$ for otherwise $y_t \in p_t^{-1}(U) \in \gamma_t$ for some $U \in \mu_t$ which contradicts the fact that $y_t \in X \backslash (\bigcup \gamma_t)$. As a consequence, $y \notin \bigcup \gamma_t$ for each $t \in T$ and therefore $y \notin \bigcup \gamma$, a contradiction. So we can fix $t_0 \in T$ with $X = \bigcup \gamma_{t_0}$ and hence $X_{t_0} = \bigcup \mu_{t_0}$. Remembering that X_{t_0} is compact, we can choose a finite $\nu \subset \mu_{t_0}$ such that $X_{t_0} = \bigcup \nu$ and hence $X = \bigcup\{p_{t_0}^{-1}(U) : U \in \nu\}$ which shows that $\{p_{t_0}^{-1}(U) : U \in \nu\}$ is a finite subcover of γ and the compactness of X is proved.

S.126. *Prove that a space X is compact if and only if X is homeomorphic to a closed subspace of $\mathbb{I}^A$ for some A with $|A| \leqslant w(X)$.*

Solution. Let us prove first that the space $\mathbb{I}$ is compact. Consider the families $\mathcal{S}_1 = \{[-1, a) : 0 < a \leqslant 1\}$ and $\mathcal{S}_2 = \{(b, 1] : -1 \leqslant b < 1\}$. It is easy to see that $\mathcal{S} = \mathcal{S}_1 \bigcup \mathcal{S}_2$ is a subbase of $\mathbb{I}$. By Problem 118(iii) it suffices to prove that every cover of $\mathbb{I}$ with the elements of $\mathcal{S}$ has a finite subcover. So assume that $\gamma \subset \mathcal{S}$ and $\bigcup \gamma = \mathbb{I}$. Let $\gamma_i = \gamma \cap \mathcal{S}_i$, $i = 1, 2$. We will need the numbers $p = \sup\{a : [-1, a) \in \gamma_1\}$ and $q = \inf\{b : (b, 1] \in \gamma_2\}$. It is easy to see that $[0, p) = \bigcup \gamma_1$ and $(q, 1] = \bigcup \gamma_2$ and hence $p > q$ for otherwise $r = \frac{p+q}{2} \notin \bigcup \gamma$. Since $q < r < p$, there are $[-1, a) \in \gamma_1$ and $(b, 1] \in \gamma_2$ such that $b < r < a$. As a consequence, $\mathbb{I} = [-1, a) \cup (b, 1]$ and therefore $\{[-1, a), (b, 1]\}$ is a finite subcover of γ. Since $\mathbb{I}$ is compact, the space $\mathbb{I}^A$ is also compact for any A by Problem 125. Any closed subspace of $\mathbb{I}^A$ is also compact by Problem 120 so, if X is homeomorphic to a closed subspace of some $\mathbb{I}^A$ then X is compact.

Now suppose that X is compact and $\kappa = w(X)$. If κ is finite then X is also finite, so any subspace of $\mathbb{I}$ with κ points is closed in *openI* and homeomorphic to X. In this case $\mathbb{I}^A$ is homeomorphic to $\mathbb{I}$ for $A = \{0\}$ so $|A| = 1 \leqslant \kappa$. If κ is infinite fix a base $\mathcal{B}$ in X of cardinality κ. Call a pair $(U, V) \in \mathcal{B} \times \mathcal{B}$ *special* if $\overline{U} \subset V$. Denote by A the set of all special pairs. Then $|A| \leqslant |\mathcal{B} \times \mathcal{B}| = \kappa$. The space X is normal by Problem 124, so given a special pair $p = (U, V)$, we can choose a continuous function $h_p : X \to \mathbb{I}$ such that $h_p(\overline{U}) = \{0\}$ and $h_p(X \backslash V) = \{1\}$. For an arbitrary $x \in X$, let $\varphi(x)(p) = h_p(x)$ for any $p \in A$. Then $\varphi(x) \in \mathbb{I}^A$. We will prove that $\varphi : X \to Y = \varphi(X) \subset \mathbb{I}^A$ is a homeomorphism. For any $p \in A$, denote by π_p the pth projection of $\mathbb{I}^A$ onto $\mathbb{I}$. Recall that $\pi_p(f) = f(p)$ for any $f \in \mathbb{I}^A$. Note that φ is continuous because, for any $p \in A$, we have $\pi_p \circ \varphi = h_p$ and the map h_p is continuous (see Problem 102). If $x \neq y$ then take any $V \in \mathcal{B}$ such that $x \in V \subset X \backslash \{y\}$. By regularity of X there is $U' \in \tau(X)$ such that $x \in U' \subset \overline{U'} \subset V$. Take any set $U \in \mathcal{B}$ such that $x \in U \subset U'$. Then the pair $p = (U, V)$ is special and $h_p(x) = 0$, $h_p(y) = 1$. As a consequence, $\varphi(x)(p) = h_p(x) \neq h_p(y) = \varphi(y)(p)$ which proves that $\varphi(x) \neq \varphi(y)$ and hence φ is a condensation. Apply Problem 123 to conclude that φ is a homeomorphism. By Problem 121 the subspace Y is closed in $\mathbb{I}^A$.

S.127. *Prove that the following properties are equivalent for any (not necessarily Tychonoff) space:*

(*i*) *X is homeomorphic to a subspace of a compact Tychonoff space.*
(*ii*) *X is homeomorphic to a subspace of a compact Hausdorff space.*
(*iii*) *X is homeomorphic to a subspace of $\mathbb{I}^A$ for some A.*
(*iv*) *X is homeomorphic to a subspace of a T_4-space.*
(*v*) *X is a Tychonoff space.*

Solution. It is evident that (i) $\Rightarrow$ (ii). Applying Problem 126 we can see that if X embeds in a compact space then it embeds in $\mathbb{I}^A$ for some A so (ii) $\Rightarrow$ (iii). The space $\mathbb{I}^A$ is compact Hausdorff and hence normal by Problem 124. This shows that (iii) $\Rightarrow$ (iv). Now, every T_4-space is Tychonoff and every subspace of a Tychonoff space is a Tychonoff space by Problem 017. This proves (iv) $\Rightarrow$ (v).

To establish that (v) $\Rightarrow$ (i) take any Tychonoff space X and let $A = C(X, \mathbb{I})$. For an arbitrary $x \in X$, let $\varphi(x)(f) = f(x)$ for each $f \in A$. Then $\varphi(x) \in \mathbb{I}^A$. We will prove that $\varphi : X \to Y = \varphi(X) \subset \mathbb{I}^A$ is a homeomorphism. For any $h \in A$, denote by π_h the hth projection of I^A onto $\mathbb{I}$. Recall that $\pi_h(y) = y(h)$ for any $y \in \mathbb{I}^A$. Note that φ is continuous because, for any $h \in A$, we have $\pi_h \circ \varphi = h$ and the map h is continuous (see Problem 102). If $x \neq y$ then by complete regularity of X there is a continuous function $h : X \to \mathbb{I}$ such that $h(x) = 1$ and $h(y) = 0$. We have $\varphi(x)(h) = h(x) = 1 \neq 0 = h(y) = \varphi(y)(h)$ which shows that $\varphi(x) \neq \varphi(y)$ and hence φ is a condensation. To finish the proof it suffices to establish that $\varphi^{-1} : Y \to X$ is continuous.

Take any $y \in Y$ and any $U \in \tau(x, X)$ where $x = \varphi^{-1}(y)$. By complete regularity of X there is a continuous function $h : X \to \mathbb{I}$ such that $h(x) = 1$ and $h|(X \backslash U) \equiv 0$. The set $W = \{z \in \mathbb{I}^A : z(h) \in (0, 1]\}$ is open in $\mathbb{I}^A$ and $y \in W$ because $y(h) = h(x) = 1$. Now take any $z \in W$. If $x' = \varphi^{-1}(z)$ then $\varphi(x') = z$ and $\varphi(x')(h) = h(x') > 0$ which shows that $x' \notin X \backslash U$ and therefore $x' \in U$. We proved that $\varphi^{-1}(W) \subset U$ and hence φ^{-1} is continuous at the point y. Thus X is homeomorphic to the subspace Y of the compact space $\mathbb{I}^A$ and the implication (v) $\Rightarrow$ (i) is proved.

S.128. *Denote by A the set of numeric sequences* $\alpha = \{\alpha_i : i \in \mathbb{N}\}$ *such that* $\alpha_i = 0$ *or* $\alpha_i = 2$ *for all i. Given* $\alpha = \{\alpha_i : i \in \mathbb{N}\} \in A$*, let* $x(\alpha) = \sum_{i=1}^{\infty} 3^{-i} \cdot \alpha_i$*. The set* $\mathbb{K} = \{x(\alpha) : \alpha \in A\}$ *is called the Cantor perfect set. Prove that*

(*i*) *the set* $\mathbb{K}$ *is a compact subset of the segment* $I = [0, 1]$.
(*ii*) $\mathbb{K}$ *is uncountable and the interior of* $\mathbb{K}$ *is empty.*
(*iii*) $\mathbb{K}$ *is homeomorphic to* $\{0, 1\}^{\mathbb{N}}$.
(*iv*) *If F is a non-empty closed subspace of* $\mathbb{K}$ *then there exists a continuous map* $r : \mathbb{K} \to F$ *such that* $r(x) = x$ *for all* $x \in F$.
(*v*) $\mathbb{K}$ *maps continuously onto any second countable compact space.*

Solution. Note first that, for any $\alpha \in A$, we have $0 \leqslant x(\alpha) \leqslant \sum_{i=1}^{\infty} 2 \cdot 3^{-i} = 1$ and hence $\mathbb{K} \subset [0, 1]$. To finish the proof of (i) we must establish that $\mathbb{K}$ is compact which is an easy consequence of (iii). So let us prove that $\mathbb{K}$ is homeomorphic to $K = \{0, 1\}^{\mathbb{N}}$. Note that K is compact by Problem 125.

Given $f \in K$, let $\varphi(f) = \sum_{i=1}^{\infty} 2 \cdot f(i) \cdot 3^{-i}$. It is clear that $\varphi(K) = \mathbb{K}$. If $f, g \in K$ and $f \neq g$ let $n = \min\{i \in \mathbb{N} : f(i) \neq g(i)\}$. Then $|\varphi(f) - \varphi(g)| = |2 \cdot 3^{-n} + \sum_{i=n+1}^{\infty} 2 \cdot ((f(i) - g(i)) \cdot 3^{-i}| \geqslant 2 \cdot 3^{-n} - \sum_{i=n+1}^{\infty} 2 \cdot 3^{-i} = 3^{-n} > 0$ and hence $\varphi(f) \neq \varphi(g)$ which proves that φ is a bijection. To see that φ is a continuous map, take any $f_0 \in K$ and $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $3^{-n} < \varepsilon$. The set $U = \{f \in K : f(i) = f_0(i)$ for all $i \leqslant n\}$ is open in K and $f_0 \in U$. For any $f \in U$, we have $|\varphi(f) - \varphi(f_0)| \leqslant \sum_{i=n+1}^{\infty} 2 \cdot 3^{-i} = 3^{-n} < \varepsilon$ and hence φ is continuous at the point f_0. Apply Problem 123 to conclude that φ is a homeomorphism and the properties (iii) and (i) hold. Since $\mathbb{K}$ is homeomorphic to K, it is uncountable so to settle (ii) we must only prove that $\mathbb{K}$ has empty interior. Given distinct $f, g \in K$, let $n = \min\{i : f(i) \neq g(i)\}$. Without loss of generality suppose that $f(n) = 0$ and $g(n) = 1$. The sets $P = \{h \in K : h(n) = 0\}$ and $Q = \{h \in K : h(n) = 1\}$ are clopen in K and disjoint. Note also that $P \cup Q = K$ and $f \in P$, $g \in Q$. Being closed in the compact space K the sets P and Q are compact.

Now assume that $0 \leqslant a < b \leqslant 1$ and $[a, b] \subset \mathbb{K}$. There are $f, g \in K$ with $\varphi(f) = a$ and $\varphi(g) = b$. Apply the preceding observation to find disjoint compact sets P and Q such that $f \in P$, $g \in Q$ and $P \cup Q = K$. The sets $P' = \varphi(P) \cap [a, b]$ and $Q' = \varphi(Q) \cap [a, b]$ are compact, disjoint, their union is $[a, b]$ and $a \in P'$, $b \in Q'$. Let $\xi = \inf Q'$. The set $Q' \subset [a, b]$ is bounded and non-empty so $\xi \in [a, b]$. The point ξ has to belong to P' or to Q'. Observe that P' and Q' are complementary closed sets in $[a, b]$ and hence they are both open in $[a, b]$. Now, if $\xi \in P'$ then, by openness of P', there is $\varepsilon > 0$ such that $[\xi, \xi + \varepsilon) \subset P'$ and hence there are no points of Q' in $[\xi, \xi + \varepsilon)$ which shows that $\inf Q' \geqslant \xi + \varepsilon$, a contradiction. If $\xi \in Q'$ then, by openness of Q', we have $(\xi - \varepsilon, \xi] \in Q'$ for some $\varepsilon > 0$. Therefore all points of $(\xi - \varepsilon, \xi]$ belong to Q' and hence $\inf Q' \leqslant \xi - \varepsilon$ and we again obtained a contradiction which proves that no non-trivial interval can be contained in $\mathbb{K}$. Hence $\mathbb{K}$ has empty interior and we established (ii).

Observe that the family $\{\{f \in K : f(i) = j\} : i \in \mathbb{N}, j \in \{0, 1\}\}$ is a countable subbase of K and consists of clopen subsets of K. Since any finite intersection of clopen sets is a clopen set, the space K (and hence $\mathbb{K}$) has a countable base $\mathcal{B}$ consisting of clopen sets. Given $x \in \mathbb{K}$, let $d(x) = \inf\{|x - y| : y \in F\}$. The function $d : \mathbb{K} \to \mathbb{R}$ is continuous (see the claim of S.019). For any $x \in \mathbb{K}\backslash F$, we have $d(x) > 0$ so there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subset \left(x - \frac{d(x)}{4}, x + \frac{d(x)}{4}\right)$. Note that $U_x \subset \mathbb{K}\backslash F$ for each $x \in \mathbb{K}\backslash F$ and hence $\mathbb{K}\backslash F = \bigcup\{U_x : x \in \mathbb{K}\backslash F\}$. Since $\mathcal{B}$ is countable, there are only countably many distinct U_x's so we can choose a set $\{x_n : n \in \mathbb{N}\} \subset \mathbb{K}\backslash F$ such that $\mathbb{K}\backslash F = \bigcup\{U_{x_n} : n \in \mathbb{N}\}$. Let $W_n = U_{x_n} \setminus \bigcup_{i=1}^{n-1} U_{x_i}$. It is clear that $\{W_n\}$ is a disjoint family of clopen sets and $\mathbb{K}\backslash F = \bigcup_{n\in\mathbb{N}} W_n$.

Choosing an appropriate subfamily and enumeration, we can assume, without loss of generality, that $W_n \neq \emptyset$ for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, choose $y_n \in F$ for which $|x_n - y_n| \leqslant \frac{5}{4}\, d(x_n)$. Define the map $r : \mathbb{K} \to F$ as follows: $r(x) = x$ for all $x \in F$ and if $x \in \mathbb{K}\backslash F$ choose the unique $n \in \mathbb{N}$ such that $x \in W_n$ and let $r(x) = y_n$. The map r is continuous at all points of $\mathbb{K}\backslash F$ because, for any $x \in W_n$, we have $r(W_n) = \{y_n\} \subset (y_n - \varepsilon, y_n + \varepsilon)$ for any $\varepsilon > 0$. If $x \in F$ and $\varepsilon > 0$, consider the open set $V = (x - \frac{\varepsilon}{3}, x + \frac{\varepsilon}{3}) \cap \mathbb{K}$. It suffices to show that $r(V) \subset (x - \varepsilon, x + \varepsilon)$ so pick any $y \in V$. If $y \in F$ then $r(y) = y \subset (x - \frac{\varepsilon}{3}, x + \frac{\varepsilon}{3}) \subset (x - \varepsilon, x + \varepsilon)$. Now take any $y \in V \cap (\mathbb{K}\backslash F)$ and $n \in \mathbb{N}$ with $y \in W_n$.

Observe first that $|x - x_n| \leqslant |x - y| + |y - x_n| < \frac{\varepsilon}{3} + \frac{d(x_n)}{4}$ due to the fact that $W_n \subset U_{x_n} \subset (x_n - \frac{d(x_n)}{4}, x_n + \frac{d(x_n)}{4})$. Therefore $d(x_n) \leqslant |x - x_n| < \frac{\varepsilon}{3} + \frac{d(x_n)}{4}$ which implies $d(x_n) < \frac{4}{9}\,\varepsilon$ and $|x - x_n| < \frac{\varepsilon}{3} + \frac{d(x_n)}{4} < \frac{\varepsilon}{3} + \frac{\varepsilon}{9} = \frac{4}{9}\,\varepsilon$. By the choice of y_n we have $|x_n - y_n| \leqslant \frac{5}{4}\, d(x_n) < \frac{5}{9}\,\varepsilon$ and therefore $|x - y_n| \leqslant |x - x_n| + |x_n - y_n| < \frac{4}{9}\,\varepsilon + \frac{5}{9}\,\varepsilon = \varepsilon$ and this proves that $|r(x) - r(y)| = |x - y_n| < \varepsilon$ for any $y \in V$ so the map r is continuous at the point x and we settled (iv).

To prove (v), note first that the map $\psi : K \to [0, 1]$ defined by the formula $\psi(f) = \sum_{i=1}^{\infty} f(i) \cdot 2^{-i}$, is continuous (the proof is identical to the proof of continuity for φ). Let us show that $\psi(K) = [0, 1]$. Take an arbitrary point $t \in [0, 1]$ and let $i_1 = \max\{i \in \{0, 1\} : \frac{i}{2} \leqslant t\}$. Suppose that we have $i_1, \dots, i_n \in \{0, 1\}$ such that, for each $k \leqslant n$ we have $i_k = \max\{i \in \{0, 1\} : \sum_{m=1}^{k-1} i_m \cdot 2^{-m} + i \cdot 2^{-k} \leqslant t\}$. Let $i_{n+1} = \max\{i \in \{0, 1\} : \sum_{m=1}^{n} i_m \cdot 2^{-m} + i \cdot 2^{-n-1} \leqslant t\}$. It is clear that we obtain a sequence $\{i_m : m \in \mathbb{N}\}$ such that $|\sum_{m=1}^{n} i_m \cdot 2^{-m} - t| \leqslant 2^{-n}$ for each $n \in \mathbb{N}$ and hence $t = \sum_{m=1}^{\infty} i_m \cdot 2^{-m}$. For the function $f \in K$ defined by $f(m) = i_m$, we have $\psi(f) = t$ finishing the proof that ψ is an onto map.

Let $\pi_n : [0, 1]^\omega \to [0, 1]$ and $p_n : K^\omega \to K$ be the relevant nth projections. Given $f \in K^\omega$, let $e(f)(n) = \psi(f(n))$ for every $n \in \omega$. Then $e(f) \in [0, 1]^\omega$ and the map $e : K^\omega \to [0, 1]^\omega$ is continuous. To see this, it is sufficient to show that $\pi_n \circ e : K^\omega \to [0, 1]$ is continuous for each $n \in \mathbb{N}$. But it is immediate that $\pi_n \circ e = \psi \circ p_n$ and the last map is continuous. Next we show that e is surjective. Indeed, if $f \in [0, 1]^\omega$ then choose any $g_n \in K$ such that $\psi(g_n) = f(n)$ for each $n \in \omega$. If $g(n) = g_n$ for all $n \in \omega$ then $g \in K^\omega$ and $e(g) = f$.

Let us observe now that there is a homeomorphism $h : K \to K^\omega$. Indeed, $K^\omega = (\{0, 1\}^{\mathbb{N}})^\omega$ is homeomorphic to $\{0, 1\}\mathbb{N}^{\times\omega}$ by Problem 103. Since there exists a bijection between $\mathbb{N}$ and $\mathbb{N} \times \omega$, we can apply Problem 104 to conclude that $\{0, 1\}\mathbb{N}^{\times\omega}$ is homeomorphic to $\{0, 1\}^\omega = K$. As a consequence, the map $w = e \circ h$ is a continuous surjection of K onto $[0, 1]^\omega$ which in turn is homeomorphic to $\mathbb{I}^\omega$ because $[0, 1]$ is homeomorphic to $\mathbb{I}$. Thus there exists a continuous onto map

$u : K \to \mathbb{I}^\omega$. Now take any second countable compact space X. By Problem 126 there exists $Y \subset \mathbb{I}^\omega$ and a homeomorphism $v : Y \to X$. The set $F = u^{-1}(Y)$ is closed in K because Y is compact. Apply (iv) to find a continuous onto map $r : K \to F$. Then $v \circ u \circ r$ maps K continuously onto X and the proof of (v) is over.

S.129. *Prove that, for any cardinal κ, the space $A(\kappa)$ is a compact Fréchet–Urysohn space of uncountable weight if $\kappa > \omega$.*

Solution. Take any $B \subset A(\kappa)$ and $x \in \overline{B}$. If x is an isolated point then $x \in B$ and letting $x_n = x$ for all n we get a sequence $\{x_n\} \subset B$ which converges to x. Now if $x = a$ and $x \in B$ we get the respective convergent sequence in the same way. Now, if $a \in \overline{B} \setminus B$ then B cannot be finite and hence we can find an infinite $C = \{x_n : n \in \omega\} \subset B$. The sequence $\{x_n\}$ converges to a because, for any $U \in \tau(a, A(\kappa))$, the set $A(\kappa)\setminus U$ is finite and hence there exists $m \in \mathbb{N}$ such that all $x_n \in U$ for all $n \geqslant m$. This proves that $A(\kappa)$ is a Fréchet–Urysohn space for each κ. Suppose that $\kappa > \omega$. If $\mathcal{B}$ is a countable base of $A(\kappa)$ then $\mathcal{B}' = \{U \in \mathcal{B} : a \in U\}$ is a local base at a. Since $A(\kappa)\setminus U$ is finite for all $U \in \mathcal{B}'$, the set $\bigcup\{A(\kappa)\setminus U : U \in \mathcal{B}'\}$ cannot cover κ. Hence there is $x \in \kappa$ such that $U \ni x$ for all $U \in \mathcal{B}$. However, $W = A(\kappa)\setminus\{x\}$ is an open set which contains a and hence $U \subset W$ for some $U \in \mathcal{B}'$ which gives a contradiction with the fact that $x \in U$. Thus, weight of $A(\kappa)$ is uncountable. To see that $A(\kappa)$ is compact, take any open cover γ of the space $A(\kappa)$. There is $U_0 \in \gamma$ with $a \in U_0$. Since $A(\kappa)\setminus U$ is finite, there are $U_1, \dots, U_n \in \gamma$ with $A(\kappa)\setminus U \subset U_1 \cup \dots \cup U_n$. Therefore the family $\{U_0, \dots, U_n\}$ is a finite subcover of γ and $A(\kappa)$ is compact.

S.130. *Given a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and a real number $r > 0$, define $B_n(x, r) = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : \sqrt{\sum_{i=1}^n (x_i - y_i)^2} < r\}$. The set $B_n(x, r)$ will be called the n-dimensional open ball of radius r centered at x. Prove that*

(*i*) *The family $\mathcal{B}$ of all open balls in $\mathbb{R}^n$ satisfies the conditions (B1) and (B2) of Problem 006 and hence it can be considered a base for some topology $\mathcal{N}_R^n$ which is called the natural (or usual) topology on $\mathbb{R}^n$;*
(*ii*) *The space $(\mathbb{R}^n, \mathcal{N}_R^n)$ is homeomorphic to the topological product of n copies of $(\mathbb{R}, \mathcal{N}_R)$.*

Solution. (i) Since $x \in B_n(x, 1)$ for any $x \in \mathbb{R}^n$, we have $\bigcup\mathcal{B} = \mathbb{R}^n$, i.e., the property (B1) holds for $\mathcal{B}$. Given $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ let $d_n(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$. Then $d_n(x, y) \geqslant 0$ and $d_n(x, y) = 0$ if and only if $x = y$. The first part of this statement is evident and it is it clear that $d_n(x, x) = 0$. Now if $d_n(x, y) = 0$ then $\sum_{i=1}^n (x_i - y_i)^2 = 0$ and hence $x_i - y_i = 0$ for all $i \leqslant n$. Therefore $x_i = y_i$ for all $i \leqslant n$ and $x = y$. It is immediate that $d_n(x, y) = d_n(y, x)$ for any $x, y \in \mathbb{R}^n$.

It is easy to check that

$$\left(\sum_{i=1}^n a_i b_i\right)^2 = \left(\sum_{i=1}^n a_i^2\right) \cdot \left(\sum_{i=1}^n b_i^2\right) - \frac{1}{2} \cdot \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2$$

for any $a_i, b_i \in \mathrm{IR}$, $i \leqslant n$. An immediate consequence is the famous inequality of Cauchy–Buniakovsky:

$$(*) \quad \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right).$$

We claim that $d_n(x, y) + d_n(y, z) \geqslant d_n(x, z)$ for any points $x, y, z \in \mathbb{R}^n$. This inequality is called *the triangle inequality*. Given any $x, y, z \in \mathrm{IR}^n$ with $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n)$ the triangle inequality for x, y y z has the following form:

$$(**) \quad \sqrt{\sum_{i=1}^{n} (z_i - x_i)^2} \leqslant \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} + \sqrt{\sum_{i=1}^{n} (z_i - y_i)^2}.$$

If $y_i - x_i = a_i$, $z_i - y_i = b_i$, we obtain $z_i - x_i = a_i + b_i$, and $(**)$ can be written as

$$(***) \quad \sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \leqslant \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}.$$

Applying $(*)$, we can see that

$$\sum_{i=1}^{n} (a_i + b_i)^2 = \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \cdot \sum_{i=1}^{n} a_i b_i$$
$$\leqslant \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \cdot \sqrt{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2} = \left(\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}\right)^2,$$

so $(***)$ is proved together with $(**)$.

Now we can show that (B2) is also fulfilled, take any point $x \in \mathbb{R}^n$ and assume that $x \in B_n(y, r) \cap B_n(z, s)$. If $t = \min\{r - d_n(x, y), s - d_n(x, z)\}$ then $B_n(x, t) \in \mathcal{B}$ and $x \in B_n(x, t) \subset B_n(y, r) \cap B_n(z, s)$. Indeed, if $x' \in B_n(x, t)$ is an arbitrary point then $d_n(x', x) < t \leqslant r - d_n(x, y)$ and hence $d_n(x', y) \leqslant d_n(x', x) + d_n(x, y) < r$, which implies $x' \in B_n(y, r)$, i.e., $B_n(x, t) \subset B_n(y, r)$. The proof of the inclusion $B_n(x, t) \subset B_n(z, s)$ is analogous.

(ii) Denote by v the topology on $\mathbb{R}^n$ which is the topology of the product of n copies of the space $(\mathbb{R}, \mathcal{N}_R)$. It suffices to show that $v = \mathcal{N}_R^n$. Take any $U \in v$. Given any $x = (x_1, \ldots, x_n) \in U$, there are $U_i \in \mathcal{N}_R$, $i \leqslant n$ such that $x \in U_1 \times \cdots \times U_n \subset U$. Thus, $x_i \in U_i$ for each i and hence there is $\varepsilon > 0$ for which $(x_i - \varepsilon, x_i + \varepsilon) \subset U_i$ for all $i \leqslant n$. We claim that $W_x = B_n(x, \varepsilon) \subset U$. Indeed, if $y = (y_1, \ldots, y_n) \in W_x$ then $|y_i - x_i| \leqslant d_n(x, y) < \varepsilon$ for each $i \leqslant n$ and therefore $y_i \in (x_i - \varepsilon, x_i + \varepsilon) \subset U_i$. Thus, $y \in U_1 \times \cdots \times U_n \subset U$ and we proved that $W_x \subset U$. Since $W_x \in \mathcal{N}_R^n$ for each $x \in U$, we have $U = \bigcup\{W_x : x \in U\} \in \mathcal{N}_R^n$ which proves that $v \subset \mathcal{N}_R^n$.

Now take any $U \in \mathcal{N}_R^n$ and any $x = (x_1, \ldots, x_n) \in U$. Since the open balls form a base of $\mathcal{N}_R^n$, there is $y \in U$ such that $x \in B_n(y, r) \subset U$ for some $r > 0$. If $s = r - d_n(x, y)$ then $B_n(x, s) \subset B_n(y, r) \subset U$. For $\delta = \frac{s}{\sqrt{n}}$ consider the set $V_x = U_1 \times \cdots \times U_n$ where $U_i = (x_i - \delta, x_i + \delta)$ for all $i \leqslant n$. Since $U_i \in \mathcal{N}_R$, we have $V_x \in \nu$. Now, if $z = (z_1, \ldots, z_n) \in V_x$ then $z_i \in (x_i - \delta, x_i + \delta)$ and therefore $|x_i - z_i| < \delta$ for each $i \leqslant n$. As a consequence, $d_n(x, z) < \sqrt{n \cdot \delta^2} = s$ which shows that $V_x \subset B_n(x, s) \subset U$ for each $x \in U$. Therefore, $U = \bigcup\{V_x : x \in U\} \in \nu$ and hence $\mathcal{N}_R^n \subset \nu$. This proves that $\nu = \mathcal{N}_R^n$ and we are done.

S.131. *Given a subset A of the space $\mathbb{R}^n$, we say that A is bounded if there is $x \in \mathbb{R}^n$ and $r > 0$ such that $A \subset B_n(x, r)$. Prove that a subspace K of the space $\mathbb{R}^n$ is compact if and only if K is a closed and bounded subset of $\mathbb{R}^n$.*

Solution. Suppose that $K \subset \mathbb{R}^n$ is compact. The space $\mathbb{R}^n$ is Hausdorff by Problems 019 and 105 so K is closed in $\mathbb{R}^n$ by Problem 121. Now, let x be the point of $\mathbb{R}^n$ with all its coordinates equal to zero. It is clear that $\mathbb{R}^n = \bigcup\{B_n(x, r) : r > 0\}$ and hence $K = \bigcup\{B_n(x, r) \cap K : r > 0\}$. The set $U_r = B_n(x, r) \cap K$ is open in K so the open cover $\gamma = \{U_r : r > 0\}$ has a finite subcover $\{U_{r_i} : i \in \{1, \ldots, m\}\}$. Now, it is evident that $K \subset B_n(x, r)$ where $r = r_1 + \cdots + r_m$ so necessity is proved.

Now suppose that K is a closed subspace of $\mathbb{R}^n$ and $K \subset B_n(y, s)$ for some $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $s > 0$. If $z = (z_1, \ldots, z_n) \in K$ then $z \in B_n(y, s)$ and $|z_i - y_i| \leqslant d_n(z, y) < s$ for each $i \leqslant n$. Thus $z_i \in [a_i, b_i]$ where $a_i = y_i - s$ and $b_i = y_i + s$ for all $i \leqslant n$. We proved in Problem 126 that $\mathbb{I}$ is compact and hence so is $[a_i, b_i]$ being homeomorphic to $\mathbb{I}$. Therefore $J = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a compact set and $K \subset J$. Now, K is closed in $\mathbb{R}^n \supset J$ and hence K is closed in J which implies that K is compact by Problem 120.

S.132. *Prove that the following conditions are equivalent for any space X:*

(*i*) *X is countably compact.*
(*ii*) *Any closed discrete subspace of X is finite.*
(*iii*) *Any infinite subset of X has an accumulation point.*
(*iv*) *If $\{F_n : n \in \omega\}$ is a sequence of closed non-empty subsets of X such that $F_{n+1} \subset F_n$ for every $n \in \omega$, then $\bigcap\{F_n : n \in \omega\} \neq \emptyset$.*
(*v*) *If γ is an open cover of the space X then there exists a finite set $A \subset X$ such that $\bigcup\{U \in \gamma : U \cap A \neq \emptyset\} = X$.*

Solution. (i) $\Rightarrow$ (ii). Suppose that D is an infinite closed discrete subspace of X. We can assume without loss of generality that $D = \{d_n : n \in \omega\}$, where $d_i \neq d_j$ if $i \neq j$. The set $U_n = X \setminus \{d_k : k \geqslant n\}$ is open in X for each n and $\bigcup\{U_n : n \in \omega\} = X$, i.e., $\gamma = \{U_n : n \in \omega\}$ is a countable open cover of X. If $\mu = \{U_{n_2}, \ldots, U_{n_k}\}$ is a finite subfamily of γ then $d_m \notin \bigcup\mu$ for any $m \geqslant n_1 + \cdots + n_k$. Therefore γ has no finite subcover which is a contradiction.

(ii) $\Rightarrow$ (iii). Take any infinite $A \subset X$. If A is not closed then any $x \in \overline{A} \setminus A$ is an accumulation point for A. If A is closed then it cannot be discrete and hence $\{x\}$ is not open in A for some $x \in A$. Thus, for any $U \in \tau(x, X)$, we have $U \cap A \neq \{x\}$ and therefore $U \cap (A\backslash\{x\}) \neq \emptyset$. As a consequence, $x \in \overline{A\backslash\{x\}}$ and x is an accumulation point of A.

(iii) ⇒ (iv). Let $\{F_n : n \in \omega\}$ be a sequence of closed non-empty subsets of X such that $F_{n+1} \subset F_n$ for every $n \in \omega$. Assume that $\bigcap\{F_n : n \in \omega\} = \emptyset$ and choose a point $x_n \in F_n$ for each $n \in \omega$. The set $A = \{x_n : n \in \omega\}$ cannot be finite because, for each $n \in \omega$, the point x_n can belong only to finitely many sets F_k and hence can coincide with at most finitely many points x_k. However, no $x \in X$ can be an accumulation point for A. Indeed, there is $k \in \omega$ such that $x \notin F_k$ and hence $U \cap A \subset \{x_1, \ldots, x_{k-1}\}$ where $U = X\backslash F_k \in \tau(x, X)$. Since X is a T_1-space, the set $W = X\backslash(\{x_1, \ldots, x_{k-1}\}\backslash\{x\})$ is open and $W \cap U$ is an open neighbourhood of x which does not meet $A\backslash\{x\}$. Hence $x \notin \overline{A\backslash\{x\}}$, i.e., x is not an accumulation point of A.

(iv) ⇒ (i). Let $\gamma = \{U_n : n \in \omega\}$ be an open cover of X. Consider the set $F_n = X\backslash(U_1 \cup \cdots \cup U_n)$ for all $n \in \omega$. It is clear that $F_{n+1} \subset F_n$ for all n and $\bigcap\{F_n : n \in \omega\} = \emptyset$. Since all F_n's are closed, one of them, say F_n, has to be empty by (iv). But this implies $U_1 \cup \cdots \cup U_n = X$ and hence $\{U_1, \ldots, U_n\}$ is a finite subcover of γ which proves that X is countably compact.

(i) ⇒ (v). Let X be countably compact. If γ is an arbitrary open cover of X, denote by γ_x the family $\{U \in \gamma : x \in U\}$ for every $x \in X$. Let us call a set $A \subset X$ *marked* if $\bigcup\{U \in \gamma : U \cap A \neq \emptyset\} = X$. We must prove that there is a marked finite set for γ. Suppose not and take any $x_0 \in X$. If we have chosen $x_0, \ldots, x_k \in X$ then $A = \{x_0, \ldots, x_k\}$ is not marked and hence there exists $x_{k+1} \in X\backslash\bigcup\{U \in \gamma : U \cap A \neq \emptyset\}$. This shows that we can construct by induction the set $D = \{x_i : i \in \omega\} \subset X$ so that $x_{k+1} \notin \bigcup\{\bigcup\gamma_{x_i} : i \leqslant k\}$. In particular, $x_i \neq x_j$ if $i \neq j$ and hence the set D is infinite. We claim that D has no accumulation points in X. Indeed, take any $x \in X$ and $U \in \gamma$ with $x \in U$. If $i < j$ and $x_i \in U$ then $U \in \gamma_{x_i}$ and hence $x_j \notin U$ by the construction of D. This shows that every point $x \in X$ has a neighbourhood which intersects at most one element of D. Therefore x cannot be an accumulation point of D which is a contradiction with (iii).

(v) ⇒ (i). Suppose that $D = \{d_n : n \in \omega\}$ is a closed discrete subset of X such that $d_n \neq d_m$ if $n \neq m$. Since X is Tychonoff, there exists a disjoint family $\{U_n : n \in \omega\} \subset \tau(X)$ such that $d_n \in U_n$ for each $n \in \omega$. The family $\gamma = \{X \setminus D\} \cup \{U_n : n \in \omega\}$ is an open cover of X. If $A \subset X$ is finite then A intersects only finitely many U_n's. Therefore, there is $m \in \omega$ such that $A \cap U_m = \emptyset$ and hence $x_m \notin \bigcup\{U \in \gamma : U \cap A \neq \emptyset\}$ which is a contradiction with the property (v) for X. Thus X is countably compact by (ii).

S.133. *Prove that a continuous image of a countably compact space is a countably compact space.*

Solution. Let $f : X \to Y$ be surjective continuous map. Suppose that X is countably compact. Given a countable open cover γ of the space Y, the family $\mu = \{f^{-1}(U) : U \in \gamma\}$ is a countable open cover of X. Since X is countably compact, there are $U_1, \ldots, U_n \in \gamma$ such that $X = f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n)$ and hence $Y = f(X) = U_1 \cup \cdots \cup U_n$, i.e., $\{U_1, \ldots, U_n\}$ is a finite subcover of γ. Therefore Y is also countably compact.

S.134. *Prove that a closed subspace of a countably compact space is a countably compact space.*

Solution. Given a countably compact space X and a closed $F \subset X$, if D is an infinite closed discrete subspace of F then D is an infinite closed discrete subspace of X which is a contradiction. Hence F is countably compact by Problem 132(ii).

S.135. *For an uncountable cardinal* κ, *define* $\Sigma = \{x \in \mathbb{I}^\kappa :$ *the set* $x^{-1}(\mathbb{I} \setminus \{0\})$ *is countable*$\}$. *Prove that*

(*i*) *The set* Σ *is dense in* $\mathbb{I}^\kappa$.
(*ii*) *If* $A \subset \Sigma$ *is countable, then* $\overline{A}$ *is compact (the closure is taken in* Σ*).*
(*iii*) Σ *Is a Fre´chet–Urysohn space.*
(*iv*) *The space* Σ *is countably compact and non-compact.*

Solution. (i). Fix any $f \in \mathbb{I}^\kappa$ and $U \in \tau(f, \mathbb{I}^\kappa)$. There exist $\alpha_1, \ldots, \alpha_n \in \kappa$ and $O_1, \ldots, O_n \in \tau(\mathbb{I})$ such that $f \in W = \{g \in \mathbb{I}^\kappa : g(\alpha_i) \in O_i$ for all $i \leqslant n\} \subset U$. Define a function $h : \kappa \to \mathbb{I}$ as follows: $h(\alpha_i) = f(\alpha_i)$ for all $i \leqslant n$ and $h(\alpha) = 0$ for all $\alpha \in \kappa \setminus \{\alpha_1, \ldots, \alpha_n\}$. It is clear that $h \in \Sigma$. Observe that $h \in W$ because $h(\alpha_i) = f(\alpha_i) \in O_i$ for all $i \leqslant n$. Consequently, $U \bigcap \Sigma \supset W \bigcap \Sigma \ni h$ and therefore $U \bigcap \Sigma \neq \emptyset$ which implies $f \in \overline{\Sigma}$. The point $f \in \mathbb{I}^\kappa$ being arbitrary we can conclude that $\overline{\Sigma} = \mathbb{I}^\kappa$.

(ii). For each $f \in A$, let $\text{supp}(f) = \{\alpha < \kappa : f(\alpha) \neq 0\}$. Since $\text{supp}(f)$ is countable for any $f \in \Sigma$, the set $S = \bigcup\{\text{supp}(f) : f \in A\}$ is countable. Let $\Sigma_S = \prod\{P_\alpha : \alpha < \kappa\}$, where $P_\alpha = \mathbb{I}$ for $\alpha \in S$ and $P_\alpha = \{0\}$ if $\alpha \in \kappa \setminus S$. It is clear that $\Sigma_S \subset \Sigma$ and $A \subset \Sigma_S$. Besides, Σ_S is a compact subspace of Σ being a product of compact spaces (Problem 125). Therefore Σ_S is closed in Σ (Problem 121) and the set $\overline{A} \subset \Sigma_S$ is compact being closed in Σ_S.

(iii). *Observation one*. Every first countable (and hence every second countable) space is Fréchet–Urysohn. Indeed, suppose that $x \in \overline{A}$ and $\{W_n : n \in \omega\}$ is a local base at x. If $U_n = \cap\{W_i : i \leqslant n\}$ for each $n \in \omega$ then the family $\{U_n : n \in \omega\}$ is also a local base at x. Choosing $x_n \in U_n \cap A$ for each $n \in \omega$ we obtain the sequence $\{x_n\} \subset A$ which converges to x.

Observation two. If a space X has a countable base then any $Y \subset X$ also has a countable base. To see this, take any base $\{U_n : n \in \omega\}$ in the space X and note that $\{U_n \cap Y : n \in \omega\}$ is a base in Y.

Observation three. The space $\mathbb{R}$ is second countable and hence so is $\mathbb{I}$ by observation two. To see that observation three holds, observe that the family $\{(a, b) : a, b \in \mathbb{Q}\}$ is a countable base in $\mathbb{R}$.

Observation four. If X is second countable and S is countable then X^S is also second countable. Indeed, if $\mathcal{B}$ is a countable base in X the family $\mathcal{C} = \{\prod_{s \in S} U_s :$ there is a finite set $A \subset S$ for which $U_s \in \mathcal{B}$ for all $s \in A$ and $U_s = X$ for all $s \in S \setminus A\}$ is a countable base for X^S. We will only show that $\mathcal{C}$ is base for X^S. All elements of $\mathcal{C}$ are standard open sets of the product so $\mathcal{C} \subset \tau(X^S)$. If $f \in U \in \tau(X^S)$ then there is a finite $A \subset S$ and $W_s \in \tau(X)$ for each $s \in A$ such that $f \in W \subset U$ where $W = \{g \in X^S : g(s) \in W_s$ for each $s \in A\}$. Since $\mathcal{B}$ is a base in X, there are $U_s \in \mathcal{B}$ such that $g(s) \in U_s \subset W_s$ for all $s \in A$. Letting $U_s = X$ for $s \in S \setminus A$ we obtain the set $\boldsymbol{C} = \prod_{s \in S} U_s$ which belongs to $\mathcal{C}$. Since $f \in \boldsymbol{C} \subset W \subset U$, this proves that $\mathcal{C}$ is a base in the space X^S.

Observation five. Given a countable $S \subset \kappa$, the space Σ_S is homeomorphic to $\mathbb{I}^S$. Here, as in (ii), $\Sigma_S = \prod\{P_\alpha : \alpha < \kappa\} \subset \Sigma$, where $P_\alpha = \mathbb{I}$ for $\alpha \in S$ and $P_\alpha = \{0\}$ if $\alpha \in \kappa\backslash S$. To prove this, consider the restriction $\pi_S : \Sigma \to \mathbb{I}^S$ given by the formula $\pi_S(f) = f|S$. The map π_S is continuous being a restriction of a continuous map to Σ (Problem 107). It is evident that $\varphi_S = \pi_S|\Sigma_S : \Sigma_S \to \mathbb{I}^S$ is a continuous bijection defined on the compact space Σ_S. Thus φ_S is a homeomorphism by Problem 123.

Observation six. If $x \in \Sigma$ and $x \in \overline{A}$ for some $A \subset \Sigma$ then there is a countable $B \subset A$ such that $x \in \overline{B}$. Let $S_0 = \text{supp}(x) \cup \omega$ (we add ω for the case when the support of x (see (ii)) is empty). Then $\pi_{S_0}(x) \in \mathbb{I}^{S_0}$ and $\mathbb{I}^{S_0}$ is second countable by Observations three and four. The map π_{S_0} is continuous so $\pi_{S_0}(x) \in \overline{\pi_{S_0}(A)}$. By Observation one, there is a countable $B_0 \subset A$ such that $\pi_{S_0}(x) \in \overline{\pi_{S_0}(B_0)}$. If we have countable sets $S_0 \subset \cdots \subset S_n \subset \kappa$ and $B_0 \subset \cdots \subset B_n \subset A$, let $S_{n+1} = S_n \cup (\bigcup\{\text{supp}(y) : y \in B_n\})$. The map $\pi_{S_{n+1}}$ is continuous and hence $\pi_{S_{n+1}}(x) \in \overline{\pi_{S_{n+1}}(A)}$. By Observation one, there is a countable $B'_{n+1} \subset A$ such that $\pi_{S_{n+1}}(x) \in \overline{\pi_{S_{n+1}}(B_{n+1})}$. Letting $B_{n+1} = B_n \cup B'_{n+1}$ we finish the inductive construction. The set $B = \cup_{n\in\omega} B_n \subset A$ is countable so it suffices to establish that $x \in \overline{B}$. Take any $U \in \tau(x, \Sigma)$. There exist $\alpha_1, \ldots, \alpha_n \in \kappa$ and $O_1, \ldots, O_n \in \tau(\mathbb{I})$ such that $x \in W \subset U$ for the open set $W = \{y \in \Sigma : y(\alpha_i) \in O_i$ for all $i \leqslant n\}$. Since $S_{i+1} \supset S_i$ for all $i \in \omega$, for the set $S = \bigcup\{S_i : i \in \omega\}$ we can find a natural m such that $S \cap \{\alpha_1, \ldots, \alpha_n\} = S_m \cap \{\alpha_1, \ldots, \alpha_n\}$. Recall that $\pi_{S_m}(x) \in \overline{\pi_{S_m}(B_m)}$ which implies that there is $z \in B_m$ such that $z(\alpha_i) \in O_i$ for any $\alpha_i \in P = S_m \cap \{\alpha_1, \ldots, \alpha_n\}$. Now, if $\alpha_i \in \{\alpha_1, \ldots, \alpha_n\}\backslash P$ then $\alpha_i \notin S$ and hence $z(\alpha_i) = 0$ because $\text{supp}(z) \subset S$. Note also that $\text{supp}(x) \subset S$ and therefore $x(\alpha_i) = 0$ for all $\alpha_i \in \{\alpha_1, \ldots, \alpha_n\}\backslash P$. This shows that $z(\alpha_i) = 0 = x(\alpha_i) \in O_i$ for all $\alpha_i \in \{\alpha_1, \ldots, \alpha_n\}\backslash P$ and $z \in W \subset U$. We proved that $U \cap B \neq \emptyset$ for any $U \in \tau(x, \Sigma)$ and hence $x \in \overline{B}$.

Now it is easy to finish the proof of (iii). Note that $B \cup \{x\} \subset \Sigma_S$ and hence the space $B \cup \{x\} \subset \Sigma_S$ is second countable by Observations four and five. By Observation one there is a sequence $\{x_n : n \in \omega\} \subset B \subset A$ which converges to x.

(iv). To see that Σ is countably compact, take any countably infinite closed discrete $D \subset \Sigma$. By (i) the set $D = \overline{D}$ is compact which is a contradiction (Problem 132(ii)). Hence all closed discrete subsets of Σ are finite and hence Σ is countably compact by Problem 132(ii). The space Σ is not compact for otherwise it would be closed in the Tychonoff space $\mathbb{I}^\kappa$. This, together with (i) would imply $\Sigma = \mathbb{I}^\kappa$. However, $x \notin \Sigma$ if $x(\alpha) = 1$ for all $\alpha \in \kappa$. The obtained contradiction proves that Σ is a countably compact non-compact space.

S.136. *Prove that the following conditions are equivalent for any space X:*

(*i*) *X is pseudocompact.*
(*ii*) *Any locally finite family of non-empty open subsets of X is finite.*
(*iii*) *Any discrete family of non-empty open subsets of X is finite.*
(*iv*) *For every decreasing sequence* $U_0 \supset U_1 \supset \cdots$ *of non-empty open subsets of X, the intersection* $\cap\{\overline{U}_n : n \in \omega\}$ *is non-empty.*
(*v*) *For every countable centered family* $\{U_n : n \in \omega\}$ *of open subsets of X, the intersection* $\cap\{\overline{U}_n : n \in \omega\}$ *is non-empty.*

Solution. (i) $\Rightarrow$ (ii). Suppose that X is pseudocompact and $\{U_n : n \in \mathbb{N}\} \subset \tau^*(X)$ is locally finite. Take $x_n \in U_n$ for all $n \in \mathbb{N}$. Since X is a Tychonoff space, there exists a continuous function $f_n : X \to [0, 1]$ such that $f_n(x_n) = 1$ and $f_n|(X\setminus U_n) \equiv 0$. The function $f = \Sigma\{n \cdot f_n : n \in \mathbb{N}\} : X \to \mathbb{R}$ is continuous. Indeed, if $x \in X$ and $\varepsilon > 0$ then there exists $U \in \tau(x, X)$ such that U intersects only finitely many sets U_i, say, $U_{k_1}, \ldots, U_{k_n}$. Since $f_i(x) = 0$ for all $x \in X \setminus U_i$, we have $f(x) = k_1 \cdot f_{k_2}, + \cdots + k_n \cdot f_{k_n}$ and hence $f|U$ is continuous being a finite sum of continuous functions (Problem 027(i)). Therefore, there is $V \in \tau(x, U)$ such that $f(V) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. Observing that V is also open in X we conclude that f is continuous at x. Thus f is continuous on X and unbounded which contradicts pseudocompactness of X.

(ii) $\Rightarrow$ (iii). This is true because every discrete family is locally finite.

(iii) $\Rightarrow$ (iv). Suppose that $\bigcap\{\overline{U}_n : n \in \omega\} = \emptyset$. Take a point $x \in U_0$. There exists $m = m(0) \in \omega$ such that $x \notin \overline{U}_m$. By regularity of X there exists an open set V_0 such that $x \in V_0 \subset \overline{V}_0 \subset U_0 \cap (X \setminus \overline{U}_m)$. Suppose that we have constructed sets $V_0, \ldots, V_n \in \tau^*(X)$ with the following properties:

(1) $V_i \subset U_i$ for all $i \leqslant n$.
(2) For each $i \leqslant n$, we have $\overline{V}_i \cap \overline{U}_m = \emptyset$ for some $m = m(i) \in \omega$.
(3) $\overline{V}_i \cap \overline{V}_j = \emptyset$ if $i \neq j$.

Let $k = m(0) + \cdots + m(n) + n$. It is clear that $\overline{V}_i \cap \overline{U}_j = \emptyset$ for any $i \leqslant n$ and any $j \geqslant k$. Take a point $x \in U_k$ and choose $m = m(n + 1)$ such that $x \notin \overline{U}_m$. Choose any $V_{n+1} \in \tau(X)$ such that $x \in V_{n+1} \subset \overline{V}_{n+1} \subset U_k \setminus \overline{U}_m$. It is evident that for the family $\{V_0, \ldots, V_n, V_{n+1}\}$ the properties (1)–(3) hold. To get a contradiction we will prove that the family $\gamma = \{V_i : i \in \omega\} \subset \tau^*(X)$ is discrete. So, let $x \in X$. There exists $m \in \omega$ such that $x \notin \overline{U}_m$. The property (1) shows that $V = X \setminus \overline{U}_m$ is neighbourhood of x which can (possibly) intersect only elements of the family $\mu = \{V_0, \ldots, V_{m-1}\}$. Since the family $\{\overline{V}_0, \ldots, \overline{V}_{m-1}\}$ is disjoint, there is $W \in \tau(x, X)$ such that W meets at most one element of μ. Then $V \cap W \in \tau(x, X)$ intersects at most one element of the family γ which shows that γ is discrete, a contradiction.

(iv) $\Rightarrow$ (v). Let $V_n = \bigcap\{U_i : i \leqslant n\}$ for all $n \in \omega$. The fact that $\{U_n : n \in \omega\}$ is centered implies that the decreasing family $\{V_n : n \in \omega\}$ consists of non-empty open sets. Applying (iv) we obtain $\bigcap\{\overline{U}_n : n \in \omega\} \supset \bigcap\{\overline{V}_n : n \in \omega\} \neq \emptyset$.

(v) $\Rightarrow$ (i). If X is not pseudocompact then fix an unbounded continuous function f on X. Then f^2 is also unbounded so we can consider that $f(x) \geqslant 0$ for all $x \in X$. The set $U_n = f^{-1}((n, +\infty))$ is open and non-empty for each $n \in \omega$. Since the family $\{U_n : n \in \omega\}$ is decreasing, it is centered. Observe also that $\overline{U}_{n+1} \subset f^{-1}([n + 1, +\infty)) \subset f^{-1}((n, +\infty)) = U_n$ and hence $\bigcap\{\overline{U}_n : n \in \omega\} = \bigcap\{U_n : n \in \omega\} = \emptyset$ which contradicts (v).

S.137. *Prove that any countably compact space is pseudocompact. Show that a pseudocompact normal space is countably compact.*

Solution. If X is countably compact and $\{U_n : n \in \omega\} \subset \tau^*(X)$ is a decreasing family then $\mu = \{\overline{U}_n : n \in \omega\}$ is also a decreasing family of non-empty closed sets of X. Apply Problem 132(iv) to conclude that $\bigcap\mu \neq \emptyset$ and hence X is pseudocompact.

Now, suppose that X is a normal pseudocompact space which is not countably compact. By Problem 132(ii) there exists a closed discrete set $D = \{x_n : n \in \omega\} \subset X$ such that $x_i \neq x_j$ if $i \neq j$. The function $f : D \to \mathbb{R}$ defined by $f(x_n) = n$ is continuous on D and hence, by normality of X, there is a continuous $F : X \to \mathbb{R}$ such that $F|D = f$ (see Problem 032). Since F is not bounded, the space X is not pseudocompact, a contradiction.

S.138. *Prove that any pseudocompact Lindelöf space is compact.*

Solution. Any Lindelöf space is normal (Problem 124) and hence any Lindelöf pseudocompact space is countably compact (Problem 137). Now, if γ is any open cover of X, then there is a countable $\gamma' \subset \gamma$ such that $X = \bigcup\gamma'$ because X is Lindelöf. Applying countable compactness of X we can conclude that the countable open cover γ' of X has a finite subcover μ. Of course, μ is also a finite subcover of γ so X is compact.

S.139. *Prove that a continuous image of a pseudocompact space is a pseudocompact space.*

Solution. Let $\varphi : X \to Y$ be a surjective continuous map. If X is pseudocompact and $f : Y \to \mathbb{R}$ is a continuous function then $f \circ \varphi$ is also a continuous real-valued function on X. Therefore $f \circ \varphi$ is bounded and hence so is f. This proves that Y is pseudocompact.

S.140. *Prove that any condensation of a pseudocompact space onto a second countable space is a homeomorphism.*

Solution. *Observation one.* Any second countable space Y is Lindelöf. To prove it, take a countable base $\mathcal{B}$ of the space Y. If γ is an open cover of Y, call $U \in \mathcal{B}$ *marked* if there is some $V \in \gamma$ such that $U \subset V$. Let $\mathcal{B}'$ be the family of all marked elements of $\mathcal{B}$. We assert that $\bigcup\mathcal{B}' = Y$. Indeed, if $y \in Y$ there is $V \in \gamma$ such that $y \in V$. Since $\mathcal{B}$ is a base, there is $U \in \mathcal{B}$ such that $x \in U \subset V$. Thus U is marked and $y \in U$. For each $U \in \mathcal{B}'$ take $O(U) \in \gamma$ with $U \subset O(U)$. The family $\{O(U) : U \in \mathcal{B}'\}$ is a countable subcover of γ.

Observation two. If Y is pseudocompact then $F = \overline{U}$ is pseudocompact for any $U \in \tau(Y)$. To prove this, take any decreasing family $\{U_n : n \in \omega\} \subset \tau^*(F)$. The family $\{U_n \cap U : n \in \omega\}$ is also decreasing and consists of non-empty open subsets of Y. Apply pseudocompactness of Y to conclude that $\bigcap\{\mathrm{cl}_F(U_n) : n \in \omega\} \supset \bigcap\{\mathrm{cl}_Y(U_n \cap U) : n \in \omega\} \neq \emptyset$ and hence $\overline{U}$ is pseudocompact.

Observation three. Any pseudocompact second countable space is compact. Indeed, it is Lindelöf by Observation one and hence compact by Problem 138.

Let $f : X \to M$ be a condensation of a pseudocompact space X onto a second countable space M. To show that f is a homeomorphism it suffices to prove that $f(F)$ is closed in M for any closed $F \subset X$. It is an easy consequence of regularity of X that $F = \bigcap\{\overline{U} : U \in \tau(F, X)\}$. Since f is a bijection, we have $f(F) = \bigcap\{f(\overline{U}) : U \in \tau(F, X)\}$. But each set $\overline{U}$ is pseudocompact by Observation two. Therefore $f(\overline{U})$ is compact by Problems 138, 139 and Observation three. Any compact subspace of M is closed in M (Problem 121) and hence $f(F)$ is closed being intersection of closed sets.

S.141. *Call a family $\mathcal{C} \subset \exp(\omega)$ almost disjoint, if every $C \in \mathcal{C}$ is infinite and $C \cap D$ is finite if C and D are distinct elements of $\mathcal{C}$. A family $\mathcal{C} \subset \omega$ is maximal almost disjoint if it is almost disjoint and, for any almost disjoint $\mathcal{D} \supset \mathcal{C}$, we have $\mathcal{D} = \mathcal{C}$. Prove that*

(*i*) *Every almost disjoint $\mathcal{C} \subset \exp(\omega)$ is contained in a maximal almost disjoint family $\mathcal{D} \subset \exp(\omega)$.*
(*ii*) *Every maximal almost disjoint infinite family on ω is uncountable.*
(*iii*) *There exists a maximal almost disjoint family $\mathcal{C} \subset \exp(\omega)$ with $|\mathcal{C}| = \mathfrak{c}$.*

Solution. (i) Let $\mathcal{P} = \{\mathcal{E} : \mathcal{C} \subset \mathcal{E}$ and $\mathcal{E}$ is almost disjoint$\}$. The partial order on $\mathcal{P}$ is the inclusion. It is clear that any maximal element of $\mathcal{P}$ will be a maximal almost disjoint family so it suffices to prove that every chain in $\mathcal{P}$ has an upper bound in $\mathcal{P}$. So take any chain $\mathbf{C}$ in $\mathcal{P}$. Let $\mathcal{U} = \bigcup \mathbf{C}$. It is clear that $\mathcal{U}$ is an upper bound for $\mathbf{C}$ and hence it is sufficient to prove that $\mathcal{U}$ belongs to $\mathcal{P}$, i.e., that $\mathcal{U}$ is almost disjoint. To see this, take distinct $A, B \in \mathcal{U}$. There exist $\mathcal{E}_1, \mathcal{E}_2 \in \mathbf{C}$ such that $A \in \mathcal{E}_1$ and $B \in \mathcal{E}_2$. Since $\mathbf{C}$ is a chain, one of the families $\mathcal{E}_1, \mathcal{E}_2$ contains the other and hence $A, B \in \mathcal{E}_i$ for some $i \in \{1, 2\}$. Since $\mathcal{E}_i$ is almost disjoint, the set $A \cap B$ is finite which proves that $\mathcal{U}$ is almost disjoint. Applying the Zorn Lemma we conclude that there exists a maximal almost disjoint family $\mathcal{D}$ which contains $\mathcal{C}$.

(ii) Suppose that an infinite almost disjoint family $\mathcal{D}$ is countable. We will show that $\mathcal{D}$ is not maximal, i.e., there is a set $A \subset \omega$ such that $A \notin \mathcal{D}$ and $\mathcal{D} \cup \{A\}$ is almost disjoint. Take some enumeration $\{D_n : n \in \omega\}$ of the set $\mathcal{D}$ such that $D_n \neq D_m$ if $n \neq m$. Observe that, for any $n \in \omega$, the set $\omega \backslash (D_0 \cup \cdots \cup D_n)$ is infinite for otherwise an infinite subset of the set D_{n+1} would be covered by the sets $D_0, \ldots, D_n$ and hence $D_i \cap D_{n+1}$ is infinite for some number $i \leqslant n$ which is a contradiction. This makes it possible to choose by induction points $\{x_i : i \in \omega\}$ in such a way that $x_i \notin (D_0 \cup \cdots \cup D_i) \cup \{x_0, \ldots, x_{i-1}\}$ for each $i \in \omega$. The set $A = \{x_i : i \in \omega\}$ does not belong to $\mathcal{D}$ and $A \cap D_n \subset \{x_0, \ldots, x_{n-1}\}$ for each $n \in \omega$ which shows that $\mathcal{D} \cup \{A\}$ is almost disjoint, a contradiction.

(iii) Since every almost disjoint family is contained in a maximal almost disjoint family, it suffices to prove that there is an almost disjoint family on ω of cardinality $\mathfrak{c}$. The sets ω and $\mathbb{Q}$ having the same cardinality it is sufficient to construct such a family on $\mathbb{Q}$. For every irrational $r \in \mathbb{R}$, take a sequence $A_r \subset \mathbb{Q}$ which converges to r. It is clear that the family $\{A_r : r \in \mathbb{R} \backslash \mathbb{Q}\}$ is almost disjoint, has cardinality $\mathfrak{c}$ and consists of subsets of $\mathbb{Q}$.

S.142. *Let $\mathcal{M}$ be an infinite (and hence uncountable) maximal almost disjoint family in ω and $M = \omega \cup \mathcal{M}$. If $x \in \omega$, let $\mathcal{B}_x = \{x\}$. Given $x \in \mathcal{M}$, define $\mathcal{B}_x = \{\{x\} \cup (x \backslash A) : A$ is a finite subset of $x\}$ (remember that for $x \in \mathcal{M}$, we can consider x to be a point of M or a subset of ω). Prove that*

(*i*) *The families $\{\mathcal{B}_x : x \in M\}$ generate a topology τ_M on M as local bases (see Problem 007).*
(*ii*) *The space (M, τ_M) (called Mrowka space) is a Fréchet–Urysohn separable space; we will further denote it by M.*

(*iii*) *The space M is locally compact (i.e., each point of M has a compact neighbourhood) and pseudocompact.*
(*iv*) *The subspace $\mathcal{M}$ is closed and discrete in M and therefore the space M is not countably compact. This also shows that a closed subspace of a pseudocompact space is not necessarily pseudocompact.*

Solution. (i) The properties 007(i)–(iii) are evidently fulfilled if $x \in \omega$. If $x \in \mathcal{M}$ then x cannot belong to $\bigcup \mathcal{B}_y$ for any $y \neq x$. This proves that (iii) holds for any point $x \in \mathcal{M}$. The property (i) is an evident consequence of the definition of $\mathcal{B}_x$ and (ii) holds because the intersection of any finite number of the elements of $\mathcal{B}_x$ belongs to $\mathcal{B}_x$ for any $x \in M$.

(ii) It is clear that, for any $x \in M$ and any $U \in \mathcal{B}_x$, we have $U \cap \omega \neq \emptyset$ so the countable set ω is dense in (M, τ_M). To prove that M is Fréchet–Urysohn, take any $x \in M$ such that $x \in \overline{A}$ for some $A \subset M$. If $x \in \omega$ then $x \in A$ and hence there is a trivial sequence in A which converges to x. If $x \notin A$ then $x \in \mathcal{M}$ and hence $A \cap x$ is infinite. Let $\{a_n : n \in \omega\}$ be some enumeration of the set $A \cap x$. If $U \in \tau(x, M)$ then there is $V \in \mathcal{B}_x$ with $V \subset U$ and hence there is a finite $B \subset \omega$ for which $V = x \backslash B \subset U$. Since $x \backslash U$ is finite, there is $m \in \omega$ such that $a_n \in x \backslash B \subset U$ for all $n \geqslant m$, i.e., the sequence $\{a_n\} \subset A$ converges to x.

(iii) If $x \in \omega$ then $\{x\}$ is a compact neighbourhood of the point x. If $x \in \mathcal{M}$ then $\{x\} \cup x$ is a neighbourhood of x which is a convergent sequence and hence compact by Problem 129. This proves that M is locally compact. To see that it is pseudocompact, suppose not and take any infinite discrete family $\{U_n : n \in \omega\}$ of open subsets of M. Since ω is dense in M, we can choose $x_n \in \omega \cap U_n$ for each $n \in \omega$. It is clear that the family $\{\{x_n\} : n \in \omega\}$ is also an infinite discrete family of open sets of M. Since $\mathcal{M}$ is a maximal almost disjoint family, it is impossible that $A = \{x_n : n \in \omega\}$ intersect each $x \in \mathcal{M}$ in a finite set for otherwise we would be able to add A to $\mathcal{M}$ obtaining a strictly larger almost disjoint family. So fix $x \in \mathcal{M}$ with $x \cap A$ infinite. The definition of the local base at x implies that every neighbourhood of x contains infinitely many points x_n and hence it intersects infinitely many open sets of the family $\{\{x_n\} : n \in \omega\}$ which is a contradiction with the fact that $\{\{x_n\} : n \in \omega\}$ is discrete. Hence M is pseudocompact.

(iv) If $x \in M \backslash \mathcal{M}$ then $x \in \omega$ and $\{x\}$ is a neighbourhood of x which does not meet $\mathcal{M}$. This proves that $\mathcal{M}$ is closed in M. Since $V_x = \{x\} \cup x$ is a neighbourhood of x and $V_x \cap \mathcal{M} = \{x\}$ for each $x \in \mathcal{M}$, the subspace $\mathcal{M}$ is discrete. Since a discrete space is pseudocompact iff it is finite, $\mathcal{M}$ is a closed subspace of M which is not pseudocompact.

S.143. *Show that a sequence $\{f_n : n \in \omega\} \subset C_p(X)$ converges to a function $f : X \to \mathbb{R}$ if and only if the numeric sequence $\{f_n(x) : n \in \omega\}$ converges to $f(x)$ for every $x \in X$.*

Solution. Suppose that $f_n \to f$. Given $\varepsilon > 0$, the set $O(f, x, \varepsilon)$ is open and contains f which implies that there is $m \in \omega$ such that $f_n \in O(f, x, \varepsilon)$ for all $n \geqslant m$. Thus, $|f_n(x) - f(x)| < \varepsilon$ for all $n \geqslant m$ and hence $\{f_n(x)\}$ converges to $f(x)$.

Now, assume that $\{f_n(x)\}$ converges to $f(x)$ for each $x \in X$. For an arbitrary set $U \in \tau(f, C_p(X))$ there are $x_1, \ldots, x_k \in X$ and $\varepsilon > 0$ with $O(f, x_1, \ldots, x_k, \varepsilon) \subset U$. Since $\{f_n(x_i)\}$ converges to $f(x_i)$ for each $i \leqslant k$, there exists $m \in \omega$ such that $|f_n(x_i) - f(x_i)| < \varepsilon$ for all $n \geqslant m$ and $i \in \{0, \ldots, k\}$. This means that we have $f_n \in O(f, x_1, \ldots, x_n, \varepsilon) \subset U$ for all $n \geqslant m$ and hence the sequence $\{f_n\}$ converges to f.

S.144. *Suppose that X is an arbitrary set. Given a family γ of subsets of X, let* $\lim \gamma = \{x \in X : |\{U \in \gamma : x \notin U\}| < \omega\}$. *We call γ an ω-cover of X, if for any finite $A \subset X$, there is $U \in \gamma$ such that $A \subset U$. Prove that the following conditions are equivalent:*

(*i*) *$C_p(X)$ is a Fréchet–Urysohn space.*
(*ii*) *For any open ω-cover γ of the space X, there is a countable $\xi \subset \gamma$ such that* $\lim \xi = X$.
(*iii*) *For any sequence $\{\gamma_n\}_{n\in\omega}$ of open ω-covers of X, one can choose $U_n \in \gamma_n$ for each n, in such a way that* $\lim\{U_n : n \in \omega\} = X$.

Solution. (i) ⇒ (ii). If $X \in \gamma$ then $\xi = \{X\}$ does the work. If not, consider the set $P = \{f \in C_p(X) : \text{supp}(f) \subset U \text{ for some } U \in \gamma\}$. Here, of course, $\text{supp}(f) = f^{-1}(\mathbb{R}\backslash\{0\})$. We claim that $u \in \overline{P} \setminus P$, where $u(x) = 1$ for all $x \in X$. Indeed, $u \notin P$ because $\text{supp}(u) = X \notin \gamma$. Now, if $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ there is $U \in \gamma$ with $\{x_1, \ldots, x_n\} \subset U$. It is an easy consequence of the Tychonoff property of X that there is $f \in C_p(X)$ such that $f(X\backslash U) = \{0\}$ and $f(x_i) = 1$ for all $i \leqslant n$. Thus, we have $f \in O(u, x_1, \ldots, x_n, \varepsilon)$ and hence $u \in \overline{P}$. By the Fréchet–Urysohn property of $C_p(X)$ there is a sequence $\{f_n : n \in \omega\} \subset P$ with $f_n \to u$. By the choice of f_n there is $U_n \in \gamma$ such that $\text{supp}(f_n) \subset U_n$ for each $n \in \omega$. We are going to prove that $\xi = \{U_n : n \in \omega\} \to X$. Indeed, if $x \in X$ then $W = O(u, x, \frac{1}{2})$ is a neighbourhood of u and hence there is $m \in \omega$ such that $f_n \in W$ for all $n \geqslant m$. This implies, in particular, that $f_n(x) \neq 0$ and hence $x \in \text{supp}(f_n) \subset U_n$ for all $n \geqslant m$. This proves that $\xi \to X$.

(ii) ⇒ (iii). Call a sequence $\{\mu_n : n \in \omega\}$ of open ω-covers of X, *special* if we can choose $V_n \in \mu_n$ for each $n \in \omega$ in such a way that $\{V_n : n \in \omega\} \to X$. We must prove that any sequence of open ω-covers is special if (ii) holds. Given families $\mathcal{U}, \mathcal{U}' \subset \exp(X)$, say that $\mathcal{U} < \mathcal{U}'$ if, for any $U \in \mathcal{U}$, there is $U' \in \mathcal{U}'$ such that $U \subset U'$. If we have families $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \exp(X)$, let $\mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n = \{U_1 \cap \cdots \cap U_n : U_i \in \mathcal{U}_i$ for all $i \leqslant n\}$. It is clear that $\mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n < \mathcal{U}_i$ for every $i \leqslant n$. Call a sequence of open covers $\{\mu_n : n \in \omega\}$ *decreasing* if $\mu_{n+1} < \mu_n$ for each $n \in \omega$.

Observation one. The family $\mathcal{U} = \mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n$ is an ω-cover of X if so is $\mathcal{U}_i$ for every $i \leqslant n$. The proof is straightforward.

Observation two. If a sequence $\{\mu_n' : n \in \omega\}$ is special and $\mu_n' < \mu_n$ for each $n \in \omega$ then the sequence $\{\mu_n : n \in \omega\}$ is also special. This follows from the evident fact that, if $\{V_n' : n \in \omega\} \to X$ and $V_n' \subset V_n$ for each $n \in \omega$ then we have $\{V_n : n \in \omega\} \to X$.

Observation three. We can reduce our considerations to the decreasing sequences of open ω-covers. More exactly, if every decreasing sequence of open ω-covers is special then every sequence of open ω-covers is special. Indeed, if $\{\mu_n : n \in \omega\}$ is a sequence of open ω-covers then the sequence $\{\mu_n' : n \in \omega\}$ is decreasing if $\mu'_n = \mu_0 \wedge \cdots \wedge \mu_n$ for all $n \in \omega$. The sequence $\{\mu'_n : n \in \omega\}$ is special and $\mu'_n < \mu_n$ for each n so applying Observation two we conclude that the sequence $\{\mu_n : n \in \omega\}$ is also special.

Observation four: A decreasing sequence $\{\mu_n : n \in \omega\}$ of open ω-covers of the space X is special if it has a special subsequence, i.e., if there is an increasing sequence $\{k_n : n \in \omega\}$ of natural numbers such that we can choose $U_{k_i} \in \mu_{k_i}$ for every $i \in \omega$ so that $\{U_{k_i} : i \in \omega\} \to X$. To see this, note that, for each $i \in \omega$ there are sets $U_j \in \mu_j$ for each $j \in \{k_i + 1, \ldots, k_{i+1} - 1\}$ for which $U_j \supset U_{j+1}$ for each $j \in \{k_i + 1, \ldots, k_{i+1} - 1\}$. These sets U_j can be constructed "backwards" using the fact that our sequence is decreasing and starting from $U_{k_{i+1}}$. The first step is to observe that $\mu_{k_{i+1}} < \mu_{k_{i+1}-1}$ and hence there is $U_{k_{i+1}-1} \in \mu_{k_{i+1}-1}$ with $U_{k_{i+1}} \subset U_{k_{i+1}-1}$. Analogously, there is $U_{k_{i+1}-2} \in \mu_{k_{i+1}-2}$ such that $U_{k_{i+1}-2} \supset U_{k_{i+1}-1}$ and so on. Now it is immediate that, after we construct $U_j \in \mu_j$ for all $j \in \omega \setminus \{k_i : i \in \omega\}$ we will have the sequence $\{U_i : i \in \omega\} \to X$ with $U_i \in \mu_i$ for all $i \in \omega$ which proves that the sequence $\{\mu_n : n \in \omega\}$ is special.

Now, take a decreasing sequence $\{\gamma_n : n \in \omega\}$ of open ω-covers of X. If X is finite then $X \in \gamma_n$ for each $n \in \omega$ so letting $U_n = X$ for all n we obtain the desired sequence $\{U_i : i \in \omega\} \to X$. If X is infinite we can choose a set $\{x_n : n \in \omega\} \subset X$ such that $x_i \neq x_j$ for $i \neq j$. Let $\mu_n = \{U \setminus \{x_n\} : U \in \gamma_n\}$ for each $n \in \omega$. It is easy to see that $\mu = \bigcup\{\mu_n : n \in \omega\}$ is an ω-cover of X. Use (ii) to choose a sequence $\{U_n : n \in \omega\} \subset \mu$ such that $\{U_i : i \in \omega\} \to X$. We have $U_i \in \mu_{k_i}$ for each $i \in \omega$. The sequence $\{k_i : i \in \omega\}$ cannot be bounded by a number m for otherwise $\{x_0, \ldots, x_m\}$ is not covered by any U_i which is a contradiction. This shows that the sequence $\{\mu_n : n \in \omega\}$ has a special subsequence. Therefore $\{\gamma_n : n \in \omega\}$ also has a special subsequence because $\mu_n < \gamma_n$ for each $n \in \omega$. Apply Observation four to conclude that $\{\gamma_n : n \in \omega\}$ is special.

(iii) $\Rightarrow$ (i). Let $u \in C_p(X)$ be the function equal to zero at all points of X. Take any $A \subset C_p(X)$ and $f \in \overline{A}$. Then $u \in \overline{A + (-f)}$ by Problem 079. If we find a sequence $S \subset A + (-f)$ convergent to u then the sequence $S + f$ is contained in A and converges to f by Problem 079. This shows that, without loss of generality, we can assume that $f = u$.

For each $n \in \mathbb{N}$, let $\gamma_n = \{g^{-1}((-\frac{1}{n}, \frac{1}{n})) : g \in A\}$. It is straightforward that γ_n is an ω-cover of X for all $n \in \mathbb{N}$. Take $U_n \in \gamma_n$ with $\{U_n : n \in \mathbb{N}\} \to X$. There is $f_n \in A$ with $U_n = f_n^{-1}((-\frac{1}{n}, \frac{1}{n}))$ for each $n \in \mathbb{N}$. To show that $f_n \to u$ it suffices, by Problem 143, to show that $f_n(x) \to u(x) = 0$ for every $x \in X$. So take any $x \in X$ and $\varepsilon > 0$. There is $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$ and $x \in U_n$ for all $n \geqslant m$. But $x \in U_n$ implies $|f_n(x)| < \frac{1}{n} \leqslant \frac{1}{m} < \varepsilon$ for all $n \geqslant m$. Therefore $f_n(x) \to 0 = u(x)$ for any $x \in X$ and hence $f_n \to u$.

S.145. *Prove that, if $C_p(X)$ is a Fréchet–Urysohn space, then $(C_p(X))^\omega$ is also a Fréchet–Urysohn space.*

Solution. Let $X_n = X$ for all $n \in \omega$ and denote the space $\bigoplus_{n\in\omega} X_n$ by Y. Then $(C_p(X))^\omega = \prod\{C_p(X_n) : n \in \omega\} = C_p(\bigoplus_{n\in\omega} X_n) = C_p(Y)$ by Problem 114 (here we use equalities to denote that the respective spaces are homeomorphic). Call a function $f \in C_p(Y)$ *a Fréchet–Urysohn point* if, for any $A \subset C_p(Y)$ with $f \in \overline{A}$, there is a sequence $\{f_n\} \subset A$ such that $f_n \to f$. It is clear that we must prove that all points of $C_p(Y)$ are Fréchet–Urysohn points. However, we may restrict ourselves to proving that only one point of $C_p(Y)$ is Fréchet–Urysohn. Indeed, if a point $g \in C_p(Y)$ is Fréchet–Urysohn and $f \in \overline{A}$ then $g \in \overline{A + (g - f)}$ by Problem 079 and hence there is a sequence $\{g_n\} \subset A + (g - f)$ such that $\{g_n\} \to g$. Then $f_n = g_n + f - g \in A$ for all $n \in \omega$ and $\{f_n\} \to f$ again by Problem 079.

Therefore, we can take any point of the space $(C_p(X))^\omega$ and establish that $(C_p(X))^\omega$ is Fréchet–Urysohn at this point. To do so, observe that $(C_p(X))^\omega$ is homeomorphic to $C_p(X, \mathbb{R}^\omega)$ (Problem 112) and consider a point $u_0 \in \mathbb{R}^\omega$ defined by $u_0(n) = 0$ for all $n \in \omega$. The function $f \in C_p(X, \mathbb{R}^\omega)$ for which we will prove the Fréchet–Urysohn property, will be defined by $f(x) = u_0$ for all $x \in X$. So take any $A \subset C_p(X, \mathbb{R}^\omega)$ with $f \in \overline{A}$. The space $\mathbb{R}^\omega$ is second countable (see Observations three and four of S.135), so it is possible to fix a countable local base $\{O_n : n \in \omega\}$ of $\mathbb{R}^\omega$ at the point u_0 such that $O_{n+1} \subset O_n$ for each $n \in \omega$. Let $\gamma_n = \{g^{-1}(O_n) : g \in A\}$ for all $n \in \omega$. It easily follows from $f \in \overline{A}$ that γ_n is an ω-cover of X for all $n \in \omega$. Apply Problem 144(iii) to conclude that we can pick $U_n \in \gamma_n$ for all $n \in \omega$ so that $\{U_n : n \in \omega\} \to X$. For each $n \in \omega$ take a function $f_n \in A$ such that $f_n^{-1}(O_n) = U_n$.

We claim that $f_n \to f$. Given arbitrary points $x_1, \dots, x_n \in X$, consider the set $W(x_1, \dots, x_n) = \{g \in C_p(X, \mathbb{R}^\omega) : g(x_i) \in O_n$ for all $i \leqslant n\}$. Note first that the family $\mathcal{B} = \{W(x_1, \dots, x_n) : n \in \mathbb{N}, x_1, \dots, x_n \in X\}$ is a local base of $C_p(X, \mathbb{R}^\omega)$ at f so if $f \in U \in \tau(C_p(X, \mathbb{R}^\omega))$ then there are $k \in \omega$ and points $x_1, \dots, x_k \in X$ such that $W(x_1, \dots, x_k) \subset U$. Since $U_n \to X$, there exists a number $m \in \omega$ such that $\{x_1, \dots, x_k\} \subset U_n = f_n^{-1}(O_n)$ for every $n \geqslant m$. This implies that $f_n \in W(x_1, \dots, x_k) \subset U$ for all $n \geqslant m$ and therefore $f_n \to f$ showing that $C_p(X, \mathbb{R}^\omega)$ is a Fréchet–Urysohn space.

S.146. *Prove that $C_p(A(\kappa))$ is a Fréchet–Urysohn space for any cardinal κ.*

Solution. Let γ be an ω-cover of the space $A(\kappa)$. Take any $U_0 \in \gamma$ with $a \in U_0$. Remember that $A(\kappa) = \kappa \cup \{a\}$ where a is the unique non-isolated point of $A(\kappa)$. The set $F_0 = A(\kappa) \backslash U_0$ is finite and hence there exists $U_1 \in \gamma$ such that $\{a\} \cup F_0 \subset U_1$. If we have $U_0, \dots, U_n \in \tau(a, A(\kappa))$ then the set $F_n = A(\kappa) \backslash (U_0 \cap \cdots \cap U_n)$ is finite and hence we can find $U_{n+1} \in \gamma$ for which $\{a\} \cup F_n \subset U_{n+1}$. Thus, the sequence $\{U_n : n \in \omega\}$ can be constructed by induction. We claim that $U_n \to X$. Indeed, take any $x \in A(\kappa)$. If $x \in \bigcap\{U_n : n \in \omega\}$ then there is nothing to prove. If $x \notin U_m$ for some $m \in \omega$ then, by construction, $x \in U_n$ for any $n > m$. This proves that $U_n \to X$. Now apply Problem 144(ii) to conclude that $C_p(A(\kappa))$ is Fréchet–Urysohn.

S.147. *Prove that $C_p(\mathbb{I})$ is not a Fréchet–Urysohn space.*

Solution. If $I = (a, b) \subset \mathbb{R}$, let $m(I) = b - a$. We will need the following statement.

Fact. Suppose that I_n is an open interval for each $n \in \mathbb{N}$ and $\mathbb{I} \subset \bigcup\{I_n : n \in \mathbb{N}\}$. Then $\sum_{n=1}^{\infty} m(I_n) \geqslant 2$, where this inequality is considered true if the sum of this series is infinite.

This fact is well known from measure theory or can be deduced easily from the Lebesgue's theorem on bounded convergence. We will give an elementary proof anyway. Since $\mathbb{I}$ is compact, we can choose a finite subfamily γ of the family $\{I_n : n \in \omega\}$ which still covers $\mathbb{I}$.

Observation. If I is an open interval, $J_1 \cup \cdots \cup J_k \subset I$, where the open intervals $J_1, \ldots, J_k$ are disjoint, then $m(J_1) + \cdots + m(J_k) \leqslant m(I)$.

We may assume that the right endpoint of J_i is less than or equal to the left endpoint of J_{i+1} for all $i \leqslant (k-1)$. Thus, $J_i = (a_i, b_i)$ and for all $i = 1, \ldots, k$, we have $a \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant a_3 < \cdots \leqslant a_{k-1} < b_{k-1} \leqslant a_k < b_k \leqslant b$.

As a consequence,

$$\sum\nolimits_{i=1}^{k} m(J_i) = \sum\nolimits_{i=1}^{k} (b_i - a_i) \leqslant a_1 - a + (b_1 - a_1) + (a_2 - b_1) \\ + (b_2 - a_2) + \cdots + (b_k - a_k) + (b - b_k) = b - a = m(I).$$

which finishes the proof of our observation.

Returning to our Fact, suppose that the points $x_0 = -1 < x_1 < \cdots < x_m = 1$ contain all endpoints of all intervals of γ which (the endpoints!) are inside $\mathbb{I}$. If $\Delta_i = (x_{i-1}, x_i)$ then $\sum_{i=1}^{m} m(\Delta_i) = 2$. Note also that each Δ_i is contained in some $U_i \in \gamma$. Let $\gamma' = \{U \in \gamma : U = U_i$ for some $i \leqslant m\}$. For each $U \in \gamma'$, let $\mu_U = \Sigma\{m(\Delta_i) : U = U_i\}$. Then by Observation, $m(U) \geqslant \mu_U$ for each $U \in \gamma'$. As a consequence, $\sum\{m(U) : U \in \gamma\} \geqslant \sum\{m(U) : U \in \gamma'\} \geqslant \sum\{\mu_U : U \in \gamma'\} = \sum_{i=1}^{m} m(\Delta_i) = 2$, and our fact is proved.

Now, it is very easy to show that the space $C_p(\mathbb{I})$ is not Fréchet–Urysohn. For each $n \in \mathbb{N}$, let $\gamma_n = \{U : U = I_1 \cup \cdots \cup I_k$ for some $k \in \mathbb{N}$, where $I_1, \ldots, I_k$ are disjoint open intervals such that $\sum_{j=1}^{k} m(I_j) < 2^{-n}\}$. It is easy to see that each $\mu_n = \{W \cap \mathbb{I} : W \in \gamma_n\}$ is an ω-cover of $\mathbb{I}$ for each $n \in \mathbb{N}$. If $C_p(\mathbb{I})$ is Fréchet–Urysohn, then we can choose $U_n \in \gamma_n$ such that $U_n \cap \mathbb{I} \to \mathbb{I}$ and, in particular, $\mathbb{I} \subset \bigcup\{U_n : n \in \mathbb{N}\}$. Now our fact says that $\Sigma\{U_n : n \in \mathbb{N}\} \geq 2$ while $\sum\{U_n : n \in \mathbb{N}\} \leqslant \sum_{i=1}^{\infty} 2^{-i} = 1$ which is a contradiction.

S.148. *Prove that the following properties are equivalent for any space X and any infinite cardinal κ:*

(i) *For every open ω-cover γ of the space X, there exists an ω-cover $\mu \subset \gamma$ of the space X such that $|\mu| \leqslant \kappa$. In other words, every open ω-cover of X has an ω-subcover of cardinality $\leqslant \kappa$.*

(ii) *$l(X^n) \leqslant \kappa$ for all $n \in \mathbb{N}$.*

Solution. (i) $\Rightarrow$ (ii). Take any $n \in \mathbb{N}$ and fix an open cover γ of the space X^n. Call a family $\mu \subset \tau(X)$ *γ-small* if, for any $U_1, \ldots, U_n \in \mu$ (which are not necessarily different), we have $U_1 \times \cdots \times U_n \subset G$ for some $G \in \gamma$. Consider the family $\delta = \{\bigcup\mu : \mu$ is a finite γ-small family$\}$. If $A \subset X$ is a finite set then, for any $x_1, \ldots, x_n \in A$, there exist $U_i \in \tau(x_i, X)$ such that $U_1 \times \cdots \times U_n \subset G$ for some $G \in \gamma$. This shows that every $x \in A$ has an open neighbourhood U_x such that $\mu = \{U_x : x \in A\}$ is a γ-small family. Since $\bigcup\mu \supset A$, we can conclude that δ is an ω-cover of X. By (i) we can find an ω-cover $\delta' \subset \delta$ with $|\,\delta'\,| \leqslant \kappa$. For any $U \in \delta'$ there exists a γ-small finite family μ_U such that $U = \bigcup \mu_U$. There also exists a finite family $\gamma_U \subset \gamma$ such that, for any $W_1, \ldots, W_n \in \mu_U$, we have $W_1 \times \cdots \times W_n \subset G$ for some $G \in \gamma_U$. The family $\gamma' = \bigcup\{\gamma_U : U \in \delta'\}$ is a subfamily of γ and $|\,\gamma'\,| \leqslant \kappa$, so we only have to prove that $\bigcup \gamma' = X^n$. If $x = (x_1, \ldots, x_n) \in X^n$ then there exists $U \in \delta'$ such that $\{x_1, \ldots, x_n\} \subset U$. Thus there are $U_1, \ldots, U_n \in \mu_U$ such that $x_i \in U_i$ for all $i \leqslant n$. By definition of γ_U there exists $G \in \gamma_U$ such that $U_1 \times \cdots \times U_n \subset G$. Therefore $x \in U_1 \times \cdots \times U_n \subset G$ so γ' is a subcover of γ of cardinality $\leqslant \kappa$.

(ii) $\Rightarrow$ (i). Take any ω-cover γ of the space X. For any $n \in \mathbb{N}$, the family $\{U^n : n \in \omega\}$ is a cover of the space X^n. Since $l(X^n) \leqslant \kappa$, there exists $\gamma'_n \subset \gamma$ with $|\gamma'_n| \leqslant \kappa$ and $\bigcup\{U^n : U \in \gamma'_n\} = X^n$. The family $\gamma' = \bigcup\{\gamma'_n : n \in \omega\}$ has cardinality $\leqslant \kappa$ so it suffices to prove that it is an ω-cover of X. Take any finite $A = \{x_1, \ldots, x_n\} \subset X$. Then $x = (x_1, \ldots, x_n) \in X^n$ and hence there is $U \in \gamma'_n$ with $x \in U^n$. As a consequence $x_i \in U$ for each $i \leqslant n$ and therefore $A \subset U$ which proves that γ' is an ω-cover of X.

S.149. *Prove that* $t(C_p(X)) = \sup\{l(X^n) : n \in \mathbb{N}\}$. *In particular, tightness of* $C_p(X)$ *is countable if and only if* X^n *is a Lindelöf space for any* $n \in \mathbb{N}$.

Solution. Let us prove first that $t(C_p(X)) \leqslant \kappa = \sup\{l(X^n) : n \in \mathbb{N}\}$. Denote by u the function for which $u(x) = 0$ for each $x \in X$. It suffices to show that, for any $A \subset C_p(X)$ with $u \in \overline{A}$ there is a set $B \subset A$ such that $u \in \overline{B}$ and $|\,B\,| \leqslant \kappa$. For each $n \in \mathbb{N}$, the family $\gamma_n = \{f^{-1}((-\frac{1}{n}, \frac{1}{n})) : f \in A\}$ is an ω-cover of X. It follows from Problem 148 that there is an ω-cover $\mu \subset \gamma_n$ with $|\,\mu\,| \leqslant \kappa$. Therefore there exists $B_n \subset A$ such that $|\,B_n\,| \leqslant \kappa$ and the family $\{f^{-1}((-\frac{1}{n}, \frac{1}{n})) : f \in B_n\}$ is an ω-cover of X. Let $B = \bigcup\{B_n : n \in \mathbb{N}\}$. It is clear that $|\,B\,| \leqslant \kappa$, so we only have to prove that $u \in \overline{B}$. Given $x_1, \ldots, x_k \in X$ and $\varepsilon > 0$, take any $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Since $\{f^{-1}((-\frac{1}{n}, \frac{1}{n})) : f \in B_n\}$ is an ω-cover of X, there exists $f \in B_n$ such that $\{x_1, \ldots, x_k\} \subset f^{-1}((-\frac{1}{n}, \frac{1}{n}))$. This means that $|\,f(x_i)\,| < \frac{1}{n} < \varepsilon$ for all $i \leqslant k$. Therefore $f \in V = O(u, x_1, \ldots, x_k, \varepsilon)$ and hence $V \cap B \neq \emptyset$ for an arbitrary basic neighbourhood V of the point u. Thus $u \in \overline{B}$.

Now let $\lambda = t(C_p(X))$. We must prove that $l(X^n) \leqslant \lambda$ for all $n \in \mathbb{N}$. By Problem 148 this is equivalent to proving that any ω-cover of X has an ω-subcover of cardinality $\leqslant \lambda$. So let γ be an ω-cover of X and $A = \{f \in C_p(X) : f^{-1}(\mathbb{R}\backslash\{0\}) \subset U$ for some $U \in \gamma\}$. It is easy to see that $w \in \overline{A}$ where $w(x) = 1$ for all $x \in X$. Since $t(C_p(X)) \leqslant \lambda$, we can find $B \subset A$ with $w \in \overline{B}$ and $|\,B\,| \leqslant \lambda$. For each $f \in B$, take $U_f \in \gamma$ with $f^{-1}(\mathbb{R}\backslash\{0\}) \subset U_f$. Then $\mu = \{U_f : f \in B\} \subset \gamma$ and $|\,\mu\,| \leqslant \lambda$. Our proof will be over if we establish that μ is an ω-cover of X. Take any finite $F \subset X$. Since $w \in \overline{B}$, there is $f \in B$ such that $f(x) > 0$ for all $x \in F$. This implies $F \subset f^{-1}(\mathbb{R}\backslash\{0\}) \subset U_f$ and hence μ is an ω-cover of X.

S.150. *Prove that* $t(C_p(X)) = t((C_p(X))^{\omega})$ *for any space* X.

Solution. Let X_i be a copy of the space X for each $i \in \omega$. Then $(C_p(X))^{\omega}$ is homeomorphic to $C_p(\bigoplus_{i \in \omega} X_i)$ by Problem 114. Now, if $t(C_p(X)) = \kappa$ then $l(X^n) \leqslant \kappa$ for each $n \in \mathbb{N}$ by Problem 149. If $Y = \bigoplus_{i \in \omega} X_i$ then Y^n is a countable union of its subspaces homeomorphic to X^n. Therefore $l(Y^n) \leqslant \kappa$. Apply Problem 149 once more to conclude that $t((C_p(X))^\omega) = t(C_p(Y)) \leqslant \kappa = t(C_p(X))$. Since the inverse equality is evident, we proved that $t(C_p(X)) = t((C_p(X))^\omega)$.

S.151. *Show that there exist spaces X and Y such that $t(C_p(X)) = \omega$ and $t(C_p(Y)) = \omega$ while $t(C_p(X) \times C_p(Y)) > \omega$.*

Solution. We will first prove some facts about the Cantor set $\mathbb{K}$.

Fact 1. For every second countable uncountable space P there exists an uncountable $P' \subset P$ such that the subspace P' has no isolated points.

Proof. Let $\gamma = \{U \subset \tau(P) : U \text{ is countable}\}$. It is easy to see that every subspace of a second countable space is second countable so $U = \bigcup\gamma$ is second countable. Every second countable space is Lindelöf (Observation one of S.140) so there are $U_i \in \gamma$, $i \in \omega$ such that $U = \bigcup\{U_i : i \in \omega\}$. The set U is countable being a union of countably many countable sets. Therefore $P' = P\backslash U$ is an uncountable subset of P. For any $p \in P'$ and any $W \in \tau(p, P')$ there exists $V \in \tau(P)$ with $V \cap P' = W$. The set V has to be uncountable (for all countable open sets are inside U) and hence $W = V\backslash U$ is also uncountable. This shows that $W \neq \{p\}$ i.e., the point p is not isolated in P' and Fact 1 is proved.

Fact 2. The cardinality of the family of all closed subsets of $\mathbb{K}$ is exactly $\mathfrak{c}$.

Proof. Note that the points of $\mathbb{K}$ give $\mathfrak{c}$ distinct closed sets and therefore the number of closed sets of $\mathbb{K}$ is greater than or equal to $\mathfrak{c}$. The number of open set is equal to the number of closed sets so it suffices to prove that $|\tau(\mathbb{K})| \leqslant \mathfrak{c}$. Fix any countable base $\mathcal{B}$ in $\mathbb{K}$ and note that any open set is a union of some subfamily of $\mathcal{B}$. Since there are $\mathfrak{c}$ subfamilies of $\mathcal{B}$, we can conclude that $|\tau(\mathbb{K})| \leqslant \mathfrak{c}$ and Fact 2 is proved.

For any $n \in \mathbb{N}$, let $p_i^n : \mathbb{K}^n \to \mathbb{K}$ be the natural projection of $\mathbb{K}^n$ onto the ith factor. Given a set $A \subset \mathbb{K}^n$, let $|A|_n = \sup\{|B| : B \subset A \text{ and } p_i^n | B \text{ is an injection for any } i \leqslant n\}$. The cardinal $|A|_n$ will be called *n-cardinality of* A. If $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$, let $b(x) = \{x_1, \ldots, x_n\} \subset \mathbb{K}$.

Fact 3. Given $A \subset \mathbb{K}^n$, suppose that n-cardinality of A is infinite. Then we have
(1) $|A|_n = \min\{|Y| : Y \subset \mathbb{K} \text{ and } A \subset \bigcup\{(p_i^n)^{-1}(Y) : i \leqslant n\}\}$ and
(2) $|A|_n = \sup\{|B| : B \subset A \text{ and } b(x) \cap b(y) = \emptyset \text{ for any distinct } x, y \in B\}$.

Proof. Denote the n-cardinality of A by κ and let κ_i be the cardinal defined in the condition (i) for $i = 1, 2$. It is clear that if $b(x) \cap b(y) = \emptyset$ for distinct $x, y \in B$ then $p_i^n | B$ is an injection for each $i \leqslant n$. This shows that $\kappa_2 \leqslant \kappa$. Now, fix a set $Y \subset \mathbb{K}$ of cardinality κ_1 with $A \subset \bigcup\{(p_i^n)^{-1}(Y) : i \leqslant n\}$. If $B \subset A$ is such that $p_i^n | B$ is an injection for all $i \leqslant n$ then, for each $x \in B$, there is $i(x) \leqslant n$ with $q(x) = p_{i(x)}^n(x) \in Y$. It is easy to see that $q : B \to Y$ and $|q^{-1}(y)| \leqslant n$ for any $y \in Y$ and therefore $|B| \leqslant |Y|$. As a consequence, $\kappa \leqslant \kappa_1$.

To finish the proof it suffices to show that $\kappa_1 \leqslant \kappa_2$. To do this, take any $Y \subset \mathbb{K}$ with $|Y| = \kappa_1$ and $A \subset \bigcup\{(p_i^n)^{-1}(Y) : i \leqslant n\}$. Choose a point $z_0 \in A$ arbitrarily. Suppose that $\alpha < \kappa_1$ and we have chosen points $\{z_\beta : \beta < \alpha\} \subset A$ so that

$b(z_\beta) \cap b(z_{\beta'}) = \emptyset$ if $\beta \neq \beta'$. The set $Y' = \bigcup\{b(z_\beta) : \beta < \alpha\}$ has cardinality $< \kappa_1$ which implies $A \not\subset \bigcup\{(p_i^n)^{-1}(Y') : i \leqslant n\}$. Take any $z_\alpha \in A \setminus \bigcup\{(p_i^n)^{-1}(Y') : i \leqslant n\}$ and observe that $b(z_\alpha) \cap Y' = \emptyset$ and hence the family $\{b(z_\beta) : \beta \leqslant \alpha\}$ is disjoint. Thus, we can construct a set $\{z_\alpha : \alpha < \kappa_1\} \subset A$ such that $\{b(z_\beta) : \beta \leqslant \kappa_1\}$ is a disjoint family. This shows that $\kappa_1 \leqslant \kappa_2$ and Fact 3 is proved.

Fact 4. Fix $m \in \mathbb{N}$ and take an arbitrary closed $F \subset \mathbb{K}^m$. If $| F |_m > \omega$ then $| F |_m = \mathfrak{c}$.

Proof. Take any continuous surjective map $\varphi : \mathbb{K} \to F$ which exists by Problem 128(v). By condition (2) of Fact 3 there exists an uncountable $A \subset F$ such that $\{b(x) : x \in A\}$ is a disjoint family. For each $x \in A$, take $z_x \in \varphi^{-1}(x)$. Observe that the open intervals with rational endpoints form a countable base in $\mathbb{R}$ and therefore $\mathbb{K} \subset \mathbb{R}$ is also second countable. Apply Fact 1 to find an uncountable $Z \subset \{z_x : x \in A\}$ with no isolated points.

Let $S_n = \{f : f$ is a function from $\{1, \ldots, n\}$ to $\{0, 1\}\}$ for each $n \in \mathbb{N}$. For each $f \in S = \bigcup\{S_n : n \in \mathbb{N}\}$ we will construct a point $t_f \in Z$ and $O_f \in \tau(t_f, \mathbb{K})$ in such a way that

(i) If f and g are distinct elements of S_n then there exists disjoint $H, G \in \tau(\mathbb{K})$ such that $\varphi(\overline{O}_f) \subset H^m$ and $\varphi(\overline{O}_g) \subset G^m$.
(ii) If $f \in S_n$, $g \in S_k$, $n < k$ and $g \mid \{1, \ldots, n\} = f$ then $\overline{O}_f \subset O_g$.

Clearly, $S_1 = \{u, w\}$ where $u(1) = 0$ and $w(1) = 1$. Choose arbitrarily distinct t_u, $t_w \in Z$ and find disjoint sets $G \in \tau(b(\varphi(t_u)), \mathbb{K})$, $H \in \tau(b(\varphi(t_w)), \mathbb{K})$. The map φ being continuous, we can find $O_u \in \tau(t_u, \mathbb{K})$, $O_w \in \tau(t_w, \mathbb{K})$ such that $\varphi(\overline{O}_u) \subset G^m$ and $\varphi(\overline{O}_w) \subset H^m$.

The sets O_u, O_w and the points t_u, t_w satisfy the condition (i) (and (ii) vacuously). Proceeding by induction suppose that we have a set O_f and a point $t_f \in Z$ for each $f \in \bigcup\{S_i : i < n\}$ with the properties (i)–(ii). Take an arbitrary $h \in S_{n-1}$ and observe that there are exactly two functions, say $h_0, h_1 \in S_n$ such that $h = h_0 \mid \{1, \ldots, n-1\} = h_1 \mid \{1, \ldots, n-1\}$. Without loss of generality, we may assume that $h_0(n) = 0$ and $h_1(n) = 1$. The set $O_h \cap Z \ni t_h$ is non-empty and cannot consist of only one point t_h because this point would be isolated in Z. Therefore it is possible to take distinct $t_{h_0}, t_{h_2} \in O_h \cap Z$. We have $b(\varphi(t_{h_0})) \cap b(\varphi(t_{h_2})) = \emptyset$ and therefore there exist disjoint sets $G_1 \in \tau(b(\varphi(t_{h_0})), \mathbb{K})$ and $H_1 \in \tau(b(\varphi(t_{h_2})), \mathbb{K})$. The mapping φ being continuous, we can find sets $O_{h_0} \in \tau(t_{h_0}, \mathbb{K}), O_{h_1} \in \tau(t_{h_1}, \mathbb{K})$ such that $\overline{O}_{h_0} \cup \overline{O}_{h_1} \subset O_h$, $\varphi(\overline{O}_{h_1}) \subset G_1^m$ and $\varphi(\overline{O}_{h_1}) \subset H_1^m$. Since $S_n = \{h_0, h_1 : h \in S_{n-1}\}$, we have constructed points t_f and sets $O_f \in \tau(t_f, \mathbb{K})$ for any $f \in \bigcup\{S_k : k \leqslant n\}$ and it is immediate that the properties (i)–(ii) are fulfilled.

To prove that $| F |_m = \mathfrak{c}$ it suffices to construct an injection $\aleph : \{0, 1\}^{\mathbb{N}} \to F$ such that $b(\aleph(x)) \cap b(\aleph(y)) = \emptyset$ for any distinct $x, y \in \{0, 1\}^{\mathbb{N}}$. So, if $x \in \{0, 1\}^{\mathbb{N}}$ let $x_n = x \mid \{1, \ldots, n\}$ and $U_n = O_{x_n}$. It an immediate consequence of (ii) that $U_1 \supset \overline{U}_2 \supset U_2 \supset \overline{U}_3 \supset \cdots$ and therefore $H_x = \bigcap\{U_n : n \in \mathbb{N}\} = \bigcap\{\overline{U}_n : n \in \mathbb{N}\}$ is a non-empty set because $\mathbb{K}$ is compact. Now, if we take any point $u_x \in H_x$ then the map $r : \{0, 1\}^{\mathbb{N}} \to F$ defined by $r(x) = \varphi(u_x)$ has the required property. Indeed, if $x, y \in \{0, 1\}^{\mathbb{N}}$ and $x \neq y$ then $x(n) \neq y(n)$ for some n whence $x_n \neq y_n$ and hence there exist disjoint open subsets $G, H \subset \mathbb{K}$ such that $\varphi(\overline{O}_{x_n}) \subset H^m$ and $\varphi(\overline{O}_{y_n}) \subset G^m$. Therefore $b(r(x)) \subset H$ and $b(r(y)) \subset G$ and consequently $b(r(x)) \cap b(r(y)) \subset H \cap G = \emptyset$ so Fact 4 is proved.

Fact 5. There exist disjoint sets $A, B \subset \mathbb{K}$ such that, for any $n \in \mathbb{N}$ and for any closed n-uncountable $F \subset \mathbb{K}^n$, we have $A^n \cap F \neq \emptyset$ and $B^n \cap F \neq \emptyset$.

Proof. Since $\mathbb{K}^n$ is homeomorphic to $\mathbb{K}$ for each natural n, apply Fact 2 to observe that there are at most $\mathfrak{c}$ closed sets which lie in some $\mathbb{K}^n$. Let $\{F_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all closed n-uncountable subsets of $\mathbb{K}^n$ for all $n \in \mathbb{N}$. For each $\alpha < \mathfrak{c}$, there is $n = n(\alpha)$ such that $F_\alpha \subset \mathbb{K}^n$. Apply Fact 4 to conclude that each F_α has n-cardinality continuum for $n = n(\alpha)$. By Fact 3, we can fix a set $H_\alpha \subset F_\alpha$ with $|H_\alpha| = \mathfrak{c}$ and $b(a) \cap b(a') = \emptyset$ for any distinct $a, a' \in H_\alpha$. Take distinct $x, y \in H_0$ and let $A_0 = b(x), B_0 = b(y)$. Suppose that $\alpha < \mathfrak{c}$ and we have constructed sets $\{A_\beta : \beta < \alpha\}$ and $\{B_\beta : \beta < \alpha\}$ with the following properties:

(iii) The family $\mu_\alpha = \{A_\beta : \beta < \alpha\} \cup \{B_\beta : \beta < \alpha\}$ is disjoint.
(iv) $A_\beta = b(p), B_\beta = b(q)$ for some distinct $p, q \in H_\beta$.

The set $P = \bigcup \mu_\alpha$ has cardinality strictly less than $\mathfrak{c}$ while $\{b(z) : z \in H_\alpha\}$ is a disjoint family of cardinality $\mathfrak{c}$. Therefore we can choose distinct $r, r' \in H_\alpha$ such that $b(r) \cup b(r')$ is disjoint from P. Letting $A_\alpha = b(r)$ and $B_\alpha = b(r')$, we obtain families $\{A_\beta : \beta \leqslant \alpha\}$ and $\{B_\beta : \beta \leqslant \alpha\}$ for which the conditions (iii)–(iv) hold so the inductive construction can go on. After we have the families $\{A_\beta : \beta < \mathfrak{c}\}$ and $\{B_\beta : \beta < \mathfrak{c}\}$ observe that the sets $A = \cup\{A_\beta : \beta < \mathfrak{c}\}$ and $B = \{B_\beta : \beta < \mathfrak{c}\}$ are disjoint by (iii). Note that, for any $x \in \mathbb{K}^n$, we have $x \in (b(x))^n$. Now take any n-uncountable $F \subset \mathbb{K}^n$. Since we enumerated all such sets, $F = F_\alpha$ for some $\alpha < \mathfrak{c}$. Therefore $A^n \cap F \supset A^n_\alpha \cap F_\alpha \neq \emptyset$ and $B^n \cap F \supset B^n_\alpha \cap F_\alpha \neq \emptyset$ by (iv) which finishes the proof of Fact 5.

Fact 6. Take a set $A \subset \mathbb{K}$ with the properties described in Fact 5. Let μ be the topology on $\mathbb{K}$ generated by the family $\mathcal{B} = \tau(\mathbb{K}) \cup \{\{x\} : x \in A\}$. It is easy to see that $\mathcal{B}$ is a base for μ. Denote by X the space $(\mathbb{K}, \mu)$. Then X is a Tychonoff space with X^n Lindelöf for all $n \in \mathbb{N}$.

Proof. We leave to the reader the trivial exercise that any topology, which contains a Hausdorff one, is Hausdorff. Since $\mu \supset \tau(\mathbb{K})$, the space X is Hausdorff. To establish complete regularity of X, take any $x \in X$ and any closed $F \subset X$ with $x \notin F$. If $x \in A$ then x is isolated in X and the function $f : X \to [0, 1]$ defined by $f(x) = 1$ and $F(X\backslash\{x\}) = \{0\}$, is continuous and equals zero on F. If $x \in X\backslash A$ then $\tau(x, \mathbb{K})$ is a local base at x and therefore $x \in U \subset X\backslash F$ for some $U \in \tau(x, \mathbb{K})$. Since $\mathbb{K}$ is a Tychonoff space, there exists a continuous map $f : \mathbb{K} \to [0, 1]$ with $f(x) = 1$ and $f(X \setminus U) \subset \{0\}$. Since $F \subset X\backslash U$, we have $f(F) \subset \{0\}$ and it is immediate that f is continuous considered as a function from X to $[0, 1]$. This proves that X is Tychonoff.

We will prove that X^n is Lindelöf by induction on n. To do this for $n = 1$ take any open cover γ of the space X. It is easy to see that without loss of generality, we may assume that $\gamma \subset \mathcal{B}$. The subspace $X\backslash A$ is homeomorphic to $\mathbb{K}\backslash A$ because the topology of $\mathbb{K}\backslash A$ has not been changed. Thus $X\backslash A$ is second countable and hence Lindelöf (Observation one of S.140). Take any countable $\gamma' \subset \gamma$ with $X\backslash A \subset \bigcup \gamma'$.

Since no element of $\mathcal{B}\backslash\tau(\mathbb{K})$ meets $X\backslash A$, we may assume that $\gamma' \subset \tau(\mathbb{K})$. But then $\cup\gamma'$ is open in $\mathbb{K}$ and hence $F = \mathbb{K}\backslash(\bigcup\gamma') \subset A$ is closed in $\mathbb{K}$ and disjoint from the set B from Fact 5. Since $B = B^1$ intersects all uncountable subsets of $\mathbb{K} = \mathbb{K}^1$ by Fact 5, the set F has to be countable. Thus there is a countable $\gamma'' \subset \gamma$ with $F \subset \bigcup\gamma''$. As a consequence, the family $\gamma' \cup \gamma''$ is a countable subcover of γ which proves that X is Lindelöf.

Suppose that we proved that X^n is Lindelöf for some $n \geqslant 1$ and take any open cover γ of X^{n+1}. Let $\mathcal{B}^{n+1} = \{U_1 \times \cdots \times U_{n+1} : U_i \in \mathcal{B}, i = 1, \ldots, (n+1)\}$. It is again an easy exercise that $\mathcal{B}^{n+1}$ is a base of X^{n+1} and hence we do not lose generality assuming that $\gamma \subset \mathcal{B}^{n+1}$. The subspace $(X\backslash A)^{n+1}$ is second countable because so is $X\backslash A$ (Observation four of S.135) and hence there is a countable $\gamma' \subset \gamma$ with $(X\backslash A)^{n+1} \subset \bigcup \gamma'$. Observe that if $x \in (X\backslash A)^{n+1} \cap (U_1 \times \cdots \times U_{n+1})$ and $U_1 \times \cdots \times U_{n+1} \in \mathcal{B}^{n+1}$ then $U_i \in \tau(\mathbb{K})$ for each i and hence we may assume that $\gamma' \subset \tau(\mathbb{K})^n$. Therefore $F = \mathbb{K}^{n+1}\backslash(\bigcup\gamma')$ is closed in $\mathbb{K}^{n+1}$ and disjoint from $B^{n+1} \subset (X\backslash A)^{n+1}$. Apply Fact 5 to conclude that F is $(n+1)$-countable and therefore there exists a countable $Z \subset \mathbb{K}$ such that $T = \bigcup\{(p_i^{n+1})^{-1}(Z) : i \leqslant n+1\}$ covers F. Note that $(p_i^{n+1})^{-1}(z)$ is homeomorphic to X^n for each $z \in Z$ and $i \leqslant n+1$. By the inductive hypothesis T is a countable union of Lindelöf subspaces of X^{n+1} and hence it is Lindelöf. Thus, there is a countable $\gamma'' \subset \gamma$ with $F \subset \bigcup \gamma''$. As a consequence, the family $\gamma' \cup \gamma''$ is a countable subcover of γ which proves that X^{n+1} is Lindelöf and Fact 6 is proved.

Fact 7. Denote by Y the subspace A of the space $\mathbb{K}$, constructed in Fact 5. The space Y is second countable and hence Y^n is Lindelöf for each $n \in \mathbb{N}$. However, $X \times Y$ is not Lindelöf.

Proof. Observation four of S.135 shows that Y^n is second countable for any $n \in \mathbb{N}$. Applying Observation one of S.140, we can conclude that Y^n is Lindelöf for each $n \in \mathbb{N}$. To prove that $X \times Y$ is not Lindelöf it suffices to find an uncountable closed discrete subspace of $X \times Y$. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the respective natural projections. Note that, as a set, Y is a subset of X and hence we can consider the set $D = \{(y, y) : y \in Y\} \subset X \times Y$. It is easy to see from the method of construction of A that it is uncountable and hence so is D. If $d = (y, y) \in D$ then $p(d) = y \in A$ is an isolated point in X and hence $U = p^{-1}(y)$ is an open set in the space $X \times Y$ such that $U \cap D = \{d\}$ which shows that D is discrete. To see that the set D is closed take any point $z = (x, y) \in (X \times Y)\backslash D$. Then $x \neq y$ and we can choose disjoint open subsets of $\mathbb{K}$ such that $x \in U$ and $y \in V$. Observe that $U \in \tau(X)$ and $V' = V \cap A$ is an open subset of Y. The set $W = U \times V'$ is open in $X \times Y$ and $z \in W$. If $d = (t, t) \in W \cap D$ then $t \in U \cap V'$, a contradiction with $U \cap V' = \emptyset$. Therefore $W \cap D = \emptyset$ and this proves that D is closed in $X \times Y$ finishing the proof of Fact 7.

To finish the solution of our problem, we will show that the spaces X and Y are as promised. Indeed, since X^n and Y^n are Lindelöf for all $n \in \mathbb{N}$, we have $t(C_p(X)) = t(C_p(Y)) = \omega$ by Problem 149. If $Z = X \bigoplus Y$ then $C_p(X) \times C_p(Y)$ is homeomorphic to $C_p(Z)$ by Problem 114 so it suffices to prove that $t(C_p(Z)) > \omega$. Observe that $X \times Y$ is a closed subspace of Z^2 and hence Z^2 is not Lindelöf. Applying Problem 149 once more we conclude that $t(C_p(Z)) > \omega$ and our solution is, at last, complete.

S.152. *Let Y be a subspace of a space X. Denote by $\pi_Y : C_p(X) \to C_p(Y)$ the restriction map, i.e., $\pi_Y(f) = f \mid Y$. Prove that*

(i) The map π_Y is continuous and $\overline{\pi_Y(C_p(X))} = C_p(Y)$.
(ii) The map π_Y is an injection if and only if Y is dense in X;
(iii) The map π_Y is a homeomorphism if and only if $Y = X$.
(iv) The subspace Y is closed in X if and only if the map $\pi_Y : C_p(X) \to \pi_Y(C_p(X))$ is open.
(v) If X is normal and Y is closed in X then $\pi_Y(C_p(X)) = C_p(Y)$.

Solution. (i) Observe first that $\pi_Y(C_p(X)) \subset C_p(Y)$ by Problem 022. The map π_Y is continuous because it coincides with the natural projection $p_Y : \mathbb{R}^X \to \mathbb{R}^Y$ restricted to $C_p(X)$ (see Problems 111 and 107). The set $C_p(X)$ is dense in $\mathbb{R}^X$ (Problem 111) so, by continuity of p_Y we have $\mathbb{R}^Y = p_Y(\mathbb{R}^X) = p_Y\left(\overline{C_p(X)}\right) \subset \overline{p_Y(C_p(X))}$ (the first bar denotes the closure in $\mathbb{R}^X$ and the second one in $\mathbb{R}^Y$). Thus the set $\pi_Y(C_p(X)) = p_Y(C_p(X))$ is dense in $\mathbb{R}^Y$ and hence in a smaller space $C_p(Y)$.

(ii) Suppose that Y is not dense in X and denote by u the function identically equal to zero on X. Take any $x \in X \backslash \overline{Y}$. By the Tychonoff property of X there exists a continuous $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f|\overline{Y} \equiv 0$. Thus $\pi_Y(f) = \pi_Y(u)$ while $u \neq f$. This shows that Y must be dense in X if π_Y is an injection. Now if Y is dense in X and $f, g \in C_p(X), f \neq g$, then $h = f - g$ is a continuous function and the open set $U = h^{-1}(\mathbb{R}\backslash\{0\})$ is non-empty. Since Y is dense in X, we can pick $y \in Y \cap U$. Then $f(y) \neq g(y)$ which implies $\pi_Y(f) = f \mid Y \neq g \mid Y = \pi_Y(g)$ and hence π_Y is an injection.

(iii) If π_Y is a homeomorphism then it is an injection so Y is dense in X by (ii). If $Y \neq X$, take any $x \in X \backslash Y$. The set $F = \{f \in C_p(X) : f(x) = 1\}$ is not dense in $C_p(X)$ because it does not intersect the open non-empty set $U = \{f \in C_p(X) : f(x) \in (2, 3)\}$. However $\pi_Y(F)$ is dense in $C_p(Y)$. Indeed, take any $g \in C_p(Y), y_1, \ldots, y_n \in Y$ and $\varepsilon > 0$. There exists $f \in C_p(X)$ such that $f(x) = 1$ and $f(y_i) = g(y_i)$ for all $i \leqslant n$ (Problem 034). Thus $\pi_Y(f) \in O(g, y_1, \ldots, y_n, \varepsilon) \cap \pi_Y(F)$ which proves that $\pi_Y(F)$ is dense in $C_p(Y)$. As a consequence, the map π_Y^{-1} is not continuous because the image of a dense set $\pi_Y(F)$ under π_Y^{-1} would be a dense set. But this image is F which is not dense in $C_p(X)$, a contradiction. We proved that if π_Y is a homeomorphism then $Y = X$. Finally, observe that if $X = Y$ then π_Y is the identity map and hence homeomorphism.

(iv) Suppose that Y is closed and denote the set $\pi_Y(C_p(X))$ by $C_p(Y \mid X)$.

Fact 1. Suppose that $y_1, \ldots, y_m \in X \backslash Y$, $r_1, \ldots, r_m \in \mathbb{R}$ and $f \in C_p(X)$. Then there exists $g \in C_p(X)$ such that $g \mid Y = f$ and $g(x_i) = r_i$ for all $i \leqslant m$.

Proof. For each $i \leqslant m$, use the Tychonoff property of X to find $g_i \in C(X, [0, 1])$ with $g_i \mid (F \cup (\{y_1, \ldots, y_m\} \backslash \{y_i\})) \equiv 0$ and $g_i(y_i) = 1$. It is easy to see that the function $g = f + \sum_{i=1}^{m} (r_i - f(y_i)) \cdot g_i$ is as promised.

Take any open $W \subset C_p(X)$. Since the standard open sets form a base in $C_p(X)$, we have $W = \bigcup\{U_\alpha : \alpha \in A\}$ where each U_α is a standard open set. Then $\pi_Y(W) = \bigcup\{\pi_Y(U_\alpha) : \alpha \in A\}$ and therefore it suffices to show that the image of any standard set under π_Y is open. Take any $x_1, \ldots, x_n \in X$ and $O_1, \ldots, O_n \in \tau(\mathbb{R})$. We will prove that the set $\pi_Y(U)$ is open where $U = [x_1, \ldots, x_n; O_1, \ldots, O_n]_X$ (the index says that the standard open set is taken in $C_p(X)$). Without loss of generality, we may assume that $x_1, \dots, x_{k-1} \in Y$ and $x_k, \dots, x_n \notin Y$ for some k 1. It suffices to show that $\pi_Y(U) = W = [x_1, \dots, x_{k-1}; O_1, \dots, O_k]_Y = \{f \in C_p(Y \mid X) : f(x_i) \in O_i$ for all $i \leqslant k-1\}$. It is evident that $\pi_Y(U) \subset W$. Now if $f \in W$ then take any $r_k, \dots, r_n \in \mathbb{R}$ with $r_i \in O_i$ for all $i = k, \dots, n$. Apply Fact 1 to find $g \in C_p(X)$ for which $g|Y = f$ and $g(x_i) = r_i$ for all $i = k, \dots, n$. Then $\pi_Y(g) = f$ and $g \in U$ which implies $W = \pi_Y(U)$. This shows that $\pi_Y : C_p(X) \to C_p(Y|X)$ is an open map.

Now suppose that π_Y is an open map and $x \in \overline{Y} \setminus Y$. We will use the following evident observation: in any space Z, if G and H are dense open subsets of Z then $G \cap H \neq \emptyset$. Let $U = [x; (0, 2)]_X$ and $V = [x; (3, 5)]_X$. To get a contradiction we will prove that the sets $G = \pi_Y(U)$ and $H = \pi_Y(V)$ are dense and disjoint in $C_p(Y \mid X)$. Therefore it is impossible that they both be open which is a contradiction with the openness of the map π_Y. Suppose that, $f \in G \cap H$. Then there is $g \in U$ and $h \in V$ with $\pi_Y(g) = \pi_Y(h) = f$. The function $w = g - h$ is continuous on X and $w(x) \neq 0$. The set $A = w^{-1}(\mathbb{R} \setminus \{0\})$ is open and contains x. Since $x \in \overline{Y}$, there is $y \in Y \cap A$. As a consequence $w(y) \neq 0$, i.e., $h(y) \neq g(y)$ which contradicts the fact that $g|Y \equiv h|Y$. This proves that the sets G and H are disjoint. Let us prove, for example, that G is dense in $C_p(Y \mid X)$; the proof for H is analogous. Given $y_1, \dots, y_n \in Y$ and $O_1, \dots, O_n \in \tau^*(\mathbb{R})$, choose $r_i \in O_i$ for each $i \leqslant n$ and observe that there exists function $f \in C_p(X)$ such that $f(x) = 1$ and $f(y_i) = r_i$ for all $i \leqslant n$ by Problem 034. It is clear that $\pi_Y(f) \in [y_1, \dots, y_n; O_1, \dots, O_n]_Y \cap G$ whence G is dense in $C_p(Y|X)$.

(v) By normality of X, for every $f \in C_p(Y)$ there is $g \in C_p(X)$ such that $\pi_Y(g) = g|Y = f$ (see Problem 032). Therefore $\pi_Y(C_p(X)) = C_p(Y)$.

S.153. *Prove that closed maps as well as open ones are quotient. Give an example*

(i) Of a quotient map which is neither closed nor open
(ii) Of a closed map which is not open
(iii) Of an open map which is not closed

Solution. Let $f : X \to Y$ be a closed map. If $U \subset Y$ and $f^{-1}(U)$ is open then consider the set $F = Y \setminus U$. Since $f^{-1}(F) = X \setminus f^{-1}(U)$, the set $f^{-1}(F)$ is closed in X. The map f being closed, the set $F = f(f^{-1}(F))$ is closed in Y and hence $U = Y \setminus F$ is open. This proves that f is quotient. Now, if f is open and $f^{-1}(U)$ is open then $U = f(f^{-1}(U))$ is open an hence f is also quotient.

(i) Let $X = \{(t, 1) : t \in [0, 2)\} \cup \{(t, 2) : t \in [1, 2]\} \subset \mathbb{R}^2$. The topology of X is induced from $\mathbb{R}^2$. For $Y = [0, 2]$ define a map $f : X \to Y$ by $f(x) = t$ if $x = (t, i)$, $i = 1, 2$. Let us prove that f is a quotient map which is neither open nor closed.

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the natural projection onto the first factor. It is clear that $f = \pi|X$ and hence f is a continuous. Let $P = \{(t, 1) : t \in [0, 2)\}$ and $Q = \{(t, 2) : t \in [1, 2]\}$. Suppose that $A \subset [0, 2]$ and $B = f^{-1}(A)$ is open in X. Then $B \cap P \in \tau(P)$ and $B \cap Q \in \tau(Q)$. If $2 \notin A$ then $A = f(B \cap P)$. The map $f|P$ is a homeomorphism of P onto $[0, 2)$ which is open in $[0, 2]$. As a consequence A is open in $[0, 2)$ and hence in $[0, 2]$. Now, if $2 \in A$ then $(2, 2) \in B \cap Q$. Since $B \cap Q$ is open in Q, there is $\varepsilon \in (0, \frac{1}{2})$ such that $\{(t, 2) : t \in (2 - \varepsilon, 2]\} \subset B \cap Q$. This implies $(2 - \varepsilon, 2] \subset A$ i.e., 2 is in the interior of A. But $A = f(B \cap P) \cup \{2\}$ and we saw already that $f(B \cap P)$ is open in [0, 2]. Thus A is a union of two open sets: $f(B \cap P)$ and the interior of A. Consequently, A is open and we proved that f is quotient.

It is immediate that both P and Q are clopen($\equiv$ closed-and-open) subsets of X. However, $f(P) = [0, 2)$ is not closed and $f(Q) = [1, 2]$ is not open which proves that f is neither closed nor open.

(ii) Let $X = [0, 2] \subset \mathbb{R}$ and $Y = [0, 1] \subset \mathbb{R}$. Given $t \in X$, let $f(t) = \min\{t, 1\}$. It is immediate that $f : X \to Y$ is a continuous map. It is also closed because X is compact (see Problem 122). However, f is not open because $U = (1, 2)$ is an open subset of X while $f(U) = \{1\}$ is not open in Y.

(iii) Let $X = \mathbb{R}^2$ and let $\pi : X \to \mathbb{R}$ be the natural projection onto the first factor. The map π is open by Problem 107. Let $F = \{(\frac{1}{n}, n) : n \in \mathbb{N}\} \subset X$. The set F is closed in X. Indeed, if $z = (x, y) \in X \setminus F$ take any $m \in \mathbb{N}$ with $m > y$ and note that $U = \{(a, b) \in X : b < m\}$ is an open set such that $z \in U$ and $U \cap F$ is finite. Thus $U \setminus F$ is an open neighbourhood of z which does not intersect F. However, $\pi(F) = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed in $\mathbb{R}$ and hence π is an open map which is not closed.

S.154. *Prove that every quotient map is* $\mathbb{R}$*-quotient. Give an example of an* $\mathbb{R}$*-quotient non-quotient map.*

Solution. Let $f : X \to Y$ be a quotient map. Suppose that $g : Y \to \mathbb{R}$ is a map such that $g \circ f$ is continuous. Given an open $U \subset \mathbb{R}$ the set $V = (g \circ f)^{-1}(U)$ is open in X. However, $V = f^{-1}(g^{-1}(U))$ and hence the inverse image of the set $g^{-1}(U)$ is open in X. The map f being quotient, the set $g^{-1}(U)$ has to be open in Y and hence g is continuous. This proves that f is $\mathbb{R}$-quotient.

To construct the promised example, we will need several auxiliary facts.

Fact 1. Let $Q = \mathbb{Q} \cap \mathbb{I}$. There exists an almost disjoint family $\mathcal{M}$ on Q such that every $\xi \in \mathcal{M}$ is a convergent sequence.

Proof. Let $\{N_\alpha : \alpha < \mathfrak{c}\}$ be some enumeration of all infinite subsets of Q. Since N_0 is a bounded infinite set, there is an infinite $M_0 \subset N_0$ such that M_0 is a sequence which converges to some $t_0 \in \mathbb{I}$. Suppose that $\alpha < \mathfrak{c}$ and we have infinite sequences $\{M_\beta : \beta < \alpha\}$ with their respective limits $\{t_\beta : \beta < \alpha\}$ chosen in such a way that the following conditions are satisfied:

(1) The family $\{M_\beta : \beta < \alpha\}$ is almost disjoint
(2) For every $\beta < \alpha$ there is $\delta < \alpha$ such that the set $N_\beta \cap M_\delta$ is infinite

If, for each $\lambda < \mathfrak{c}$, there is $\beta < \alpha$ such that the set $N_\lambda \cap M_\beta$ is infinite, the inductive construction stops. If not, let $\delta = \min\{\lambda < \mathfrak{c} : N_\lambda \cap M_\beta$ is finite for all $\beta < \alpha\}$. Then $\delta \geqslant \alpha$ and the bounded sequence N_δ has an infinite subsequence M_α which converges to a point $t_\alpha \in \mathbb{I}$. It is clear that the properties (1)–(2) hold the family $\{M_\beta : \beta \leqslant \alpha\}$ and hence the inductive construction can go on. If it stops at some $\alpha < \mathfrak{c}$ then, for every $\lambda < \mathfrak{c}$ there is some $\beta < \alpha$ such that the set $N_\lambda \cap M_\beta$ is infinite. This means that the family $\mu = \{M_\beta : \beta < \alpha\}$ is maximal almost disjoint. Indeed, if some infinite set

$N \subset Q$ has finite intersections with all elements of μ then $N = N_\lambda$ for some $\lambda < \mathfrak{c}$ and hence there is $\beta < \alpha$ with $N_\lambda \cap M_\beta = N \cap M_\beta$ infinite which is a contradiction.

Now if the inductive process did not stop until $\mathfrak{c}$ then we have the family $\mu = \{M_\beta : \beta < \mathfrak{c}\}$ which is also almost disjoint. Indeed, if some infinite set $N \subset Q$ has finite intersections with all elements of μ then $N = N_\lambda$ for some $\lambda < \mathfrak{c}$. If $\alpha = \lambda + 1$ then by (2) there is $\beta < \alpha$ with $N_\lambda \cap M_\beta = N \cap M_\beta$ infinite, a contradiction which concludes the proof of Fact 1.

Fact 2. There is a maximal almost disjoint family $\mathcal{M}$ on ω such that the respective Mrowka space M (see Problem 142) can be mapped continuously onto $\mathbb{I}$.

Proof. Take any bijection $\varphi : \omega \to Q = \mathbb{Q} \cap \mathbb{I}$ and let $\mathcal{M} = \{\varphi^{-1}(\xi) : \xi \in \mathcal{N}\}$ where $\mathcal{N}$ is the maximal almost disjoint family constructed in Fact 1. For each $N \in \mathcal{N}$ denote by l_N the limit of the convergent sequence N. Given any $\delta \in \mathcal{M}$, let $f(\delta) = l_{\varphi(\delta)}$ and let $f(n) = \varphi(n)$ for each $n \in \omega$. We will prove that the map $f : M \to \mathbb{I}$ is continuous and surjective. Since every point of ω is isolated, we must only prove continuity at any $\delta \in \mathcal{M}$. If $\varepsilon > 0$ then there is a finite $A \subset \varphi(\delta)$ such that $\varphi(\delta) \setminus A \subset (f(x) - \varepsilon, f(x) + \varepsilon)$ because the sequence $\varphi(\delta)$ converges to $l_{\varphi(\delta)} = f(x)$. The set $B = \varphi^{-1}(A)$ is finite and hence $U = \delta \setminus B$ is an open neighbourhood of the point δ for which $f(U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. Thus f is continuous at δ. Observe that $f(M)$ is a pseudocompact second countable space by Problem 139 and hence $f(M)$ is compact. By Problem 121 the set $f(M)$ is closed in $\mathbb{I}$. Since $Q \subset F(M)$, we have $\mathbb{I} = \overline{Q} \subset f(M)$ and Fact 2 is proved.

Fact 3. If X is a pseudocompact space and Y is a second countable space then every continuous surjective map $f : X \to Y$ is $\mathbb{R}$-quotient.

Proof. Take any function $g : Y \to \mathbb{R}$ such that $g \circ f$ is continuous. Fix any closed set $F \subset \mathbb{R}$. We are going to prove that $G = g^{-1}(F)$ is closed in Y. For any $x \in \mathbb{R}$, let $d_F(x) = \inf\{|x - y| : y \in F\}$. The function $d_F : \mathbb{R} \to [0, +\infty)$ is continuous (see Claim of S.019) and hence the set $U_n = d_F^{-1}\left(\left[0, \frac{1}{n}\right)\right)$ is open in X and contains F. Besides, $\overline{U}_{n+1} \subset d_F^{-1}\left(\left[0, \frac{1}{n+1}\right]\right) \subset d_F^{-1}\left(\left[0, \frac{1}{n}\right)\right) = U_n$ for each $n \in \mathbb{N}$. Consequently, $F = \bigcap\{U_n : n \in \mathbb{N}\} = \bigcap\{\overline{U}_n : n \in \mathbb{N}\}$. The set $H = (g \circ f)^{-1}(F)$ is closed in X and $H = \bigcap\{V_n : n \in \mathbb{N}\} = \bigcap\{\overline{V}_n : n \in \mathbb{N}\}$, where $V_n = (g \circ f)^{-1}(U_n)$ for all numbers $n \in \mathbb{N}$. Therefore $G = \bigcap\{f(\overline{U}_n) : n \in \mathbb{N}\}$.

Note that every $\overline{U}_n$ is pseudocompact. Indeed, if γ is a locally finite family of non-empty open sets of $\overline{U}_n$ then $\gamma' = \{U \cap U_n : U \in \gamma\}$ is a locally finite family of non-empty open subsets of X. Since X is pseudocompact, the family γ' is finite and hence so is γ. Now Problem 136(ii) implies that $\overline{U}_n$ is pseudocompact. Every $f(\overline{U}_n)$ is compact being Lindelöf and pseudocompact (see Problem 138) and hence $f(\overline{U}_n)$ is closed in Y for each n. Finally, G is closed being the intersection of closed sets. This proves that g is continuous and Fact 3 is proved.

Now let $f : M \to \mathbb{I}$ be a continuous surjective map of the Mrowka space constructed in Fact 2. The set ω is dense in M and hence $C = f(\omega)$ is a countable dense subset of $\mathbb{I}$. The set C cannot be open because every open subset of $\mathbb{I}$ contains a non-empty interval which is uncountable. Observe that $f^{-1}(C)$ contains ω and hence $M \setminus f^{-1}(C)$ is contained in $\mathcal{M}$ which is closed and discrete. Since every subset of a closed discrete set is closed, the set $f^{-1}(C)$ is open. Thus the set C witnesses the fact that f is not a quotient map. By Fact 3 and Problem 142(iii) the map f is $\mathbb{R}$-quotient and our solution is complete.

S.155. *Prove that any* $\mathbb{R}$*-quotient condensation is a homeomorphism.*

Solution. Suppose that $f : X \to Y$ is an $\mathbb{R}$-quotient condensation. By Problem 127 we may consider that X is a subspace of $\mathbb{I}^A$ for some A. Let $p_a : I^A \to \mathbb{I}$ be the natural projection onto the ath factor. The map $f^{-1} : Y \to X$ can be considered as a map from Y to $\mathbb{I}^A$ and it suffices to prove that $f^{-1} : Y \to \mathbb{I}^A$ is continuous. By Problem 102 it is sufficient to show that $g_a = p_a \circ f^{-1}$ is continuous for each $a \in A$. Observe that $g_a \circ f = p_a \circ f^{-1} \circ f = p_a$ is a continuous map. Applying the fact that f is $\mathbb{R}$-quotient, we conclude that g_a is continuous for each a and hence f^{-1} is a continuous map, i.e., f is a homeomorphism.

S.156. *For any space X prove that*

(i) $c(X) \leqslant d(X) \leqslant nw(X) \leqslant w(X)$.
(ii) $c(X) \leqslant s(X)$ *and* $\text{ext}(X) \leqslant l(X) \leqslant nw(X)$.
(iii) $\psi(X) \leqslant \chi(X)$ *and* $\psi(X) \leqslant iw(X) \leqslant nw(X)$.
(iv) $t(X) \leqslant \chi(X) \leqslant w(X)$ *and* $t(X) \leqslant nw(X)$.

Solution. (i) Any base is also a network so $nw(X) \leqslant w(X)$. If $\mathcal{N} \subset \exp(X) \setminus \{\emptyset\}$ is a network of X, then pick a point $x(P) \in P$ for any $P \in \mathcal{N}$. It is immediate that $\{x(P) : P \in \mathcal{N}\}$ is dense in X whence $d(X) \leqslant nw(X)$. Now if $D \subset X$ is a dense set, then for any disjoint family $\gamma \subset \tau^*(X)$ we can choose $y(U) \in U$ for any $U \in \gamma$. The map $U \to y(U)$ is clearly an injection of γ to D which proves that $c(X) \leqslant d(X)$.

(ii) If $\gamma \subset \tau^*(X)$ is a disjoint family, choose $y(U) \in U$ for each $U \in \gamma$. The subspace $D = \{y(U) : U \in \gamma\}$ is discrete and $|D| = |\gamma|$. Consequently, $c(X) \leqslant s(X)$. Assume that $l(X) \leqslant \kappa$ and take any closed discrete $D \subset X$. For each $d \in D$ choose $U_d \in \tau(d, X)$ with $U_d \cap D = \{d\}$. The family $\mu = \{U_d : d \in D\} \cup \{X \setminus D\}$ is an open cover of X. It is easy to see that every subcover μ' of the cover μ has to contain the family $\{U_d : d \in D\}$ and hence $|\mu'| = |\mu|$. This shows that $D = |\mu| \leqslant \kappa$ whence $\text{ext}(X) \leqslant l(X)$. Suppose finally that $\mathcal{N}$ is a network of X, $|\mathcal{N}| = \kappa$ and $\gamma \subset \tau(X)$ is an open cover of X. Let $\mathcal{N}' = \{P \in \mathcal{N}$: there is $U(P) \in \gamma$ with $P \subset U(P)\}$. The family $\gamma' = \{U(P) : P \in \mathcal{N}'\}$ is a subcover of γ of cardinality $\leqslant \kappa$. Indeed, we only have to prove that γ' is a cover. For any $x \in X$, there is $U \in \gamma$ with $x \in U$. The family $\mathcal{N}$ being a network, there is $P \in \mathcal{N}$ with $x \in P \subset U$. This shows that $P \in \mathcal{N}'$ and hence $x \in P \subset U(P)$. Thus γ' is a cover of X and we proved that $l(X) \leqslant nw(X)$.

(iii) If $\mathcal{B}$ is a local base at a point $x \in X$ then, for any $y \neq x$, the set $X \setminus \{y\}$ is an open neighbourhood of X (don't forget that all spaces are Tychonoff and hence T_1). Thus, there exists $U \in \mathcal{B}$ such that $x \in U \subset X \setminus \{y\}$. As a consequence, $\bigcap \mathcal{B} = \{x\}$ and therefore $\psi(X) \leqslant \chi(X)$. Let $f : X \to Y$ be a condensation such that $w(Y) \leqslant \kappa$. Then $\psi(Y) \leqslant \chi(Y) \leqslant w(Y) \leqslant \kappa$. Now, if $x \in X$, let $y = f(x)$ and fix a family $\mathcal{B} \subset \tau(Y)$ such that $\bigcap \mathcal{B} = \{y\}$. Then $\mathcal{B}' = \{f^{-1}(U) : U \in \mathcal{B}\} \subset \tau(X)$ and $\bigcap \mathcal{B}' = \{x\}$ which proves that $\psi(X) \leqslant \kappa$. Thus, we established the inequality $\psi(X) \leqslant iw(X)$. To show that $iw(X) \leqslant nw(X)$ take any network $\mathcal{N}$ of the space X with $|\mathcal{N}| = nw(X) = \kappa$. It is an easy consequence of regularity of X that if we take the closures of the elements of $\mathcal{N}$, we will still have a network in X. Therefore, we can assume that all elements of $\mathcal{N}$ are closed.

Fact 1. If a space M is second countable and A is an infinite set then, for any $Y \subset M^A$, we have $w(Y) \leqslant |A|$.

Proof. Fix a countable base $\mathcal{B}$ in M. Given $a_1, \ldots, a_n \in A$ and $O_1, \ldots, O_n \in \mathcal{B}$, let $[a_1, \ldots, a_n; O_1, \ldots, O_n] = \{x \in M^A : x(a_i) \in O_i \text{ for all } i \leqslant n\}$. The family $\mathcal{C} = \{[a_1, \ldots, a_n; O_1, \ldots, O_n] : n \in \mathbb{N}, a_i \in A \text{ and } O_i \in \mathcal{B} \text{ for all } i \leqslant n\}$ is a base in M^A and $|\mathcal{C}| \leqslant |A|$ so $w(M^A) \leqslant |A|$. It is evident that $\mathcal{C}' = \{U \cap Y : U \in \mathcal{C}\}$ is a base in Y and $|\mathcal{C}'| \leqslant |\mathcal{C}| \leqslant |A|$ so Fact 1 is proved.

Fact 2. There exists a set $A \subset C(X, [0, 1])$ with $|A| \leqslant \kappa$ such that, for any distinct $x, y \in X$, there is $f \in A$ for which $f(x) \neq f(y)$.

Proof. Let us call a pair $p = (M, N) \in \mathcal{N} \times \mathcal{N}$ *marked* if there exist a function $f = f_p \in C(X, [0, 1])$ such that $f(M) \subset [0, \frac{1}{3}]$ and $f(N) \subset [\frac{2}{3}, 1]$. Let $A = \{f_p : p \text{ is a marked pair}\}$. It is clear that $|A| \leqslant |\mathcal{N} \times \mathcal{N}| \leqslant \kappa$. Given distinct $x, y \in X$, by the Tychonoff property of X, there is $g \in C(X, [0, 1])$ such that $g(x) = 1$ and $g(y) = 0$. If $U = g^{-1}([0, \frac{1}{3}))$ and $V = g^{-1}((\frac{2}{3}, 1])$ then $U \in \tau(x, X)$ and $V \in \tau(y, X)$. As a consequence there are $M, N \in \mathcal{N}$ such that $x \in M \subset U$ and $y \in N \subset V$. Since $g(M) \subset [0, \frac{1}{3}]$ and $g(N) \subset [\frac{2}{3}, 1]$, the pair $p = (M, N)$ is marked and hence $f = f_p \in A$ so $f(x) \leqslant \frac{1}{3}$ and $f(y) \geqslant \frac{2}{3}$ which implies $f(x) \neq f(y)$ so Fact 2 is proved.

To finish the proof of the inequality $iw(X) \leqslant nw(X)$, take the set A given by Fact 2 and consider the map $\varphi : X \to [0, 1]^A$ defined by $\varphi(x)(f) = f(x)$ for all $x \in X$ and $f \in A$. If $f \in A$ and $\pi_f : [0, 1]^A \to [0, 1]$ is the respective natural projection then the composition $\pi_a \circ \varphi = f$ is a continuous function on X which proves that φ is a continuous map by Problem 102. If $Y = \varphi(X)$ then $w(Y) \leqslant |A| \leqslant \kappa$ by Fact 1. Thus, we only have to prove that φ is an injection. To see this take distinct $x, y \in X$. By Fact 2 there is $f \in A$ with $f(x) \neq f(y)$. Then $\varphi(x)(f) = f(x) \neq f(y) = \varphi(y)(f)$ and therefore $\varphi(x) \neq \varphi(y)$ and the proof of (iii) is complete.

(iv) If $\mathcal{B}$ is a base of the space X then, for any $x \in X$, the family $\mathcal{B}_x = \{U \in \mathcal{B} : x \in U\}$ is a local base at x and $|\mathcal{B}_x| \leqslant |\mathcal{B}|$ which proves the inequality $\chi(X) \leqslant w(X)$. Now suppose that $\chi(X) \leqslant \kappa$ and take any $A \subset X$ with $x \in \overline{A}$. Fix a local base $\mathcal{B}$ at the point x such that $|\mathcal{B}| \leqslant \kappa$. For any $U \in \mathcal{B}$, pick any point $x_U \in U \cap A$ and let $B = \{x_U : U \in \mathcal{B}\}$. It is clear that $B \in A$ and $|B| \leqslant \kappa$. To see that $x \in \overline{B}$, take any $W \in \tau(x, X)$. Since $\mathcal{B}$ is a local base at x, there is $U \in \mathcal{B}$ with $U \subset W$. Therefore $x_U \in W \cap A$ and hence $W \cap A \neq \emptyset$ for any open $W \ni x$. Thus $x \in \overline{B}$ and we proved that $t(X) \leqslant \chi(X)$. Now suppose that $nw(X) \leqslant \kappa$ and take any $A \subset X$ with some $x \in \overline{A}$. Let $\mathcal{N}' = \{N \in \mathcal{N} : N \cap A \neq \emptyset\}$ and choose a point $a_N \in N \cap A$ for each $N \in \mathcal{N}'$. It is clear that $B = \{a_N : N \in \mathcal{N}'\} \subset A$ and $|B| \leqslant |\mathcal{N}| \leqslant \kappa$. We claim that $x \in \overline{B}$. For if not, take any $U \in \tau(x, X)$ with $U \cap B = \emptyset$ and note that $B \cap A \neq \emptyset$ because $x \in \overline{A}$. Pick any $a \in A \cap U$ and $N \in \mathcal{N}$ such that $a \in N \subset U$. It is clear that $N \in \mathcal{N}'$ and hence $a_N \in U \cap B$ which is a contradiction. Therefore $x \in \overline{B}$ and we proved that $t(X) \leqslant nw(X)$.

S.157. *Prove that, for any space X, if Y is a continuous image of X, then*

(i) $c(Y) \leqslant \mathbf{c}(X)$.
(ii) $d(Y) \leqslant d(X)$.
(iii) $nw(Y) \leqslant nw(X)$.
(iv) $s(Y) \leqslant s(X)$.
(v) $\text{ext}(Y) \leqslant \text{ext}(X)$.
(vi) $l(Y) \leqslant l(X)$.

Solution. Let $f : X \to Y$ be a continuous onto map.

(i) If γ is a disjoint family of non-empty open subsets of Y then the family $\gamma' = \{f^{-1}(U) : U \in \gamma\} \subset \tau^*(X)$ is also disjoint and $|\gamma'| = |\gamma|$. This shows that $c(Y) \leqslant c(X)$.

(ii) If A is a dense subset of X, then $f(A)$ is dense in Y, $|f(A)| \leqslant |A|$ and hence $d(Y) \leqslant d(X)$.

(iii) If $\mathcal{N}$ is a network in X, then $\mathcal{M} = \{f(P) : P \in \mathcal{N}\}$ is a network in Y, which together with $|\mathcal{M}| \leq |\mathcal{N}|$ implies $nw(Y) \leqslant nw(X)$.

(iv)–(v) If D is a (closed) discrete subspace of the space Y, then we can choose $x(d) \in f^{-1}(d)$ for each $d \in D$. It is immediate that the set $E = \{x(d) : d \in D\}$ is (closed) discrete and $|E| = |D|$. This shows that $s(Y) \leqslant s(X)$ and $\text{ext}(Y) \leqslant \text{ext}(X)$.

(vi) If $\gamma \subset \tau(Y)$ is a cover of Y, then the family $\mu = \{f^{-1}(U) : U \in \gamma\}$ is an open cover of X. Take any $\mu' \subset \mu$ such that $\bigcup \mu' = X$ and $|\mu'| \leqslant \kappa = l(X)$. Then $\{f(U) : U \in \mu'\}$ is a subcover of γ of cardinality $\leqslant \kappa$. Therefore $l(Y) \leqslant l(X)$.

S.158. *Show that, for any* $\varphi \in \{$*weight, character, pseudocharacter, i-weight, tightness*$\}$ *there exist spaces X and Y such that Y is a continuous image of X and* $\varphi(Y) > \varphi(X)$.

Solution. We will need the following statement.
Fact 1. Let T be a dense subset of a space Z. Then, for each $t \in T$, we have $\chi(t,T) = \chi(t, Z)$.

Proof. If $\mathcal{B}$ is a local base at t in Z then $\{U \cap T : U \in \mathcal{B}\}$ is a local base at t in T and therefore $\chi(t, T) \leqslant \chi(t, Z)$. Now take a local base $\mathcal{C}$ of t in T. For each $U \in \mathcal{C}$ choose $O(U) \in \tau(Z)$ such that $O(U) \cap T = U$. We claim that the family $\mathcal{B} = \{O(U) : U \in \mathcal{C}\}$ is a local base at t in Z. Indeed, assume that $t \in W \in \tau(Z)$. By regularity of Z there is $V \in \tau(t, Z)$ such that $\overline{V} \subset W$ (the bar will denote the closure in Z). Take $U \in \mathcal{C}$ with $U \subset V \cap T$. Then $\overline{O(U)} = \overline{U} \subset \overline{V} \subset W$ and hence $O(U) \subset W$ which proves that $\mathcal{B}$ is a local base at t in Z. Since $|\mathcal{B}| \leqslant |\mathcal{C}|$, we have $\chi(t, Z) \leqslant \chi(t, T)$ concluding the proof of Fact 1.

Since $d(\mathbb{R}) \leqslant w(\mathbb{R}) = \omega$ by Problem 156(i), we have $d(\mathbb{R}^{\mathfrak{c}}) \leqslant \omega$ by Problem 108. Take any dense countable subspace $Y \subset \mathbb{R}^{\mathfrak{c}}$. Then $\chi(y, Y) > \omega$ for every $y \in Y$. Indeed, if $\chi(y, Y) = \omega$ for some $y \in Y$ then $\chi(y, \mathbb{R}^{\mathfrak{c}}) = \omega$ by Fact 1. It is evident that $\mathbb{R}^{\mathfrak{c}}$ is homeomorphic to $C_p(D(\mathfrak{c}))$ and hence, for any $f, g \in \mathbb{R}^{\mathfrak{c}}$ there is a homeomorphism $\varphi : \mathbb{R}^{\mathfrak{c}} \to \mathbb{R}^{\mathfrak{c}}$ with $\varphi(f) = g$ (see Problem 079). As a consequence $\chi(\mathbb{R}^{\mathfrak{c}}) = \omega$. Observe that $|\mathbb{R}| = \mathfrak{c}$ and hence $C_p(\mathbb{R})$ is homeomorphic to a subspace of $\mathbb{R}^{\mathfrak{c}}$ so $\chi(C_p(\mathbb{R})) = \omega$ which is a contradiction with S.046. Thus we have $\chi(y, Y) > \omega$

for any $y \in Y$ and, in particular, $w(Y) > \omega$. Now if X is a discrete countably infinite space then Y is a continuous image of X (any surjection of X onto Y will do) while $w(X) = \chi(X) = \omega$.

To see that tightness is not preserved by continuous images observe that the space $Y = C_p(D(\omega_1))$ has uncountable tightness because $D(\omega_1)$ is not Lindelöf (see Problem 149). If X is the set $C_p(D(\omega_1))$ with the discrete topology then $t(X) = \omega$ and Y is a continuous image of X while $t(Y) > \omega$.

Now if $Y = A(\omega_1)$ then $\psi(Y) > \omega$. Indeed, take any countable $\gamma \subset \tau(a, A(\omega_1))$. Then $\omega_1 \setminus U$ is finite for each $U \in \gamma$ and hence $M = \bigcup\{A(\omega_1) \setminus U : U \in \gamma\}$ is countable. Since any $\alpha \in \omega_1 \setminus M$ belongs to $\bigcap \gamma$, we have $\bigcap \gamma \neq \{a\}$ and hence $\psi(a, A(\omega_1)) > \omega$. Note that also $iw(A(\omega_1)) \geqslant \psi(A(\omega_1)) > \omega$ (see Problem 156(iii)). The space $X = D(\omega_1)$ has the same cardinality as Y and hence Y is a continuous image of X. However, $\psi(X) = \omega$ and $iw(X) = \omega$. The first equality is evident so let us show that X can be condensed onto a second countable space. Since $|X| \leqslant |\mathbb{R}|$, there is an bijection $\varphi : X \to Z \subset \mathbb{R}$. Since every map of a discrete space is continuous, φ is a condensation onto a second countable space Z. Thus $iw(X) \leqslant \omega$ and our solution is complete.

S.159. *Suppose that X is a space and $Y \subset X$. Prove that*

(i) $w(Y) \leqslant w(X)$.
(ii) $nw(Y) \leqslant nw(X)$.
(iii) $\psi(Y) \leqslant \psi(X)$.
(iv) $s(Y) \leqslant s(X)$.
(v) $iw(Y) \leqslant iw(X)$.
(vi) $t(Y) \leqslant t(X)$.
(vii) $\chi(Y) \leqslant \chi(X)$.

Solution. If $\mathcal{B}$ is a base (network) in X then $\mathcal{B}' = \{U \cap Y : U \in \mathcal{B}\}$ is a base (network) in Y and $|\mathcal{B}'| \leqslant |\mathcal{B}|$ so (i) and (ii) are proved. If $y \in Y$, $\mathcal{P} \subset \tau(X)$ and $\bigcap \mathcal{P} = \{x\}$ then $\mathcal{P}' = \{U \cap Y : U \in \mathcal{P}\} \subset \tau(Y)$ and $\bigcap \mathcal{P}' = \{x\}$ so (iii) is established. Any discrete subspace of Y is also a discrete subspace of X so (iv) holds. Assume that $iw(X) = \kappa$ and take any condensation $f : X \to Z$ for which $w(Z) \leqslant \kappa$. Then $f \mid Y : Y \to f(Y)$ is a condensation and $w(f(Y)) \leqslant w(Y) \leqslant \kappa$. Therefore $iw(Y) \leqslant \kappa$ and (v) is proved. Suppose that $t(X) \leqslant \kappa$ and take any $A \subset Y$ such that $y \in \mathrm{cl}_Y(A)$. Then $y \in \mathrm{cl}_X(A)$ and hence we can find $B \subset A$ with $|B| \leqslant \kappa$ and $y \in \mathrm{cl}_X(B)$. It is immediate that $y \in \mathrm{cl}_Y(B)$ and (vi) is settled. If $y \in Y$ and $\mathcal{C}$ is a local base at y in X then $\mathcal{C}' = \{U \cap Y : U \in \mathcal{C}\}$ is a local base at y in Y and $|\mathcal{C}'| \leqslant |\mathcal{C}|$ which settles (vii).

S.160. *Let $\varphi \in \{$Souslin number, density, extent, Lindelöf number$\}$. Show that there exist spaces X and Y such that $Y \subset X$ and $\varphi(Y) > \varphi(X)$.*

Solution. The space $\mathbb{I}^A$ has the Souslin property for any A by Problem 109. The space $Y = D(\omega_1)$ is Tychonoff and hence it is homeomorphic to a subspace of $X = \mathbb{I}^A$ for some A by Problem 127. Thus we have $c(X) = \omega$ while $c(Y) = \omega_1 > c(X)$.

The space $Y = A(\omega_1)$ is compact and $w(Y) = d(Y) = \omega_1$. By Problem 126 the space Y is homeomorphic to a subspace of $X = \mathbb{I}^{\omega_1}$. The space X is separable by Problem 108 and hence $d(Y) = \omega_1 > d(X) = \omega$.

Now if $X = A(\omega_1)$ then $\mathrm{ext}(X) = l(X) = \omega$ because X is compact. The subspace $Y = \omega_1 \subset X$ is discrete and $l(Y) = \mathrm{ext}(Y) = \omega_1 > l(X) = \mathrm{ext}(X)$.

S.161. *Suppose that $f: X \to Y$ be an open map. Prove that $w(Y) \leqslant w(X)$ and $\chi(Y) \leqslant \chi(X)$.*

Solution. If $\mathcal{B}$ is a base of X (a local base at a point $x \in X$) then the family $\mathcal{B}' = \{f(U) : U \in \mathcal{B}\}$ is a base in Y (a local base at the point $f(x)$ in Y, respectively) and $|\mathcal{B}'| \leqslant |\mathcal{B}|$ which proves that $w(Y) \leqslant w(X)$ and $\chi(Y) \leqslant \chi(X)$.

S.162. *Let $f: X \to Y$ be a quotient map. Prove that $t(Y) \leqslant t(X)$.*

Solution. We will first prove the following lemma for further references.

Lemma. Given an infinite cardinal κ and a space Z, we have $t(Z) \leqslant \kappa$ if and only if, for any non-closed $A \subset Z$, there is $B \subset A$ such that $|B| \leqslant \kappa$ and $\overline{B} \setminus A \neq \emptyset$.

Proof. If $t(Z) \leqslant \kappa$ then, for any $z \in \overline{A} \setminus A$ we have $B \subset A$ with $|B| \leqslant \kappa$ such that $z \in \overline{B}$. Thus $\overline{B} \setminus A \ni z$ and necessity is proved. To establish sufficiency, take any $A \subset Z$ and let $[A]_\omega = \bigcup\{\overline{B} : B \subset A \text{ and } |B| \leqslant \kappa\}$. We must prove that $[A]_\omega = \overline{A}$. If it is not so, then $[A]_\omega$ is not closed and hence we have $C \subset [A]_\omega$ such that $|C| \leqslant \kappa$ and $\overline{C} \setminus A]_\omega \neq \emptyset$. For each $c \in C$ fix a set $B_c \subset A$ with $|B_c| \leqslant \kappa$ and $c \in \overline{B}_c$. The set $B = \bigcup\{B_c : c \in C\}$ has cardinality $\leqslant \kappa$ and $c \in \overline{B}$ for every $c \in C$. Therefore $\overline{C} \subset \overline{B} \subset [A]_\omega$, this contradiction finishes the proof of our lemma.

Suppose that $t(X) = \kappa$ and take any non-closed $A \subset Y$. Since the map f is quotient, the set $A' = f^{-1}(A)$ is non-closed in X and hence there is $B' \subset A'$ such that $|B'| \leqslant \kappa$ and $\overline{B}' \setminus A' \neq \emptyset$. If $B = f(B')$ then $B \subset A$, $|B| \leqslant \kappa$ and $\overline{B} \setminus A \neq \emptyset$. Applying our lemma, we can conclude that $t(Y) \leqslant \kappa = t(X)$.

S.163. *Let X and Y be topological spaces. Given a continuous map $r: X \to Y$, define the dual map $r^*: C_p(Y) \to C_p(X)$ by $r^*(f) = f \circ r$ for any $f \in C_p(Y)$. Prove that*

(i) The map r^* is continuous.
(ii) If $r(X) = Y$, then r^* is a homeomorphism of $C_p(Y)$ onto $r^*(C_p(Y))$.
(iii) If $r(X) = Y$, then the set $r^*(C_p(Y))$ is closed in $C_p(X)$ if and only if r is an $\mathbb{R}$-quotient map.
(iv) If $r(X) = Y$, then the set $r^*(C_p(Y))$ is dense in $C_p(X)$ if and only if r is a condensation.
(v) If $r(X) = Y$, then the set $r^*(C_p(Y))$ coincides with $C_p(X)$ if and only if r is a homeomorphism.
(vi) If $r(X) = Y$ and $s: X \to Z$ is a continuous onto map, then there exists a continuous map $t: Z \to Y$ with $t \circ s = r$ if and only if $r^*(C_p(Y)) \subset s^*(C_p(Z))$.

Solution. (i) Fix an arbitrary function $h_0 \in C_p(Y)$ and a basic neighbourhood $U = O_X(g_0, x_1, \ldots, x_n, \varepsilon)$ of the function $g_0 = r^*(h_0)$ (the index X shows that the basic neighbourhood is taken in $C_p(X)$). Here $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$.

Then $W = O_Y(h_0, r(x_1), \ldots, r(x_n), \varepsilon) \in \tau(h_0, C_p(Y))$ and $r^*(W) \subset U$. Indeed, $h \in W$ implies $|h(r(x_i)) - h_0(r(x_i))| = |r^*(h)(x_i) - g_0(x_i)| < \varepsilon$ for all $i \leqslant n$ and therefore $r^*(h) \in U$. This proves that r^* is continuous at an arbitrary point $h_0 \in C_p(Y)$ so (i) is proved.

(ii) To see that r^* is an injection take distinct $f, g \in C_p(Y)$. Then $f(y) \neq g(y)$ for some $y \in Y$. If $x \in r^{-1}(y)$ then $r^*(f)(x) = f(y) \neq g(y) = r^*(g)(x)$ and hence $r^*(f) \neq r^*(g)$. Let us show that $(r^*)^{-1}$ is continuous. Take any $f \in C_p(Y)$ and any basic neighbourhood $U = O_Y(f, y_1, \ldots, y_n, \varepsilon)$ of the function f. Choose $x_i \in r^{-1}(y_i)$ for all $i \leqslant n$ and let $V = O_X(g, x_1, \ldots, x_n, \varepsilon) \cap r^*(C_p(Y))$, where $g = r^*(f)$. The set V is an open neighbourhood of g in the space $r^*(C_p(Y))$. If $h \in V$ then $h = r^*(h')$ for some $h' \in C_p(Y)$ and $h(x_i) = r^*(h')(x_i) = h'(r(x_i)) = h'(y_i)$ for all $i \leqslant n$. Thus, $h' = (r^*)^{-1}(h) \in U$ for any $h \in V$ and hence $(r^*)^{-1}(V) \subset U$ which proves continuity of $(r^*)^{-1}$ at an arbitrary point f. Hence the mapping $r^* : C_p(Y) \to r^*(C_p(Y))$ is a homeomorphism.

(iii) Suppose that r is $\mathbb{R}$-quotient. Given any $f \in \overline{r^*(C_p(Y))}$, observe that $f|(r^{-1}(y))$ is a constant function for any $y \in Y$. Indeed, if $x_1, x_2 \in (r^*)^{-1}(y)$ and $f(x_1) \neq f(x_2)$ then for $\varepsilon = \frac{1}{2} \cdot |f(x_1) - f(x_2)| > 0$, take an arbitrary function $g \in O_X(f, x_1, x_2, \varepsilon) \cap r^*(C_p(Y))$. Since $g \in O_X(f, x_1, x_2, \varepsilon)$, we have $g(x_1) \neq g(x_2)$ while $g = r^*(g')$ for some $g' \in C_p(Y)$ and hence $g(x_1) = g'(r(x_1)) = g'(y) = g'(r(x_2)) = g(x_2)$ which is a contradiction. As a consequence, there exists a function $f' : Y \to \mathbb{R}$ such that $f = f' \circ r$. Since r is $\mathbb{R}$-quotient and f is continuous, the map f' has to be continuous, i.e., $f = r^*(f') \in r^*(C_p(Y))$ which proves that $r^*(C_p(Y))$ is closed in $C_p(X)$.

Now assume that $r^*(C_p(Y))$ is closed in $C_p(X)$. To show that r is $\mathbb{R}$-quotient, take any $f : Y \to \mathbb{R}$ such that $g = f \circ r \in C_p(X)$. We claim that $g \in \overline{r^*(C_p(Y))}$. Indeed, if $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ there exists $h \in C_p(Y)$ such that $h(r(x_i)) = g(x_i) = f(r(x_i))$ for each $i \leqslant n$ (note that here we used the fact that $g = f \circ r$ and hence g is constant on the fibres of r). Then $r^*(h) \in O_X(x_1, \ldots, x_n, \varepsilon) \cap r^*(C_p(Y))$ and therefore $g \in \overline{r^*(C_p(Y))} = r^*(C_p(Y))$. This means that $g = r^*(g') = r^*(f)$ for some $g' \in C_p(Y)$. The map $r^* : \mathbb{R}^Y \to \mathbb{R}^X$ being an injection we can conclude that $f = g'$, i.e., the map f is continuous which proves that r is $\mathbb{R}$-quotient.

(iv) Suppose that r is a condensation. Given $f \in C_p(X), x_1, \ldots, x_n \in X$ and $\varepsilon > 0$, there exists $g \in C_p(Y)$ such that $g(r(x_i)) = f(x_i)$ for all $i \leqslant n$. It is clear that $r^*(g) \in O_X(f, x_1, \ldots, x_n, \varepsilon)$ and hence $f \in \overline{r^*(C_p(Y))}$ for each $f \in C_p(X)$ whence $\overline{r^*(C_p(Y))} = C_p(X)$. Now if r is not injective then take distinct $x_1, x_2 \in X$ for which $r(x_1) = r(x_2)$. Observe that $U = [x_1, x_2; (0, 1), (2, 3)]_X$ is a non-empty open set in $C_p(X)$ with $U \cap r^*(C_p(Y)) = \emptyset$ and hence $r^*(C_p(Y))$ is not dense in $C_p(X)$.

(v) If r is a homeomorphism then, for any $f \in C_p(X)$, the function $g = f \circ r^{-1}$ belongs to $C_p(Y)$ and $r^*(g) = f$ which proves that $r^*(C_p(Y)) = C_p(X)$. Now suppose that $r^*(C_p(Y)) = C_p(X)$. Then the map r is a condensation by (iv) and $\mathbb{R}$-quotient by (iii). Thus r is a homeomorphism by Problem 155.

(vi) Assume first that there exists a continuous map $t : Z \to Y$ with $t \circ s = r$. Then, for any $f \in C_p(Y)$ we have $t^*(f) \in C_p(Z)$ and $s^*(t^*(f)) = (f \circ t) \circ s = f \circ (t \circ s) = f \circ r = r^*(f)$. This proves that $r^*(C_p(Y)) \subset s^*(C_p(Z))$. Now assume that $r^*(C_p(Y)) \subset s^*(C_p(Z))$. Observe first that r is constant on $s^{-1}(z)$ for any $z \in Z$. Indeed, so is $s^*(f)$

for any $f \in C_p(Z)$ and hence $r^*(g)$ must be constant on $s^{-1}(z)$ for each $g \in C_p(Y)$ and $z \in Z$. Now, if $x, y \in s^{-1}(z)$ and $r(x) \neq r(y)$ then there is $g \in C_p(Y)$ with $g(r(x)) = 1$ and $g(r(y)) = 0$ which implies that $r^*(g)$ is not constant on $s^{-1}(z)$, a contradiction. We proved that r is constant on the fibres of s and hence there exists a map $t : Z \to Y$ with $t \circ s = r$ so we only have to show that t is continuous.

Take any $B \subset Z$ and any $z \in \overline{B}$. It suffices to show that $t(z) \in \overline{t(B)}$. If this is not true then there is $f \in C_p(Y)$ with $f(t(z)) = 1$ and $f(t(B)) \subset \{0\}$. Pick any point $x \in s^{-1}(z)$ and observe that $r(x) = t(z)$ which implies $r^*(f)(x) = 1$. There exists $g \in C_p(Z)$ such that $s^*(g) = r^*(f)$. We have $1 = r^*(f)(x) = s^*(g)(x) = g(s(x)) = g(z)$. Now if $b \in B$, take any $a \in s^{-1}(b)$ and note that $r(a) = t(s(a)) = t(b) \in t(B)$ and therefore $r^*(f)(a) = f(t(b)) = 0$. Thus $0 = r^*(f)(a) = s^*(g)(a) = g(s(a)) = g(b)$ for each $b \in B$ and therefore $g(B) \subset \{0\}$. We have $z \in \overline{B}$, $g(z) = 1$ and $g(B) \subset \{0\}$ which contradicts continuity of g. The obtained contradiction shows that the function t is continuous and we are done.

S.164. *Let X be a separable space with* $\text{ext}(X) \geqslant \mathfrak{c}$. *Prove that X cannot be normal.*

Solution. Fix a closed discrete $D \subset X$ of cardinality $\mathfrak{c}$ and a countable dense $S \subset X$. If X is normal then $\mathbb{R}^D = C_p(D)$ is a continuous image of $C_p(X)$ by Problem 152(v) and therefore $|C_p(X)| \geqslant |\mathbb{R}^D| = 2^{\mathfrak{c}} > \mathfrak{c}$. Now the map $\pi_S : C_p(X) \to C_p(S)$ is an injection by Problem 152(ii) and hence $|C_p(X)| \leqslant |C_p(S)| \leqslant |\mathbb{R}^S| \leqslant \mathfrak{c}^\omega = \mathfrak{c}$ which is a contradiction.

S.165. *Consider the family $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$. Check that $\mathcal{B}$ has the properties (B1) and (B2) formulated in Problem 006 and hence can be considered a base for a unique topology τ_s on the set $\mathbb{R}$. The space $S = (\mathbb{R}, \tau_s)$ is called the Sorgenfrey line. Prove that*

(i) Any subspace of $\mathcal{S}$ is Lindelöf.
(ii) Any subspace of the space $\mathcal{S}$ is separable.
(iii) No uncountable subspace of $\mathcal{S}$ has a countable network.
(iv) The space $\mathcal{S} \times \mathcal{S}$ is not normal and has a closed discrete subspace of power $\mathfrak{c}$.
(v) Prove that the space $C_p(S)$ has a closed discrete subspace of cardinality $\mathfrak{c}$ *Deduce from this fact that $C_p(S)$ is not normal.*

Solution. It is clear that $\bigcup \mathcal{B} = \mathbb{R}$ and $U \cap V \in \mathcal{B}$ for any $U, V \in \mathcal{B}$. Therefore the properties (B1) and (B2) hold for $\mathcal{B}$.

(i) Let X be any subspace of S. We leave it to the reader to verify that X is Lindelöf if and only if, for any $\gamma \subset \mathcal{B}$ such that $\bigcup \gamma \supset X$, there is a countable $\mu \subset \gamma$ with $\bigcup \mu \supset X$. So take any $\gamma \subset \mathcal{B}$ with $X \subset \bigcup \gamma$. Given $U = [a, b) \in \gamma$, let $U' = (a, b)$. We claim that the set $X' = \bigcup \{U' : U \in \gamma\}$ covers almost all points of X, i.e., $Y = X \setminus X'$ is at most countable. Indeed, if $x \in Y$ then there exists $r(x) > x$ such that $[x, r(x)) \in \gamma$. Note that, for distinct $x, y \in Y$, we have $[x, r(x)) \cap [y, r(y)) = \emptyset$ for otherwise $x \in (y, r(y)) \subset X'$ or $y \in (x, r(x)) \subset X'$. Since weight of $\mathbb{R}$ is countable, its Souslin number is also countable by Problem 156(i) and therefore $\mathbb{R}$ has no uncountable disjoint family of open intervals. Since the family $\{(x, r(x)) : x \in Y\}$

is disjoint, the set Y has to be countable. As a consequence, there is a countable $\mu' \subset \gamma$ such that $Y \subset \bigcup \mu'$. Evidently, the family $\mu \cup \mu'$ is a countable subfamily of γ which covers X. This proves that X is Lindelöf.

(ii) Take any $X \subset S$ and denote by Y the set X with the topology induced from $\mathbb{R}$. Then $d(Y) \leqslant w(Y) = \omega$ by Problem 156(i) and hence there is a countable subset $A \subset X$ which is dense in Y. Denote by B the closure of the set A in X. We claim that the set $C = X \setminus B$ is countable. Indeed, if $x \in C$ then there is $r(x) > x$ such that $[x, r(x)) \cap A = \emptyset$. Now, if $V = (x, r(x)) \cap X \neq \emptyset$ then V is an open non-empty subset of Y. Since A is dense in Y, we have $V \cap A \neq \emptyset$, a contradiction which shows that $[x, r(x)) \cap X = \{x\}$. Therefore $[x, r(x)) \cap [y, r(y)) = \emptyset$ for distinct $x, y \in C$ for otherwise $x \in (y, r(y))$ or $y \in (x, r(x))$. Any disjoint collection of non-empty open intervals of $\mathbb{R}$ is countable (see (i)). Since the family $\{(x, r(x)) : x \in C\}$ is disjoint, the set C has to be countable. It is clear that $C \cup A$ is a countable dense subset of X and we proved that X is separable.

(iii) Let X be an uncountable subspace of S. Assume that $\mathcal{N} = \{P_n : n \in \omega\}$ is a countable network in X. For each $n \in \omega$, let $q_n = \inf P_n$ if the $\inf P_n > -\infty$. Since X is uncountable, we can choose $x \in X$ such that $x \neq q_n$ for any $n \in \omega$. Since $\mathcal{N}$ is a network of X, there exists $n \in \omega$ such that $x \in P_n \subset [x, x+1)$. But this implies that $q_n = x$ which is a contradiction.

(iv) Let us show that $D = \{(t, -t) : t \in \mathbb{R}\}$ is a closed discrete subspace of $S \times S$. For any $d = (t, -t) \in D$ the set $U = [t, t+1) \times [-t, -t+1)$ is open in $S \times S$ and $U \cap D = \{d\}$ which proves that D is discrete. Now, if $z = (x, y) \notin D$ then $\varepsilon = |x + y| > 0$ and therefore $W = [x, x + \frac{\varepsilon}{2}) \times [y, y + \frac{\varepsilon}{2})$ is an open neighbourhood of z in $S \times S$ with $W \cap D = \emptyset$. Thus D is closed in $S \times S$ whence $\mathrm{ext}(S \times S) = \mathfrak{c}$. Since $S \times S$ is separable (this is an easy exercise for the reader), we can apply Problem 164 to conclude that $S \times S$ is not normal.

(v) Let $r : \mathbb{R} \to \mathbb{R}$ be the identity map, i.e., $r(t) = t$ for all $t \in \mathbb{R}$. It is clear that $r : S \to \mathbb{R}$ is a condensation and hence $r^*(C_p(\mathbb{R}))$ is a dense subspace of $C_p(S)$ homeomorphic to $C_p(\mathbb{R})$ (see Problem 163(iv)). If A is a countable dense subspace of $C_p(\mathbb{R})$ (which exists by Problem 047) then $r^*(A)$ is a countable dense subspace of $C_p(S)$ i.e., $C_p(S)$ is separable.

Given an "arrow" $[a, b) \subset S$, observe that it is a clopen subset of S. Let $f(t) = t - a$ and $g(t) = \frac{1}{b-a} \cdot t$ for all $t \in \mathbb{R}$. It is immediate that $h = g \circ f : S \to S$ is a homeomorphism and $h([a, b)) = [0, 1)$. This shows that every two arrows in S are homeomorphic subspaces of S. Note also that $S = \bigcup\{A_k : k \in \omega\}$ where $A_{2i} = [-i-1, -i)$ and $A_{2i+1} = [i, i+1)$ for all $i \in \omega$. Applying Problem 113(iii), we can see that $S = \bigoplus\{A_k : k \in \omega\}$. Now, $[0, 1) = \bigcup\{[p_i, q_i) : i \in \omega\}$ where $p_i = \frac{i}{i+1}$ and $q_i = \frac{i+1}{i+2}$ for each $i \in \omega$. Applying Problem 113(iii) again, we verify that $[0, 1) = \bigoplus\{B_k : k \in \omega\}$ where each B_k is an arrow. As a consequence S is homeomorphic to any of its arrows so we will consider the space $S' = [0, 1)$ from now on. For each $t \in (0, 1]$ let $f_t(x) = 1$ if $0 \leqslant x < t$ and $f_t(x) = 0$ for all $x \in [t, 1)$. It is immediate that $f_t \in C_p(S')$ for all $t \in (0, 1]$.

Fact 1. The set $F = \{f_t : t \in (0, 1]\}$ is closed in $C_p(S')$.

Proof. Given $f \notin F$ suppose first that $f(x) \notin \{0, 1\}$ for some $x \in S'$. Then, for $\varepsilon = \min\{|f(x)|, |f(x) - 1|\}$ the set $[x; (f(x) - \varepsilon, f(x) + \varepsilon)]$ is an open neighbourhood of f which does not meet the set F. Now, if $f(S') \subset \{0, 1\}$ and $f \in \overline{F}$ then $f(x) = 1$ implies $f(y) = 1$ for any $y < x$. Indeed, if $f(y) = 0$ for some $y < x$ then $[y, x; (-1/2, 1/2), (1/2, 3/2)]$ is an open neighbourhood of f which does not meet F because all functions from F are non-increasing. Analogously, $f(x) = 0$ implies $f(y) = 0$ for any $y > x$. Thus, there is $t \in S'$ such that $f(x) = 1$ for all $x < t$ and $f(x) = 0$ for all $x > t$. If $f(t) = 1$ then f discontinuous so $f(t) = 0$ and hence $f = f_t \in F$. This shows that F is closed in $C_p(S')$.

Fact 2. The map $\varphi : S' \to F$ defined by $\varphi(s) = f_{1-s}$ is a homeomorphism.

Proof. It is clear that φ is a bijection. Since $C_p(S')$ is a subspace of the product space $\mathbb{R}^{S'}$, to prove continuity of φ, it suffices to show that $h_t = p_t \circ \varphi$ is continuous for each p_t defined by $p_t(f) = f(t)$ for all $f \in C_p(S')$ and $t \in [0, 1)$ (see Problem 102). Since $h_t(s) = f_{1-s}(t) = 1$ for $s < 1 - t$ and $h_t(s) = 0$ for $s \geqslant 1 - t$, the function h_t is continuous for all $t \in S'$. To finish the proof it suffices to show that the mapping φ is open. Since the arrows form a base in S', it is sufficient to establish that $\varphi(U)$ is open in F for any $U = [a, b) \subset S'$. To see this, observe that $\varphi(U) = \{f_t : 1 - b < t \leqslant 1 - a\} = [1 - b, 1 - a; (1/2, 3/2), (-1/2, 1/2)] \cap F$ and we are done.

Fact 3. The spaces $C_p(S')$ and $C_p(S') \times C_p(S')$ are homeomorphic.

Proof. Since $S' = [0, 1/2) \cup [1/2, 1)$ and S' is homeomorphic to every arrow, the space S' is homeomorphic to $S' \oplus S'$. Thus $C_p(S')$ is homeomorphic to $C_p(S' \oplus S')$ which in turn is homeomorphic to $C_p(S') \times C_p(S')$ by Problem 114.

It follows from Facts 1 and 2 that F is a closed subspace of $C_p(S')$ homeomorphic to S'. Therefore $C_p(S)$ has a closed subspace T which is homeomorphic to S. The space $T \times T$ is a closed subspace of $C_p(S) \times C_p(S)$. By (iv), there is a closed discrete $D \subset T \times T$ with $|D| = \mathfrak{c}$. It is clear that D is also a closed discrete subspace of $C_p(S) \times C_p(S)$. Apply Fact 3 to conclude that $C_p(S)$ is homeomorphic to $C_p(S) \times C_p(S)$ and hence it also has a closed discrete subspace of cardinality continuum. Apply Problem 164 to see that $C_p(S)$ is not normal.

S.166. *Suppose that X is an arbitrary space and $F \subset C_p(X)$. For any $x \in X$, define the function $e_x^F : F \to \mathbb{R}$ by the formula $e_x^F(f) = f(x)$ for any $f \in F$; observe that e_x^F belongs to $C_p(F)$. Let $E^F(x) = e_x^F$ for any $x \in X$; then $E^F : X \to C_p(F)$ is called the evaluation map. Prove that*

(i) E^F is continuous for any $F \subset C_p(X)$.
(ii) E^F is injective if and only if F separates the points of X, i.e., for any distinct $x, y \in X$ there is $f \in F$ with $f(x) \neq f(y)$.
(iii) E^F is an embedding if and only if F generates the topology of X, i.e., the family $\mathcal{U}_F = \{f^{-1}(U) : f \in F, U \in \tau(\mathbb{R})\}$ is a subbase of X.
(iv) E^F is an embedding if F separates the points and the closed subsets of X, i.e., for any $x \in X$ and any closed $G \subset X$ such that $x \notin G$, we have $f(x) \notin \overline{f(G)}$ for some $f \in F$.
(v) The set $X' = E^F(X) \subset C_p(F)$ generates the topology of F and hence F embeds in $C_p(X')$.

Solution. Given $x \in X$, the natural projection $p_x : \mathbb{R}^X \to \mathbb{R}$ to the factor indexed by x, is defined by $p_x(f) = f(x)$ for all $f \in \mathbb{R}^X$. Therefore $e_x^F = p_x|F$ is a continuous mapping, i.e., we proved that $e_x^F \in C_p(F)$ for every $x \in X$.

To see that (i) holds let us show that E^F is continuous at any $x \in X$. Take any basic neighbourhood $U = O(e_x^F, f_1, \ldots, f_n, \varepsilon)$ of the function e_x^F in $C_p(F)$. The set $V = \bigcap\{f_i^{-1}((f_i(x) - \varepsilon, f_i(x) + \varepsilon)) : i \leqslant n\}$ is an open neighbourhood of the point x and it is immediate that $E^F(V) \subset U$. This shows that E^F is a continuous map and hence (i) is proved.

If F separates the points of X then, for any distinct $x, y \in X$ there is $f \in F$ such that $e_x^F(f) = f(x) \neq f(y) = e_y^F(f)$ which shows that the maps e_x^F and e_y^F do not coincide, i.e., $E^F(x) \neq E^F(y)$. Thus E^F is an injection. This settles necessity in (ii). Now, if the mapping E^F is injective then, for any distinct $x, y \in X$ we have $e_x^F = E^F(x) \neq E^F(y) = e_y^F$ and hence there is $f \in F$ for which $f(x) = e_x^F(f) \neq e_y^F(f) = f(y)$, i.e., F separates the points of X and hence (ii) is proved.

To tackle with (iii) assume that E^F is an embedding and hence we can consider that the map E^F is a homeomorphism between X and $X' = E^F(X)$. Thus $\{(E^F)^{-1}(B) : B \in \mathcal{B}\}$ is a subbase in X whenever $\mathcal{B}$ is a subbase in X'. For any $f \in F$ and $U \in \tau(\mathbb{R})$ the set $[f, U] = \{\varphi \in C_p(F) : \varphi(f) \in U\}$ is open in $C_p(F)$ and the family $\mathcal{B}' = \{[f, U] : f \in F, U \in \tau(\mathbb{R})\}$ is a subbase in $C_p(F)$. As a consequence $\mathcal{B} = \{W \cap X' : W \in \mathcal{B}'\}$ is a subbase in X' so the family $\mathcal{S} = \{(E^F)^{-1}([f, U] \cap X') : f \in F, U \in \tau(\mathbb{R})\}$ is a subbase in X. Next observe that $e_x^F \in [f, U]$ if and only if $x \in f^{-1}(U)$ which shows that $(E^F)^{-1}([f, U] \cap X') = f^{-1}(U)$ so the family $\{f^{-1}(U) : U \in \tau(\mathbb{R}), f \in F\}$ is a subbase in X because it coincides with $\mathcal{S}$. Thus, F generates the topology of X and we proved necessity in (iii).

Now, assume that F generates the topology of X. Given distinct $x, y \in X$ there are $f_1, \ldots, f_n$ and $U_1, \ldots, U_n \in \tau(\mathbb{R})$ such that $x \in W = \bigcap\{f_i^{-1}(U_i) : i \leqslant n\}$ and $y \notin W$. As a consequence, $f_i(x) \neq f_i(y)$ for some $i \leqslant n$ which proves that F separates the points of X so E^F is injective by (ii) and hence we can consider E^F to be a bijection between X and $X' = E^F(X)$. Let us prove that the map $g = (E^F)^{-1}$ is continuous at every $y \in X'$. Let $x = h(y)$ and take any $W \in \tau(x, X)$. Since the family $\mathcal{U}_F$ is a subbase in X, there are $f_1, \ldots, f_n \in F$ and $U_1, \ldots, U_n \in \tau(\mathbb{R})$ such that $x \in G = \bigcap\{f_i^{-1}(U_i) : i \leqslant n\} \subset W$.

The set $H = \bigcap\{[f_i, U_i] \cap X' : i \leqslant n\}$ is open in X' and $y \in H$. It is straightforward that $g(H) = G \subset W$ and hence H witnesses continuity of $g = (E^F)^{-1}$ at the point y. Applying (i) we conclude that $E^F : X \to X'$ is a homeomorphism so (iii) is proved.

Now suppose that F separates the points and closed subsets of X. Given $x \in X$ and $W \in \tau(x, X)$ let $G = X \setminus W$; there is $f \in F$ such that $f(x) \notin \overline{f(G)}$ so $U = \mathbb{R} \setminus \overline{f(G)} \in \tau(f(x), \mathbb{R})$ and hence $x \in f^{-1}(U) \subset W$. This proves that $\mathcal{U}_F$ is a base in X so E^F is an embedding by (iii) and hence we settled (iv).

Finally, let $G(x, U) = \{f \in C_p(X) : f(x) \in U\}$ for any $x \in X$ and $U \in \tau(\mathbb{R})$; then the family $\{G(x, U) \cap F : x \in X, U \in \tau(\mathbb{R})\}$ is a subbase in F. Observe that $G(x, U) \cap F = (e_x^F)^{-1}(U)$ for any $x \in X$ and $U \in \tau(\mathbb{R})$ which shows that $\mathcal{E} = \{(e_x^F)^{-1}(U) : x \in X, U \in \tau(\mathbb{R})\}$ is a subbase in F; it is obvious that we have the equality $\mathcal{E} = \{\varphi^{-1}(U) :$

$\varphi \in X'$, $U \in \tau(\mathbb{R})\}$ so the set X', indeed, generates the topology of F. This finishes the proof of (v) and completes our solution.

S.167. *Let X be an arbitrary space. For each point $x \in X$, define the function $e_x : C_p(X) \to \mathbb{R}$ by the formula $e_x(f) = f(x)$ for all $f \in C_p(X)$. For any $x \in X$, let $E(x) = e_x$.*

(i) Show that the map $E : X \to C_p(C_p(X))$ is an embedding.
(ii) Prove that $E(X)$ is closed in $C_p(C_p(X))$.

As a consequence, any space X can be canonically identified with the closed subspace $E(X)$ of the space $C_p(C_p(X))$.

Solution. (i) Taking $F = C_p(X)$ in Problem 166, we can see that $E = E^F$ is a continuous map. By the Tychonoff property of X, for any $x \in X$ and any closed $G \subset X$ there is $f \in C_p(X)$ such that $f(X) = 1$ and $f(G) \subset \{0\}$ and hence $f(x) \notin \overline{f(G)}$. Applying Problem 166 again we can conclude that E is a homeomorphism.

(ii) Denote by f_0 the function equal to zero at all points of X and take any continuous function $\varphi : C_p(X) \to \mathbb{R}$ with $\varphi \in \overline{E(X)} \setminus E(X)$. It is straightforward that any $e_x \in E(X)$ is a linear functional. The set of linear functionals is closed in $C_p(C_p(X))$ by Problem 078 which implies that φ is linear and hence $\varphi(f_0) = 0$. By continuity of φ there exists a basic neighbourhood $W = O(f_0, x_1, \ldots x_n, \varepsilon)$ such that $\varphi(W) \subset (-\frac{1}{2}, \frac{1}{2})$. The space $C_p(C_p(X))$ is Tychonoff and hence there are disjoint open sets $U_i \in \tau(x_i, X)$, $i \leqslant n$ such that $\varphi \notin \overline{\bigcup\{E(U_i) : i \leqslant n\}}$. For each $i \leqslant n$ fix a continuous function $f_i : X \to [0, 1]$ with $f(x_i) = 1$ and $f|(X \setminus U_i) \equiv 0$. Then the function $f = 1 - (f_1 + \cdots + f_n)$ is continuous, $f(x_i) = 0$ for all $i \leqslant n$ and $f|(X \setminus U) \equiv 1$ for $U = \bigcup_{i \leqslant n} U_i$. Since $f \in W$, we have $\varphi(f) \in (-\frac{1}{2}, \frac{1}{2})$. On the other hand $e_x(f) = f(x) = 1$ for any $x \in X \setminus U$ which shows that the set $V = \{\psi \in C_p(C_p(X)) : \psi(f) \in (-\frac{1}{2}, \frac{1}{2})\} \in \tau(\varphi, C_p(C_p(X)))$ does not intersect the set $E(X \setminus U)$. As a consequence, $\varphi \notin \overline{E(X \setminus U)}$ and therefore φ does not belong to the closure of the set $E(X \setminus U) \cup \bigcup\{E(U_i) : i \leqslant n\} = E(X)$ which is a contradiction. This proves that $E(X)$ is closed in $C_p(C_p(X))$.

S.168. *Prove that, for any continuous function $f : E(X) \to \mathbb{R}$, there exists a continuous function $F : \mathbb{R}^{C_p(X)} \to \mathbb{R}$ such that $F \setminus E(X) = f$. Identifying X and $E(X)$, it is possible to say that each continuous real-valued function on X extends continuously to $\mathbb{R}^{C_p(X)}$ and hence to $C_p(C_p(X))$.*

Solution. The function $f' = f \circ E$ belongs to $C_p(X)$. For each $\varphi \in \mathbb{R}^{C_p(X)}$ let $F(\varphi) = \varphi(f')$. The function $F : \mathbb{R}^{C_p(X)} \to \mathbb{R}$ is continuous being the natural projection of $\mathbb{R}^{C_p(X)}$ onto the factor indexed by f'. Given $e_x \in E(X)$, we have $F(e_x) = e_x(f') = f'(x) = f(e_x)$ and hence $F|_{E(X)} = f$ as promised.

S.169. *Prove that we have $|X| = \chi(C_p(X)) = w(C_p(X))$ for any infinite space X. In particular, weight of $C_p(X)$ is countable if and only if X is countable.*

Solution. The family $\mathcal{B} = \{[x_1, \ldots, x_n; O_1, \ldots, O_n] : n \in \mathbb{N}, x_i \in X$ and O_i is a rational interval for all $i \leqslant n\}$ is a base of $C_p(X)$ by Problem 056 and $|\mathcal{B}| \leqslant |X|$.

Therefore $w(C_p(X)) \leqslant |X|$. Denote by u the function equal to zero at all points of X and take any local base $\mathcal{C}$ at the point u in the space $C_p(X)$. For any $U \in \mathcal{C}$, we can choose a standard open set $O(U)$ such that $u \in O(U) \subset U$. It is clear that the family $\{O(U) : U \in \mathcal{C}\}$ is also a local base at u and therefore we can assume without loss of generality that all elements of $\mathcal{C}$ are standard. Given an arbitrary $U = [x_1, \ldots, x_n; O_1, \ldots, O_n] \in \mathcal{C}$, let $\text{supp}(U) = \{x_1, \ldots, x_n\}$. Consider the set $Y = \bigcup\{\text{supp}(U) : U \in \mathcal{C}\}$. If $x \in X \setminus Y$ then there is $U = [x_1, \ldots, x_n; O_1, \ldots, O_n] \in \mathcal{C}$ with $U \subset W = [x; (-1, 1)]$. However, $\text{supp}(U) = \{x_1, \ldots, x_n\} \subset Y \subset X \setminus \{x\}$ and therefore there exists a function $f \in C_p(X)$ such that $f(x_i) \in O_i$ and $f(x) = 1$ by Problem 034. It is clear that $f \in U \setminus W$ which is a contradiction.

As a consequence, we must have $Y = X$ and hence $|X| = |Y| \leqslant |\chi(C_p(X))|$. Now, the inequalities $w(C_p(X)) \leqslant |X| \leqslant \chi(C_p(X)) \leqslant w(C_p(X))$ (see Problem 156(iv)) show that $|X| = w(C_p(X)) = \chi(C_p(X))$.

S.170. *Let X be an arbitrary space. Suppose that there exists a compact subspace K of $C_p(X)$ such that $\chi(K, C_p(X)) \leqslant \omega$. Prove that X is countable.*

Solution. Given a standard set $U = [x_1, \ldots, x_n; O_1, \ldots, O_n] \in \tau(C_p(X))$, let supp $(U) = \{x_1, \ldots, x_n\}$. Let $\{O_n : n \in \omega\}$ be a base of neighbourhoods of K in $C_p(X)$. Fix $n \in \omega$. For every $f \in K$ pick a standard U_f such that $f \in U_f \subset O_n$. Taking any finite subcover $\{U_{f_1}, \ldots, U_{f_m}\}$ of the open cover $\{U_f : f \in K\}$ of the compact set K we obtain a set $W_n = U_{f_1} \cup \cdots \cup U_{f_m}$ with $K \subset W_n \subset O_n$ and the set $A_n = \text{supp}(U_{f_1}) \cup \cdots \cup \text{supp}(U_{f_m})$. It is evident that the set $A = \bigcup\{A_n : n \in \omega\}$ is countable so it suffices to prove that $A = X$.

Suppose that $x \in X \setminus A$. The map $e_x : C_p(X) \to \mathbb{R}$ defined by $e_x(f) = f(x)$ is continuous and therefore the set $e_x(K)$ is bounded in $\mathbb{R}$. Choose any $r > 0$ such that $|f(x)| < r$ for all $f \in K$ and observe that $W = [x; (-r, r)]$ is an open neighbourhood of K. There exists $n \in \omega$ such that $K \subset O_n \subset W$ and hence $W_n = U_{f_1} \cup \cdots \cup U_{f_m} \subset W$. This implies $U_{f_1} = [x_1, \ldots, x_n; O_1, \ldots, O_n] \subset W$ while $x \notin \{x_1, \ldots, x_n\}$. Apply Problem 034 to find $g \in C_p(X)$ such that $g(x_i) \in O_i$ for all $i \leqslant n$ and $g(x) = r$. It is immediate that $g \in W_n \setminus W$ which is a contradiction. Thus $A = X$ and hence X is countable.

S.171. *Given a space X and $x \in X$, call a family $\mathcal{B} \subset \tau^*(X)$ a π-base of X at x if for any $U \in \tau(x, X)$ there is $V \in \mathcal{B}$ such that $V \subset U$. Note that the elements of a π-base at x need not contain the point x. Suppose that $C_p(X)$ has a countable π-base at some of its points. Prove that X is countable.*

Solution. Given a standard set $U = [x_1, \ldots, x_n; O_1, \ldots, O_n] \in \tau(C_p(X))$, let supp $(U) = \{x_1, \ldots, x_n\}$. By homogeneity of $C_p(X)$ (see Problem 079), if $C_p(X)$ has a countable π-base at some point then it has a countable π-base $\{U_n : n \in \omega\}$ at the point $u \equiv 0$. For each $n \in \omega$ choose a non-empty standard open set $V_n \subset U_n$ and let $A = \bigcup\{\text{supp}(V_n) : n \in \omega\}$. If $x \in X \setminus A$ then $W = [x; (-1, 1)]$ is an open neighbourhood of u and there exists $n \in \omega$ such that $V_n \subset U_n \subset W$. Observe that if $V_n = [y_1, \ldots, y_m; O_1, \ldots, O_m]$ then $x \notin \{y_1, \ldots, y_m\}$ and hence there exists

$f \in C_p(X)$ such that $f(y_i) \in O_i$ for each $i \leqslant n$ and $f(x) = 1$ (Problem 034). It is evident that $f \in V_n \setminus W$ which is a contradiction. Thus $X = A$ is countable.

S.172. *Prove that, for any space X, we have $nw(X) = nw(C_p(X))$. In particular, the space $C_p(X)$ has a countable network if and only if X has one.*

Solution. Let $\mathcal{N}$ be a network in the space X with $|\mathcal{N}| = \kappa = nw(X)$. For any collection $N_1, \dots, N_k \in \mathcal{N}$ and any open intervals $I_1, \dots, I_k$ with rational end-points, let $M(N_1, \dots, N_k; I_1, \dots, I_k) = \{f \in C_p(X) : f(N_j) \subset I_j \text{ for all } j \leqslant k\}$. It is evident that the family $\mathcal{M} = \{M(N_1, \dots, N_k; I_1, \dots, I_k) : k \in \mathbb{N}, N_1, \dots, N_k \in \mathcal{N}$ and I_j is a rational interval for all $j \leqslant k\}$ has cardinality $\leqslant \kappa$. To prove that $\mathcal{M}$ is a network in $C_p(X)$, take any $f \in C_p(X)$ and any $U \in \tau(f, C_p(X))$. There is a canonical open set $V = [x_1, \dots, x_k; O_1, \dots, O_k]$ with $f \in V \subset U$. Without loss of generality we can consider the points $x_1, \dots, x_k$ to be distinct. Continuity of the function f makes it possible to choose disjoint open sets $U_1, \dots, U_k$ with $x_i \in U_i$ for each $i \leqslant k$ and rational intervals $I_1, \dots, I_k$ such that $f(U_j) \subset I_j \subset O_j$ for all $j \leqslant k$. There exist $N_1, \dots, N_k \in \mathcal{N}$ such that $x_j \in N_j \subset U_j$ for every $j \leqslant k$. It is easy to see that $f \in M(N_1, \dots, N_k; I_1 \dots, I_k) \subset V \subset U$ which proves that $\mathcal{M}$ is a network in $C_p(X)$. Since $|\mathcal{M}| \leqslant \kappa = nw(X)$, we have $nw(C_p(X)) \leqslant nw(X)$ for all spaces X. Thus $nw(C_p(C_p(X))) \leqslant nw(C_p(X)) \leqslant nw(X)$ for any space X. However, X embeds in $C_p(C_p(X))$ (see Problem 167) which implies $nw(X) \leqslant nw(C_p(C_p(X)))$. The obtained inequalities show that $nw(X) = nw(C_p(X))$.

S.173. *Prove that $d(X) = \psi(C_p(X)) = \Delta(C_p(X)) = iw(C_p(X))$ for any space X. In particular, $C_p(X)$ condenses onto a second countable space if and only if X is separable.*

Solution. Suppose that we are given a space Y and a family $\gamma \subset \tau(Y \times Y)$ such that $\Delta_Y = \bigcap \gamma$. Fix $y \in Y$ and, for any $U \in \gamma$ choose $O_U \in \tau(y, Y)$ such that $O_U \times O_U \subset U$. Then $\bigcap \{O_U : U \in \gamma\} = \{y\}$ which proves that $\psi(Y) \leqslant \Delta(Y)$.

Now suppose that $f : Z \to Y$ is a condensation and $\mathcal{B}$ is a base of Y with $|\mathcal{B}| = \kappa = iw(X)$. Then $l((Y \times Y) \setminus \Delta_Y) \leqslant w((Y \times Y) \setminus \Delta_Y) \leqslant \kappa$. For each $z \in (Y \times Y) \setminus \Delta_Y$ choose $U_z \in \tau(z, Y \times Y)$ such that $\overline{U}_z \cap \Delta_Y = \emptyset$. The open cover $\{U_z : z \in (Y \times Y) \setminus \Delta_Y\}$ of the space $(Y \times Y) \setminus \Delta_Y$ has a subcover μ with $|\mu| \leqslant \kappa$. Therefore, $(Y \times Y) \setminus \Delta_Y = \bigcup \{\overline{U} : U \in \mu\}$ is a union of $\leqslant \kappa$ closed sets and hence $\psi(\Delta_Y, Y \times Y) \leqslant \kappa$. Since f is a condensation, we have $\psi(\Delta_Z, Z \times Z) \leqslant \kappa$ and hence $\Delta(Z) \leqslant iw(Z)$.

Given a standard set $V = [x_1, \dots, x_n; O_1, \dots, O_n] \in \tau(C_p(X))$, let $\text{supp}(V) = \{x_1, \dots, x_n\}$. Take any $\gamma \subset \tau(C_p(X))$ such that $\{u\} = \bigcap \gamma$, where $u \equiv 0$ and $|\gamma| = \kappa$. For each $U \in \gamma$ take a standard set $O_U = [x_1, \dots, x_n; O_1, \dots, O_n]$ such that $u \in O_U \subset U$. It is easy to see that the set $Y = \bigcup \{\text{supp}(O_U) : U \in \gamma\}$ has cardinality $\leqslant \kappa$. If $X \neq \overline{Y}$, pick any $x \in X \setminus \overline{Y}$ and find a function $f \in C_p(X)$ such that $f(x) = 1$ and $f(Y) \subset \{0\}$. It is straightforward that $f \in \cap \gamma$ which is a contradiction with $f \neq u$ and $\bigcap \gamma = \{u\}$. Hence Y is dense in X and therefore $d(X) \leqslant |Y| \leqslant \kappa$. This proves that $d(X) \leqslant \psi(C_p(X))$ and applying the inequalities we proved above, we have $d(X) \leqslant \psi(C_p(X)) \leqslant \Delta(C_p(X)) \leqslant iw(C_p(X))$. To finish the proof let us show that $iw(C_p(X)) \leqslant d(X)$. Fix a set $Y \subset X$ such that $|Y| = d(X)$ and $X = \overline{Y}$. The map

$\pi_Y : C_p(X) \to Z = \pi_Y(C_p(X)) \subset C_p(Y)$ is a condensation by Problem 152(ii) and $w(Z) \leqslant w(C_p(Y)) = |Y| = d(X)$. This shows that $iw(C_p(X)) \leqslant d(X)$ and our proof is complete.

S.174. *Prove that, for any space X, we have $iw(X) = d(C_p(X))$. In particular, the space $C_p(X)$ is separable if and only if X condenses onto a second countable space.*

Solution. Observe that $iw(X) \leqslant iw(C_p(C_p(X))) \leqslant d(C_p(X))$. The first inequality holds because X embeds in $C_p(C_p(X))$ (Problem 167) so Problem 159(v) is applicable. The second inequality follows from Problem 173.

On the other hand, if $iw(X) \leqslant \kappa$, take a condensation $f : X \to Y$ such that $w(Y) \leqslant \kappa$. Then f^* embeds $C_p(Y)$ into $C_p(X)$ as a dense subspace (Problem 163(iv)). Let $Z = f^*(C_p(Y))$. Then $d(Z) \leqslant nw(Z) = nw(C_p(Y)) = nw(Y) \leqslant w(Y) \leqslant \kappa$ and therefore $d(C_p(X)) \leqslant d(Z) \leqslant \kappa$ which establishes that $d(C_p(X)) \leqslant iw(X)$ and we are done.

S.175. *Suppose that $\psi(K, C_p(X)) \leqslant \omega$ for some compact subspace K of the space $C_p(X)$. Prove that X is separable.*

Solution. Since $C_p(X)$ is homogeneous (i.e., for any f, $g \in C_p(X)$ there is a homeomorphism $\varphi : C_p(X) \to C_p(X)$ such that $\varphi(f) = g$ (see Problem 079)), we can assume that $u \in K$ where $u \equiv 0$. Given a standard set $U = [x_1, \dots, x_n; O_1, \dots, O_n] \in \tau(C_p(X))$, let $\operatorname{supp}(U) = \{x_1, \dots, x_n\}$. Let $\gamma = \{O_n : n \in \omega\}$ be a family of neighbourhoods of K such that $\bigcap\gamma = K$. Fix $n \in \omega$. For every $f \in K$ pick a standard open set U_f such that $f \in U_f \subset O_n$. Taking any finite subcover $\{U_{f_1}, \dots, U_{f_m}\}$ of the open cover $\{U_f : f \in K\}$ of the compact set K, we obtain a set $W_n = U_{f_1} \cup \dots \cup U_{f_m}$ with $K \subset W_n \subset O_n$ and the set $A_n = \operatorname{supp}(U_{f_1}) \cup \dots \cup \operatorname{supp}(U_{f_m})$. It is evident that the set $A = \bigcup\{A_n : n \in \omega\}$ is countable so it suffices to prove that $\overline{A} = X$.

Suppose that $x \in X \setminus \overline{A}$. The map $e_x : C_p(X) \to \mathbb{R}$ defined by $e_x(f) = f(x)$ is continuous and therefore the set $e_x(K)$ is bounded in $\mathbb{R}$. Choose any $r > 0$ such that $|f(x)| < r$ for all $f \in K$ and find some $g \in C_p(X)$ such that $g(x) = r$ and $g(A) \subset \{0\}$. It follows from $g(x) = r$ that $g \notin K$. However, $g|A = u|A$ implies $g \in \bigcap \gamma$ which contradicts the fact that $\bigcap \gamma = K$.

S.176. *Prove that $s(X) \leqslant s(C_p(X))$ for any space X. Give an example of a space X with $s(X) < s(C_p(X))$.*

Solution. Let D be a discrete subspace of X. Fix a family $\{U_d : d \in D\} \subset \tau(X)$ such that $U_d \cap D = \{d\}$ for each $d \in D$. There is $f_d \in C_p(X)$ with $f_d(d) = 1$ and $f_d(X \setminus U_d) \subset \{0\}$. We claim that the set $E = \{f_d : d \in D\}$ is discrete. Indeed, the set $V_d = [d, (0, 2)]$ is open in $C_p(X)$ and $V_d \cap E = \{f_d\}$ which proves the discreteness of E and the inequality $s(X) \leqslant s(C_p(X))$.

Now if X is the Sorgenfrey line (see Problem 165) then $s(X) = \omega$ because every subspace of X is Lindelöf (Problem 165(i)) and a Lindelöf discrete space is countable. However, $s(C_p(X)) \geqslant \operatorname{ext}(C_p(X)) \geqslant \mathfrak{c}$ by Problem 165(v) whence $s(X) < s(C_p(X))$.

S.177. *Suppose that X is homeomorphic to $Y \times \mathbb{R}$ for some space Y. Prove that $C_p(X)$ is linearly homeomorphic to $(C_p(X))^\omega$, i.e., there exists a homeomorphism $\xi : C_p(X) \to (C_p(X))^\omega$ such that $\xi(f+g) = \xi(f) + \xi(g)$ and $\xi(\lambda f) = \lambda\xi(f)$ for all $f, g \in C_p(X)$ and $\lambda \in \mathbb{R}$.*

Solution. We will need the discreate subspace $= \{0, \pm1, \pm2, \dots\}$ of the space $\mathbb{R}$ and the space $X_0 = Y \times \ \subset X$. Given $r \in \mathbb{R}$, the maximal integer which does not exceed r is denoted by $[r]$ and $\langle r\rangle = r - [r]$. If $x = (y, r) \in X$ then $x^- = (y, [r]) \in X_0$ and $x^+ = (y, [r]+1) \in X_0$. We are going to construct a continuous linear mapping $\varphi : C_p(X_0) \to C_p(X)$ which extends the functions from $C_p(X_0)$, i.e., $\varphi(f)|X_0 = f$ for all $f \in C_p(X_0)$. To do so, for any $x = (y, r) \in X$ and $f \in C_p(X_0)$, let $\varphi(f)(y, r) = \varphi(f)(x) = \langle r\rangle \cdot f(x^+) + (1 - \langle r\rangle) \cdot f(x^{-1})$.

Now, if $x = (y, r) \in X_0$ then $r \in$ and $\langle r\rangle = 0$ which implies $x^- = x$ and $\varphi(f)(x) = f(x)$, i.e., the function $\varphi(f)$ extends f for any $f \in C_p(X_0)$. Let us prove that $\varphi(f)$ is continuous for every $f \in C_p(X_0)$. First, fix $n \in$ and take any $x = (y, r) \in Y \times (n, n+1)$ and $\varepsilon > 0$. Let $f_k(z) = f(z, k)$ for any $k \in$ and $z \in Y$. It is evident that $f_k \in C(Y)$ for each $k \in$. There exists $U \in \tau(y, Y)$ and $M > 1$ such that $|f_n(z)| + |f_{n+1}(z)| < M$ and $|f_n(z) - f_n(y)| + |f_{n+1}(z) - f_{n+1}(y)| < \frac{\varepsilon}{4M}$ for all $z \in U$. Now, let $W = (U \times (r - \frac{\varepsilon}{4M}, r + \frac{\varepsilon}{4M})) \cap (Y \times (n, n+1))$. We claim that $\varphi(f)(W) \subset (\varphi(f)(x) - \varepsilon, \varphi(f)(x) + \varepsilon)$. Indeed, take any $x_1 = (y_1, r_1) \in W$. Then $y_1 \in U$ and

$$\begin{aligned}|\varphi(f)(x_1) - \varphi(f)(x)| &= |\langle r_1\rangle f_{n+1}(y_1) + (1-\langle r_1\rangle)f_n(y_1) - \langle r\rangle f_{n+1}(y) - (1-\langle r\rangle)f_n(y)|\\ &\leqslant |\langle r_1\rangle f_{n+1}(y_1) - \langle r\rangle f_{n+1}(y)| + |(1-\langle r_1\rangle)f_n(y_1) - (1-\langle r\rangle)f_n(y)|.\end{aligned}$$

Now observe that $|\langle r_1\rangle f_{n+1}(y_1) - \langle r\rangle f_{n+1}(y)| \leqslant |\langle r_1\rangle| \cdot |f_{n+1}(y_1) - f_{n+1}(y)| + |\langle r_1\rangle - \langle r\rangle| \cdot |f_{n+1}(y)| \leqslant 1 \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{4M} \cdot M < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$.

Analogously, $|(1 - \langle r_1\rangle)f_n(y_1) - (1 - \langle r\rangle)f_n(y)| \leqslant |1 - \langle r_1\rangle| \cdot |f_n(y_1) - f_n(y)| + |\langle r_1\rangle - \langle r\rangle| \cdot |f_n(y)| \leqslant 1 \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{4M} \cdot M < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$ which shows that, for any $x_1 \in W$, we have $|\varphi(f)(x_1) - \varphi(f)(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and continuity of $\varphi(f)$ at the point x is proved.

Now take an arbitrary point $x = (y, n) \in Y \times \{n\}$ and $\varepsilon > 0$. There exists $U \in \tau(y, Y)$ and $M \in \mathbb{R}$, $M > 1$ such that $|f_{n-1}(z)| + |f_n(z)| + |f_{n+1}(z)| < M$ and $|f_n(z) - f_n(y)| < \frac{\varepsilon}{6M}$ for all $z \in U$. Now, let $W = U \times (n - \frac{\varepsilon}{6M}, n + \frac{\varepsilon}{6M})$. Take any $x_1 = (y_1, r) \in W$. If $r \leqslant n$, we have

$$\begin{aligned}|\varphi(f)(x_1) - \varphi(f)(x)| &= |\langle r\rangle f_n(y_1) + (1 - \langle r\rangle)f_{n-1}(y_1) - f_n(y)|\\ &\leqslant \langle r\rangle |f_n(y_1) - f_n(y)| + (1 - \langle r\rangle)\,(|f_{n-1}(y_1)| + |f_n(y)|)\\ &\leqslant 1 \cdot \tfrac{\varepsilon}{6M} + \tfrac{\varepsilon}{6M}(M + M) \leqslant \tfrac{\varepsilon}{6} + \tfrac{\varepsilon}{3} = \tfrac{\varepsilon}{2}.\end{aligned}$$

Analogously, if $r > n$ then

$$\begin{aligned}|\varphi(f)(x_1) - \varphi(f)(x)| &= |\langle r\rangle f_{n+1}(y_1) + (1 - \langle r\rangle)f_n(y_1) - f_n(y)|\\ &\leqslant \langle r\rangle\,(|f_{n+1}(y_1)| + |f_n(y)|) + (1 - \langle r\rangle)\,(|f_n(y_1) - f_n(y)|)\\ &\leqslant \tfrac{\varepsilon}{6M}(M + M) + 1 \cdot \tfrac{\varepsilon}{6M} \leqslant \tfrac{\varepsilon}{6} + \tfrac{\varepsilon}{3} = \tfrac{\varepsilon}{2}\end{aligned}$$

and we finished the proof of continuity of $\varphi(f)$.

It is evident that φ is a linear map, so let us check that φ is continuous. Let $U = O_X(\varphi(f), x_1, \dots, x_n, \varepsilon)$ be an arbitrary basic neighbourhood of $\varphi(f)$ in $C_p(X)$, where $x_i = (y_i, r_i)$ for each $i \leqslant n$. Then the set $V = O_{X_0}(f, x_1^-, x_1^+, \dots, x_n^-, x_n^+, \varepsilon)$ is an open neighbourhood of f in $C_p(X_0)$ and $\varphi(V) \subset U$. Indeed, if we have $|g(x_i^+) - f(x_i^+)| < \varepsilon$ and $|g(x_i^-) - f(x_i^-)| < \varepsilon$ for each $i \leqslant n$ then $|g(x_i) - f(x_i)| = |\langle r_i\rangle(g(x_i^+) - f(x_i^+)) + (1 - \langle r_i\rangle)(g(x_i^-) - f(x_i^-))| < \varepsilon\ (\langle r_i\rangle + 1 - \langle r_i\rangle) = \varepsilon$ for each $i \leqslant n$ and we are done.

Fact 1. The space $C_p(X)$ is linearly homeomorphic to the space $C_p(X_0) \times L$ where $L = \{g \in C_p(X) : g(X_0) = \{0\}\}$.

Proof. Given an arbitrary $(f, g) \in C_p(X_0) \times L$, let $\psi(f, g) = \varphi(f) + g$. It is clear that $\psi : C_p(X_0) \times L \to C_p(X)$ is a continuous linear map. It is easy to check that ψ is a bijection. Now, given $h \in C_p(X)$ the formula $\chi(h) = (h|X_0, h - \varphi(h|X_0))$ defines a continuous map from $C_p(X)$ to $C_p(X_0) \times L$ and its is easy to see that $\chi = \psi^{-1}$ whence ψ is a linear homeomorphism and Fact 1 is proved.

Since X_0 is a discrete union of ω copies of the space Y (see Problem 113), we can apply Problem 114 to conclude that $C_p(X)$ is homeomorphic to $(C_p(Y))^\omega$. In fact these spaces are even linearly homeomorphic (check that the proof of Problem 114 for the case $Y = \mathbb{R}$ gives a linear homeomorphism). For any $n \in$, let $L_n = \{g \in C_p(Y \times [n, n+1]) : g(Y \times \{n\}) = g(Y \times \{n+1\}) = \{0\}\}$. Let $\pi_n(f) = f|L_n$ for each $f \in L$ and $n \in$. We leave to the reader an easy verification that the map $\pi : L \to \prod\{L_n : n \in \}$ defined by $\pi(f)(n) = \pi_n(f)$, is a linear homeomorphism. So, if we denote by $\approx$ the relationship of being linearly homeomorphic, then $C_p(X_0) \approx (C_p(Y))^\omega$ and $L \approx (L_0)^\omega$ because $L_0 \approx L_n$ for all $n \in$. Finally,

$$C_p(X) \approx C_p(X_0) \times L \approx (C_p(Y))^\omega \times (L_0)^\omega \approx (C_p(Y) \times L_0))^\omega \approx ((C_p(Y) \times L_0))^\omega)^\omega$$
$$\approx ((C_p(Y))^\omega \times (L_0)^\omega)^\omega \approx (C_p(X_0) \times L)^\omega \approx (C_p(X))^\omega.$$

Note that we used here Problems 103 and 104 in their linear forms for which the same proofs will do if the factors are linear spaces.

S.178. *Let $a(X) = \sup\{|Y| : Y \subset X$ and Y is homeomorphic to the Alexandroff one-point compactification of an infinite discrete space$\}$. Prove that we have the equality $p(X) = a(C_p(X))$ for any infinite space X.*

Solution. Suppose that $\gamma \subset \tau^*(X)$ is an infinite point-finite family. For each $U \in \gamma$ fix a point $x_U \in U$ and $f_U \in C_p(X)$ such that $f_U(x_U) = 1$ and $f_U(U) \subset \{0\}$. Denote by u the function equal to zero at all points of X. Since the family γ is point-finite, the set $\{V \in \gamma : f_V(x_U) = 1\}$ is finite for any $U \in \gamma$. Therefore, the set $\{V \in \gamma : f_V = f_U\}$ is also finite for any $U \in \gamma$. As a consequence the set $A = \{f_U : U \in \gamma\}$ has the same cardinality as the family γ.

Given any $W \in \tau(u, C_p(X))$, there are $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that $G = O(u, x_1, \dots, x_n, \varepsilon) \subset W$. The family $\mu = \{U \in \gamma : x_i \in U$ for some $i \leqslant n\}$ is finite and hence $f_U(x_i) = 0$ for all $i \leqslant n$ if $U \in \gamma \setminus \mu$. Thus $f_U \in G \subset W$ for all but finitely many U. This proves that the space $\{u\} \cup A$ is homeomorphic to $A(\kappa)$ where $\kappa = |\gamma|$. Therefore $p(X) \leqslant a(C_p(X))$.

Now assume that $k = p(X) < a(C_p(X))$ and fix an Alexandroff one-point compactification of a discrete space $P \subset C_p(X)$ with $|P| = \kappa^+$. Without loss of generality, we may assume that $P = A \cup \{u\}$ and u is the unique non-isolated point of P. Let $O_n = \mathbb{R} \setminus [-\frac{1}{n}, \frac{1}{n}]$ for each $n \in \mathbb{N}$. Since $W_x = [x; (-\frac{1}{n}, \frac{1}{n})]$ is an open neighbourhood of u for any $x \in X$, the set $P \setminus W_x$ is finite and hence the family $\gamma_n = \{f^{-1}(O_n) : f \in A\}$ is point-finite for any $n \in \mathbb{N}$. For any $f \in A$ there is $n = n_f \in \mathbb{N}$ such that $f^{-1}(O_n) \neq \emptyset$ and therefore there is $m \in \mathbb{N}$ for which there are κ^+ non-empty elements of γ_m. Some of them can coincide, but, given $U = f^{-1}(O_n) \in \gamma_m$, the family $\{g \in A : g^{-1}(O_n) = U\}$ is finite because γ_m is point-finite. Thus $|\gamma_m| = \kappa^+ > p(X)$ which is a contradiction showing that $a(C_p(X)) \leqslant p(X)$.

S.179. *Prove that, for an arbitrary space X, we have $c(X) \leqslant p(X) \leqslant d(X)$ and $p(X) \leqslant s(X)$. Give an example of a space Y such that $c(Y) < p(Y)$. Is it possible for such a space Y to be a space $C_p(X)$ for some X?*

Solution. Any disjoint family is point-finite and hence $c(X) \leqslant p(X)$. Now if $D \subset X$ is a dense subspace with $|D| = d(X)$ take any point-finite family $\gamma \subset \tau^*(X)$. For each $U \in \gamma$ pick $x_U \in U \cap D$ and observe that, for each $d \in D$, there are only finitely many $U \in \gamma$ for which $d = d_U$. Therefore $|\gamma| \leqslant |D| = d(X)$ which implies $p(X) \leqslant d(X)$.

Suppose that $s(X) = \kappa < p(X)$ and choose a point-finite $\gamma \subset \tau^*(X)$ with $|\gamma| = \kappa^+$. For any $U \in \gamma$ pick $x_U \in U$ and consider the subspace $Y = \{x_U : U \in \gamma\}$. Given $n \in \mathbb{N}$ say that $\text{ord}(U) = n$ if the point x_U belongs to exactly n elements of γ. Then $Y = \bigcup\{Y_n : n \in \mathbb{N}\}$ where $Y_n = \{x_U : \text{ord}(U) = n\}$ for each $n \in \mathbb{N}$. If $x \in Y_n$ then $x \in V = V_1 \cap \cdots \cap V_n$ for some distinct $V_1, \ldots, V_n \in \gamma$. The set $V \cap Y_n$ is finite because $x_U \in V$ implies $U \in \{V_1, \ldots, V_n\}$. This proves that each Y_n is discrete (or empty). We have $|Y_n| = |Y| = \kappa^+$ for some $n \in \mathbb{N}$ which is a contradiction with $s(X) \leqslant \kappa$. This proves that $p(X) \leqslant s(X)$.

Finally, let $Y = C_p(A(\omega_1))$. Then $A(\omega_1)$ embeds in $C_p(Y) = C_p(C_p(A(\omega_1)))$ by Problem 167 so we can conclude that $p(Y) = a(C_p(Y)) \geqslant \omega_1$. To finish the proof, apply Problem 111 to see that $c(Y) = c(C_p(A(\omega_1))) = \omega < p(Y)$ and we are done.

S.180. *Prove that, for an arbitrary space X, any locally finite family of non-empty open subsets of the space $C_p(X)$ is countable. Is it possible to say the same about point-finite families of non-empty open subsets of $C_p(X)$?*

Solution. In the solution of Problem 179, we gave an example of a space $C_p(X)$ for which $p(C_p(X)) > \omega$ and hence $C_p(X)$ has a point-finite uncountable family of non-empty open sets. Now assume that $\gamma \subset \tau^*(C_p(X))$ is a locally finite family. Let μ be a maximal disjoint family of non-empty open sets each one of which meets at most finitely many elements of γ. It follows from local finiteness of γ that $\bigcup\mu$ is dense in $C_p(X)$. Since $c(C_p(X)) = \omega$ (Problem 111), the family μ is countable. Each $U \in \gamma$ has to intersect some $V \in \mu$ because $\bigcup\mu$ is dense in X. If γ were uncountable some $U \in \gamma$

would have to intersect uncountably many elements of μ which is a contradiction. Hence $|\gamma| \leqslant \omega$.

S.181. *Prove that, if a space X is pseudocompact, then* $c(X) = a(C_p(X))$, *where* $a(C_p(X)) = \sup\{\kappa : A(\kappa)$ *embeds in* $C_p(X)\}$.

Solution. It follows from Problem 178 that it suffices to prove that $c(X) = p(X)$. If this is not true then there exists a point-finite family $\gamma \subset \tau^*(X)$ such that $|\gamma| = \kappa^+$ where $\kappa = c(X)$. Let $X_n = \{x \in X : x$ belongs to at most n elements of $\gamma\}$. If $y \in X \setminus X_n$ then there are distinct $U_1, \ldots, U_{n+1} \in \gamma$ such that $y \in U = U_1 \cap \cdots \cap U_{n+1}$ and hence $x \in U \subset X \setminus X_n$. This shows that X_n is a closed set for each $n \in \omega$. Since γ is point-finite, we have $X = \bigcup\{X_n : n \in \omega\}$.

Fact 1. The set $O = \bigcup\{\mathrm{Int}(X_n) : n \in \omega\}$ is dense in X.

Proof. Suppose not. Then there is an open non-empty $U \subset X$ such that $U \cap O = \emptyset$. Since X_0 can cover no open non-empty subset of U, there is $U_0 \in \tau^*(X)$ such that $\overline{U}_0 \subset U$ and $U_0 \cap X_0 = \emptyset$. Suppose that we have $U_0, \ldots, U_n$ such that $\overline{U}_{i+1} \subset U_i$ and $U_i \cap X_i = \emptyset$ for all $i \leqslant n - 1$. Since X_{n+1} can cover no non-empty open subset of U_n, there is $U_{n+1} \in \tau^*(X)$ such that $\overline{U}_{n+1} \subset U_n$ and $U_{n+1} \cap X_{n+1} = \emptyset$. Since X is pseudocompact, we have $P = \bigcap\{U_n : n \in \omega\} = \bigcap\{\overline{U}_n : n \in \omega\} \neq \emptyset$ by Problem 136(iv). However, $P \cap X_n = \emptyset$ for all $n \in \omega$ which contradicts $X = \bigcup\{X_n : n \in \omega\}$. Hence O is dense in X and Fact 1 is proved.

Since $U \cap O \neq \emptyset$ for all $U \in \gamma$ by Fact 1, there is $m \in \omega$ such that the cardinality of the family $\{U \in \gamma : U \cap \mathrm{Int}(X_m) \neq \emptyset\}$ is equal to κ^+. Let $\mu = \{U \cap \mathrm{Int}(X_m) : U \in \gamma$ and $U \cap \mathrm{Int}(X_m) \neq \emptyset\}$. Then $\mu \subset \tau^*(X)$, $|\mu| = \kappa^+$ and $\bigcup\mu \subset \mathrm{Int}(X_m)$. Let v be the maximal family of non-empty open subsets of $O_m = \mathrm{Int}(X_m)$ each one of which meets at most finitely many elements of μ. We claim that $\bigcup v$ is dense in O_m. Indeed, if $W = O_m \setminus \overline{\bigcup v} \neq \emptyset$ then there is $x \in W$ which belongs to a maximal number $j \leqslant m$ of the elements of μ. If $U_1, \ldots, U_j$ are distinct elements of μ with $x \in V = U_1 \cap \cdots \cap U_j$ then the set V can only intersect elements from $\{U_1, \ldots, U_j\}$ (for otherwise the relevant intersection would have a point which belongs to $> j$ elements, a contradiction). Therefore $\{V\} \cup v$ is still a disjoint family each element of which intersects at most finitely many elements of μ which is a contradiction with the maximality of v. This proves that $\bigcup v$ is dense in O_m. As a consequence each element of μ intersects some element of v. Since each $W \in v$ meets but finitely many elements of μ, we have $|v| = \mu = \kappa^+$ which is a contradiction with $c(X) \leqslant \kappa$ because $v \subset \tau^*(X)$ is a disjoint family. This contradiction shows that $p(X) \leqslant c(X)$ and we are done.

S.182. *Let X be a space. Given* $x \in X$, *let* $C_x = \{f \in C_p(X) : f(x) = 0\}$. *Prove that* $C_p(X)$ *is homeomorphic to* $C_x \times \mathbb{R}$.

Solution. Define a map $\varphi : C_x \times \mathbb{R} \to C_p(X)$ by $\varphi(z) = f + t$ for any element $z = (f, t) \in C_x \times \mathbb{R}$. Given any $g \in C_p(X)$, let $\psi(g) = (g - g(x), g(x)) \in C_x \times \mathbb{R}$. This defines a map $\psi : C_p(X) \to C_x \times \mathbb{R}$. It is evident that the maps φ and ψ are continuous and it is easy to check that they are bijections and $\psi = \varphi^{-1}$. Therefore φ is a homeomorphism.

S.183. *Prove that compact spaces X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isomorphic (the isomorphism between $C(X)$ and $C(Y)$ need not be topological). Show that there exists a compact space X and a non-compact space Y such that $C(X)$ is isomorphic to $C(Y)$.*

Solution. Let X be homeomorphic to Y. Fix some homeomorphism $\varphi : X \to Y$ and let $\varphi^*(f) = f \circ \varphi$ for any function $f \in C(Y)$. It follows from Problem 163(v) that the mapping $\varphi^* : C(Y) \to C(X)$ is a bijection. To prove that the map φ is an isomorphism, take any $f, g \in C(Y)$ and $x \in X$. Then $\varphi^*(f + g)(x) = (f + g)(\varphi(x)) = f(\varphi(x)) + g(\varphi(x)) = \varphi^*(f)(x) + \varphi^*(g)(x)$ which shows that $\varphi^*(f + g) = \varphi^*(f) + \varphi^*(g)$. Analogously, $\varphi^*(f \cdot g)(x) = (f \cdot g)(\varphi(x)) = f(\varphi(x)) \cdot g(\varphi(x)) = \varphi^*(f)(x) \cdot \varphi^*(g)(x)$ and therefore $\varphi^*(f \cdot g) = \varphi^*(f) \cdot \varphi^*(g)$ whence φ^* is an isomorphism.

Given a space Z call a set $I \subset C(Z)$ *an ideal* if I has at least two distinct elements, $I \neq C(Z)$ and, for any $f, g \in I$ and $h \in C(Z)$, we have $f + g \in I$ and $f \cdot h \in I$. Call an ideal $I \subset C(Z)$ *a maximal ideal* if, for any ideal $J \subset C(Z)$, we have $J = I$ whenever $I \subset J$.

Fact 1. Given a compact space Z, for any $z \in Z$, the set $I_z = I_z^Z = \{f \in C(Z) : f(z) = 0\}$ is an ideal in $C(Z)$ and, for any ideal $J \subset C(Z)$, there exists $z \in Z$ such that $J \subset I_z$.

Proof. It is straightforward that the set I_z is an ideal in $C(Z)$. Given $f \in J$, let $Z_f = f^{-1}(0)$. We claim that the family $\gamma = \{Z_f : f \in J\}$ is centered. Indeed, if not, we can find $f_1, \ldots, f_n \in J$ such that $\bigcap\{Z_{f_i} : i \leqslant n\} = \emptyset$. Since J is an ideal, we have $f_i^2 \in J$ for each $i \leqslant n$ and hence $f = \sum_{i=1}^n f_i^2 \in J$. It is clear that $f(z) > 0$ for any $z \in Z$. It follows from $f \in J$ that $h = f \cdot \frac{1}{f} \in J$. Now, since $h(z) = 1$ for each $z \in Z$, we have $g = g \cdot h \in J$ for each $g \in C(Z)$. It turns out that $J = C(Z)$ which is a contradiction. The family γ being centered we have $\bigcap \gamma \neq \emptyset$ (Problem 118(viii)). It is clear that, for each $z \in \bigcap\gamma$, we have $J \subset I_z$ and Fact 1 is proved.

Fact 2. Given a compact space Z, the ideal $I_z = I_z^Z$ is maximal for any $z \in Z$. Besides, the correspondence $z \mapsto I_z$ is a bijection between Z and the family $\mathcal{M}$ of all maximal ideals in $C(Z)$.

Proof. Suppose that $J \subset C(Z)$ is an ideal with $I_z \subset J$. By Fact 1 there is $y \in Z$ such that $J \subset I_y$. If $y \neq z$ then there exists $f \in C(Z)$ such that $f(z) = 0$ and $f(y) = 1$. This implies $f \in I_z \setminus I_y$ which is a contradiction. Hence $y = z$ and $I_z \subset J \subset I_z$ whence $J = I_z$. Therefore I_z is a maximal ideal. As a consequence, the map $\varphi(z) = I_z$ sends Z into $\mathcal{M}$. If $y \neq z$ then, for any $f \in C(Z)$ with $f(z) = 0$ and $f(y) = 1$ we have $f \in I_z \setminus I_y$ which implies $I_z \neq I_y$ so φ is an injection. Now, if $J \subset \mathcal{M}$ then $J \subset I_z$ for some $z \in Z$ by Fact 1. Since J is maximal, we have $J = I_z = \varphi(z)$ and Fact 2 is proved.

Fact 3. Let Z be a (not necessarily compact) space. For any $A \subset Z$ and $z \in Z$, we have $z \in \overline{A}$ if and only if $I_z \supset \bigcap\{I_y : y \in A\}$.

Proof. If $z \in \overline{A}$ and $f \in \bigcap\{I_y : y \in A\}$ then $f(y) = 0$ for all $y \in A$. Since the function f is continuous, we have $f(z) = 0$, i.e., $f \in I_z$. This proves necessity. Now suppose that $\bigcap\{I_y : y \in A\} \subset I_z$. If $z \notin \overline{A}$ then there exists $f \in C(Z)$ such that $f(z) = 1$ and $f(A) = \{0\}$. Thus $f \in \bigcap\{I_y : y \in A\} \setminus I_z$ which is a contradiction. Fact 3 is proved.

Returning to the proof of sufficiency, assume that $\varphi : C(X) \to C(Y)$ is an isomorphism. Observe that the notion of a maximal ideal is defined in algebraic terms and hence $\varphi(I)$ is a maximal ideal in $C(Y)$ for any maximal ideal $I \subset C(X)$. For any $x \in X$, the set I_x^X is a maximal ideal of $C(X)$ by Fact 2; hence $\varphi(I_x)$ is a maximal ideal of $C(Y)$. Applying Fact 2 again, we conclude that there is $y \in Y$ such that $\varphi(I_x) = I_y^Y$. Letting $y = f(x)$ we obtain a function $f : X \to Y$. Since φ is an isomorphism, the map $I_x \mapsto \varphi(I_x)$ is a bijection between the families of maximal ideals in $C(X)$ and $C(Y)$. Applying Fact 2 we see that f is a bijection. Finally, let $A \subset X$. Given $x \in \overline{A}$, we have $I_x^X \supset \bigcap\{I_z^X : z \in A\}$ by Fact 3. Since φ is a bijection, we have $I_{f(x)}^Y \supset \bigcap\{I_z^Y : z \in f(A)\}$ and hence $f(x) \in \overline{f(A)}$ by Fact 3. This proves that $f(\overline{A}) \subset \overline{f(A)}$ and hence f is continuous. Since f is a bijection, we can apply Problem 123 to see that f is a homeomorphism.

S.184. *Suppose that we are given a function $f_n \in C_p(X)$ for all $n \in \omega$. Prove that, if $f : X \to \mathbb{R}$ and $f_n \rightrightarrows f$, then $f_n \to f$. Give an example of a sequence $\{f_n : n \in \omega\} \subset C_p(\mathbb{I})$ such that $f_n \to f$ for some $f \in C_p(\mathbb{I})$ and $\{f_n\}$ does not converge uniformly to f.*

Solution. If $f_n \rightrightarrows f$, take any $U \in \tau(f, C_p(X))$. There are $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $O(f, x_1, \ldots, x_n, \varepsilon) \subset U$. By uniform convergence of f_n to f, there exists $m \in \omega$ such that $|f_n(y) - f(y)| < \varepsilon$ for all $n \geqslant m$ and $y \in X$. In particular, $|f_n(x_i) - f(x_i)| < \varepsilon$ for all $n \geqslant m$ and $i \leqslant n$. As a consequence, $f_n \in O(f, x_1, \ldots, x_n, \varepsilon) \subset U$ for all $n \geqslant m$ and hence $f_n \to f$.

To give the required example, for an arbitrary $n \in \omega$, let $f_n(t) = 0$ for all $t \in [-1, \frac{n}{n+1}]$. Now, $f_n(t) = (n+1)(n+2)(t - \frac{n}{n+1})$ for all $t \in [\frac{n}{n+1}, \frac{n+1}{n+2}]$ and $f_n(t) = (n+2)(1-t)$ if $t \in [\frac{n+1}{n+2}, 1]$. It is easy to see that $\{f_n : n \in \omega\} \subset C_p(\mathbb{I})$. Observe that, for any $t_1, \ldots, t_k \in \mathbb{I}$ there exists $m \in \omega$ such that $f_n(t_i) = 0$ for all $n \geqslant m$ and $i \leqslant k$. This shows that $f_n \in O(f, t_1, \ldots, t_k, \varepsilon)$ for any $\varepsilon > 0$ and $n \geqslant m$ if $f(t) = 0$ for all $t \in \mathbb{I}$. Therefore $f_n \to f$. However, the sequence $\{f_n\}$ does not converge uniformly to f, because $f_n(\frac{n+1}{n+2}) = 1$ for each $n \in \omega$ and hence the definition of the uniform convergence is not satisfied for $\varepsilon = 1$.

S.185. *(The Dini theorem). Let X be a pseudocompact space. Suppose that $f_n \in C_p(X)$, $f_{n+1}(x) \geqslant f_n(x)$ for all $x \in X$ and $n \in \omega$. Prove that if there exists $f \in C_p(X)$ such that $f_n \to f$ then the sequence $\{f_n\}$ converges uniformly to the function f.*

Solution. Take an arbitrary number $\varepsilon > 0$. For any $n \in \mathbb{N}$, consider the open set $U_n = \{x \in X : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\}$. Observe that, for any point $x \in X$ we have $|f_n(x) - f(x)| = f(x) - f_n(x) \geqslant f(x) - f_{n+1}(x)$ due to the inequalities $f_n(x) \leqslant f_{n+1}(x) \leqslant f(x)$. This implies $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. If $U_n \neq \emptyset$ for all $n \in \mathbb{N}$ then, by pseudocompactness of X, we have $\bigcap\{\overline{U}_n : n \in \mathbb{N}\} \neq \emptyset$ (Problem 136(iv)). Pick any $x \in \bigcap\{\overline{U}_n : n \in \mathbb{N}\}$. If $g_n = f - f_n$ for every n, then $g_n(x) \in g_n(\overline{U}_n) \subset \overline{g_n(U_n)} \subset \overline{(\frac{\varepsilon}{2}, +\infty)} = [\frac{\varepsilon}{2}, +\infty)$. Thus $g_n(x) \geqslant \frac{\varepsilon}{2}$ for each $n \in \mathbb{N}$ which is a contradiction because $g_n(x) \to 0$ due to the fact that $f_n(x) \to f(x)$.

Therefore, $U_m = \emptyset$ for some $m \in \mathbb{N}$ which implies $U_n = \emptyset$ for every $n \geqslant m$ and hence $|f_n(x) - f(x)| \leqslant \frac{\varepsilon}{2} < \varepsilon$ for each $n \geqslant m$ and $x \in X$. This shows that $f_n \rightrightarrows f$.

S.186. *Prove that the following are equivalent for any non-empty space X:*

(i) $C_p(X)$ *is σ-compact.*
(ii) $C_p(X)$ *is σ-countably compact.*
(iii) $C_p(X)$ *is locally compact, i.e., every $f \in C_p(X)$ has a compact neighbourhood.*
(iv) $C_p(X)$ *is locally countably compact, i.e., every $f \in C_p(X)$ has a countably compact neighbourhood.*
(v) $C_p(X)$ *is locally pseudocompact, i.e., every $f \in C_p(X)$ has a pseudocompact neighbourhood.*
(vi) The space X is finite.

Solution. For any $A \subset X$ let $\pi_A : C_p(X) \to C_p(A)$ be the restriction map. If X is finite then $C_p(X)$ is homeomorphic to $\mathbb{R}^n$, where $n = |\, X \,|$. Take any $x \in \mathbb{R}^n$, $r > 0$ and $y \in \overline{B_n(x,r)}$. For any $\varepsilon > 0$ there exist a point $z \in B_n(y, \varepsilon) \cap B_n(x, r)$ and hence $d_n(y,x) \leqslant d_n(y,z) + d_n(z,x)$ (see S.130(i) for the definition of d_n and the proof of the last inequality) which implies that $d_n(y, x) < r + \varepsilon$. This proves that $\overline{B_n(x,r)} \subset B_n(x, r + \varepsilon)$ and therefore the set $\overline{B_n(x,r)}$ is compact by Problem 131. As a consequence the space $\mathbb{R}^n$ is locally compact. This proves (vi) $\Rightarrow$ (iii). It is evident that we also have (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Note also that $\mathbb{R}^n = \bigcup\{\overline{B_n(x,k)} : k \in \mathbb{N}\}$ for any $x \in \mathbb{R}^n$ and therefore $\mathbb{R}^n$ is σ-compact. This proves (vi) $\Rightarrow$ (i).

Now suppose that $C_p(X)$ is locally pseudocompact and fix a pseudocompact neighbourhood U of the function $h \equiv 0$. There exist $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $O(h, x_1, \ldots, x_n, \varepsilon) \subset U$. If X is infinite then there is $y \in X \setminus \{x_1, \ldots, x_n\}$. The map $e_y : C_p(X) \to \mathbb{R}$, defined by $e_y(f) = f(y)$ for all $f \in C_p(X)$, is continuous (Problem 166). For any $r \in \mathbb{R}$ there is $f \in C_p(X)$ such that $f(y) = r$ and $f(x_i) = 0$ for all $i \leqslant n$ (Problem 034). Since $f \in O(h, x_1, \ldots, x_n, \varepsilon) \subset U$ and $e_y(f) = r$, the function e_y is not bounded on U which contradicts the pseudocompactness of U. Hence X has to be finite and (v) $\Rightarrow$ (vi).

To finish the proof, it suffices to show that (ii) $\Rightarrow$ (vi).

Fact 1. If X is not pseudocompact then $C_p(X)$ maps continuously onto $\mathbb{R}^\omega$.

Proof. If X is not pseudocompact then, by Problem 136, there exists a discrete family $\mathcal{U} = \{U_n : n \in \omega\} \subset \tau^*(X)$. Take a point $x_n \in U_n$ for each $n \in \omega$ and fix a function $f_n \in C(X, [0, 1])$ such that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) = \{0\}$. It is clear that $D = \{x_n : n \in \omega\}$ is a discrete and closed subspace of X. If $f : D \to \mathbb{R}$ then the function $g = \Sigma_{n\in\omega} f_n \cdot f(x_n)$ is continuous on X. Indeed, if $y \in X$ then there is $U \in \tau(y, X)$ which meets at most one element of $\mathcal{U}$, say, U_k. Then $g|U = f_k \cdot f(x_k)|U$ is a continuous function. This implies continuity of g at the point y and hence g is a continuous. Since $g|D = f$, the restriction map $\pi_D : C_p(X) \to C_p(D) = \mathbb{R}^D$ is onto. Since $\mathbb{R}^D$ is homeomorphic to $\mathbb{R}^\omega$, Fact 1 is proved.

Fact 2. The space $\mathbb{R}^\omega$ is not σ-pseudocompact and hence it is neither σ-countably compact nor σ-compact.

Proof. Let $\pi_n : \mathbb{R}^\omega \to \mathbb{R}$ be the natural projection onto the nth factor. Assume that $\mathbb{R}^\omega = \bigcup\{F_n : n \in \omega\}$ where each F_i is pseudocompact. Since $\mathbb{R}^\omega$ is second countable, so is F_n for each n and hence each F_n is compact by Problem 138.

The set $\pi_n(F_n)$ is bounded in $\mathbb{R}$ by Problem 131, so there is $x_n \in \mathbb{R} \setminus \pi_n(F_n)$. Now it is immediate that the point $x \in \mathbb{R}^\omega$ defined by $x(n) = x_n$ does belong to $\bigcup\{F_n : n \in \omega\}$ which is a contradiction. Fact 2 is proved.

Fact 3. If $C_p(X)$ is σ-countably compact then X is pseudocompact.

Proof. If not then $C_p(X)$ maps continuously onto $\mathbb{R}^\omega$ by Fact 1. It is easy to see that any continuous image of a σ-countably compact space is a σ-countably compact space so $\mathbb{R}^\omega$ is σ-countably compact which contradicts Fact 2 and Fact 3 is proved.

Any countable intersection of open sets of a space X is called *a G_δ-subset of* X. Call a space X *a P-space* if any G_δ-subset of X is open.

Fact 4. For any space X if $A \subset X$ and the set $C_A = \pi_A(C_p(X))$ is σ-countably compact then A is a P-space. In particular, if $C_p(X)$ is σ-countably compact then X is a P-space.

Proof. Assume that C_A is σ-countably compact. If A is not a P-space then there exists $x \in A$ and a family $\{F_n : n \in \omega\}$ of closed subsets of A such that $x \notin F_n$ and $F_n \subset F_{n+1}$ for each $n \in \omega$ while $x \in \mathrm{cl}_A(\bigcup\{F_n : n \in \omega\})$. Let $F = \bigcup\{F_i : i \in \omega\}$; it is clear that $x \in \overline{F}$.

The set $I_x = \{f \in C_A : f(x) = 0\}$ is closed in C_A and hence it is σ-countably compact; therefore $I_x = \bigcup\{K_n : n \in \omega\}$ where each K_n is countably compact. We claim that, for each $n \in \omega$ and $\varepsilon > 0$, there is $k_n \in \omega$ such that for every $f \in K_n$ there is $z \in F_{k_n}$ with $f(z) < \varepsilon$. If it were not true then, for each $i \in \omega$, there is $f_i \in K_n$ such that $f_i(y) \geqslant \varepsilon$ for every $y \in F_i$. Since K_n is countably compact, the set $\{f_i : i \in \omega\}$ has an accumulation point $f \in K_n$. If $y \in F = \bigcup\{F_i : i \in \omega\}$ then $y \in F_m$ for some $m \in \omega$ and hence $f_i(y) \geqslant \varepsilon$ for all $i \geqslant m$. It is immediate that this implies $f(y) \geqslant \varepsilon$. Thus we have $f(y) \geqslant \varepsilon$ for all $y \in F$ while $f(x) = 0$ which contradicts continuity of f on A and the fact that $x \in \mathrm{cl}_A(F)$.

Therefore, we can fix a sequence $\{k_n : n \in \omega\} \subset \omega$ with the following properties:

(1) $k_{n+1} > k_n$ for each $n \in \omega$;
(2) For every $f \in K_n$ there is $y \in F_{k_n}$ such that $f(y) < \frac{1}{2^n}$.

The set F_n being closed in A it follows from $x \in A$ that $x \notin \overline{F}_n$ for every $n \in \omega$. Thus, we can apply the Tychonoff property of X to choose a continuous function $g_n : X \to [0, \frac{1}{2^n}]$ such that $g_n(x) = 0$ and $g_n(F_{k_n}) = \{\frac{1}{2^n}\}$ for each $n \in \omega$. The function $g = \Sigma_{n\in\omega}\, g_n$ is a uniform limit of the sequence $\{g_0 + \cdots + g_n\}_{n\in\omega}$ and hence $g \in C_p(X)$. It is evident that $g(x) = 0$ so $h = g \mid A \in I_x$. However, we have $h(y) = g(y) \geqslant g_n(y) \geqslant \frac{1}{2^n}$ for each $y \in F_{k_n}$ whence $h \notin K_n$ for all $n \in \omega$. Therefore $h \in I_x \setminus (\bigcup\{K_n : n \in \omega\})$; this contradiction shows that Fact 4 is proved.

To finish the proof of (ii) $\Rightarrow$ (vi) it suffices to establish that each pseudocompact P-space X is finite (see Facts 3 and 4). Suppose that X is infinite. Take any distinct $x, y \in X$ and disjoint $U \in \tau(x, X)$, $V \in \tau(y, X)$. Since $X = (X \setminus U) \cup (X \setminus V)$, one of the sets $X \setminus U$, $X \setminus V$ has to be infinite. If, for example, $\mid X \setminus U \mid \geqslant \omega$ then we can apply regularity of X and find $W \in \tau(x, X)$ with $\overline{W} \subset U$. It is clear that $X \setminus \overline{W}$ is infinite. Thus we have proved, that in any infinite space X there is $W \in \tau^*(X)$ such that $X \setminus \overline{W}$ is infinite.

Call such W *a small subset of* X. So, choose a small $U_0 \subset X$ and if we have $U_0, \cdots, U_n \in \tau^*(X)$ with $X_n = X \setminus (\overline{U}_0 \cup \cdots \cup \overline{U}_n)$ infinite, let U_{n+1} be a small subset of X_n. By induction, we can construct a sequence $\{U_n : n \in \omega\}$ of disjoint non-empty open subsets of X. Pick $x_n \in U_n$ for all n and use regularity of X to construct open sets $\{U_n^i : i \in \omega\}$ so that $U_n^0 = U_n$ and $\overline{U_n^{i+1}} \subset \overline{U_n^i}$ for each $i \in \omega$. Since X is a P-space, the set $V_n = \bigcap\{U_n^i : i \in \omega\} = \bigcap\{\overline{U_n^i} : i \in \omega\}$ is clopen in X and $V_n \subset U_n$ for each $n \in \omega$. Using once more the P-property of X we can conclude that the set $V = X \setminus (\bigcup\{V_n : n \in \omega\})$ is also open in X and therefore the family $\{V_n : n \in \omega\}$ is discrete which contradicts pseudocompactness of X. Hence X is finite and our proof is complete.

S.187. *Prove that $C_p(X)$ is locally Lindelöf ($\equiv$ each $f \in C_p(X)$ has a Lindelöf neighbourhood) if and only if $C_p(X)$ is Lindelöf.*

Solution. If $C_p(X)$ is Lindelöf then it is a Lindelöf neighbourhood of any of its points. To prove sufficiency, assume that $C_p(X)$ is locally Lindelöf and fix a Lindelöf neighbourhood U of the function $h \equiv 0$. There exist $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $\overline{V} \subset U$ where $V = O(h, x_1, \ldots, x_n, \varepsilon)$. It is an easy exercise that a closed subspace of a Lindelöf space is Lindelöf so $\overline{V}$ is also Lindelöf. It is an easy consequence of Problem 116 that the map $\varphi_k : C_p(X) \to C_p(X)$ defined by $\varphi_k(f) = k \cdot f$, is continuous. We can apply Problem 157(vi) to conclude that $\varphi_k(\overline{V})$ is Lindelöf for any $k \in \mathbb{N}$. Now, if $f \in C_p(X)$ then there is $k \in \mathbb{N}$ such that $k \cdot \varepsilon > f(x_i)$ for all $i \leqslant n$. As a consequence $f \in \varphi_k(V)$. The function f being arbitrary, we proved that $\bigcup\{\varphi_k(\overline{V}) : k \in \mathbb{N}\} = C_p(X)$. Another easy exercise is to prove that a countable union of Lindelöf spaces is also a Lindelöf space. Thus, $C_p(X)$ is Lindelöf and we are done.

S.188. *Assume that $C_p(X)$ is Lindelöf. Prove that any discrete family $\gamma \subset \tau^*(X)$ is countable.*

Solution. Any subfamily of a discrete family is discrete so if there is some uncountable discrete family of non-empty open subsets of X, then we can find a discrete family $\gamma = \{U_\alpha : \alpha < \omega_1\} \subset \tau^*(X)$. Pick $x_\alpha \in U_\alpha$ for each $\alpha < \omega_1$ and let $D = \{x_\alpha : \alpha < \omega_1\}$. Fix a function $f_\alpha \in C_p(X, [0, 1])$ such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(X \setminus U_\alpha) = \{0\}$. Given any $f : D \to \mathbb{R}$, let $g = \sum_{\alpha<\omega_1} f_\alpha \cdot f(\alpha)$. It is clear that $g|D = f$. Besides, the function g is continuous because for any $x \in X$ there is $W \in \tau(x, X)$ such that W intersects at most one element of γ, say U_β. Then $g|W = f_\beta \cdot f(x_\beta)|W$ is a continuous function and hence g is continuous at x. The point x being arbitrary, the function g is continuous. Hence the restriction map $\pi_D : C_p(X) \to C_p(D)$ is continuous and onto (see Problem 152). Hence the space $C_p(D)$ is Lindelöf by Problem 156 (vi). The space D is clearly discrete so $C_p(D)$ is homeomorphic to $\mathbb{R}^{\omega_1}$. To obtain the desired contradiction, it suffices to prove that $\mathbb{R}^{\omega_1}$ is not Lindelöf.

Assume that the space $\mathbb{R}^{\omega_1}$ is Lindelöf. The space $(\mathbb{R}^{\omega_1})^n$ is homeomorphic to $\mathbb{R}^{\omega_1}$ by Problem 103 and hence $t(C_p(\mathbb{R}^{\omega_1})) = \omega$ by Problem 149. Let

$\varphi_\alpha : \mathbb{R}^{\omega_1} \to \mathbb{R}$ be the natural projection onto the αth factor for each $\alpha < \omega_1$. Denote by h the function on $\mathbb{R}^{\omega_1}$ which is identically zero. We claim that h belongs to the closure of the set $A = \{\varphi_\alpha - \varphi_\beta : \alpha < \beta < \omega_1\}$. It suffices to prove that for arbitrary $x_1, \ldots, x_n \in \mathbb{R}^{\omega_1}$ and $\varepsilon > 0$, we have $O(h, x_1, \ldots, x_n, \varepsilon) \cap A \neq \emptyset$.

Define a map $\theta : \omega_1 \to \mathbb{R}^n$ by the formula $\theta(\alpha) = (x_1(\alpha), \ldots, x_n(\alpha)) \in \mathbb{R}^n$ for each $\alpha < \omega_1$. If $\theta(\alpha) = \theta(\beta)$ for some $\alpha < \beta$ then $\varphi_\alpha(x_i) - \varphi_\beta(x_i) = x_i(\alpha) - x_i(\beta) = 0 < \varepsilon$ for each $i \leqslant n$ and therefore $\varphi_\alpha - \varphi_\beta \in O(h, x_1, \ldots, x_n, \varepsilon) \cap A$. Now, if θ is an injection then the set $\theta(\omega_1) \subset \mathbb{R}^n$ is uncountable and we can apply Fact 1 of S.151 to conclude that there is $P \subset \omega_1$ such that $\theta(P)$ has no isolated points. Pick any $\alpha \in P$. Since $\theta(\alpha)$ is not isolated, there is $\beta \in P \setminus \{\alpha\}$ such that $\theta(\beta) \in B_n(\theta(\alpha), \varepsilon)$ (see Problem 130). There will be no loss of generality to assume that $\alpha < \beta$. It is easy to see that $\theta(\beta) \in B_n(\theta(\alpha), \varepsilon)$ implies $|\, x_i(\alpha) - x_i(\beta)\,| < \varepsilon$ for any $i \leqslant n$ and hence $\varphi_\alpha - \varphi_\beta \in O(h, x_1, \ldots, x_n, \varepsilon) \cap A$. This shows that $h \in \overline{A}$.

Since we have the equality $t(C_p(\mathbb{R}^{\omega_1})) = \omega$, there is a countable set $B \subset A$ such that $h \in \overline{B}$. Take a countably infinite set $C \subset \omega_1$ such that $B \subset \{\varphi_\alpha - \varphi_\beta : \alpha, \beta \in C$ and $\alpha < \beta\}$. Choose any enumeration $\{\delta_i : i \in \mathbb{N}\}$ of the set C and define $x \in \mathbb{R}^{\omega_1}$ as follows: $x(\delta_i) = i$ for all $i \in \omega$ and $x(\gamma) = 0$ for all $\gamma \in \omega_1 \setminus C$. We claim that $V \cap B = \emptyset$ where $V = O(h, x, 1)$. Indeed, if $\psi \in B$ then $\psi = \varphi_{\delta_i} - \varphi_{\delta_j}$ for some $i, j \in \mathbb{N}$, $i \neq j$. Therefore $|\, \psi(x)\,| = |\, x(\delta_i)\,| - x(\delta_j)\,| = |\, i - j\,| \geqslant 1$ and hence $\psi \notin V$ which is a contradiction with the fact that $h \in \overline{B}$. This contradiction shows that $t(C_p(\mathbb{R}^{\omega_1})) > \omega$ and hence $\mathbb{R}^{\omega_1}$ is not Lindelöf so our proof is over.

S.189. *(Asanov's theorem) Prove that $t(X^n) \leqslant l(C_p(X))$ for any space X and $n \in \mathbb{N}$. In particular, if $C_p(X)$ is a Lindelöf space, then $t(X^n) \leqslant \omega$ for all $n \in \mathbb{N}$.*

Solution. Let $l(C_p(X)) \leqslant \kappa$. Fix any $n \in \mathbb{N}$ to prove that $t(X^n) \leqslant \kappa$. Take any $x = (x_1, \ldots, x_n) \in X^n$ and any $A \subset X^n$ with $x \in \overline{A}$. Choose $O_i \in \tau(x_i, X)$ in such a way that $O_i \cap O_j = \emptyset$ if $x_i \neq x_j$ and $O_i = O_j$ if $x_i = x_j$. The set $O = O_1 \times \cdots \times O_n$ is a neighbourhood of x and $x \in \overline{A \cap O}$ which makes it possible to assume that $A \subset O$. The set $\Phi = \{f \in C_p(X) : f(x_i) = 1$ for all $i \leqslant n\}$ is closed in $C_p(X)$ and hence $l(\Phi) \leqslant \kappa$. Given $y = (y_1, \ldots, y_n) \in A$, let $U_y = \{g \in C_p(X) : g(y_i) > 0$ for all $i \leqslant n\}$. Now, if $f \in \Phi$ then $U_i = f^{-1}((0, +\infty)) \in \tau(x_i, X)$ for all $i \leqslant n$. Since $x \in \overline{A}$, there is $y = (y_1, \ldots, y_n) \in A \cap (U_1 \times \cdots \times U_n)$. Thus $f(y_i) > 0$ for each $i \leqslant n$ and hence $f \in U_y$. This proves that $\Phi \subset \bigcup\{U_y : y \in A\}$. Since $l(\Phi) \leqslant \kappa$, there exists $B \subset A$ such that $|\, B\,| \leqslant \kappa$ and $\Phi \subset \bigcup\{U_y : y \in B\}$. We claim that $x \in \overline{B}$. If not, then $V \cap B = \emptyset$ for some $V = V_1 \times \cdots \times V_n$ where $x_i \in V_i \in \tau(O_i)$ for each $i \leqslant n$ and $V_i = V_j$ if $x_i = x_j$.

It is easy to see that there exists a function $g \in \Phi$ such that $g(z) = 0$ for any $z \in X \setminus (\bigcup\{V_i : i \leqslant n\})$. We have $g \in U_y$ for some $y = (y_1, \ldots, y_n) \in B$. Now $y \in A$ implies $y_i \in U_i$ for all $i \leqslant n$. Recall that $g(y_i) > 0$ and therefore $y_i \in V_i$ for otherwise $g(y_i) = 0$. As a consequence $y \in (V_1 \times \cdots \times V_n) \cap B$ which is a contradiction.

S.190. *For a space X, let $A \subset C^*(X)$ be an algebra which is closed with respect to uniform convergence. Prove that $f, g \in A$ implies* $\max(f, g) \in A$ *and* $\min(f, g) \in A$.

Solution. Denote by I the closed interval $[0, 1] \subset \mathbb{R}$. Let $s(t) = \sqrt{t}$ for any $t \in I$.

Fact 1. There exists a sequence $\{p_i\}_{i\in\omega}$ of polynomials on I such that $p_n \rightrightarrows s$.

Proof. Let $p_0(t) = 0$ for all $t \in I$. If we have the polynomial p_i for some $i \geqslant 0$, let $p_{i+1}(t) = p_i(t) + \frac{1}{2}(t - p_i^2(t))$ for every $t \in I$.

Let us show by induction that $p_i(t) \leqslant \sqrt{t}$ for all $t \in I$ and $i \in \omega$. Evidently, this inequality holds for $i = 0$, so suppose that $p_i(t) \leqslant \sqrt{t}$. Since

$$\sqrt{t} - p_{i+1}(t) = \sqrt{t} - p_i(t) - \tfrac{1}{2}(t - p_i^2(t)) = (\sqrt{t} - p_i(t))(1 - \tfrac{1}{2}(\sqrt{t} + p_i(t))),$$

the inductive hypothesis and the inequality $t \leqslant 1$ imply

$$\sqrt{t} - p_{i+1}(t) \geqslant (\sqrt{t} - p_i(t))(1 - \tfrac{1}{2} \cdot 2 \cdot \sqrt{t}) \geqslant 0,$$

which completes the proof of the fact that $p_i \leqslant s$ for all $i \in \omega$. For any $t \in I$ the sequence $\{p_i(t)\}$ is bounded by $\sqrt{t}$ and $p_{i+1}(t) \geqslant p_i(t)$ for all $i \in \omega$. Therefore $p_i(t)$ converges to some $p(t) \in I$. Therefore it is possible to pass to the limit in the equality $p_{i+1}(t) = p_i(t) + \frac{1}{2}(t - p_i^2(t))$, when $i \to \infty$. This gives us $p(t) = p(t) - \frac{1}{2}(t - p^2(t))$ and hence $p(t) = \sqrt{t}$. We are now in condition to apply the Dini theorem (Problem 185) to conclude that $p_i \rightrightarrows \sqrt{t} = s$ so Fact 1 is proved.

Returning to the main proof observe that $\min(f, g) = \frac{1}{2}(f + g - |f - g|)$ and $\max(f, g) = \frac{1}{2}(f + g + |f - g|)$. Since A is an algebra, we have $f + g \in A$ and $f - g \in A$ so it suffices to prove that $|f + g| \in A$ and $|f - g| \in A$. In other words, it suffices to show that $h \in A$ implies $|h| \in A$. Let $h_i(x) = p_i((h(x))^2)$ for all $x \in X$ and $i \in \omega$. Here p_i is the respective polynomial from Fact 1. Since A is an algebra, we have $h_i \in A$ for all $i \in \omega$. By Fact 1 the sequence $\{h_i\}$ converges uniformly to $\sqrt{h^2} = |h|$. Since A is closed with respect to uniform convergence, we have $|h| \in A$ and our proof is complete.

S.191. *(The Stone–Weierstrass theorem). Let X be a compact space. Suppose that A is an algebra in $C(X)$ which separates the points of X and is closed with respect to uniform convergence. Prove that $A = C(X)$. Deduce from this fact that if A is an algebra in $C(X)$ which separates the points of X then, for any $f \in C(X)$, there is a sequence $\{f_n\}_{n\in\omega} \subset A$ such that $f_n \rightrightarrows f$.*

Solution. Let $\max(f_0, \dots, f_n)(x) = \max\{f_0(x), \dots, f_n(x)\}$ for any point $x \in X$ and functions $f_0, \dots, f_n \in C(X)$. This defines the function $\max(f_0, \dots, f_n) \in C(X)$. Analogously, letting $\min(f_0, \dots, f_n)(x) = \min\{f_0(x), \dots, f_n(x)\}$ for any $x \in X$, we define the function $\min(f_0, \dots, f_n) \in C(X)$.

Fact 1. For any $f \in C(X)$ and $\varepsilon > 0$, there exists a function $f_\varepsilon \in A$ such that $|f_\varepsilon(x) - f(x)| < \varepsilon$ for any $x \in X$.

Proof. For every pair of distinct points $a, b \in X$ we can find a function $h \in A$ such that $h(a) \neq h(b)$. Since A is an algebra, the function g defined by the formula $g(x) = (h(x) - h(a))(h(b) - h(a))^{-1}$ for every $x \in X$, belongs to A. It is immediate that $g(a) = 0$ and $g(b) = 1$. Now let $f_{a,b}(x) = (f(b) - f(a))g(x) + f(a)$ for each $x \in X$.

Of course, $f_{a,b} \in A$ and $f_{a,b}(a) = f(a)$ and $f_{a,b}(b) = f(b)$. The sets $U_{a,b} = \{x \in X : f_{a,b}(x) < f(x)+\varepsilon\}$ and $V_{a,b} = \{x \in X : f_{a,b}(x) > f(x)-\varepsilon\}$ are open neighbourhoods of the points a and b, respectively. Fix any $b \in X$ and extract a finite subcover $\{U_{a_i,b} : i \in \{0, \ldots, n\}\}$ of the open cover $\{U_{a,b} : a \in X\}$ of the compact space X. Apply Problem 190 to conclude that the function $f_b = \min(f_{a_0,\, b}, \ldots, f_{a_n,\, b})$ belongs to A. It is easy to see that $f_b(x) < f(x)+\varepsilon$ for all $x \in X$ and $f_b(x) > f(x)-\varepsilon$ for any $x \in V_b = \bigcap\{V_{a_i,b} : i \leqslant n\}$. Since X is compact, we can choose a finite subcover $\{V_{b_i} : 0 \leqslant i \leqslant k\}$ of the open cover $\{V_b : b \in X\}$ of the space X. Apply Problem 190 once more to observe that the function $f_\varepsilon = \max(f_{b_0}, \ldots, f_{b_k})$ belongs to A and we have $|\, f_\varepsilon(x) - f(x)\,| < \varepsilon$ for all $x \in X$ so Fact 1 is proved.

To finish our proof take an arbitrary $f \in C(X)$ and find a function $f_n \in A$ such that $|f_n(x) - f(x)| < \frac{1}{n}$ for all $x \in X$. The existence of such f_n is guaranteed by Fact 1. It is obvious that $f_n \rightrightarrows f$ and therefore $f \in A$ because A is closed with respect to uniform convergence. Being $f \in C(X)$ an arbitrary function, we proved that $A = C(X)$.

Suppose finally that A is an algebra which separates the points of X.

Fact 2. The set $B = \{f \in C(X) : f_n \rightrightarrows f$ for some sequence $\{f_n\}_{n\in\omega} \subset A\}$ is also an algebra.

Proof. If $f, g \in B$, fix sequences $\{f_n\}, \{g_n\} \subset A$ such that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$. Then $\{f_n + g_n\} \subset A$ because A is algebra and $f_n + g_n \rightrightarrows f + g$ by Problem 035. This shows that $f + g \in B$. The sequence $\{f_n \cdot g_n\}$ also lies in A because A is an algebra. We will prove that $f_n \cdot g_n \rightrightarrows f \cdot g$.

Let us prove first that there exists $K \in \mathbb{R}$ such that $|\,f(x)\,| \leqslant K$, $|\,f_n(x)\,| \leqslant K$ and $|\,g_n(x)\,| \leqslant K$ for all $n \in \omega$ and $x \in X$. Since f is continuous and X is compact, the functions f and g are bounded on X, i.e., there exists $M \in \mathbb{R}$ such that $|\,f(x)\,| \leqslant M$ and $|\,g(x)\,| \leqslant M$ for all $x \in X$. Applying the relevant uniform convergences we can find $m \in \omega$ such that $|\,f_n(x) - f(x)\,| < 1$ and $|\,g_n(x) - g(x)\,| < 1$ for all $n \geqslant m$ and $x \in X$. The functions $f_1, \ldots, f_m$ and $g_1, \ldots, g_m$ are bounded on X which implies that there is $N \in \mathbb{R}$ such that $|\,f_i(x)\,| + |\,g_i(x)\,| \leqslant N$ for all $i \leqslant m$ and $x \in X$. It is easy to verify that the number $K = M + N + 1$ is as promised.

Given an arbitrary $\varepsilon > 0$, we can find $l \in \omega$ such that $|\,f_n(x) - f(x)\,| < \frac{\varepsilon}{2K}$ and $|\,g_n(x) - g(x)\,| < \frac{\varepsilon}{2K}$ for all $n \geqslant l$ and $x \in X$. Then

$$\begin{aligned}|f_n(x)g_n(x) - f(x)g(x)| &= |g_n(x)(f_n(x) - f(x)) + f(x)(g_n(x) - g(x))| \\ &\leqslant |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)| < K\cdot\tfrac{\varepsilon}{2K} + K\cdot\tfrac{\varepsilon}{2K} = \varepsilon\end{aligned}$$

for all $n \geqslant l$ and $x \in X$ which proves that $f_n \cdot g_n \rightrightarrows f \cdot g$ and hence $f \cdot g \in B$. Since B contains A and A contains all constant functions, the set B also contains all constant functions and hence B is an algebra.

Note that B is the closure of A in the space $C_u(X)$ (see Problem 084) and hence it is closed in $C_u(X)$, i.e., B is an algebra closed with respect to uniform convergence. We have proved already that B has to be equal to $C(X)$ and this implies precisely that any $f \in C(X)$ is a uniform limit of a sequence from A.

S.192. *Let X be an arbitrary space. Prove that, if $A \subset C_p(X)$ is an algebra which separates the points of X, then A is dense in $C_p(X)$.*

Solution. Take any finite $P \subset X$. Observe that $\pi_P(C_p(X)) = C_p(P) = \mathbb{R}^P$ by Problem 034. Here $\pi_P : C_p(X) \to C_p(P)$ is the restriction map. It is immediate that $B = \pi_P(A)$ is an algebra in $C_p(P)$ which contains all constant functions and separates the points of P. The space P being compact, we can apply Problem 191 to conclude that the set B is uniformly dense in $\mathbb{R}^P$ i.e., for any $f \in \mathbb{R}^P$ and any $\varepsilon > 0$, there is $g \in B$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in P$.

To prove that A is dense in $C_p(X)$ fix $f \in C_p(X)$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$. Applying the observation of the first paragraph to the set $P = \{x_1, \ldots, x_n\}$ we can find $g \in A$ such that $|\pi_P(g)(x_i) - \pi_P(f)(x_i)| < \varepsilon$ for all $i \leqslant n$. Therefore $|g(x_i) - f(x_i)| < \varepsilon$ for all $i \leqslant n$ and hence $g \in O(f, x_1, \ldots, x_n, \varepsilon) \cap A$ which proves that $f \in \overline{A}$. The function f being taken arbitrarily, we have $\overline{A} = C_p(X)$ i.e., A is dense in $C_p(X)$.

S.193. *Prove that, for any $f \in C([a, b])$, there exists a sequence of polynomials $\{p_n : n \in \omega\}$ such that $p_n \rightrightarrows f$ on $[a, b]$.*

Solution. The space $[a, b]$ is compact (Problem 131) and the set P of all polynomials on $[a, b]$ is, evidently, an algebra in $C([a, b])$. Observe that P separates the points of $[a, b]$ because even the polynomial $p(x) = x$ separates them. Therefore Problem 191 is applicable to conclude that P is uniformly dense in $C([a, b])$.

S.194. *Prove that, for any $f \in C_p(\mathbb{R})$, there exists a sequence of polynomials $\{p_n : n \in \omega\}$ such that $p_n \to f$.*

Solution. Apply Problem 193 to find a polynomial p_n such that $|p_n(x) - f(x)| < \frac{1}{n}$ for all $x \in [-n, n]$. Given $x \in \mathbb{R}$ and $\varepsilon > 0$ there is $m \in \omega$ such that $|x| \leqslant m$ and $\frac{1}{m} < \varepsilon$. Then $x \in [-n, n]$ for each $n \geqslant m$ and therefore $|p_n(x) - f(x)| < \frac{1}{n} \leqslant \frac{1}{m} < \varepsilon$ which shows that the numeric sequence $\{p_n(x)\}$ converges to $f(x)$. The point x being taken arbitrarily, we can apply Problem 143 to conclude that $p_n \to f$.

S.195. *Is it true that, for any $f \in C_p(\mathbb{R})$, there exists a sequence of polynomials $\{p_n : n \in \omega\}$ such that $p_n \rightrightarrows f$?*

Solution. No, this is not true. To see this, let $f(t) = 0$ for all $t \in \mathbb{R} \setminus \mathbb{I}$. If $t \in [-1, 0]$ then $f(t) = t + 1$ and $f(t) = 1 - t$ for all $t \in [0, 1]$. It is clear that $f \in C_p(\mathbb{R})$. It turns out that f cannot be a uniform limit of polynomials. To prove this, we will use the following fact well known in algebra (we will prove it anyway).

Fact 1. For any $\varepsilon > 0$, if a polynomial is bounded on $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$ then it is a constant function.

Proof. Suppose not and let $n \geqslant 1$ be the minimal possible degree of a polynomial p which is bounded on $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$. If $n = 1$ then the function $p(x) = ax + b$, $a \neq 0$ cannot be bounded on $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$ so $n \geqslant 2$. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$. Since $p(x)$ is bounded on $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$, the function $p(x) - a_0 = x(a_1 + \cdots + a_n x^{n-1})$ is also bounded on the same set. The function $g(x) = \frac{1}{x}$ being bounded on $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$ as well, the polynomial $q(x) = g(x) \cdot (p(x) - a_0)$ is also bounded on $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$ and deg $(q) \leqslant n - 1$ which is a contradiction with n being the minimal degree of a bounded polynomial on $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$. Fact 1 is proved.

To finish our proof, suppose that $p_n \rightrightarrows f$ for some sequence $\{p_n\}$ of polynomials. There is $m \in \mathbb{N}$ such that $| p_n(x) - f(x) | < 1$ for all $x \in \mathbb{R}$ and $n \geqslant m$. Therefore $| p_n(x) | \leqslant | f(x) | + 1 \leqslant 2$ for all $x \in \mathbb{R}$ and $n \geqslant m$. Thus the polynomials p_n are bounded for all $n \geqslant m$ and hence they are all constant by Fact 1. Let $\boldsymbol{c}_n$ be the constant value of the polynomial p_n. Since $p_n(x) \to f(x) = 0$ for any $x \geqslant 2$ (see Problems 184 and 143), we have $\boldsymbol{c}_n \to 0$. However, $p_n(0) \to f(0) = 1$ and hence $c_n \to 1$ which is a contradiction.

S.196. *Let us call a function $\varphi : C_p(X) \to \mathbb{R}$ a linear functional if we have $\varphi(\alpha f + \beta g) = \alpha\varphi(f) + \beta\varphi(g)$ for any $f, g \in C_p(X)$ and $\alpha, \beta \in \mathbb{R}$. The functional φ is called trivial if $\varphi(f) = 0$ for any $f \in C_p(X)$. Prove that, for any $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, the function $\lambda_2 e_{x_1} + \cdots + \lambda_n e_{x_n}$ is a linear continuous functional on $C_p(X)$. Recall that $e_x : C_p(X) \to \mathbb{R}$ is defined by $e_x(f) = f(x)$ for all $f \in C_p(X)$.*

Solution. Note first that, for any point $x \in X$, we have the equalities $e_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = (\alpha f)(x) + (\beta g)(x) = \alpha f(x) + \beta g)(x) = \alpha e_x(f) + \beta e_x(g)$ and therefore e_x is a linear functional for any $x \in X$. Continuity of e_x was proved in Problem 166. To finish the proof observe that any finite sum of continuous linear functionals is a continuous linear functional and $\lambda\varphi$ is a continuous linear functional whenever φ is a continuous linear functional with $\lambda \in \mathbb{R}$.

S.197. *Prove that, for any continuous linear functional $\varphi : C_p(X) \to \mathbb{R}$, there exist $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\varphi = \lambda_2 e_{x_2} + \cdots + \lambda_n e_{x_n}$.*

Solution. Since φ is continuous, there exist distinct $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $\varphi(O(h, x_1, \ldots, x_n, \varepsilon)) \subset (-1, 1)$. Here $h \equiv 0$ and hence $\varphi(h) = 0 \in (-1, 1)$. Observe that $\varphi(f) = 0$ for any $f \in C_p(X)$ with $f(x_i) = 0$ for all $i \leqslant n$. Indeed, for each $k \in \mathbb{N}$, we have $k \cdot f \in O(h, x_1, \ldots, x_n, \varepsilon)$ and hence $| \varphi(kf) | < 1$ whence $| \varphi(f) | < \frac{1}{k}$. Since this is true for any $k \in \mathbb{N}$, we have $\varphi(f) = 0$. An evident consequence of this fact and the linearity of φ is the equality $\varphi(f) = \varphi(g)$ for any $f, g \in C_p(X)$ with $f(x_i) = g(x_i)$ for all $i \leqslant n$.

Choose disjoint open sets $U_1, \ldots, U_n$ such that $x_i \in U_i$ for all $i \leqslant n$. There exist functions $f_i \in C(X, [0, 1])$ such that $f_i(x_i) = 1$ and $f_i(X \setminus U_i) = \{0\}$ for all $i \leqslant n$. Let $\lambda_i = \varphi(f_i)$ for all $i \leqslant n$. We will prove that $\varphi = \lambda_2 e_{x_2} + \cdots + \lambda_n e_{x_n}$. Take any $f \in C_p(X)$ and note that, for any $i \leqslant n$, we have $g(x_i) = f(x_i)$ where $g = \sum_{i=1}^n f(x_i) \cdot f_i$. Therefore $\varphi(f) = \varphi(g) = \sum_{i=1}^n f(x_i)\varphi(f_i) = \sum_{i=1}^n \lambda_i f(x_i) = (\lambda_2 e_{x_2} + \cdots + \lambda_n e_{x_n})(f)$ and hence $\varphi = \lambda_2 e_{x_2} + \cdots + \lambda_n e_{x_n}$.

S.198. *Give an example of a (discontinuous) linear functional $\varphi : C_p(\mathbb{R}) \to \mathbb{R}$ which cannot be represented as $\lambda_2 e_{x_1} + \cdots + \lambda_n e_{x_n}$ for any points $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.*

Solution. No discontinuous linear functional $\varphi : C_p(\mathbb{R}) \to \mathbb{R}$ can be represented as $\lambda_1 e_{x_1} + \cdots + \lambda_n e_{x_n}$ by Problem 196 so it suffices to give an example of a discontinuous linear functional $\varphi : C_p(\mathbb{R}) \to \mathbb{R}$. Let $\varphi(f) = \int_0^3 f(t)\mathrm{d}t$ for any $f \in C_p(\mathbb{R})$. The linearity of φ is evident.

Given $n \in \mathbb{N}$, let $f_n(t) = 0$ for all $t \in [\frac{2}{n}, +\infty) \cup (-\infty, 0)$. If $t \in [0, \frac{1}{n}]$ we let $f_n(t) = n^2 t$ and $f_n(t) = -n^2(t - \frac{2}{n})$ for all $t \in [\frac{1}{n}, \frac{2}{n}]$. This gives us a set $A = \{f_n : n \in \mathbb{N}\} \subset C_p(\mathbb{R})$ and $h \equiv 0$ belongs to the closure of A. However, $\varphi(f_n) = \int_0^3 f_n(t)\mathrm{d}t = 1$ for all $n \in \mathbb{N}$ and hence $0 = \varphi(h) \notin \overline{\varphi(A)} = \overline{\{1\}} = \{1\}$. Now apply Problem 009(vi) to see that φ is not continuous.

S.199. *A map $\xi : C_p(X) \to \mathbb{R}$ is called a linear multiplicative functional if $\xi(f + g) = \xi(f) + \xi(g)$ and $\xi(f \cdot g) = \xi(f) \cdot \xi(g)$ for all $f, g \in C_p(X)$. Prove that, for any continuous non-trivial linear multiplicative functional $\xi : C_p(X) \to \mathbb{R}$, there exists a point $x \in X$ such that $\xi = e_x$.*

Solution. Since ξ is a continuous linear functional on $C_p(X)$, there exist distinct $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\xi = \lambda_1 e_{x_2} + \cdots + \lambda_n e_{x_n}$ (see Problem 197). If $\lambda_i \neq 0 \neq \lambda_j$ for some distinct $i, j \leqslant n$ then take functions $f, g \in C_p(X)$ such that $f(x_i) = 1, f(x_k) = 0$ for all $k \neq i$ and $g(x_j) = 1$, $g(x_k) = 0$ for all $k \neq j$. Then $(f \cdot g)(x_k) = 0$ for all $k \leqslant n$ and it is easy to see that $\xi(f) = \lambda_i$, $\xi(g) = \lambda_j$ while $\xi(f \cdot g) = 0 \neq \xi(f) \cdot \xi(g) = \lambda_i \lambda_j$ so ξ is not multiplicative, a contradiction. Thus $\xi = \lambda e_x$ for some $x \in X$ and $\lambda \in \mathbb{R}$. Since ξ is non-trivial, we have $\lambda \neq 0$. Let $h(x) = 1$ for all $x \in X$. Then $h \cdot h = h$ and therefore $\xi(h \cdot h) = \xi(h)\xi(h) = \lambda^2 = \xi(h) = \lambda$. This shows that $\lambda = 1$ and hence $\xi = e_x$.

S.200. *(Theorem of J. Nagata). Prove that spaces X and Y are homeomorphic if and only if the algebras $C_p(X)$ and $C_p(Y)$ are topologically isomorphic.*

Solution. Suppose that X is homeomorphic to Y and fix some homeomorphism $\varphi : X \to Y$; let $\varphi^*(f) = f \circ \varphi$ for any $f \in C_p(Y)$. It follows from Problem 163(v) that the mapping $\varphi^* : C_p(Y) \to C_p(X)$ is a homeomorphism. To prove that the map φ is also an isomorphism, take any functions $f, g \in C_p(Y)$ and $x \in X$. Then $\varphi^*(f + g)(x) = (f + g)(\varphi(x)) = f(\varphi(x)) + g(\varphi(x)) = \varphi^*(f)(x) + \varphi^*(g)(x)$ which shows that $\varphi^*(f + g) = \varphi^*(f) + \varphi^*(g)$. Analogously, we have the equalities $\varphi^*(f \cdot g)(x) = (f \cdot g)(\varphi(x)) = f(\varphi(x)) \cdot g(\varphi(x)) = \varphi^*(f)(x) \cdot \varphi^*(g)(x)$ and therefore $\varphi^*(f \cdot g) = \varphi^*(f) \cdot \varphi^*(g)$ whence φ^* is a topological isomorphism.

Now suppose that $i : C_p(X) \to C_p(Y)$ is a topological isomorphism. The map $i^* : C_p(C_p(Y)) \to C_p(C_p(X))$ defined by $i^*(\varphi) = \varphi \circ i$, is a homeomorphism by Problem 163 so it suffices to show that $i^*(E(Y)) = E(X)$ (see Problem 167), because $E(X)$ is homeomorphic to X and $E(Y)$ is homeomorphic to Y by Problem 167. Note that any $\xi \in E(Y)$ is a continuous multiplicative linear functional on $C_p(Y)$. Since i is a topological isomorphism, the map $i^*(\xi) = \xi \circ i$ is also a linear continuous multiplicative functional on $C_p(X)$ and hence $i^*(\xi) \in E(X)$ by Problem 199. This proves that $i^*(E(Y)) \subset E(X)$. Now if $\psi \in E(X)$ then $\psi \circ i^{-1}$ is a linear continuous multiplicative functional on $C_p(Y)$ and therefore $\varphi = \psi \circ i^{-1} \in E(Y)$ whence $\psi = i^*(\varphi)$. Thus $i^*(E(Y)) = E(X)$ and our proof is complete.

S.201. *Let* (X, d) *be a metric space. Show that*

(i) The open balls form a base of $(X, \tau(d))$.
(ii) $(X, \tau(d))$ *is Hausdorff and hence* T_1.

As a consequence, every metrizable space is Hausdorff and hence T_1.

Solution. (i) By the definition of $\tau(d)$, if $x \in U \in \tau(d)$ then $B(x, r) \subset U$ for some $r > 0$ so we only have to prove that every ball $B(x, r)$ is open in $(X, \tau(d))$. Take any $y \in B(x, r)$ and let $s = r - d(x, y) > 0$. It suffices to show that $B(y, s) \subset B(x, r)$. Given $z \in B(y, s)$, we have $d(z, x) \leqslant d(z, y) + d(y, x) < r - d(x, y) + d(x, y) = r$ and therefore $z \in B(x, r)$ which proves that $B(y, s) \subset B(x, r)$ and hence $B(x, r) \in \tau(d)$.

(ii) If $x, y \in X$ and $x \neq y$ then $r = \frac{d(x,y)}{2} > 0$. If $z \in B(x, r) \cap B(y, r)$ then $d(x, y) \leqslant d(x, z) + d(z, y) < r + r = 2r = d(x, y)$ which is a contradiction. Therefore $B(x, r) \cap B(y, r) = \emptyset$ so for the open sets $U = B(x, r)$, $V = B(y, r)$, we have $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

S.202. *Let* (X, d) *be a metric space. Considering that* X *has the topology* $\tau(d)$, *prove that the metric is a continuous function on* $X \times X$. *Deduce from this fact that any metrizable space is Tychonoff.*

Solution. We will denote by X both spaces (X, d) and $(X, \tau(d))$. Take an arbitrary point $z_0 = (x_0, y_0) \in X \times X$ and $\varepsilon > 0$. The set $U = B(x_0, \frac{\varepsilon}{4}) \times B(y_0, \frac{\varepsilon}{4})$ is an open neighbourhood of the point z_0. To prove continuity of the metric d at the point z_0 it suffices to establish that $d(U) \subset (d(x_0, y_0) - \varepsilon, d(x_0, y_0) + \varepsilon)$. So, take any $z = (x, y) \in U$. Then $d(x, x_0) < \frac{\varepsilon}{4}$ and $d(y, y_0) < \frac{\varepsilon}{4}$ and therefore

$$\begin{aligned} d(x,y) &\leqslant d(x,x_0) + d(x_0,y_0) + d(y_0,y) < d(x_0,y_0) + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} \\ &= d(x_0,y_0) + \tfrac{\varepsilon}{2} < d(x_0,y_0) + \varepsilon. \end{aligned}$$

Analogously,

$$\begin{aligned} d(x_0,y_0) &\leqslant d(x_0,x) + d(x,y) + d(y,y_0) < d(x,y) + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} \\ &= d(x,y) + \tfrac{\varepsilon}{2} < d(x,y) + \varepsilon, \end{aligned}$$

and therefore $d(x, y) > d(x_0, y_0) - \varepsilon$. Thus $d(z) \in (d(x_0, y_0) - \varepsilon, d(x_0, y_0) + \varepsilon)$ and hence $d(U) \subset (d(x_0, y_0) - \varepsilon, d(x_0, y_0) + \varepsilon)$.

An easy consequence of continuity of d is continuity of the function $d_y : X \to \mathbb{R}$ defined by $d_y(x) = d(y, x)$ for all $x \in X$. To show that X is Tychonoff we must only prove complete regularity of X (Problem 201). Take any $x \in X$ and any closed $F \subset X$ with $x \notin F$. There is $r > 0$ such that $B(x, r) \cap F = \emptyset$. The function $g = \frac{1}{r} \cdot d_x$ is continuous on X and $g(x) = 0$. It is easy to see that $g(y) \geqslant 1$ for all $y \in F$. Denote by g_1 the function which is identically 1 on X and let $h = \min(g, g_1)$. The function $h : X \to [0, 1]$ is continuous on X by Problem 028. Evidently, we have $h(x) = 0$ and $h(F) \subset \{1\}$. Finally, for the function $f : X \to [0, 1]$ defined by $f(x) = 1 - h(x)$ for all $x \in X$, we have $f(x) = 1$ and $f(F) \subset \{0\}$ which proves that X is Tychonoff.

S.203. *Let (X, d) be a metric space. Given a subspace $Y \subset X$, prove that the function $d_Y = d\,|(Y \times Y)$ is a metric on Y which generates on Y the topology of the subspace of the space $(X, \tau(d))$.*

Solution. It is immediate that the axioms (MS1)–(MS3) hold for d_Y so it is a metric on Y. Let us prove that the topology $\tau(d_Y)$ coincides with the topology μ induced on Y by $\tau(d)$. Take any $U \in \tau(d_Y)$. For any $y \in U$ fix $r_y > 0$ such that $B_{d_Y}(y, r_y) \subset U$. We proved in Problem 201 that all balls are open in metric spaces. Therefore the set $V = \bigcup\{B_d(y, r_y) : y \in U\}$ is open in X being a union of balls. If $z \in V \cap U$ then $z \in Y$ and $z \in B_d(y, r_y)$ for some $y \in U$. This implies $d_Y(y, z) = d(y, z) < r_y$ and therefore $z \in B_{d_Y}(y, r_y) \subset U$. This shows that $V \cap Y \subset U$. Since it is evident that $U \subset V$, we have $V \cap Y = U$ and hence $U \in \mu$ so $\tau(d_Y) \subset \mu$.

To see that $\mu \subset \tau(d_Y)$, take any $U \in \mu$. Then $U = V \cap Y$ for some $V \in \tau(d)$. As a consequence, for any $y \in U$ we have $y \in V$ and therefore there exists $r > 0$ such that $B_d(y, r) \subset V$. Now if $z \in B_{d_Y}(y, r)$ then $d(z, y) = d_Y(z, y) < r$ and hence $z \in V \cap Y = U$. Thus $B_{d_Y}(y, r) \subset U$ which proves that $U \in \tau(d_Y)$ and $\mu = \tau(d_Y)$.

S.204. *Let X be a discrete space. Prove that the function defined by the formula $d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$ is a complete metric on X which generates the topology of X. Hence every discrete space is completely metrizable.*

Solution. The axioms (MS1) and (MS2), clearly, hold for the function d. Given points $x, y, z \in X$, if $d(x, z) = 0$ then $d(x, z) \leqslant d(x, y) + d(y, z)$. If $d(x, z) = 1$ then x and z are distinct and hence it is impossible that $y = x$ and $y = z$. As a consequence, $d(x, y) + d(y, z) \geqslant 1 = d(x, z)$ so the triangle inequality also holds for d. Since $B(x, 1) = \{x\}$ for any $x \in X$, all points of X are open and hence $(X, \tau(d))$ is discrete, i.e., the metric d generates the topology of X.

The last thing we have to prove is that d is a complete metric on X. Given a Cauchy sequence $s = \{x_n : n \in \omega\} \subset X$, there exists $m \in \omega$ such that $d(x_i, x_j) < 1$ for all $i, j \geqslant m$. This means $x_i = x_m$ for all $i \geqslant m$ and hence the sequence s converges to the point x_m.

S.205. *For any points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of the space $\mathbb{R}^n$ let $\rho_n(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$. Prove that ρ_n is a complete metric on $\mathbb{R}^n$ which generates the natural topology on $\mathbb{R}^n$. Hence $\mathbb{R}^n$ is completely metrizable.*

Solution. We proved in S.130 that ρ_n is a metric on $\mathbb{R}^n$ such that the open balls generate the natural topology μ_n on $\mathbb{R}^n$. Thus every $U \in \mu_n$ is a union of balls and hence $U \in \tau(\rho_n)$. On the other hand, any $U \in \tau(\rho_n)$ is a union of some family of balls which belong to μ_n. Therefore $U \in \mu_n$ and we proved that $\tau(\rho_n) = \mu_n$.

To show that ρ_n is a complete metric on $\mathbb{R}^n$ let us prove first that $\mathbb{R}$ is complete with the metric $\rho_1(x, y) = |\,x - y\,|$. Assume that $s = \{x_n\}$ is a fundamental sequence in $\mathbb{R}$. There exists a number $m \in \omega$ such that $|\,x_n - x_k\,| < 1$ for all $n, k \geqslant m$.

If $K = 1 + \sum_{i=1}^{m} |x_i|$ then $|\ x_i\ | < K$ for any $i \leqslant m$. If $i > m$ then $|\ x_i - x_m\ | < 1$ and therefore $|\ x_i\ | < |\ x_m\ | + 1 < K$. This proves that the sequence s is bounded.

The set $A = \{t \in \mathbb{R} : (-\infty, t] \cap s \text{ is finite}\}$ is non-empty and has an upper bound K which implies that there exists $x = \sup A$. Given $\varepsilon > 0$ there are only finitely many elements of s in $(-\infty, x - \varepsilon]$. Thus there exists $m_1 \in \mathbb{N}$ such that $x_n > x - \varepsilon$ for all $n \geqslant m_1$. Take any $m_2 \in \mathbb{N}$ with $|\ x_n - x_k\ | < \frac{\varepsilon}{2}$ for all $n, k \geqslant m_2$. The set $(-\infty, x + \frac{\varepsilon}{2}] \cap s$ is infinite and hence there is $k > m_2$ for which $x_k < x + \frac{\varepsilon}{2}$. Therefore, for any $n \geqslant m_2$ we have $x_n < x_k + \frac{\varepsilon}{2} < x + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = x + \varepsilon$. Now, if $n \geqslant m = m_1 + m_2$ then $|\ x_n - x\ | < \varepsilon$ which proves that $x_n \to x$ and hence $\mathbb{R}$ is complete.

Now fix $n \in \mathbb{N}$ to prove that $\mathbb{R}^n$ is complete. Let $s = \{x_k\}$ be a fundamental sequence in $\mathbb{R}^n$. Fix $i \leqslant n$ and consider the sequence $\{x_k(i)\}$ of ith coordinates of the elements of the sequence s. Given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ for which $d_n(x_k, x_l) < \varepsilon$ for all $k, l \geqslant m$. Then $|\ x_k(i) - x_l(i)\ | \leqslant d(x_k, x_l) < \varepsilon$ for all $k, l \geqslant m$ and therefore the sequence $\{x_k(i)\} \subset \mathbb{R}$ is fundamental. The completeness of $\mathbb{R}$ being proved, the sequence $\{x_k(i)\}$ converges to some $x_i \in \mathbb{R}$. Let us show that $x_k \to x = (x_1, \ldots x_n)$. Given $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $|x_k(i) - x_i| < \frac{\varepsilon}{\sqrt{n}}$ for all $k \geqslant m$. As a consequence $d_n(x_k, x) = \sqrt{\sum_{i=1}^{n} (x_k(i) - x_i)^2} < \sqrt{n \cdot \frac{\varepsilon^2}{n}} = \varepsilon$, and hence $x_k \to x$.

S.206. *Let (X, d) be a metric space. Given $x, y \in X$ consider the function* $d^*(x, y) = \begin{cases} d(x,y) & \text{if } d(x,y) \leqslant 1, \\ 1, & \text{if } d(x,y) > 1. \end{cases}$ *Prove that*

(i) d^* is a metric on X which generates the same topology on X; hence the metrics d and d^* are equivalent.
(ii) If d is a complete metric then d^* is also complete.

As a consequence, every (complete) metric space has an equivalent (complete) metric which is bounded by 1.

Solution. (i) The axioms (MS1) and (MS2), evidently, hold for the function d^*. Take any points $x, y, z \in X$. If $d(x, y) < 1$ and $d(y, z) < 1$ then it follows grom $d^*(x, z) \leqslant d(x, z) \leqslant d(x, y) + d(y, z) = d^*(x, y) + d^*(y, z)$ that the triangle inequality holds for this case. Now, if $d(x, y) \geqslant 1$ or $d(y, z) \geqslant 1$ then $d^*(x, y) = 1$ or $d^*(y, z) = 1$ and therefore $d^*(x, z) \leqslant 1 \leqslant d^*(x, y) + d^*(y, z)$ and hence d^* is a metric. To see that d^* is equivalent to d, observe that, for any $r \in (0, 1)$ the balls $B_d(x, r)$ and $B_{d^*}(x, r)$ coincide for each $x \in X$. Another observation is that any $U \in \tau(d)$ is a union of d-balls of radius <1 and hence each one of these balls belongs to $\tau(d^*)$ whence $U \in \tau(d^*)$. Analogously, any $U \in \tau(d^*)$ is a union of d^*-balls of radius <1 and hence each one of these balls belongs to $\tau(d)$ whence $U \in \tau(d)$. This shows that $\tau(d) = \tau(d^*)$, i.e., the metrics d and d^* are equivalent.

(ii) Suppose that d is a complete metric and take an arbitrary d^*-Cauchy sequence $\{x_n\} \subset X$. Given $\varepsilon > 0$, take any $\delta > 0$ with $\delta < \min\{\varepsilon, 1\}$. There exists a number $m \in \mathbb{N}$ such that $d^*(x_n, x_k) < \delta$ for all $n, k \geqslant m$. Since $\delta < 1$, we have $d(x_n, x_k) = d^*(x_n, x_k) < \delta < \varepsilon$ which shows that the sequence $\{x_n\}$ is also Cauchy with respect to the metric d. The metric d being complete, the sequence $\{x_n\}$ converges to some point x. But we proved in (i) that $\tau(d) = \tau(d^*)$ and the convergence is a topological notion so the convergence of $\{x_n\}$ proves that d^* is also complete.

S.207. *Let (X_n, d_n) be a (complete) metric space such that $d_n(x, y) \leqslant 1$ for all $n \in \mathbb{N}$ and $x, y \in X_n$. For arbitrary points $x, y \in X = \prod\{X_n : n \in \mathbb{N}\}$, consider the function $d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \cdot d_n(x(n), y(n))$. Prove that d is a (complete) metric on X which generates the product topology on X.*

Solution. Since $0 \leqslant d_n(x(n), y(n)) \leqslant 1$ for all $n \in \mathbb{N}$, the series in the definition of $d(x, y)$ converges being bounded by the convergent series $\sum_{i=1}^{\infty} 2^{-i}$. This shows that the function d is well defined. To check that d is a metric, note that $d(x, y) \geqslant 0$ because all terms of the respective series are non-negative. If $d(x, y) = 0$ then $d_n(x(n), y(n)) = 0$ and hence $x(n) = y(n)$ for each $n \in \mathbb{N}$. This means, of course, that $x = y$ so (MS1) holds. It is evident that the axiom of symmetry also holds for d. Now, if $x, y, z \in X$ then $2^{-n}\, d_n(x(n), z(n)) \leqslant 2^{-n}\, d_n(x(n), y(n)) + 2^{-n}\, d_n(y(n), z(n))$ by the triangle inequality for each d_n. Summing the respective series we obtain the triangle inequality for d so d is a metric.

Denote by τ the product topology on X. Take any $U \in \tau$. Let us prove that $U \in \tau(d)$, i.e., for any $x \in U$ there is $r > 0$ with $B_d(x, r) \subset U$. There exist $m \in \mathbb{N}$ and $U_i \in \tau(X_i)$, $i \leqslant n$ such that $x \in W = U_1 \times \cdots \times U_m \times \prod\{X_n : n > m\} \subset U$. Since $\tau(X_i) = \tau(d_i)$ for all $i \leqslant n$, there is $s > 0$ such that $B_{d_i}(x(i), s) \subset U_i$ for every $i \leqslant n$. Let $r = s \cdot 2^{-n}$ and take any $y \in B_d(x, r)$. For any $i \leqslant n$ we have $d_i(x(i), y(i)) \leqslant 2^i\, d(x, y) \leqslant 2^n\, d(x, y) < 2^n \cdot r = s$. As a consequence $y(i) \in B_{d_i}(x(i), s) \subset U_i$ for all $i \leqslant n$. Thus $y \in U_1 \times \cdots \times U_m \times \prod\{X_n : n > m\} = W$ and therefore $B_d(x, r) \subset W \subset U$ so $U \in \tau(d)$ and we proved that $\tau \subset \tau(d)$.

Suppose now that $x \in U \in \tau(d)$. Fix $r > 0$ such that $B_d(x, r) \subset U$ and choose $k \in \mathbb{N}$ with $2^{-k} < \frac{r}{2}$. The set $V_i = B_{d_i}(x(i), \frac{r}{2})$ is open in X_i for each $i \leqslant k$ and therefore $V_x = V_1 \times \cdots \times V_k \times \prod\{X_i : i > k\} \in \tau$. Given $y \in V_x$, we have $d_i(x(i), y(i)) < \frac{r}{2}$ for all $i \leqslant k$. As a consequence,

$$
\begin{aligned}
d(x,y) &= \sum_{i=1}^{k} 2^{-i} d_i(x(i), y(i)) + \sum_{i=k+1}^{\infty} 2^{-i} d_i(x(i), y(i)) \\
&< \tfrac{r}{2}\Big(\sum_{i=1}^{k} 2^{-i}\Big) + \sum_{i=k+1}^{\infty} 2^{-i} = \tfrac{r}{2}(1 - 2^{-k}) + 2^{-k} < \tfrac{r}{2} + \tfrac{r}{2} = r,
\end{aligned}
$$

which proves that $y \in B_d(x, r) \subset U$. It turns out that, for any $x \in U$ there is $V_x \in \tau$ such that $x \in V_x \subset U$. Therefore $U = \bigcup\{V_x : x \in U\}$ belongs to τ being a union of τ-open sets. This shows that $\tau = \tau(d)$, i.e., the metric d generates the topology of the product on X.

Suppose finally that each metric d_n is complete and take a Cauchy sequence $\{x_n\} \subset X$. Given $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $d(x_i, x_j) < \frac{\varepsilon}{2^k}$ for all $i, j \geqslant m$. Therefore $d_k(x_i(k), x_j(k)) \leqslant 2^k \cdot d(x_i, x_j) < \varepsilon$ which shows that the sequence $\{x_i(k)\}_{i \in \mathbb{N}}$ is fundamental for every $k \in \mathbb{N}$. The space (X_k, d_k) being complete for each k, we can find $w_k \in X_k$ such that $x_i(k) \to w_k$ when $i \to \infty$. Letting $x(k) = w_k$ for all $k \in \mathbb{N}$ we obtain a point $x \in X$. To prove that $x_n \to x$ take any $U \in \tau(x, X)$. There is $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$. Fix $k \in \mathbb{N}$ with $2^{-k} < \frac{\varepsilon}{2}$. Since $x_n(i) \to x(i)$ for each

$i \leqslant k$, there is $m \in \mathbb{N}$ such that $d_i(x_n(i), x(i)) < \frac{\varepsilon}{2}$ whenever $n \geqslant m$ and $i \leqslant k$. As a consequence,

$$d(x_n, y) = \sum\nolimits_{i=1}^{k} 2^{-i} d_i(x_n(i), x(i)) + \sum\nolimits_{i=k+1}^{\infty} 2^{-i} d_i(x_n(i), x(i))$$
$$< \tfrac{\varepsilon}{2}\Big(\sum\nolimits_{i=1}^{k} 2^{-i}\Big) + \sum\nolimits_{i=k+1}^{\infty} 2^{-i} = \tfrac{\varepsilon}{2}(1 - 2^{-k}) + 2^{-k} < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

which proves that $x_n \in B_d(x, \varepsilon) \subset U$ for all $n \geqslant m$ and therefore $x_n \to x$.

S.208. *Show that any countable or finite product of (completely) metrizable spaces is a (completely) metrizable space.*

Solution. Let (X_n, ρ_n) be a metric space for all $n \in \mathbb{N}$. If $d_n = \rho_n^*$ then the topology generated by d_n coincides with $\tau(\rho_n)$ (Problem 206) and $d_n(x, y) \leqslant 1$ for all $x, y \in X_n$. If ρ_n is a complete metric then d_n is complete too: this was also proved in Problem 206. It follows from Problem 207 that the topology of $X = \prod\{X_n : n \in \mathbb{N}\}$ is generated by the metric d introduced in Problem 207 which is complete if all d_n's are complete. Thus X is metrizable (by a complete metric if all metrics ρ_n's are complete). To see that the same is true for finite products, observe that any finite product $X_1 \times \cdots \times X_n$ is homeomorphic to the countable product $\prod\{X_i : i \in \mathbb{N}\}$ where X_i is a one-point space for all $i > n$. Since any one-point space is metrizable by a complete metric (Problem 204), the product $X_1 \times \cdots \times X_n$ is metrizable (by a complete metric if all metrics $d_1, \ldots, d_n$ are complete).

S.209. *Prove that the following conditions are equivalent for any infinite space X and an infinite cardinal κ:*

(i) $w(X) \leqslant \kappa$.
(ii) *X embeds in $\mathbb{I}^\kappa$.*
(iii) *X embeds in $\mathbb{R}^\kappa$.*

Deduce from these equivalencies that any second countable space is metrizable.

Solution. It is clear that $\mathbb{R}^\kappa$ is homeomorphic to $C_p(D(\kappa))$. Apply Problem 169 to conclude that $w(\mathbb{R}^\kappa) = |D(\kappa)| = \kappa$. Now apply Problem 159(i) to see that any subspace of $\mathbb{R}^\kappa$ has weight $\leqslant \kappa$. This proves (iii) $\Rightarrow$ (i). Since $\mathbb{I} \subset \mathbb{R}$, the space $\mathbb{I}^\kappa$ embeds in $\mathbb{R}^\kappa$. This settles (ii) $\Rightarrow$ (iii).

To prove that (i) $\Rightarrow$ (ii) fix a base $\mathcal{B}$ in X of cardinality $\leqslant \kappa$. Call a pair $p = (U, V) \in \mathcal{B} \times \mathcal{B}$ *special* if there exists $h_p \in C(X, [0, 1])$ such that $h_p(U) \subset \{0\}$ and $h_p(X \setminus V) \subset \{1\}$. Denote by A the set of all special pairs. Then $|A| \leqslant |\mathcal{B} \times \mathcal{B}| \leqslant \kappa$. For an arbitrary $x \in X$, let $\varphi(x)(p) = h_p(x)$ for any $p \in A$. Then $\varphi(x) \in \mathbb{I}^A$. We will prove that $\varphi : X \to Y = \varphi(X) \subset \mathbb{I}^A$ is a homeomorphism. For any $p \in A$, denote by π_p the pth projection of $\mathbb{I}^A$ onto $\mathbb{I}$. Recall that $\pi_p(f) = f(p)$ for any $f \in \mathbb{I}^A$. Note that φ is continuous because, for any $p \in A$, we have $\pi_p \circ \varphi = h_p$ and the map h_p is continuous (see Problem 102).

If $x \neq y$ then take any set $V \in \mathcal{B}$ such that $x \in V \subset X \setminus \{y\}$. There exists a function $f \in C(X, [0, 1])$ such that $f(x) = 0$ and $f(X \setminus V) = \{1\}$. Choose any $U \in \mathcal{B}$ such that $x \in U \subset f^{-1}([0, \frac{1}{2}))$. Let $r(t) = 0$ if $t \in [0, \frac{1}{2}]$ and $r(t) = 2t - 1$ for $t \in (\frac{1}{2}, 1]$. Then $r : [0, 1] \to [0, 1]$ is a continuous function and hence $g = r \circ f$ is also continuous. It is straightforward that $g(U) \subset \{0\}$ and $g(X \setminus V) = \{1\}$ so the pair $p = (U, V)$ is special. Therefore $\varphi(x)(p) = h_p(x) = 0$ and $\varphi(y)(p) = h_p(y) = 1$ which proves that $\varphi(x) \neq \varphi(y)$ and φ is a bijection.

To show that φ^{-1} is continuous, take any point $y \in Y$ and let $x = \varphi^{-1}(y)$. Given a set $W \in \tau(x, X)$ there exists $V \in \mathcal{B}$ such that $x \in V \subset W$. There exists a function $f \in C(X, [0, 1])$ such that $f(x) = 0$ and $f(X \setminus V) = \{1\}$. Choose any $U \in \mathcal{B}$ such that $x \in U \subset f^{-1}([0, \frac{1}{2}))$. The function $g = r \circ f : X \to [0, 1]$ is continuous (see the previous paragraph) and it is straightforward that $g(U) \subset \{0\}$ and $g(X \setminus V) = \{1\}$ so the pair $p = (U, V)$ is special. The set $O = \pi_p^{-1}([0, 1)) \cap Y$ is open in Y and $y \in O$ because $\pi_p(y) = h_p(x) = 0$. If $z \in \varphi^{-1}(O)$ then $\varphi(z) \in O$ which implies $h_p(z) < 1$ and therefore $z \in V \subset W$. The point z was chosen arbitrarily, so $\varphi^{-1}(O) \subset W$ and hence φ^{-1} is continuous at the point y.

To finish our solution, observe that $\mathbb{R}^\omega$ is metrizable by Problems 205 and 208. If X has countable weight then it is metrizable because it embeds into the metrizable space $\mathbb{R}^\omega$.

S.210. *Prove that any metrizable space is first countable. As a consequence, $C_p(X)$ is metrizable if and only if X is countable.*

Solution. Let (X, d) be a metric space. Given any $x \in X$ it suffices to prove that the family $\mathcal{B}_x = \{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at the point x. Indeed, if $x \in U \in \tau(X)$ then there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. For any $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$ we have $B(x, \frac{1}{n}) \subset B(x, \varepsilon) \subset U$ and therefore $\mathcal{B}_x$ is a local base at x. If $C_p(X)$ is metrizable then $\chi(C_p(X)) = \omega$ so we can apply Problem 169 to conclude that X has to be countable.

S.211. *Given an arbitrary space X and functions $f, g \in C_u(X)$, let $d(f, g) = 1$ if $|f(x) - g(x)| \geqslant 1$ for some $x \in X$. If $|f(x) - g(x)| < 1$ for all $x \in X$ then let $d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. Prove that d is a metric on $C_u(X)$ which generates the topology of $C_u(X)$. In particular, the space $C_u(X)$ is metrizable for any space X.*

Solution. It is clear that $d(f, g) \geqslant 0$ for all $f, g \in C_u(X)$. If $d(f, g) = 0$ then $|f(x) - g(x)| = 0$ for all $x \in X$ and hence $f = g$ so the first axiom of metric is fulfilled. It is clear from the definition that always $d(f, g) = d(g, f)$ so we only have to check the triangle inequality. Take any $f, g, h \in C_u(X)$. Observe first that $d(f, h) \leqslant 1$ so $d(f, h) \leqslant d(f, g) + d(g, h)$ if $d(f, g) = 1$ or $d(g, h) = 1$. Now if $d(f, g) < 1$ and $d(g, h) < 1$ then, for any $x \in X$, we have

$$|f(x) - h(x)| \leqslant |f(x) - g(x)| + |g(x) - h(x)| \leqslant d(f, g) + d(g, h).$$

Thus $d(f, h) \leqslant \sup\{|f(x) - h(x)| : x \in X\} \leqslant d(f, g) + d(g, h)$ (the second inequality is true because $|f(x) - h(x)| \leqslant d(f, g) + d(g, h)$ for all $x \in X$). This proves that d is a metric on $C_u(X)$.

Given $U \in \tau_u$, to prove that $U \in \tau(d)$ we must show that for any $f \in U$ there exists $r > 0$ such that $B_d(f, r) \subset U$. Suppose not. Then, for any $n \in \mathbb{N}$, we can find a function $f_n \in B_d(f, \frac{1}{n}) \cap (C_u(X) \setminus U)$. We have $|f_n(x) - f(x)| \leqslant d(f, f_n) < \frac{1}{n}$ for each $x \in X$ and hence $f_n \rightrightarrows f$ which implies $f \in \overline{C_u(X)/U}^u$ which is a contradiction. Therefore $\tau_u \subset \tau(d)$.

Assume now that $U \in \tau(d)$. To prove that $U \in \tau_u$, it suffices to show that the set $C_u(X) \setminus U$ is τ_u-closed. Striving for contradiction suppose that it is not closed. Then there is a sequence $\{f_n : n \in \mathbb{N}\} \subset C_u(X) \setminus U$ such that $f_n \rightrightarrows f$ for some $f \in U$. Since U is $\tau(d)$-open, we can find $r \in (0, 1)$ such that $B_d(f, r) \subset U$. Choose any $m \in \mathbb{N}$ with $|f_m(x) - f(x)| < \frac{r}{2}$ for all $x \in X$. It is evident that $d(f_m, f) \leqslant \frac{r}{2} < r$ and hence $f_m \in B_d(f, r) \cap (C_u(X) \setminus U)$, a contradiction. This proves that $\tau_u = \tau(d)$ and our solution is complete.

S.212. *Show that, for any metrizable space X, the following are equivalent:*

(i) X is compact.
(ii) X is countably compact.
(iii) X is pseudocompact.
(iv) There exists a complete and totally bounded metric d on X with $\tau(d) = \tau(X)$.
(v) X embeds as a closed subset into $\mathbb{I}^\omega$.

Solution. It is clear that (i) ⇒(ii) ⇒ (iii).

Fact 1. Take any metric d on X with $\tau(d) = \tau(X)$. Given any set $A \subset X$, let $d_A(x) = \inf\{d(x, a) : a \in A\}$. Then the map $d_A : X \to \mathbb{R}$ is continuous.

Proof. For any $x_0 \in X$ and $\varepsilon > 0$, let $U = B(x_0, \frac{\varepsilon}{2})$. It is sufficient to show that $d_A(U) \subset (r_0 - \varepsilon, r_0 + \varepsilon)$ where $r_0 = d_A(x_0)$. To do so, take any $x \in U$. The infimum condition in the definition of d_A implies the existence of a point $y \in A$ such that $d(x_0, y) < d_A(x_0) + \frac{\varepsilon}{2}$. Then

$$d(x, y) \leqslant d(x, x_0) + d(x_0, y) < \tfrac{\varepsilon}{2} + d_A(x_0) + \tfrac{\varepsilon}{2} = d_A(x_0) + \varepsilon.$$

Therefore $d_A(x) \leqslant d(x, y) < r_0 + \varepsilon$. To prove that $d_A(x) > r_0 - \varepsilon$ suppose not. Then $d_A(x) < r_0 - \frac{\varepsilon}{2}$ and hence we can find $z \in A$ such that $d(x, z) < r_0 - \frac{\varepsilon}{2}$. Now, $d(x_0, z) \leqslant d(x_0, x) + d(x, z) < \frac{\varepsilon}{2} + r_0 - \frac{\varepsilon}{2} = r_0$ and, as a consequence, $d_A(x_0) \leqslant d(x, z) < r_0$ which is a contradiction. Thus $d_A(x) \in (r_0 - \varepsilon, r_0 + \varepsilon)$ and Fact 1 is proved.

Fact 2. Every metrizable space is normal.

Proof. Take any closed disjoint non-empty $F, G \subset X$. Fact 1 implies that the function $\varphi = d_F - d_G : X \to \mathbb{R}$ is continuous. Letting $U = \varphi^{-1}((-\infty, 0))$ and $V = \varphi^{-1}((0, +\infty))$, we obtain open disjoint sets U, V such that $F \subset U$ and $G \subset V$.

Returning to our solution, note that (iii) ⇒(ii) because X is normal (Fact 2) and every normal pseudocompact space is countably compact (Problem 137).

Next we will prove that (ii) ⇒ (iv). Suppose that X is countably compact and take any metric d on X which generates the topology of X. If d is not totally bounded then there is $\varepsilon > 0$ such that $\bigcup\{B(x, \varepsilon) : x \in A\} \neq X$ for any finite $A \subset X$. This makes it possible to construct by induction the set $S = \{x_n : n \in \omega\} \subset X$ such that $x_{n+1} \notin \bigcup\{B(x_i, \varepsilon) : i \leqslant n\}$ for each $n \in \omega$. As a consequence, $d(x_i, x_j) \geqslant \varepsilon$ whenever $i \neq j$. For any $x \in X$ the set $B(x, \frac{\varepsilon}{2})$ can contain at most one point of the set S. Indeed, if x_i, $x_j \in B(x,\frac{\varepsilon}{2})$ then $d(x_i, x_j) \leqslant d(x_i, x) + d(x, x_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ which is a contradiction when $i \neq j$. This implies that S is closed in X. Since $\{x_i\} = B(x_i, \varepsilon) \cap S$, the set S is also discrete which is a contradiction with countable compactness of X (Problem 132(ii)). This contradiction proves that the metric d is totally bounded.

To see that d is complete, take any fundamental sequence $\{x_n : n \in \omega\} \subset X$. The sets $F_n = \overline{\{x_i : i \geqslant n\}}$ are closed, non-empty and $F_{n+1} \subset F_n$ for any $n \in \omega$. By countable compactness of X, there is $x \in \bigcap\{F_n : n \in \omega\}$ (Problem 132(iv)). Fix $r > 0$ and choose $m \in \omega$ such that $d(x_i, x_j) < \frac{r}{2}$ for all $i, j \geqslant m$. Since $x \in F_m$, we have $B(x, \frac{r}{2}) \cap \{x_i : i \geqslant m\} \neq \emptyset$. Pick any $k \geqslant m$ with $x_k \in B(x, \frac{r}{2})$. If $n \geqslant k$ then $d(x_n, x_k) < \frac{r}{2}$ and therefore $d(x, x_n) \leqslant d(x, x_k) + d(x_k, x_n) < \frac{r}{2} + \frac{r}{2} = r$ which shows that $x_n \in B(x, r)$ for all $n \geqslant k$. Thus $x_n \to x$ and the proof of the implication (ii) $\Rightarrow$ (iv) is complete.

Now we take to the proof of (iv) $\Rightarrow$ (i).

Fact 3. Any totally bounded metric space Z is second countable.

Proof. Fix a totally bounded metric ρ on Z with $\tau(\rho) = \tau(Z)$. For each $n \in \mathbb{N}$ we can find a finite set $A_n \subset Z$ such that $\bigcup\{B(a, \frac{1}{n}) : a \in A_n\} = Z$. The set $A = \bigcup\{A_n : n \in \mathbb{N}\}$ is countable and dense in Z. Indeed, if $z \in Z$ and $\varepsilon > 0$ then take any $n > \frac{1}{\varepsilon}$ and $a \in A_n$ with $z \in B(a,\frac{1}{n})$. Then $d(z, a) < \frac{1}{n} < \varepsilon$ and hence $B(z, \varepsilon) \cap A \neq \emptyset$. As a consequence, $U \cap A \neq \emptyset$ for any $U \in \tau(z, Z)$ and therefore $z \in \overline{A}$. The point $z \in Z$ was taken arbitrarily so $\overline{A} = Z$.

We claim that the family $\mathcal{B} = \{B(a, r) : a \in A, r \in \mathbb{Q} \cap (0, +\infty)\}$ is a base of Z. To see this, take any $z \in Z$ and $U \in \tau(z, Z)$. There is $\varepsilon > 0$ with $B(z, \varepsilon) \subset U$ and a point $a \in B(z, \frac{\varepsilon}{2}) \cap A$. Take any rational $r \in (d(a, z), \frac{\varepsilon}{2})$ and observe that $z \in V = B(a, r) \in \mathcal{B}$. If $w \in V$ then $d(w, z) \leqslant d(w, a) + d(a, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and therefore $w \in B(z, \varepsilon)$ which implies $V \subset B(z, \varepsilon) \subset U$. This proves that $\mathcal{B}$ is a base of Z. Since $\mathcal{B}$ is countable, Fact 3 is proved.

Returning to the proof of (iv) $\Rightarrow$ (i), observe that X is second countable by Fact 3 and hence Lindelöf by Observation one of S.140. By Problem 138 it suffices to show that X is countably compact.

Assume that X is not countably compact and fix an infinite closed discrete $D \subset X$.

Fact 4. The set D can contain no non-trivial convergent sequence.

Proof. If $S = \{x_n\} \subset D$ and $x_n \to x$ then $T = S \setminus \{x\}$ is a non-closed (in X) subset of D. But D is closed and discrete which implies that T is closed in D and hence in X, a contradiction.

Fact 5. For every infinite $F \subset D$ and any $\varepsilon > 0$ there exists an infinite $G \subset F$ such that $\text{diam}(G) < \varepsilon$.

Proof. By total boundedness of the space X, there exists a finite set $A \subset X$ such that $X = \bigcup\{B(a, \frac{\varepsilon}{3}) : a \in A\}$. Since $F \subset \bigcup\{B(a, \frac{\varepsilon}{3}) : a \in A\}$, there is $a \in A$ such that $G = B(a, \frac{\varepsilon}{3}) \cap F$ is infinite. Given any points $x, y \in B(a, \frac{\varepsilon}{3})$ we have the inequalities $d(x, y) \leqslant d(x, a) + d(a, y) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$ which imply that $\mathrm{diam}(G) \leqslant \mathrm{diam}(B(a, \frac{\varepsilon}{3})) \leqslant \frac{2\varepsilon}{3} < \varepsilon$ so Fact 4 is proved.

Using Fact 5 it is easy to construct by induction a sequence $\{D_i : i \in \omega\}$ of infinite sets such that $D = D_0 \supset D_1 \supset \cdots \supset D_n \supset \ldots$, and $\mathrm{diam}(D_n) < \frac{1}{n}$ for each $n \in \mathbb{N}$. Pick an arbitrary $d_0 \in D_0$. If we have distinct points $d_i \in D_i$ for all $i \leqslant n$, choose any $d_{n+1} \in D_{n+1}$ $\{d_0, \ldots, d_n\}$. This choice is possible because D_{n+1} is infinite. The sequence $S = \{d_i\}$ is non-trivial and contained in D. Given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. If $n, k \geqslant m$ then $x_n, x_k \in D_m$ and therefore $d(x_n, x_k) \leqslant \mathrm{diam}(D_m) < \frac{1}{m} < \varepsilon$. As a consequence, the sequence S is fundamental and hence convergent because the metric d is complete. The contradiction with Fact 4 finishes the proof of (iv) $\Rightarrow$ (i).

Now that we proved that all properties (i)–(iv) are equivalent, let us show that (iv) $\Rightarrow$ (v). Assume that (iv) holds for X. By Fact 3 X is second countable and compact by (iv) $\Rightarrow$ (i). Now apply Problem 126 to conclude that X is homeomorphic to a closed subspace of $\mathbb{I}^A$ for some countable A. Of course, $\mathbb{I}^A$ is homeomorphic to $\mathbb{I}^\omega$ and therefore (iv) $\Rightarrow$ (v) is established. Finally, observe that any closed subspace of $\mathbb{I}^\omega$ is compact by Problems 131, 125 and 120. Thus (v) $\Rightarrow$ (i) and our solution is complete.

S.213. *Let X be a compact space. Prove that X is metrizable if and only if $C_p(X)$ is separable.*

Solution. If X is metrizable then it is second countable by Problem 212. Applying Problem 174, we can see that $d(C_p(X)) = iw(X) \leqslant w(x) = \omega$ and hence the space $C_p(X)$ is separable. Now if $C_p(X)$ is separable then $iw(X) = d(C_p(X)) \leqslant \omega$ by Problem 174. Thus there is a condensation $f : X \to Y$ onto a second countable space Y. Apply Problem 123 to conclude that f is a homeomorphism and hence $w(X) \leqslant \omega$. Now Problem 209 implies that X is metrizable.

S.214. *Prove that* $\mathrm{ext}(X) = s(X) = \boldsymbol{c}(X) = d(X) = nw(X) = w(X) = l(X)$ *for any metrizable space X. Hence, for a metrizable space X being Lindelöf or separable or having the Souslin property, is equivalent to X being second countable.*

Solution. Fix a metric d on X with $\tau(d) = \tau(X)$. Suppose that $\mathrm{ext}(X) \leqslant \kappa$. Given a discrete $D \subset X$, let $F = \overline{D} \setminus D$. It is an easy exercise that F is closed in X. For any $x \in X$ let $d_F(x) = \inf\{d(x, y) : y \in F\}$. The function $d_F : X \to \mathbb{R}$ is continuous (see Fact 1 of S.212) and $x \in F$ if and only if $d_F(x) = 0$. As a consequence, $D = \bigcup\{D_n : n \in \mathbb{N}\}$ where $D_n = \{d \in D : d_F(d) \geqslant \frac{1}{n}\}$. Since the set $\overline{D} \setminus D_n = d_F^{-1}((-\frac{1}{n}, \frac{1}{n})) \cap \overline{D}$ is open in $\overline{D}$, the set D_n is closed in $\overline{D}$ and hence in X. Being the set D_n closed and discrete, we have $|D_n| \leqslant \kappa$ for each $n \in \mathbb{N}$ and therefore $|D| \leqslant \omega \cdot \kappa = \kappa$. This proves that

(1) $s(X) \leqslant \mathrm{ext}(X)$.

Suppose that $c(X) \leqslant \kappa$. For each $n \in \mathbb{N}$, let $\mathcal{B}_n = \{B(x, \frac{1}{n}) : x \in X\}$ and choose a maximal disjoint subfamily γ_n of the family $\mathcal{B}_n$. Given $U = B(x,\frac{1}{n}) \in \gamma_n$, let $a(U) = x$ and $A_n = \{a(U) : U \in \gamma_n\}$. We claim that the set $A = \bigcup\{A_n : n \in \omega\}$ is dense in X. Indeed, if $z \in W \in \tau(z, X)$ take $r > 0$ with $B(z, r) \subset U$ and $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{r}{2}$. Since $B(z, \frac{1}{n}) \in \mathcal{B}_n$, there is $V \in \gamma_n$ with $V \cap B(z, \frac{1}{n}) \neq \emptyset$ because γ_n is a maximal disjoint subfamily of $\mathcal{B}_n$. Now $V = B(y, \frac{1}{n})$ and there is $w \in B(y, \frac{1}{n}) \cap B(z, \frac{1}{n})$. Consequently, $d(z, y) \leqslant d(z, w) + d(w, y) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} < r$. Thus $y = a(V) \in A \cap U$ which proves that $z \in \overline{A}$. Since the point $z \in X$ was chosen arbitrarily, we have $\overline{A} = X$. Observe that the map $U \to a(U)$ is a surjection of γ_n onto A_n. Since γ_n is disjoint, we have $|A_n| \leqslant |\gamma_n| \leqslant \kappa$ for each $n \in \mathbb{N}$. Therefore $|A| \leqslant \omega \cdot \kappa = \kappa$ which shows that $d(X) \leqslant \kappa$ and hence we have

(2) $d(X) \leqslant c(X)$.

Now assume that $d(X) \leqslant \kappa$. Fix a dense $A \subset X$ with $|A| \leqslant \kappa$ and let $\mathcal{B} = \{B(x, r) : x \in A, r \in \mathbb{Q} \cap (0, +\infty)\}$. It is immediate that $|\mathcal{B}| \leqslant \kappa$. Let us prove that the family $\mathcal{B}$ is a base in the space X. To see this, take any $x \in X$ and $U \in \tau(x, X)$. There is $\varepsilon > 0$ with $B(x, \varepsilon) \subset U$ and a point $a \in B(x, \frac{\varepsilon}{2}) \cap A$. Take any rational $r \in (d(a, x), \frac{\varepsilon}{2})$ and observe that $x \in V = B(a, r) \in \mathcal{B}$. If $w \in V$ then $d(w, x) \leqslant d(w, a) + d(a, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and therefore $w \in B(x, \varepsilon)$ which implies $V \subset B(x, \varepsilon) \subset U$. This proves that $\mathcal{B}$ is a base of Z and we have

(3) $w(X) \leqslant d(X)$.

Recall that we have the inequalities $\text{ext}(X) \leqslant l(X) \leqslant nw(X) \leqslant w(X)$ and $\boldsymbol{c}(X) \leqslant s(X)$ for any space X (see Problem 156). Applying the properties (1)–(3) we convince ourselves that $w(X) \leqslant d(X) \leqslant \boldsymbol{c}(X) \leqslant s(X) \leqslant \text{ext}(X) \leqslant w(X)$ which shows that $\text{ext}(X) = l(X) = nw(X) = w(X) = d(X) = \boldsymbol{c}(X) = s(X)$ and this is what we had to prove.

S.215. *Let X be a metrizable space. Prove that the following properties are equivalent:*

(i) *$C_p(X)$ is Lindelöf.*
(ii) *$C_p(X)$ is normal.*
(iii) *The extent of $C_p(X)$ is countable.*
(iv) *All compact subspaces of $C_p(X)$ are metrizable.*
(v) *X is second countable.*

Solution. Suppose that the space X is second countable. By Problem 172 we have $nw(C_p(X)) = nw(X) \leqslant w(X) = \omega$. Apply Problem 156(ii) to verify that $C_p(X)$ is Lindelöf and hence $\text{ext}(C_p(X)) = \omega$. By Problem 124 the space $C_p(X)$ is normal. If K is a compact subset of $C_p(X)$ then $iw(K) \leqslant nw(K) \leqslant nw(C_p(X)) = \omega$ and hence K condenses onto a second countable space. But every condensation of K is a homeomorphism (Problem 123) so X has countable weight. Applying Problem 209 we conclude that K is metrizable. This shows that (v) $\Rightarrow$ (i) $\Rightarrow$ (ii), (v) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (iv). Fix a metric d on X with $\tau(d) = \tau(X)$.

Fact 1. Let (X, d) be a metric space. If $\text{ext}(X) > \omega$ then $\mathbb{R}^{\omega_1}$ embeds in $C_p(X)$ as a closed subspace.

Proof. Take a closed discrete $A \subset X$ with $|A| = \omega_1$. For any $a \in A$ there is $r = r(a) > 0$ such that $B(a, r) \cap A = \{a\}$. Then $A = \bigcup\{A_n : n \in \mathbb{N}\}$ where $A_n = \{a \in A : r(a) > \frac{1}{n}\}$. As a consequence, $|A_n| = \omega_1$ for some $n \in \mathbb{N}$. It is clear that $d(x, y) > \frac{1}{n}$ for distinct x, $y \in A_n$. Given $a \in A_n$, let $U_a = B(a, \frac{1}{5n})$. We claim that the family $\gamma = \{U_a : a \in A_n\}$ is discrete. Indeed, if $x \in X$ then $U_x = B(x, \frac{1}{5n})$ cannot intersect more than one element of the family γ. To see this, take any points $a, b \in A_n$, $a \neq b$ and suppose $y \in U_x \cap U_a$, $z \in U_x \cap U_b$. Then $d(a, b) \leqslant d(a, y) + d(y, x) + d(x, z) + d(z, b) < \frac{4}{5n} < \frac{1}{n}$, a contradiction which shows that γ is discrete.

For any $a \in D = A_n$, fix a continuous function $f_a : X \to [0, 1]$ such that $f_a(a) = 1$ and $f_a(X \setminus U_a) \subset \{0\}$. Given an arbitrary function $f \in \mathbb{R}^D$ and $x \in X$, let $\varphi(f)(x) = \sum\{f(a) \cdot f_a(x) : a \in D\}$. Let us check that $\varphi(f)$ is a continuous function for each $f \in \mathbb{R}^D$. Given $x \in X$, the open set U_x contains x and intersects at most one element of γ, say U_a. Then $\varphi(f)|U_x = (f(a) \cdot f_a) \mid U_x$ is a continuous function on U_x and this easily implies continuity of $\varphi(f)$ at the point x. We claim that $\varphi : \mathbb{R}^D \to C_p(X)$ is a continuous map. Since $C_p(X) \subset \mathbb{R}^X$, it suffices to show that $e_x \circ \varphi$ is continuous for any $x \in X$. Here $e_x(f) = f(x)$ for any $f \in C_p(X)$, i.e., e_x is the projection onto the factor determined by x. We saw already that there is at most one $a \in D$ with $x \in U_a$ and hence $\varphi(f)(x) = f(a) \cdot f_a(x)$. If $p_a : \mathbb{R}^D \to \mathbb{R}$ is the projection onto the factor determined by a then $e_x \circ\varphi = f_a(x) \cdot p_a$ is continuous being a product of the continuous map p_a and a constant $f_a(x)$. This proves that φ is a continuous map. It is immediate that $\pi_D(\varphi(f)) = f$ for any $f \in \mathbb{R}^D$, where $\pi_D : C_p(X) \to C_p(D) = \mathbb{R}^D$ is the restriction map. Therefore φ is an embedding.

Let us prove that $E = \varphi(\mathbb{R}^D)$ is closed in $C_p(X)$. Take any $f \in C_p(X)\setminus E$. Then $g = \varphi(\pi_D(f)) \in E$ and there are open disjoint sets $U, V \subset C_p(X)$ such that $f \in U$, $g \in V$ and $\varphi(\pi_D(U)) \subset V$. If $h \in U \cap E$ then $h = \varphi(\pi_D(h)) \in \varphi(\pi_D(U)) \subset V$ which is a contradiction with $U \cap V = \emptyset$. Hence $U \cap E = \emptyset$ and E is closed in $C_p(X)$. Since E is homeomorphic to $\mathbb{R}^D$ which in turn is homeomorphic to $\mathbb{R}^{\omega_1}$, Fact 1 is proved.

Fact 2. The spaces $\mathbb{N}^{\omega_1}$ and $\mathbb{R}^{\omega_1}$ are not normal.

Proof. Since $\mathbb{N}^{\omega_1}$ is a closed subspace of $\mathbb{R}^{\omega_1}$, it suffices to prove that $\mathbb{N}^{\omega_1}$ is not normal. Let $F = \{x \in \mathbb{N}^{\omega_1} : x^{-1}(i)$ has at most one element for each $i \neq 1\}$ and $G = \{x \in \mathbb{N}^{\omega_1} : x^{-1}(i)$ has at most one element for each $i \neq 2\}$. The sets F and G are closed; let us establish this for F, the proof for G is identical. If $x \in \mathbb{N}^{\omega_1}\setminus F$ then there are distinct $\alpha, \beta \in \omega_1$ such that $x(\alpha) = x(\beta) = i \neq 1$. Then the set $U_x = \{y \in \mathbb{N}^{\omega_1} : y(\alpha) = y(\beta) = i\}$ is open in $\mathbb{N}^{\omega_1}$, $x \in U_x$ and $U_x \cap F = \emptyset$.

The sets F and G are disjoint, for if $x \in F \cap G$ then $x \in F$ and hence there are distinct $\alpha, \beta \in \omega_1$ such that $x(\alpha) = x(\beta) = 1$ which shows that $x \notin G$, a contradiction. Let us show that F and G cannot be separated by disjoint open sets, i.e., there exist no $U, V \in \tau(\mathbb{N}^{\omega_1})$ such that $F \subset U$, $G \subset V$ and $U \cap V = \emptyset$. Assume, on the contrary, that such U and V exist. Call a set $W \subset \mathbb{N}^{\omega_1}$ *standard* if there exist $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \omega_1$ and $i_1, \ldots, i_n \in \mathbb{N}$ such that $W = [\alpha_1, \ldots, \alpha_n; i_1, \ldots, i_n] = \{x \in \mathbb{N}^{\omega_1} : x(\alpha_k) = i_k$ for all $k \leqslant n\}$. It is clear that standard sets constitute a base of the space $\mathbb{N}^{\omega_1}$.

Let γ be a maximal disjoint family of standard sets contained in U. It is evident that $\overline{\bigcup\gamma} = \overline{U}$. Since $c(\mathbb{N}^{\omega_1}) = \omega$ (Problem 109), the family γ is countable. If $W = [\alpha_1, \dots, \alpha_n; i_1, \dots, i_n] \in \gamma$, let $\mathrm{supp}(W) = \{\alpha_1, \dots, \alpha_n\}$. The set $A = \bigcup\{\mathrm{supp}(W) : W \in \gamma\}$ is countable. Denote by $p_A : \mathbb{N}^{\omega_1} \to \mathbb{N}^A$ the natural projection defined by $p_A(x) = x|A$ for all $x \in \mathbb{N}^{\omega_1}$.

We claim that $p_A^{-1}(p_A(\overline{U})) = \overline{U}$. It is evident that the only non-trivial inclusion is $p_A^{-1}(p_A(\overline{U})) \subset \overline{U}$, so take any point $x \in \mathbb{N}^{\omega_1}$ such that $p_A(x) \in p_A(\overline{U})$. Fix any standard set $W = [\alpha_1, \dots, \alpha_n; i_1, \dots, i_n] \ni x$. Pick some $y \in \overline{U}$ such that $p_A(y) = p_A(x)$. Without loss of generality we may assume that $\alpha_1, \dots, \alpha_k \in A$ and $\alpha_{k+1}, \dots, \alpha_n \in \omega_1 A$ for some $k \in \{0, \dots, n\}$. The set $O = [\alpha_1, \dots, \alpha_k; i_1, \dots, i_k]$ is an open neighbourhood of $y \in \overline{U} = \overline{\bigcup\gamma}$ and hence there is $H \in \gamma$ such that $O \cap H \neq \emptyset$, i.e., there exists $z \in H$ with $z(\alpha_m) = i_m$ for all $m \leqslant k$. Define $z' \in \mathbb{N}^{\omega_1}$ as follows: $z' \mid A = z$, $z'(\alpha_m) = i_m$ for all $m = k+1, \dots, n$ and $z'(\alpha) = 1$ for all $\alpha \in \omega_1 \setminus (A \cup \mathrm{supp}(W))$. Observe now that $\mathrm{supp}(H) \subset A$ implies $p_A^{-1}(p_A(H)) = H$ and therefore $z' \in H$. It follows from $z \mid A = z' \mid A$ that $z'(\alpha_m) = i_m$ for all $m \leqslant k$ as well and therefore $z' \in H \cap W \subset W \cap U$ which proves that $x \in \overline{U}$ and hence the proof of the equality $p_A^{-1}(p_A(\overline{U})) = \overline{U}$ is complete.

Let $\{\alpha_n : n \in \omega\}$ be a faithful enumeration of the set A. Define $x \in \mathbb{N}^{\omega_1}$ as follows: $x(\beta) = 2$ for all $\beta \in \omega_1 A$ and $x(\alpha_n) = n$ for each $n \in \omega$. It is clear that $x \in G$. Now, let $y(\beta) = 1$ for all $\beta \in \omega_1 A$ and $y(\alpha_n) = n$ for each $n \in \omega$. We have $y \in F \subset U$. Therefore $y \in \overline{U}$ and $p_A(y) = p_A(x)$. It follows from the equality $p_A^{-1}(p_A(\overline{U})) = \overline{U}$ that $x \in \overline{U}$. However, $x \in G \subset V$ which implies $U \cap V \neq \emptyset$, a contradiction. This contradiction shows that F and G cannot be separated by disjoint open set so $\mathbb{N}^{\omega_1}$ is not normal and Fact 2 is proved.

Fact 3. We have $\mathrm{ext}(\mathbb{N}^{\omega_1}) = \mathrm{ext}(\mathbb{R}^{\omega_1}) = \omega_1$.

Proof. Observe that the space $\mathbb{R}^{\omega_1}$ is homeomorphic to $C_p(D(\omega_1))$ and hence $w(\mathbb{R}^{\omega_1}) = |D(\omega_1)| = \omega_1$ by Problem 169. Since $\mathbb{N}^{\omega_1}$ is a closed subspace of $\mathbb{R}^{\omega_1}$, it suffices to prove that $\mathrm{ext}(\mathbb{N}^{\omega_1}) \geqslant \omega_1$.

Given an ordinal $\xi \in \omega_1 \setminus \omega$, the space N_ξ is the set $\{\alpha : \alpha < \xi\}$ endowed with the discrete topology. It is clear that $\mathbb{N}^{\omega_1}$ is homeomorphic to the space $N = \prod\{N_\xi : \omega \leqslant \xi < \omega_1\}$. For each $\xi \in \omega_1 \setminus \omega$, fix an injection $f_\xi : N_{\xi+1} \to \omega$. For every $\xi \in \omega_1 \setminus \omega$, define $d_\xi \in N$ as follows: $d_\xi(\alpha) = f_\xi(\alpha)$ for every $\alpha \in (\xi+1) \setminus \omega$ and $d_\xi(\alpha) = \xi$ for all $\alpha \in \omega_1 \setminus (\xi+1)$. We claim that the set $D = \{d_\xi : \xi \in \omega_1 \setminus \omega\}$ is closed and discrete in N. To prove this, observe first that it suffices to show that, for any $x \in N$ there is $U \in \tau(x, N)$ such that U contains at most one element of D. So, fix $x \in N$. We have $x(\alpha) \in N_\alpha$ for each $\alpha \in \omega_1 \setminus \omega$ and hence x is a function from $\omega_1 \setminus \omega$ to ω_1 such that $x(\alpha) < \alpha$ for all $\alpha \in \omega_1 \setminus \omega$. If x is injective then let $P_0 = \omega$ and $P_{n+1} = x^{-1}(P_n)$ for each $n \in \omega$. It follows from the injectivity of x that the set $P = \bigcup\{P_n : n \in \omega\}$ is countable. Take any $\alpha_0 \in \omega_1 \setminus P$ and let $\alpha_{n+1} = x(\alpha_n)$ for each $n \in \omega$. It follows from $\alpha \notin P$ that $\alpha_n > \omega$ for each $n \in \omega$ and hence $\alpha_0 > \alpha_1 > \dots$ is an infinite decreasing sequence of ordinals which cannot exist, a contradiction. This contradiction shows that there are distinct $\alpha, \beta \in \omega_1 \setminus \omega$ such that $x(\alpha) = x(\beta) = \delta$. The set $U = \{y \in N : y(\alpha) = y(\beta) = \delta\}$ is open in N and contains x. If $d_\xi \in U$ then $d_\xi(\alpha) = d_\xi(\beta)$ which implies $\alpha > \xi$ and $\beta > \xi$ because d_ξ is an injection on $(\xi + 1) \setminus \omega$ and $d_\xi(\gamma') \neq d_\xi(\gamma)$ for any $\gamma \leqslant \xi$ and $\gamma' > \xi$. As a consequence, $d_\xi(\alpha) = \xi = d_\xi(\beta) = \delta$ and hence the only ξ for which $d_\xi \in U$ is possible, is $\xi = \delta$. Thus D is a closed discrete subspace of N of cardinality ω_1 and Fact 3 is proved.

Returning to our solution, let us prove that (ii) $\Rightarrow$ (v). By Problem 214 it is sufficient to show that $\text{ext}(X) \leqslant \omega$. Suppose not. Then $\mathbb{R}^{\omega_1}$ embeds in $C_p(X)$ as a closed subspace. Since $C_p(X)$ is assumed to be normal, the space $\mathbb{R}^{\omega_1}$ has to be normal too, a contradiction with Fact 2. Thus $\text{ext}(X) \leqslant \omega$ and the implication (ii) $\Rightarrow$ (v) is established.

To see that (iii) $\Rightarrow$ (v) suppose that $\text{ext}(X) > \omega$. Then $\mathbb{R}^{\omega 1}$ embeds in $C_p(X)$ as a closed subspace. Since $C_p(X)$ is assumed to have countable extent, the space $\mathbb{R}^{\omega 1}$ has to have countable extent too, a contradiction with Fact 3. Thus $\text{ext}(X) \leqslant \omega$ and we have the implication (iii) $\Rightarrow$ (v).

To finally prove that (iv) $\Rightarrow$ (v) observe that $A(\omega_1)$ is a compact space (Problem 129) and $w(A(\omega_1)) = \omega_1$ (this is an easy exercise). Applying Problem 126, we conclude that $A(\omega_1)$ embeds in $\mathbb{R}^{\omega 1}$. Besides, the space $A(\omega_1)$ is not metrizable by Problem 212. Now, if $\text{ext}(X) > \omega$ then $\mathbb{R}^{\omega 1}$ embeds in $C_p(X)$ (Fact 1) and hence a non-metrizable compact space $A(\omega_1)$ embeds in $C_p(X)$ which is a contradiction.

S.216. *Let X be a metrizable space such that $C_p(X)$ is separable. Is it true that X must be second countable?*

Solution. No, this is not true because $X = D(\mathfrak{c})$ is a metrizable space (Problem 204) which is not second countable while $C_p(X)$ is homeomorphic to $\mathbb{R}^{\mathfrak{c}}$ which is separable by Problem 108.

S.217. *Suppose that Z is a space and Y is a dense subspace of Z. Prove that, for any point $y \in Y$, we have $\chi(y, Y) = \chi(y, Z)$. Deduce from this fact that, if $C_p(X)$ has a dense metrizable subspace, then it is metrizable and hence X is countable.*

Solution. If $\mathcal{B}'$ is a local base of Z at y then $\mathcal{B} = \{U \cap Y : U \in \mathcal{B}'\}$ is a local base of Y at y with $|\mathcal{B}| \leqslant |\mathcal{B}'|$ which shows that $\chi(y, Y) \leqslant \chi(y, Z)$. Note that here we used neither density of Y in Z nor the Tychonoff property of the space Z.

Fact 1. If X is any space and D is a dense subspace of X then $\overline{U} = \overline{U \cap D}$ for any $U \in \tau(X)$.

Proof. We only have to prove that $\overline{U} \subset \overline{U \cap D}$. Take any point $x \in \overline{U}$ and any set $W \in \tau(x, X)$. Then $W \cap U \neq \emptyset$ because $x \in \overline{U}$. Since D is dense in X, we have $(W \cap U) \cap D = W \cap (U \cap D) \neq \emptyset$ and therefore $x \in \overline{U \cap D}$. The point $x \in \overline{U}$ having been chosen arbitrarily, we have $\overline{U} = \overline{U \cap D}$ and Fact 1 is proved.

Now take any local base $\mathcal{B}$ of Y at the point y. For each $U \in \mathcal{B}$ there is $U' \in \tau(Z)$ such that $U' \cap Y = U$. We claim that the family $\mathcal{B}' = \{U' : U \in \mathcal{B}\}$ is a local base of Z at y. Indeed, assume that $y \in W \in \tau(Z)$. By regularity of X there is $V \in \tau(y, Z)$ such that $\overline{V} \subset W$. We have $V \cap Y \in \tau(y, Y)$ and therefore there exists $U \in \mathcal{B}$ with $U \subset V \cap Y$. Then $\overline{U'} = \overline{U}$ by Fact 1 and therefore $U' \subset \overline{U'} = \overline{U} \subset \overline{V} \subset W$ which proves that $\mathcal{B}'$ is a local base of Z at y. Since $|\mathcal{B}'| \leqslant |\mathcal{B}|$, we conclude that $\chi(y, Z) \leqslant \chi(y, Y)$ and hence $\chi(y, Z) = \chi(y, Y)$.

Suppose, finally, that M is a dense metrizable subspace of $C_p(X)$. Since $\chi(f, M) = \omega$ for any $f \in M$ (Problem 210), we have $\chi(f, C_p(X)) = \omega$. As a consequence $\chi(C_p(X)) = \omega$ and hence $|X| = w(C_p(X)) = \omega$ (169). Finally, apply Problem 209 to conclude that $C_p(X)$ is metrizable.

S.218. *(The Stone Theorem) Prove that every open cover of a metrizable space has an open refinement which is σ-discrete and locally finite at the same time. In particular, every metrizable space is paracompact.*

Solution. Take any metric space (X, d) and an open cover $\mathcal{S} = \{U_s : s \in S\}$ of the space X. We consider that a well order $<$ is fixed on the set S. We will need the set $H_s = U_s \setminus (\bigcup\{U_t : t < s\})$ for each $s \in S$. Given $i \in \omega$ and $s \in S$, let $B_{s,i} = \{c \in H_s : B(c, 3/2^i) \subset U_s\}$. Next, we define by induction on $i \in \omega$ the sets $V_{s,i}$ for all $s \in S$. The first step is to define $V_{s,0} = \bigcup\{B(c,1) : c \in B_{s,0}\}$ for all $s \in S$. If we have constructed $V_{s,j}$ for each $j < i$ and $s \in S$, consider the sets $V_{s,i} = \bigcup\{B(c, 1/2^i) : c \in B_{s,i} \setminus (\bigcup\{V_{s,j} : s \in S, j < i\})\}$ for all $s \in S$. Observe that $V_{s,i} \subset U_s$ is an open set for all $s \in S$ and $i \in \omega$. Let $\mathcal{B}_i = \{V_{s,i} : s \in S\}$ and $\mathcal{B} = \bigcup\{\mathcal{B}_i : i \in \omega\}$. For any $x \in X$ there is a minimal $s \in S$ with $x \in U_s$. This implies $x \in H_s$. Pick any $i \in \omega$ such that $B(x, 3/2^i) \subset U_s$; for this i we have $x \in B_{s,i}$. Now, if $x \in \bigcup\{V_{s,j} : s \in S, j < i\}$ then $x \in \bigcup\mathcal{B}$. If not, then $B(x, 1/2^i) \subset V_{s,i}$ and again $x \in \bigcup\mathcal{B}$. This yields $X = \bigcup\mathcal{B}$ and hence $\mathcal{B}$ is a refinement of $\mathcal{S}$.

We will prove that $\mathcal{B}$ is σ-discrete and locally finite; this will finish our solution. To establish σ-discreteness of $\mathcal{B}$ it suffices to show that each $\mathcal{B}_i$ is discrete.

Claim. If $x \in V_{s,i}$, $y \in V_{t,i}$ where $s < t$ then $d(x, y) > 1/2^i$.
Proof of the claim. There exists $c \in B_{s,i}$ such that $x \in B(c, 1/2^i)$ and $c' \in B_{t,i}$ with $y \in B(c', 1/2^i)$. We have $B(c, 3/2^i) \subset U_s$ while $c' \notin U_s$. Thus $d(c, c') \geqslant 3/2^i$ and if $d(x, y) \leqslant 1/2^i$ then

$$d(c,c') \leqslant d(c,x) + d(x,y) + d(y,c') < 1/2^i + 1/2^i + 1/2^i = 3/2^i;$$

this is a contradiction which proves our claim.

Now take any point $z \in X$ and $U_z = B(z, 1/2^{i+1}) \in \tau(z, X)$. If there exist $s,t \in S$ such that $s < t$ and $U_z \cap V_{s,i} \neq \emptyset \neq U_z \cap V_{t,i}$ then pick any points $x \in U_z \cap V_{s,i}$ and $y \in U_z \cap V_{t,i}$; our claim implies that $d(x, y) > 1/2^i$. However, $d(x, y) \leqslant d(x, z) + d(z, y) \leqslant 1/2^{i+1} + 1/2^{i+1} = 1/2^i$ which is a contradiction. As a consequence, each $z \in X$ has a neighbourhood U_z which intersects at most one element of $\mathcal{B}_i$. Therefore each $\mathcal{B}_i$ is discrete.

Finally, to see that $\mathcal{B}$ is locally finite, fix any $z \in X$. There exist $k,j \in \omega$ such that $B(z, 1/2^k) \subset V_{t,j}$ for some $t \in S$. Since each $\mathcal{B}_m$ is discrete, it suffices to prove that $B(z, 1/2^{k+j+1}) \cap (\bigcup\mathcal{B}_i) = \emptyset$ for all $i \geqslant j + k + 1$ because this implies that the set $W_z = B(z, 1/2^{k+j+1})$ intersects at most $j + k+1$ elements of $\mathcal{B}$. So take any $i \geqslant k + j + 1$ and $s \in S$. If $W_z \cap V_{s,i} \neq \emptyset$ then there is $c \in B_{s,i}$ with $B(c, 1/2^i) \cap W_z \neq \emptyset$. However, $c \notin V_{t,j}$ and $B(z, 1/2^k) \subset V_{t,j}$ which implies $d(z, c) \geqslant 1/2^k$. On the other hand, if we pick any point $y \in W_z \cap V_{s,i}$ then $d(z, c) \leqslant d(z, y) + d(y, c) < 1/2^{k+j+1} + 1/2^{k+j+1} \leqslant 1/2^{k+1} + 1/2^{k+1} = 1/2^k$ which is a contradiction.

S.219. *Let X be an arbitrary space. Prove that $C_p(X)$ is paracompact if and only if it is Lindelöf.*

Solution. We will use the following statement.

Fact 1. Every Lindelöf space is paracompact.

Proof. Let Z be a Lindelöf space. If $\mathcal{U}$ is an open cover of Z, for each $z \in Z$ find U_z, $V_z \in \tau(z, Z)$ such that $\overline{V}_z \subset U_z \subset W$ for some $W \in \mathcal{U}$. Since Z is Lindelöf, we can choose a countable set $A = \{z_i : i \in \omega\} \subset Z$ such that $Z = \bigcup\{V_{z_i} : i \in \omega\}$. Letting $W_0 = U_{z_0}$ and $W_n = U_{z_n} \backslash (\overline{V}_{z_0} \cup \cdots \cup \overline{V}_{z_{n-1}})$ for each $n \in \mathbb{N}$ we obtain a family $\mathcal{W} = \{W_n : n \in \omega\} \subset \tau(Z)$. It is clear that, for each $n \in \omega$, there is $W \in \mathcal{U}$ with $W_n \subset W$ so it suffices to prove that $\mathcal{W}$ is a locally finite cover of Z. Given $z \in Z$, let $n = \min\{k \in \omega : z \in \overline{V}_{z_k}\}$. It is immediate that $z \in W_n$ and hence $\mathcal{W}$ covers Z, i.e., $\mathcal{W}$ is a refinement of $\mathcal{U}$. There exists $m \in \omega$ such that $z \in V_{z_m}$; we have $V_{z_m} \cap W_k = \emptyset$ for each $k > m$ which shows that $V_{z_m} \in \tau(z, Z)$ meets at most $(m + 1)$-many elements of $\mathcal{W}$. Therefore $\mathcal{W}$ is locally finite so Fact 1 is proved.

Fact 2. If Z is a paracompact space with $c(Z) = \omega$ then Z is Lindelöf.

Proof. If γ is an open cover of Z choose a locally finite open refinement μ of the cover γ. It is sufficient to show that μ is countable. Let δ be a maximal disjoint family of open sets each one of which intersects only finitely many elements of μ. The set $\bigcup\delta$ is dense in X because, otherwise we can take $x \in U = X \setminus \overline{\bigcup\delta}$ and $V \in \tau(x, X)$ which intersects only finitely many elements of μ. The disjoint family $\delta' = \delta \cup \{U \cap V\}$ is strictly larger than δ and every element of δ' intersects only finitely many elements of μ, a contradiction with the maximality of δ. The family δ is countable because $c(Z) = \omega$ and the family $\mathcal{A}_U = \{V \in \mu : V \cap U \neq \emptyset\}$ is finite for each $U \in \delta$. It follows from density of $\bigcup\delta$ that $\mu = \bigcup\{\mathcal{A}_U : U \in \delta\}$ and hence μ is countable. Fact 2 is proved.

By Fact 1, if $C_p(X)$ is Lindelöf then it is paracompact. Now assume that $C_p(X)$ is paracompact. Since $C_p(X)$ has the Souslin property, we can apply Fact 2 to conclude that $C_p(X)$ is Lindelöf.

S.220. *Suppose that $C_p(X)$ has a dense paracompact subspace. Must $C_p(X)$ be Lindelöf?*

Solution. Not necessarily. If $X = D(\omega_1)$ then $C_p(X) = \mathbb{R}^{\omega_1}$ is separable (Problem 108) so there is a countable dense $Y \subset C_p(X)$. It is evident that Y is Lindelöf and hence paracompact (Fact 1 from S.219). However, $C_p(X) = \mathbb{R}^{\omega_1}$ is not paracompact because, otherwise it would be Lindelöf by problem 219 and hence normal by Problem 124. However, $\mathbb{R}^{\omega_1}$ is not normal (Fact 2 from S.215). This contradiction shows that our solution is complete.

S.221. *Prove that the following conditions are equivalent for any space X:*

(i) *X is metrizable.*
(ii) *X has a σ-discrete base.*
(iii) *X has a σ-locally finite base.*

The equivalence (i) $\Leftrightarrow$ (ii) is known as the Bing metrization theorem. The statement (i) $\Leftrightarrow$ (iii) is the Nagata–Smirnov metrization theorem.

Solution. Since every discrete family is locally finite, we have (ii) $\Rightarrow$ (iii). If X is a metrizable space, take any metric d on X such that $\tau(d) = \tau(X)$ and consider the family $\gamma_n = \{B(x, \frac{1}{n}) : x \in X\}$ for each $n \in \mathbb{N}$. Apply Problem 218 to find a σ-discrete open refinement $\mathcal{B}_n$ of the cover γ_n. It is evident that the family $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\} \subset \tau(X)$ is σ-discrete. Let us prove that $\mathcal{B}$ is a base of the space X. If $x \in U \in \tau(X)$ then $B(x, r) \subset U$ for some $r > 0$. Pick any $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{r}{2}$. Since $\mathcal{B}_n$ is a cover of X, there is $V \in \mathcal{B}_n$ such that $x \in V$. There is $W = B(y, \frac{1}{n}) \in \gamma_n$ for which $V \subset W$. If $z \in W$ then $d(x, z) \leqslant d(x, y) + d(y, z) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} < r$ which shows that we have $x \in V \subset W \subset B(x, r) \subset U$. Since $V \in \mathcal{B}$ we proved that $\mathcal{B}$ is a σ-discrete base of X and hence (i) $\Rightarrow$ (ii) holds.

The proof of (iii) $\Rightarrow$ (i) is difficult and will be split in several steps.

Fact 1. Given an arbitrary (not necessarily Tychonoff) space X, suppose that for any closed $F \subset X$ and any $W \in \tau(F, X)$ there exists a family $\{W_i : i \in \omega\} \subset \tau(X)$ such that $F \subset \bigcup\{W_n : n \in \omega\}$ and $\overline{W}_n \subset W$ for each $n \in \omega$. Then X is normal.

Proof. Take any closed $F, G \subset X$ such that $F \cap G = \emptyset$. For the set $W = X \setminus G \supset F$ find a sequence $\{U_i : i \in \omega\}$ of open sets such that $F \subset \bigcup\{U_i : i \in \omega\}$ and $\overline{U}_i \cap G = \emptyset$ for all $i \in \omega$. If we let $W = X \setminus F \supset G$ then we can find a sequence $\{V_i : i \in \omega\} \subset \tau(X)$ for which $G \subset \bigcup\{V_i : i \in \omega\}$ and $\overline{V}_i \cap F = \emptyset$ for all $i \in \omega$. Let $U'_n = U_n \setminus (\bigcup\{\overline{V}_i : i \leqslant n\})$ and $V'_n = V_n \setminus (\bigcup\{\overline{U}_i : I \leqslant n\})$ for all $n \in \omega$. The sets $U = \bigcup\{U'_n : n \in \omega\}$ and $V = \bigcup\{V'_n : n \in \omega\}$ are what we are looking for, i.e., they are open, $F \subset U$, $G \subset V$ and $U \cap V = \emptyset$. It is clear that $U, V \in \tau(X)$. For any $x \in F$, there is $n \in \omega$ with $x \in U_n$; it is evident that $x \in U'_n \subset U$ which proves that $F \subset U$. If $y \in G$ then $y \in V_\kappa$ for some $k \in \omega$ so $y \in V'_n \subset V$ and therefore $G \subset V$.

To finally prove that $U \cap V = \emptyset$, suppose not. Then $U'_k \cap V'_l \neq \emptyset$ for some $k, l \in \omega$. If $k \leqslant l$ then $V'_l \subset X \setminus \overline{U}_k \subset X \setminus U'_k$ which is a contradiction. If $l \leqslant k$ then $U'_k \subset X \setminus \overline{V}_l \subset X \setminus V'_l$ and this contradiction shows that $U \cap V = \emptyset$. Fact 1 is proved.

Fact 2. A locally finite family in any (not necessarily Tychonoff) space is closure-preserving.

Proof. Let $\{A_s : s \in S\}$ be a locally finite family in a space X. If $T \subset S$ and we have a point $x \in \overline{\bigcup\{A_s : s \in T\}}$ then find a set $U \in \tau(x, X)$ such that the set $P_x = \{s \in S : U \cap A_s \neq \emptyset\}$ is finite. It is clear that $x \notin \overline{\bigcup\{A_s : s \in T \setminus P_x\}}$ and therefore $x \in \overline{\bigcup\{A_s : s \in P_x \cap T\}}$. The set $P_x \cap T$ being finite, we have $x \in \overline{A}_s$ for some $s \in P_x \cap T$ and the proof of Fact 2 is complete.

Fact 3. If a regular (not necessarily Tychonoff) space X has a σ-locally finite base then X normal.

Proof. Fix a base $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ in X such that all $\mathcal{B}_n$'s are locally finite. Take any closed $F \subset X$ and any $W \in \tau(F, X)$. For every point $x \in F$ there is a number $n(x) \in$

ω and $U_x \in \mathcal{B}_{n(x)}$ such that $x \in U_x \subset \overline{U}_x \subset W$. For each $n \in \omega$ let $U_n = \bigcup\{U_x : n(x) = n\}$. This gives us a sequence $\{U_n : n \in \omega\}$ of open sets with $F \subset \bigcup\{U_n : n \in \omega\}$ and $\overline{U}_n = \bigcup\{\overline{U}_x : n(x) = n\} \subset W$. The last equality is true because the family $\{U_x : n(x) = n\}$ is closure-preserving by Fact 2. Thus X is normal by Fact 1 and Fact 3 is proved.

Call a function $d : X \times X \to \mathbb{R}$ *a pseudometric* if $d(x, y) \geqslant 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$. In other words the axioms of a pseudometric are those of a metric except that the distance equal to zero does not imply coincidence of the respective points.

Fact 4. Assume that $\mathcal{D} = \{d_i : i \in \omega\}$ is a family of pseudometrics on a space X with the following properties:

(1) The function $d_i : X \times X \to \mathbb{R}$ is continuous for all $i \in \omega$.
(2) $d_i(x, y) \leqslant 1$ for all $x, y \in X$ and $i \in \omega$.
(3) For every $x \in X$ and every non-empty closed $A \subset X$ with $x \notin A$ there exists $i \in \omega$ such that $d_i(x, A) = \inf\{d_i(x, a) : a \in A\} > 0$.

Then the space X is metrizable and the function $d(x, y) = \sum_{i=0}^{\infty} (1/2^i) d_i(x, y)$ is a metric on X which generates $\tau(X)$.

Proof. It is evident that $d(x, y) \geqslant 0$ for all $x, y \in X$. If $x = y$ then all summands in the definition of d are equal to zero and hence $d(x, y) = 0$. If $x \neq y$ then $F = \{y\}$ is a closed set with $x \notin F$ so (3) is applicable and $d_i(x, F) = d_i(x, y) > 0$ for some $i \in \omega$, therefore $d(x, y) > 0$ and we checked the axiom (MS1) for d. The axioms (MS2) and (MS3) hold for d because they hold for every d_i and summing preserves them. Hence d is a metric on X.

It follows easily from (2) that the convergence of the series in the definition of d is uniform so (1) implies that $d : X \times X \to \mathbb{R}$ is a continuous function. Thus, for any $x \in X$, the function $d_x : X \to \mathbb{R}$ defined by $d_x(y) = d(x, y)$, is also continuous. This means that $B(x, r) = d_x^{-1}((-1, r))$ is an open set in X. Since any $U \in \tau(d)$ is a union of balls, any $U \in \tau(d)$ is open, i.e., $\tau(d) \subset \tau(X)$.

Let us prove that, for each $A \subset X$, the function $d_A : X \to \mathbb{R}$ defined by the formula $d_A(x) = \inf\{d(x, a) : a \in A\}$ for each $x \in X$, is also continuous. By Fact 1 of S.212 the function d_A is continuous on the space $(X, \tau(d))$. Thus, for any $U \in \tau(\mathbb{R})$, we have $d_A^{-1}(U) \in \tau(d) \subset \tau(X)$ and hence $d_A^{-1}(U)$ is open in X, i.e., the function d_A is continuous.

To prove that $\tau(X) \subset \tau(d)$ take any $U \in \tau(X)$ and $x \in U$. Then $F = X \setminus U$ is a closed set and $x \notin F$. The property (3) implies that $d_i(x, F) = r > 0$ for some $i \in \omega$. As a consequence $d(x, F) \geqslant \frac{r}{2^i}$ and therefore $B_d(x, \frac{r}{2^i}) \subset U$ which proves that $U \in \tau(d)$. Thus $\tau(X) = \tau(d)$ and Fact 4 is proved.

Returning to our solution, take a base $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ in the space X such that every $\mathcal{B}_n = \{U_s : s \in S_n\}$ is a locally finite family. Fix any numbers $i, j \in \omega$ and, for any index $s \in S_i$, let $\gamma_s = \{W \in \mathcal{B}_j : \overline{W} \subset U_s\}$. The family γ_s is closure-preserving by Fact 2 and hence we have the inclusion $F_s = \overline{\bigcup \gamma_s} \subset U_s$. The space X is normal by Fact 3 so there exists a continuous function $f_s : X \to [0, 1]$ such that $f_s(F_s) \subset \{1\}$ and $f_s(X \setminus U_s) \subset \{0\}$. Define a function $g_{i,j} : X \times X \to \mathbb{R}$ as follows:

$g_{i,j}(x_1, x_2) = \sum_{s \in S_i} |f_s(x_1) - f_s(x_2)|$ for every $(x_1, x_2) \in X \times X$. Observe that this definition makes sense because, for any $x \in X$ there is $O(x) \in \tau(x, X)$ and a finite set $S(x)$ such that $U_s \cap O(x) = \emptyset$ whenever $s \in S_i \setminus S(x)$. This implies $f_s(x) = 0$ for all $s \in S_i \setminus S(x)$ and therefore $g_{i,j}(x_1, x_2) = \sum_{s \in S(x_1) \cup S(x_2)} |f_s(x_1) - f_s(x_2)|$ which makes sense because the last sum is finite.

Note also that the equality $g_{i,j}(x) = \sum_{s \in S(x_1) \cup S(x_2)} |f_s(x_1) - f_s(x_2)|$ holds for all $x = (x_1, x_2) \in O = O(x_1) \times O(x_2)$ and hence $g_{i,j}$ is continuous at the point x. The point x having been taken arbitrarily, we proved that $g_{i,\,j}$ is a continuous function.

It is easy to check that $g_{i,j}$ is a pseudometric on X and hence the function $d_{i,j} : X \times X \to \mathbb{R}$ defined by $d_{i,j}(x, y) = \min\{1, g_{i,j}(x, y)\}$ is also a continuous pseudometric on X (this is proved exactly as was proved in S.206 the same fact for metrics). The family $\mathcal{D} = \{d_{i,j} : i, j \in \omega\}$ is countable and consists of continuous pseudometrics bounded by 1. Thus the properties (1) and (2) of Fact 4 are satisfied for $\mathcal{D}$. To prove that X is metrizable it suffices to show that $\mathcal{D}$ satisfies (3) as well.

Take any $x \in X$ and any closed $A \subset X$ such that $x \notin A$. There exist sets $U, V \in \mathcal{B}$ with $x \in V \subset \overline{V} \subset U \subset X \setminus A$. It is clear that $U = U_s \in \mathcal{B}_i$ and $V \in \mathcal{B}_j$ for some $i,j \in \omega$. Therefore $x \in F_s$ and we have $g_{i,j}(x, A) \geqslant 1$ because $f_s(x) = 1$ and $f_s(y) = 0$ for all $y \in A$. As a consequence, $d_{i,j}(x, A) = 1 > 0$ and we checked the condition (3) of Fact 4 for the family $\mathcal{D}$. This proves that X is metrizable settling the implication (iii) $\Rightarrow$ (i) so our solution is complete.

S.222. *Let $I_\alpha = (0, 1] \times \{\alpha\}$ for each $\alpha < \kappa$ and $J(\kappa) = \bigcup\{I_\alpha : \alpha < \kappa\} \cup \{0\}$. Given $x, y \in J(\kappa)$, $x = (t, \alpha)$, $y = (s, \beta)$, let $\rho(x, y) = |\,t - s\,|$ if $\alpha = \beta$. If $\alpha \neq \beta$ then $\rho(x, y) = t + s$. Let $\rho(x, 0) = t$, $\rho(0, y) = s$ and $\rho(0, 0) = 0$. Prove that*

(*i*) *$(J(\kappa), \rho)$ is a complete metric space (called Kowalsky hedgehog with κ spines).*
(*ii*) *Any metrizable space embeds into $(J(\kappa))^\omega$ for some κ.*

Solution. (i) We omit the trivial verification of the fact that ρ is a metric on $J(\kappa)$. Let us prove that ρ is complete. Take any fundamental sequence $S = \{x_n\} \subset J(\kappa)$. It is an easy exercise to show that if a fundamental sequence has a convergent subsequence then it is convergent. So let us assume that no subsequence of S is convergent. In particular, no subsequence of S converges to 0. Therefore there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \cap S$ is finite. Thus, without loss of generality we may assume that $S \cap B(0, \varepsilon) = \emptyset$. Suppose first that the set $\{\alpha : S \cap I_\alpha \neq \emptyset\}$ is infinite. Then there is a subsequence $\{x_{n_k}\} \subset S$ such that x_{n_k} and x_{n_m} do not belong to the same I_α if $m \neq k$. This implies $\rho(x_{n_k}, x_{n_m}) \geqslant \varepsilon$ for all $k \neq m$, a contradiction with the fact that S is fundamental. This contradiction shows that there is a subsequence $S' \subset S$ which lies in I_α for some α.

Observe that the map $i_\alpha : I_\alpha \cup \{0\} \to [0, 1]$ given by $i_\alpha((t, \alpha)) = t$ for $t > 0$ and $i_\alpha(0) = 0$, is an isometry. The space $I = [0, 1]$ is complete being closed in a complete space $\mathbb{R}$. Since $I_\alpha \cup \{0\}$ is isometric to I, it is also complete and therefore the fundamental sequence S' is convergent. This implies convergency of S, this last contradiction proving that ρ is complete.

(ii) If X is a metrizable space, fix a base $\mathcal{B} \bigcup\{\mathcal{B}_n : n \in \omega\}$ in X such that every $\mathcal{B}_n$ is discrete (see Problem 221). For the cardinal $\kappa = |\mathcal{B}|$ choose $A_n \subset \kappa$ and a faithful enumeration $\{U^n_\alpha : \alpha \in A_n\}$ for each $\mathcal{B}_n$. Fix $n, k \in \omega$. For each $U^n_\alpha \in \mathcal{B}_n$ let $V_\alpha = \bigcup\{V : V \in \mathcal{B}_k \text{ and } \overline{V} \subset U^n_\alpha\}$. Since the family $\mathcal{B}_k$ is discrete, we have $\overline{V}_\alpha \subset U^n_\alpha$ and hence there exists $f_\alpha \in C(X, [0, 1])$ such that $f_\alpha(\overline{V}_\alpha) \subset \{1\}$ and $f_\alpha(X \backslash U^n_\alpha) \subset \{0\}$. Define a function $g_{n,k} : X \to J(\kappa)$ as follows: if there exists $\alpha \in A_n$ such that $f_\alpha(x) \neq 0$ then $g_{n,k}(x) = (f_\alpha(x),\alpha) \in I_\alpha$ and $g_{n,k}(x) = 0$ if $f_\alpha(x) = 0$ for all $\alpha \in A_n$. Note that this definition is consistent because, for any $x \in X$, there can be at most one $\alpha \in A_n$ with $f_\alpha(x) \neq 0$.

To see that $g_{n,k}$ is continuous at any point $x \in X$, take any $U_x \in \tau(x, X)$ which intersects at most one element of $\mathcal{B}_n$, say, U^n_α. Then $f_\beta(U^n_\alpha) \subset \{0\}$ for all $\beta \neq \alpha$ and therefore $g_{n,k}(U_x) \subset I_\alpha \cup \{0\}$ which shows that $g_{n,k}|U_x = (i^{-1}_\alpha \circ f_\alpha)|U_x$. Since $i^{-1}_\alpha \circ f_\alpha$ is continuous, the map $g_{n,k}$ is continuous at the point x.

Let $\{h_m : m \in \omega\}$ be some enumeration of the countable set $\{g_{n,k} : n, k \in \omega\}$. Given $x \in X$, let $h(x)(m) = h_m(x)$ for each $m \in \omega$. Then $h(x) \in (J(\kappa))^\omega$ and the map $h : X \to (J(\kappa))^\omega$ is continuous because $\pi_m \circ h = h_m$ where $\pi_m : (J(\kappa))^\omega \to J(\kappa)$ is the natural projection onto the mth factor. Let us prove that $h : X \to Y = h(X)$ is a homeomorphism.

If we are given distinct points $x, y \in X$ then there exist $U, V \in \mathcal{B}$ such that $x \in V \subset \overline{V} \subset U \subset X \backslash \{y\}$. There exist $n, k \in \omega$ such that $V \in \mathcal{B}_k$ and $U = U^n_\alpha \in \mathcal{B}_n$ for some $\alpha \in A_n$. Since $V \subset V_\alpha$, we have $g_{n,k}(x) = (1, \alpha)$ and $g_{n,k}(y) = 0 \neq g_{n,k}(x)$. This proves that h is a bijection.

To see that h^{-1} is continuous, take any $y \in Y$ and any $O \in \tau(x, X)$ where $x = h^{-1}(y)$. Pick $U, V \in \mathcal{B}$ for which $x \in V \subset \overline{V} \subset U \subset O$. There exist $n, k \in \omega$ such that $V \in \mathcal{B}_k$ and $U = U^n_\alpha \in \mathcal{B}_n$ for some $\alpha \in A_n$. There is $m \in \omega$ such that $g_{n,k} = h_m$. Let $W' = \{z \in J((\kappa))^\omega : z(m) \in I_\alpha\}$. The set W' is open in $(J(\kappa))^\omega$ and hence $W = W' \cap Y$ is open in Y. Note that $y(m) = h_m(x) = g_{n,k}(x) = (1, \alpha) \in I_\alpha$ and therefore $y \in W$. To finish our proof it suffices to show that $h^{-1}(W) \subset O$. So take any $z \in W$. If $t = h^{-1}(z)$ then we have $g_{n,k}(t) = z(m) \in I_\alpha$. As a consequence $f_\alpha(t) \neq 0$ and hence $t \in U^n_\alpha = U \subset O$. The point $z \in W$ having been taken arbitrarily, we proved that $h^{-1}(W) \subset O$ and hence h^{-1} is continuous at the point y. This, of course, implies that h^{-1} is continuous and hence h is a homeomorphism. Our solution is complete.

S.223. *Show that a space is first countable if and only if it is an open continuous image of a metrizable space.*

Solution. Our first step is to establish the following fact.

Fact 1. Any open image of a first countable space is first countable.

Proof. Assume that Y is first countable and $f : Y \to Z$ is an open map. Given $z \in Z$ and any $y \in Y$ with $f(y) = z$ take any countable local base $\mathcal{B}$ of Y at the point y. Observe that $\mathcal{C} = \{f(U) : U \in \mathcal{B}\}$ is a countable family of open subsets of Z so it suffices to show that $\mathcal{C}$ is a local base of Z at z. To do so, take any $W \in \tau(z, Z)$. Then $W' = f^{-1}(W)$ is open in Y and $y \in W'$. Now if $U \in \mathcal{B}$ and $U \subset W'$ then $V = f(U) \in \mathcal{C}$ and $V \subset W$. Hence $\mathcal{C}$ is a countable local base at z and therefore Z is first countable. Fact 1 is proved.

Any metrizable space is first countable (Problem 210) so if X is an open image of a metrizable space then X is first countable by Fact 1.

Now suppose that X is a first countable space and denote by T the set $\tau(X)$ with the discrete topology. Then T is metrizable as well as T^ω (Problems 204 and 208). Let $M = \{f \in T^\omega : \{f(n) : n \in \omega\}$ is a base at some point $x \in X\}$. Given $f \in M$, let $\varphi(f) \in X$ be the point at which the family $\{f(n) : n \in \omega\}$ is a local base. It is clear that a given family can be a local base at one point at most, so the definition of φ is consistent.

Let us prove that $\varphi : M \to X$ is an open map. Given $x \in X$, take any countable local base $\{U_n : n \in \omega\}$ at the point x and let $f(n) = U_n$ for every $n \in \omega$. Then $f \in M$ and $\varphi(f) = x$ whence φ is surjective. If $\varphi(f) = x \in U \in \tau(X)$ then there is $n \in \omega$ such that $f(n) \subset U$ because $\{f(n) : n \in \omega\}$ is a local base at x. The set $V = \{g \in M : g(n) = f(n)\}$ is open in M and $f \in V$. For any $g \in V$ we have $\varphi(g) \in \bigcap\{g(k) : k \in \omega\} \subset g(n) = f(n) \subset U$ and therefore $\varphi(V) \subset U$ which proves continuity of the map φ.

To prove that $\varphi(O)$ is open for any $O \in \tau(M)$ observe that it suffices to establish that $\varphi(f) \in \mathrm{Int}(\varphi(O))$ for any $f \in O$. Recalling the definition of the product topology, we can see that there is $n \in \omega$ such that $W = \{g \in M : g(i) = f(i)$ for all $i \leqslant n\} \subset O$. It suffices to prove that $V = f(0) \cap \cdots \cap f(n) \subset \varphi(O)$ because V is open in X and hence $\varphi(f) \in V \subset \mathrm{Int}(\varphi(O))$. Take any $y \in V$ and choose a local base $\{V_n : n \in \omega\}$ at the point y in such a way that $V_i = f(i)$ for all $i \leqslant n$. This is possible because we can always add the sets $\{f(0), \ldots, f(n)\}$ to any given countable local base at y and choose a relevant enumeration of the obtained family. If $g(n) = V_n$ for all $n \in \omega$ then $g \in W \subset O$ and $\varphi(g) = y$ which shows that $\varphi(O) \supset V$ so our solution is complete.

S.224. *Show that a space is sequential if and only if it is a quotient image of a metrizable space.*

Solution. Let us first prove that sequentiality behaves well with respect to quotient maps.

Fact 1. Any quotient image of a sequential space is sequential.

Proof. Suppose that Y is a sequential space and $f : Y \to Z$ is a quotient map. If $A \subset Z$ is not closed then $B = f^{-1}(A)$ is not closed in Y and hence there is a sequence $\{y_n\} \subset B$ such that $y_n \to y \in Y \setminus B$. It is clear that $\{f(y_n)\} \subset A$ and $f(y_n) \to f(y) \in Z \setminus A$ so Z is sequential. Fact 1 is proved.

Fact 2. Suppose that X_t is a metrizable space for each $t \in T$. Then the space $X = \bigoplus\{X_t : t \in T\}$ is metrizable (see Problem 113 for the definition of the discrete union).

Proof. Fix a metric d_t on the set X_t with $\tau(d_t) = \tau(X_t)$. We can assume without loss of generality that $d_t(x, y) \leqslant 1$ for all $t \in T$ and $x, y \in X_t$ (Problem 206). We will identify X_t with the respective open subspace of X and consider that X is a disjoint union of X_t's where each X_t is closed and open in X (see Problem 113(iii)). Given $x, y \in X$, let $d(x, y) = d_t(x, y)$ if $x, y \in X_t$. In case when $x \in X_t$ and $y \notin X_t$, we let $d(x, y) = 1$. We leave to the reader the simple verification of the fact that d is a metric on X. Since $d|(X_t \times X_t) = d_t$, the topology induced from $(X, \tau(d))$ on X_t coincides with $\tau(X_t)$ so it suffices to prove that each X_t is open in $(X, \tau(d))$. Given $x \in X_t$ the set $U_x = B_d(x, 1)$ is open in $(X, \tau(d))$ and $y \in U_x$ implies $y \in X_t$ for otherwise $d(x, y) = 1$. Thus $U_x \subset X_t$ and hence X_t is open in $(X, \tau(d))$ so Fact 2 is proved.

Observe that any metrizable space is first countable (Problem 210) and hence sequential. Thus, if X is a quotient image of a metrizable space then X is sequential by Fact 1. To prove necessity, suppose that X is sequential and consider the space $M = \bigoplus \{S : S \subset X \text{ is a convergent sequence (with its limit included)}\}$. We will again identify each convergent sequence $S \subset X$ with the respective open subset of M. It is immediate that any convergent sequence is homeomorphic to the subspace $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of the space $\mathbb{R}$. Since $\mathbb{R}$ is metrizable (Problem 205), every convergent sequence $S \subset X$ is metrizable. Applying Fact 2 we convince ourselves that M is a metrizable space.

If $S \subset X$ is a convergent sequence then every $x \in S$ also belongs to M so if $x \in S \subset M$, we denote by x' is twin in X. This makes it possible to define a map $\varphi : M \to X$ by $\varphi(x) = x'$ for each $x \in M$. To finish our solution it suffices to prove that φ is a quotient map. Given $y \in X$ note that $x_n \to y$ if $x_n = y$ for all $n \in \omega$. Hence $S = \{x_n : n \in \omega\}$ is a convergent sequence in X so $x_n \in M$ and $\varphi(x_n) = y$ for each n which proves that φ is onto.

To see that φ is continuous, take a closed $F \subset X$ and $x \in M \backslash \varphi^{-1}(F)$. We have $x \in S$ where S is open in M and S is a convergent sequence of X. If x is an isolated point of S then $\{x\}$ is an open neighbourhood of x which does not meet $\varphi^{-1}(F)$. If x is the limit of S then $A = S \cap F$ must be finite for otherwise $S \cap F \to \varphi(x)$ and hence $\varphi(x) \in \overline{F} \backslash F$, a contradiction. Thus $S \backslash A$ is an open neighbourhood of x which does not meet $\varphi^{-1}(F)$. We proved that $\varphi^{-1}(F)$ is closed in M and hence φ is continuous.

Finally, take a non-closed $P \subset X$. Since X is sequential, there is a sequence $\{p_n : n \in \omega\} \subset P$ with $p_n \to x \in X \backslash P$. It is clear that $S = \{p_n : n \in \omega\} \cup \{x\}$ is a subspace of M and $\varphi^{-1}(P) \cap S = \{p_n : n \in \omega\}$ whence $x \in \overline{\varphi^{-1}(P)} \backslash \varphi^{-1}(P)$ so $\varphi^{-1}(P)$ is not closed in M. This proves that φ is a quotient map and our solution is complete.

S.225. *A continuous onto map $f : X \to Y$ is called pseudo-open if, for any $y \in Y$ and any $U \in \tau(X)$ such that $f^{-1}(y) \subset U$, we have $y \in \mathrm{Int}(f(U))$. Show that*

(*i*) *A map $f : X \to Y$ is pseudo-open if and only if it is hereditarily quotient, i.e., $f|(f^{-1}(Z)) : f^{-1}(Z) \to Z$ is quotient for any $Z \subset Y$.*
(*ii*) *A composition of pseudo-open maps is a pseudo-open map.*
(*iii*) *Any open map as well as any closed one is pseudo-open.*
(*iv*) *If X is a Fréchet–Urysohn space and $f : X \to Y$ is a pseudo-open map then Y is Fréchet–Urysohn.*
(*v*) *A space is Fréchet–Urysohn if and only if it is a pseudo-open image of a metrizable space.*

Solution. (i) Suppose that f is pseudo-open, fix any $Z \subset Y$ and let $T = f^{-1}(Z)$. Denote the map $f|T$ by f_T. Note first that $f_T^{-1}(A) = f^{-1}(A)$ for any $A \subset Z$. To prove that f_T is quotient, take any $W \subset Z$ such that $U = f_T^{-1}(W) \in \tau(Z)$. Take any $z \in W$. Then $f^{-1}(z) = f_T^{-1}(z) \subset U$. Take any $V \in \tau(X)$ such that $V \cap T = U$. Since f is pseudo-open, we have $z \in O = \mathrm{Int}(f(V))$ (the interior is taken in Y). Since $V \cap f^{-1}(Z \backslash W) = \emptyset$, we have $z \in O_z = O \cap Z \subset W$. It turns out that every $z \in W$ is contained in W together with its neighbourhood O_z. Hence W is open and the map f_T is proved to be quotient.

Now assume that f is hereditarily quotient. Take any point $y \in Y$ and any set $U \in \tau(f^{-1}(y), X)$. It suffices to prove that $y \notin \overline{Y \backslash f(U)}$. Assume the contrary and let $Z = (Y \backslash f(U)) \cup \{y\}$. The map $g = f|T : T = f^{-1}(Z) \to Z$ is quotient and the set $Y \backslash f(U)$ is not closed in Z. Therefore $g^{-1}(Y \backslash f(U)) = f^{-1}(Y \backslash f(U))$ is not closed in T which implies $f^{-1}(y) \cap \overline{f^{-1}(Y \backslash f(U))} \neq \emptyset$, a contradiction with $f^{-1}(y) \subset U \subset X \backslash f^{-1}(Y \backslash f(U))$.

(ii) Assume that $f : X \to Y$ and $g : Y \to Z$ are pseudo-open maps. Given a point $z \in Z$ and a set $U \in \tau((g \circ f)^{-1}(z), X)$, we have $f^{-1}(y) \subset U$ for any $y \in g^{-1}(z)$ and therefore $y \in \mathrm{Int}_Y(f(U))$. Since we have chosen a point $y \in g^{-1}(z)$ arbitrarily, we can conclude that $g^{-1}(z) \subset V = \mathrm{Int}_Y(f(U))$. As an immediate consequence, $y \in \mathrm{Int}_Z(g(V)) \subset \mathrm{Int}_Z(g(f(U))) = \mathrm{Int}_Z((g \circ f)(U))$ which proves that $g \circ f$ is pseudo-open.

(iii) Let $f : X \to Y$ be an open map. If $y \in Y$ and $U \in \tau(f^{-1}(y), X)$ then $y \in f(U) = \mathrm{Int}_Y(f(U))$ which proves that f is pseudo-open. If f is closed, then $F = X \backslash U$ is a closed set which does not meet $f^{-1}(y)$. Therefore $y \notin G = f(F)$ and hence $y \in V = Y \backslash G \subset \mathrm{Int}_Y(f(U))$ and we are done.

(iv) Take any $A \subset Y$ and $y \in \overline{A}$. If $y \in A$ then there is nothing to prove. If $y \in \overline{A} \backslash A$ then A is not closed in $Z = A \cup \{y\}$. The map f is hereditarily quotient by (i) and hence $f^{-1}(A)$ is not closed in $f^{-1}(A) \cup f^{-1}(y)$. This implies that there is $x \in \overline{f^{-1}(A)} \cap f^{-1}(y)$. Since X is Fréchet–Urysohn, we can choose a sequence $\{x_n\} \subset f^{-1}(A)$ which converges to the point x. It is immediate that $\{f(x_n)\} \subset A$ and $f_n(x) \to y$ which proves that Y is a Fréchet–Urysohn space.

(v) Any metric space is Fréchet–Urysohn, so any pseudo-open image of any metric space is a Fréchet–Urysohn space by (iv). This proves sufficiency. Now take any Fréchet–Urysohn space X and consider the space $M = \bigoplus \{S : S \subset X$ is a convergent sequence (with its limit included)$\}$. We will identify each convergent sequence $S \subset X$ with the respective open subset of M. It is immediate that any convergent sequence is homeomorphic to the subspace $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of the space $\mathbb{R}$. Since $\mathbb{R}$ is metrizable (Problem 205), every convergent sequence $S \subset X$ is metrizable. Applying Fact 2 of S.224 we convince ourselves that M is a metrizable space.

If $S \subset X$ is a convergent sequence then every $x \in S$ also belongs to M so if $x \in S \subset M$, we denote by x' is twin in X. This makes it possible to define a map $\varphi : M \to X$ by $\varphi(x) = x'$ for each $x \in M$. To finish our solution it suffices to prove that φ is a pseudo-open map. Given $y \in X$ note that $x_n \to y$ if $x_n = y$ for all $n \in \omega$. Hence $S = \{x_n : n \in \omega\}$ is a convergent sequence in X so $x_n \in M$ and $\varphi(x_n) = y$ for each n which proves that φ is onto.

To see that φ is continuous, take a closed $F \subset X$ and $x \in M \backslash \varphi^{-1}(F)$. We have $x \in S$ where S is open in M and S is a convergent sequence of X. If x is an isolated point of S then $\{x\}$ is an open neighbourhood of x which does not meet $\varphi^{-1}(F)$. If x is the limit of S then $A = S \cap F$ must be finite for otherwise $S \cap F \to \varphi(x)$ and hence $\varphi(x) \in$

$\overline{F}\backslash F$, a contradiction. Thus $S\setminus A$ is an open neighbourhood of x which does not meet $\varphi^{-1}(F)$. We proved that $\varphi^{-1}(F)$ is closed in M and hence φ is continuous.

To show that φ is pseudo-open, take any $x \in X$ and any $U \in \tau(\varphi^{-1}(x), M)$. If $x \notin \text{Int}_X\,(f(U))$ then $x \in \overline{X\backslash f(U)}$ and hence there is a sequence $\{x_n\} \subset X\backslash f(U)$ with $x_n \to x$. The sequence $S = \{x_n\} \cup \{x\}$ is also contained in M; denote its twin sequence by $\{y_n\} \cup \{y\}$. We have $x = y'$ and $x_n = y'_n$ for any n. Observe that $\varphi(y) = x$ and hence $y \in U$. However, $\varphi(y_n) = x_n \notin f(U)$ for all n and hence $y_n \notin U$ for each n which is a contradiction with the fact that $y_n \to y$ and $U \in \tau(y, M)$. The obtained contradiction shows that $x \in \text{Int}_X\,(f(U))$ and hence f is pseudo-open.

S.226. *Prove that a perfect image of a metrizable space is a metrizable space.*

Solution. Our first step is to prove the following fact.

Fact 1. Let $g : Z \to T$ be a closed map. Given $t \in T$ and $U \in \tau(g^{-1}(t), Z)$, let $g^{\#}(U) = T\setminus g(X\setminus U)$. Then $V = f^{\#}(U)$ is an open set, $y \in V$ and $g^{-1}(V) \subset U$.

Proof. The set $X\setminus U$ is closed and hence so is $g(X\setminus U)$. This shows that $g^{\#}(U)$ must be open. Since $g^{-1}(y) \subset U$, we have $y \notin g(X\setminus U)$ whence $y \in V$. Finally, if $z \in V$ then $z \notin g(X\setminus U)$ and therefore $g^{-1}(z) \subset U$. Fact 1 is proved.

Fact 2. Let Z be a paracompact space. Given an open cover $\mathcal{U} = \{U_s : s \in S\}$, there exists a closed locally finite cover $\{F_s : s \in S\}$ such that $F_s \subset U_s$ for each $s \in S$.

Proof. Given any point $z \in Z$, there is $s \in S$ with $z \in U_s$. Use regularity of Z to find $W_z \in \tau(z, Z)$ with $\overline{W_z} \subset U_z$. By paracompactness of Z there exists a locally finite refinement γ of the open cover $\{W_z : z \in Z\}$ of the space Z. It is easy to see that the family $\mathcal{F} = \{\,\overline{W} : W \in \gamma\}$ is a locally finite refinement of $\mathcal{U}$. For each $F \in \mathcal{F}$ choose $s = s(F) \in S$ such that $F \subset U_s$. Let $F_s = \bigcup\{F \in \mathcal{F} : s(F) = s\}$ for each $s \in S$. It is straightforward that the family $\{F_s : s \in S\}$ is as promised and hence Fact 2 is proved.

Fact 3. If every open cover of a space Z has a locally finite closed refinement then Z is paracompact.

Proof. Let $\mathcal{U}$ be an open cover of the space Z; take a locally finite refinement $\mathcal{A} = \{A_s : s \in S\}$ of $\mathcal{U}$ and for every $z \in Z$ fix $V_z \in \tau(z, Z)$ which intersects only finitely many elements of $\mathcal{A}$. Find a locally finite closed refinement $\mathcal{F}$ of the open cover $\{V_z : z \in Z\}$ and let $W_s = X\backslash\bigcup\{F \in \mathcal{F} : F \cap A_s = \emptyset\}$ for any $s \in S$. Observe that

$(*)$ for any $s \in S$ and any $F \in \mathcal{F}$ we have $W_s \in \tau(A_s, Z)$ and $W_s \cap F \neq \emptyset$ if and only if $A_s \cap F \neq \emptyset$.

For every $s \in S$ choose $U(s) \in \mathcal{U}$ such that $A_s \subset U(s)$ and let $V_s = U(s) \cap W_s$. The family $\{V_s : s \in S\}$ is an open refinement of $\mathcal{U}$. If $z \in Z$ then, there is $O \in \tau(z, Z)$ which intersects only finitely many elements of $\mathcal{F}$, say $F_1, \ldots, F_n$. Each F_i intersects only finitely many elements of $\mathcal{A}$ and by $(*)$ it intersects but finitely many elements of $\{W_s : s \in S\}$. As a consequence, the set O intersects only finitely many elements of the cover $\{V_s : s \in S\}$ and Fact 3 is proved.

Fact 4. A perfect image of a paracompact space is a paracompact space.

Proof. Suppose that $h : Z \to T$ is a perfect map. Take an arbitrary open cover $\mathcal{V} = \{V_s : s \in S\}$ of the space T. Then $\{h^{-1}(V_s) : s \in S\}$ is an open cover of the space X. Apply Fact 2 to find a locally finite closed cover $\mathcal{F} = \{F_s : s \in S\}$ of X such that $F_s \subset h^{-1}(V_s)$ for each $s \in S$. We have $f(F_s) \subset V_s$ for each $s \in S$ and hence $\mathcal{G} = \{f(F_s) : s \in S\}$ is a closed refinement of $\mathcal{V}$. By Fact 3 it suffices to show that $\mathcal{G}$ is locally finite. So take any $t \in T$. For any $z \in h^{-1}(t)$ there is $O_z \in \tau(z, Z)$ such that O_z meets only finitely many elements of $\mathcal{F}$. By compactness of $h^{-1}(t)$ there are $z_1, \ldots, z_n \in h^{-1}(t)$ such that $h^{-1}(t) \subset O = O_{z1} \cup \cdots \cup O_{zn}$. Now, we can apply Fact 1 to find $W \in \tau(t, T)$ with $h^{-1}(W) \subset O$. It is immediate that W intersects only finitely many elements of $\mathcal{G}$ and hence $\mathcal{G}$ is locally finite. Fact 4 is proved.

Returning to our solution let $f : X \to Y$ be a perfect map of a metrizable space X onto a space Y. Then the space Y is paracompact by Problem 218 and Fact 4. Fix a metric d on the space X such that $\tau(d) = \tau(X)$. Given a point $y \in Y$ let $U_i(y) = \{x \in X : d_{f^{-1}(y)}(x) = \inf\{d(x, z) : z \in f^{-1}(y)\} < 1/i\}$. The function $d_{f^{-1}(y)} : X \to \mathbb{R}$ is continuous (Fact 1 of S.212) and hence $U_i(y)$ is an open set for any $y \in Y$ and $i \in \mathbb{N}$.

The family $\mathcal{B}_y = \{U_i(y) : i \in \mathbb{N}\}$ is an outer base of the set $f^{-1}(y)$. Indeed, let $U \in \tau(f^{-1}(y), X)$. For $F = X \setminus U$ the function d_F is continuous and $d_F(x) > 0$ for any $x \in f^{-1}(y)$. The subspace $f^{-1}(y)$ being compact, there is $\varepsilon > 0$ such that $d_F(x) > \varepsilon$ for each $x \in f^{-1}(y)$. This implies $U_i(y) \subset U$ for any $i > \frac{1}{\varepsilon}$ which proves that $\mathcal{B}_y$ is an outer base of $f^{-1}(y)$.

Now, let $W_i(y) = f^{\#}(U_i(y))$ and $V_i(y) = f^{-1}(W_i(y)) \subset U_i(y)$ for each $y \in Y$ and $i \in \mathbb{N}$. Observe that $U_j(y) \subset U_i(y)$, $W_j(y) \subset W_i(y)$ and $V_j(y) \subset V_i(y)$ whenever $j \geqslant i$. Besides, the family $\{W_i(y) : i \in \mathbb{N}\}$ is a local base in Y at any $y \in Y$. To see this, take any $V \in \tau(y, Y)$, find $i \in \mathbb{N}$ with $U_i(y) \subset f^{-1}(V)$ and observe that $W_i(y) \subset f(U_i(y)) \subset V$. The following property is crucial.

$(**)$ For every $y \in Y$ and $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $W_j(z) \subset W_i(y)$ whenever $y \in W_j(z)$.

To prove it, find $j \geqslant 2i$ such that $U_j(y) \subset V_{2i}(y)$; this is possible because $\mathcal{B}_y$ is an outer base of $f^{-1}(y)$. Take any $z \in Y$ such that $y \in W_j(z)$. Then $f^{-1}(y) \subset V_j(z) \subset U_j(z)$, i.e., there exist $x \in f^{-1}(z)$ and $x' \in f^{-1}(y)$ such that $d(x, x') < 1/j$. As a consequence, $U_j(y) \cap f^{-1}(z) \neq \emptyset$ and $f^{-1}(z) \subset V_{2i}(y)$ because the last set contains the fibre $f^{-1}(f(x))$ together with any point $x \in V_{2i}(y)$.

To prove that $W_j(z) \subset W_i(y)$ take any $t \in W_j(z)$. Then $f^{-1}(t) \subset U_j(z)$ and hence for any $x \in f^{-1}(t)$ there is $x' \in f^{-1}(z)$ such that $d(x, x') < 1/j \leqslant 1/(2i)$. We have shown that $f^{-1}(z) \subset V_{2i}(y) \subset U_{2i}(y)$ which implies that there exists $x'' \in f^{-1}(y)$ such that $d(x', x'') < 1/(2i)$. By the triangle inequality, we have $d(x, x'') < 1/i$ and hence $x \in U_i(y)$. The point $x \in f^{-1}(t)$ having been chosen arbitrarily, we have $f^{-1}(t) \subset U_i(y)$ whence $t \in W_i(y)$ and the proof of $(**)$ is concluded.

For each $i \in \mathbb{N}$ the family $\mathcal{W}_i = \{W_i(y) : y \in Y\}$ is an open cover of Y and hence there is a locally finite open refinement $\mathcal{B}_i$ of the cover $\mathcal{W}_i$. We claim that the family $\mathcal{B} = \bigcup\{\mathcal{B}_i : i \in \mathbb{N}\}$ is a base of Y. To prove this assume that $y \in U \in \tau(Y)$. There exists $i \in \mathbb{N}$ such that $W_i(y) \subset U$. By $(**)$ there is $j \in \mathbb{N}$ such that $y \in W_j(z)$

implies $W_j(z) \subset W_i(y)$. Take $W \in \mathcal{B}_j$ such that $y \in W$. Since $\mathcal{B}_j$ is a refinement of $\mathcal{W}_j$, there is $z \in Y$ for which $W \subset W_j(z)$. Therefore $y \in W_j(z)$ and hence $y \in W \subset W_j(z) \subset W_i(y) \subset U$ and we are done. As a consequence $\mathcal{B}$ is a σ-locally finite base of Y and hence Y is metrizable by Problem 121(iii).

S.227. *Show that a closed image of a countable second countable space is not necessarily a metrizable space.*

Solution. Let $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ be the usual convergent sequence. The space $X = S \times \mathbb{N}$ is countable and second countable (the topology of $\mathbb{N}$ is discrete). Now, let the underlying set of Y be $\{(0,0)\} \cup (\mathbb{N} \times \mathbb{N})$. Denote by T_m the set $\mathbb{N} \times \{m\} \subset Y$ for all $m \in \mathbb{N}$. All points of $\mathbb{N} \times \mathbb{N}$ are isolated in Y and a set $U \ni \xi = (0, 0)$ is open in Y if and only if $T_m \backslash U$ is finite for all $m \in \mathbb{N}$. The space X is Tychonoff being a product of Tychonoff spaces. To show that Y is Tychonoff, observe first that Y is Hausdorff. Indeed, if $x, y \in Y$ are distinct points then one of them, say x, is isolated. Then $U = \{x\}$ and $V = Y\backslash\{x\}$ are disjoint open sets which separate the points x and y. Given $A \subset Y$, denote by $\chi_A : Y \to \{0,1\}$ the characteristic function of A defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

Suppose that $x \in Y$ and $F \not\ni x$ is closed in Y. If x is isolated then the continuous function $\chi_{\{x\}}$ separates x and F. If $x = \xi$ then the set F is open and hence the function $\chi_{X\backslash F}$ is continuous and separates x from F. This proves that Y is Tychonoff.

Now, if $x = (\frac{1}{n}, m) \in X$ then let $f(x) = (n, m) \in Y$. For any $x = (0, n)$ we let $f(x) = \xi$. The map $f : X \to Y$ is evidently onto, so let us check that f is continuous. It suffices to verify continuity at all points of the set $L = \{x_m = (0, m) : m \in \mathbb{N}\}$ because all other points of X are isolated. So take any $x_m \in L$ and any open $O \ni f(x_m) = \xi$. The set $A = T_m\backslash O$ is finite and hence there is $p \in \mathbb{N}$ such that $(n, m) \in O$ for all $n \geqslant p$. The set $U = \{x_m\} \cup \left\{\left(\frac{1}{n}, m\right) : n \geqslant p\right\}$ is open in X and $f(U) \subset O$ which proves continuity at the point x_m. Therefore, f is continuous.

To see that f is a closed map, note first that every set $A \subset Y$ with $\xi \in A$, is closed in Y. Therefore, $f(F)$ is closed in Y for any F which intersects the set L. Now, if $F \cap L = \emptyset$ then $F \cap (S \times \{m\})$ has to be finite for any $m \in \mathbb{N}$. This means that $f(F) \cap T_m$ is finite for any $m \in \mathbb{N}$ and therefore $Y\backslash f(F)$ is open in Y, i.e., $f(F)$ is closed in Y.

Thus the space Y is a closed image of the countable second countable space X. To finish our solution it suffices to show that Y is not first countable and hence not metrizable (Problem 210). Let $\mathcal{B} = \{O_n : n \in \mathbb{N}\} \subset \tau(\xi, Y)$. For any $m \in \mathbb{N}$ the set $T_m\backslash O_m$ is finite and hence we can choose $s_m \in O_m \cap T_m$ for each $m \in \mathbb{N}$. It is straightforward that $W = Y\backslash\{s_m : m \in \mathbb{N}\}$ is an open set such that $\xi \in W$. However, $s_m \in O_m\backslash W$ for each $m \in \mathbb{N}$ which shows that no O_m is contained in W. As a consequence, $\mathcal{B}$ is not a base of Y at ξ and our solution is complete.

S.228. *Suppose that $C_p(X)$ is a closed image of a metrizable space (that is, there is a metrizable space M and a closed map $\varphi : M \to C_p(X)$). Prove that $C_p(X)$ is metrizable and hence X is countable.*

Solution. Call a continuous onto map $h : Y \to Z$ *irreducible* if, for any closed $F \subset Y$ with $F \neq Y$, we have $h(F) \neq Z$. For any $U \subset Y$ let $h^\#(U) = Z \setminus h(Y \setminus U)$. A family $\mathcal{B} \subset \tau^*(Y)$ is called *a* π-*base* in Y if, for any $U \in \tau^*(Y)$, there is $V \in \mathcal{B}$ such that $V \subset U$. The minimal cardinality of all possible π-bases of Y is called π-*weight of* Y and is denoted by $\pi\omega(Y)$.

Fact 1. Assume that $h : Y \to Z$ is an irreducible closed map. Then $\boldsymbol{c}(Y) = \boldsymbol{c}(Z)$ and $\pi w(Y) = \pi w(Z)$.

Proof. Apply Problem 157(i) to see that $c(Z) \leqslant c(Y)$. If $\gamma \subset \tau^*(Y)$ is a disjoint family then $\mu = \{h^\#(U) : U \in \gamma\} \subset \tau(Z)$ is also disjoint (see Fact 1 of S.226) and $h^\#(U) \neq \emptyset$ for each $U \in \gamma$ because h is irreducible. This shows that $c(Y) \leqslant c(Z)$ and hence $c(Y) = c(Z)$.

Take any π-base $\mathcal{B}$ in Y. Then $\mathcal{B}' = \{h^\#(U) : U \in \mathcal{B}\}$ consists of non-empty open subsets of Z because h is irreducible. Given $W \in \tau^*(Z)$, find $U \in \mathcal{B}$ with $U \subset h^{-1}(W)$. Then $U' = h^\#(U) \in \mathcal{B}'$ and $U' \subset W$ which proves that $\mathcal{B}'$ is a π-base in Z. As a consequence $\pi w(Z) \leqslant \pi w(Y)$. Given a π-base $\mathcal{B}'$ in Z, let $\mathcal{B} = \{h^{-1}(U) : U \in \mathcal{B}'\}$. It is evident that $\mathcal{B} \subset \tau^*(Y)$. Given any open non-empty $W \subset Y$, the set $h^\#(W)$ is open in Z and non-empty because h is irreducible. Find any $U \in \mathcal{B}'$ with $U \subset h^\#(W)$. Then $V = h^{-1}(U) \in \mathcal{B}$ and $V \subset h^{-1}(h^\#(W)) \subset W$ (see Fact 1 of S.226) and hence $\mathcal{B}$ is a π-base of Y. This establishes that $\pi w(Y) \leqslant \pi w(Z)$ whence $\pi w(Y) = \pi w(Z)$ and Fact 1 is proved.

Fact 2. Let Y be a metrizable space. Then, for any closed discrete $D \subset X$ there exists a discrete family $\{U_d : d \in D\} \subset \tau(Y)$ such that $d \in U_d$ for any $d \in D$.

Proof. Fix a metric ρ on Y with $\tau(\rho) = \tau(X)$. For any $d \in D$ find $\varepsilon = \varepsilon(d) > 0$ such that $B(d, \varepsilon) \cap D = \{d\}$ and let $V_d = B\left(d, \frac{\varepsilon(d)}{3}\right)$ for each $d \in D$. Note first that $\rho(d, c) \geqslant \max\{\varepsilon(c), \varepsilon(d)\}$ for any distinct $c, d \in D$. The family $\{V_d : d \in D\}$ is disjoint for if $x \in V_c \cap V_d$ for some distinct $c, d \in D$ then $\rho(c, d) \leqslant \rho(c, x) + \rho(x, d)$ which implies $\rho(c, d) < \frac{2}{3}\max\{\varepsilon(c), \varepsilon(d)\}$ which is a contradiction. The sets D and $F = Y \setminus (\bigcup\{V_d : d \in D\})$ are closed and disjoint. Since the space Y is normal (Fact 2 of S.212), we can find $G, H \in \tau(Y)$ such that $D \subset G$, $F \subset H$ and $G \cap H = \emptyset$. Finally, let $U_d = V_d \cap G$ for all $d \in D$. We claim that the family $\{U_d : d \in D\}$ is discrete. Indeed, take any $x \in Y$. If $x \in F$ then $x \notin \overline{G}$ and hence $Y \setminus \overline{G}$ is a neighbourhood of x which does not intersects any of U_d's. If $x \notin F$ then $x \in V_d$ for some $d \in D$ and hence V_d is a neighbourhood of x which does not intersect any U_c with $c \neq d$. Fact 2 is proved.

Fact 3. Let Y be a metrizable space. Suppose that Z is a space in which any point is a limit of a non-trivial convergent sequence. Then any closed map $h : Y \to Z$ is irreducible on some closed subset of Y, i.e., there is a closed $F \subset Y$ such that $h(F) = Z$ and $h_F = h|F$ is irreducible.

Proof. For every $y \in Z$ fix a sequence $S_y = \{y_n : n \in \omega\} \subset Z \setminus \{y\}$ converging to y. We will prove first that the set $P_y = h^{-1}(y) \cap \overline{\bigcup\{h^{-1}(y_n) : n \in \omega\}}$ is compact for every $y \in Z$. Indeed, if for some $y \in Z$ the set P_y is not compact, then it is not countably compact (Problem 212) and therefore there is a countably infinite closed discrete set $D = \{x_n : n \in \omega\} \subset P_y$. Apply Fact 2 to find a discrete family $\gamma = \{U_n : n \in \omega\} \subset \tau^*(Y)$ with $x_n \in U_n \cap P_y$ for all $n \in \omega$.

If A is an arbitrary finite subset of ω, then for each natural number n, we have $U_n \cap (\bigcup\{h^{-1}(z_k) : k \in \omega\backslash A\}) \neq \emptyset$. This makes it possible to choose a point $z_n \in U_n \cap (\bigcup\{h^{-1}(z_k) : k \in \omega\})$ in such a way that $h(z_m) \neq h(z_n)$ if $n \neq m$.

The family γ being discrete the set $D = \{z_n : n \in \omega\}$ is closed and discrete in Y. The set $h(D)$ is also closed because h is a closed map. Note that $h(D)$ has also to be discrete because $h(C)$ is closed for any $C \subset D$. However, $h(D)$ is a non-trivial sequence converging to y, a contradiction with the fact that $h(D)$ is closed and discrete. This proves P_y is compact for all $y \in Z$.

Claim. Suppose that H is a closed subset of Y such that $h(H) = Z$. Then $H \cap P_y \neq \emptyset$ for all $y \in Z$.

Proof of the claim. Fix $y \in Z$ with $H \cap P_y = \emptyset$. It follows from $h(H) = Z$ that it possible to choose $t_n \in H \cap h^{-1}(y_n)$ for all $n \in \omega$. The map h is closed, so $\overline{\{t_n : n \in \omega\}} \cap h^{-1}(y) \neq \emptyset$. But $H \supset \overline{\{t_n : n \in \omega\}}$ and $\overline{\{t_n : n \in \omega\}} \cap h^{-1}(y) \subset P_y$. Therefore, we have $H \cap P_y \neq \emptyset$ and the claim is proved.

Suppose that we have a family $\mathcal{F}$ of closed subsets of Y such that $\mathcal{F}$ is totally ordered by inclusion and $h(H) = Z$ for every $H \in \mathcal{F}$. Then $h(\bigcap\mathcal{F}) = Z$. Indeed, $H \cap P_y \neq \emptyset$ for any $y \in Z$ and $H \in \mathcal{F}$. We proved that the set P_y is compact so $(\bigcap\mathcal{F}) \cap h^{-1}(y) \supset (\bigcap\mathcal{F}) \cap P_y \neq \emptyset$ for all $y \in Z$ and we are done. Finally, use Zorn's lemma to find a closed $F \subset Y$ which is maximal (with respect to the inverse inclusion) in the family of all closed sets $H \subset Y$ such that $h(H) = Z$. It is evident that h_F is irreducible and Fact 3 is proved.

Now we are ready to present the solution. Given $f \in C_p(X)$ observe that the sequence $\{f + \frac{1}{n}\}$ is non-trivial and converges to f. Therefore Fact 3 is applicable to the map $\varphi : M \to C_p(X)$ to obtain a closed $F \subset M$ such that $\varphi(F) = C_p(X)$ and $\varphi|F$ is irreducible. Apply Fact 1 to conclude that $c(F) = c(C_p(X)) = \omega$. As a consequence $\omega(F) = \omega$ by Problem 214. Use Fact 1 once more to conclude that $\pi\omega(C_p(X)) = \pi\omega(F) \leqslant w(F) = \omega$. Observe that any π-base of $C_p(X)$ is also a π-base at any point of $C_p(X)$ (see Problem 171). Hence $C_p(X)$ has a countable π-base at any of its points and therefore X is countable by Problem 171. To finish the solution observe that $C_p(X)$ is metrizable by Problem 210.

S.229. *Suppose that $C_p(X)$ is an open image of a metrizable space (that is, there is a metrizable space M and an open map $\varphi : M \to C_p(X)$). Prove that $C_p(X)$ is metrizable and hence X is countable.*

Solution. Apply Problem 223 to show that $C_p(X)$ is first countable. This means X is countable (Problem 169). Applying Problem 210 we can conclude that $C_p(X)$ is metrizable.

S.230. *Prove that the following conditions are equivalent for any space X.*

(*i*) *X is paracompact.*
(*ii*) *Every open cover of X has a (not necessarily open) locally finite refinement.*

(*iii*) *Every open cover of X has a closed locally finite refinement.*
(*iv*) *Every open cover of X has a σ-locally finite open refinement.*
(*v*) *Every open cover of X has a σ-discrete open refinement.*
(*vi*) *Every open cover of X has an open closure-preserving refinement.*
(*vii*) *Every open cover of X has a closure-preserving refinement.*
(*viii*) *Every open cover of X has a closed closure-preserving refinement.*
(*ix*) *Every open cover of X has a σ-closure-preserving open refinement.*
(*x*) *Every open cover of X has a barycentric open refinement.*
(*xi*) *Every open cover of X has an open star refinement.*

Solution. The implication (i) ⇒ (ii) is obvious. If (ii) holds and $\mathcal{U}$ is an open cover of X then, for each $x \in X$, take $U \in \mathcal{U}$ with $x \in U$. By regularity of X there is $U_x \in \tau(x, X)$ such that $\overline{U}_x \subset U$. If $\mathcal{F}$ is a locally finite refinement of the cover $\{U_x : x \in X\}$ then $\mathcal{G} = \{\overline{A} : A \in \mathcal{F}\}$ is a locally finite closed refinement of $\mathcal{U}$ and (ii) ⇒ (iii) is proved. The implication (iii) ⇒ (i) is precisely Fact 3 from S.226.

The implication (i) ⇒ (iv) is evident. Suppose that (iv) holds and take any open cover $\mathcal{U}$ of the space X. Let $\mathcal{B} = \bigcup\{\mathcal{B}_i : I \in \omega\}$ be an open refinement of $\mathcal{U}$ such that all $\mathcal{B}_i$'s are locally finite. For any $n \in \omega$ let $U' = U\backslash(\bigcup\{\bigcup\mathcal{B}_i : i < n\})$ for every $U \in \mathcal{B}_n$. Let $\mathcal{B}'_n = \{U' : U \in \mathcal{B}_n\}$ and $\mathcal{B}' = \bigcup\{\mathcal{B}'_n : n \in \omega\}$. We claim that $\mathcal{B}$' is a locally finite refinement of $\mathcal{U}$. To see this, take any $x \in X$ and let $n = \min\{i \in \omega : x \in \bigcup \mathcal{B}_i\}$. There is $U \in \mathcal{B}_n$ with $x \in U$. It is clear that $x \in U$' $\in \mathcal{B}'$ and hence $\mathcal{B}'$ is a refinement of $\mathcal{U}$. To see that $\mathcal{B}$' is locally finite take any $x \in X$. If $x \in U \in \mathcal{B}_n$ then U is an open neighbourhood of x with $U \cap (\bigcup \mathcal{B}'_m) = \emptyset$ for any $m > n$. Since each $\mathcal{B}'_i$ is locally finite, there is $W \in \tau(x, X)$ such that W meets only finitely many elements of each $\mathcal{B}_i$, $i \leqslant n$. It is clear that $V = U \cap W \in \tau(x, X)$ and V intersects only finitely many elements of $\mathcal{B}'$. This settles (iv) ⇒ (ii) and hence the properties (i)–(iv) are equivalent.

Now suppose that X is paracompact and take any open cover $\mathcal{U}$ of the space X. Let $\{F_t : t \in T\}$ be a closed locally finite refinement of $\mathcal{U}$. For every $t \in T$ pick any $U_t \in \mathcal{U}$ with $F_t \subset U_t$. For any $x \in X$ denote by $T(x)$ the (finite) set $\{t \in T : x \in F_t\}$. Let $U_x = \bigcap\{U_t : t \in T(x)\}\backslash(\bigcup\{F_t : t \in T\backslash T(x)\})$. It is clear that $U_x \in \tau(x, X)$ and hence the family $\gamma = \{U_x : x \in X\}$ is an open refinement of $\mathcal{U}$. To show that γ is a barycentric refinement of $\mathcal{U}$ take any $x_0 \in X$ and any $t_0 \in T(x_0)$. If $x_0 \in U_x$ for some $x \in X$ then $t_0 \in T(x)$ and hence $U_x \subset U_{t_0}$. This proves that $\mathrm{St}(x_0, \gamma) \subset U_{t_0} \in \mathcal{U}$. Thus γ is a barycentric refinement of $\mathcal{U}$ and we proved (i) ⇒(x).

To show that (x) ⇒(xi) take any open cover $\mathcal{U}$ of the space X. Choose an open barycentric refinement $\mathcal{C}$ of the cover $\mathcal{U}$ and an open barycentric refinement $\mathcal{B}$ of the cover $\mathcal{C}$. We claim that $\mathcal{B}$ is a star refinement of $\mathcal{U}$. Indeed, pick any $W \in \mathcal{B}$ and $x \in W$. There is $U \in \mathcal{U}$ with $\mathrm{St}(x, \mathcal{C}) \subset U$. Now, if $W' \in \mathcal{B}$ and $W' \cap W \neq \emptyset$, take any $y \in W' \cap W$ and observe that $W \cup W' \subset \mathrm{St}(y, \mathcal{B})$. There exists $G \in \mathcal{C}$ such that $\mathrm{St}(y, \mathcal{B}) \subset G$ which implies $x \in W \cup W' \subset \mathrm{St}(y, \mathcal{B}) \subset G$ and therefore $G \subset \mathrm{St}(x, \mathcal{C}) \subset U$. As a consequence, $W' \subset U$. Since the set $W' \in \mathcal{B}$ with $W' \cap$

$W \neq \emptyset$ has been chosen arbitrarily, we have $\mathrm{St}(W, \mathcal{B}) \subset U$ and hence (x)⇒(xi) is settled.

Let us prove that (xi) ⇒(v). Take any open cover $\mathcal{U} = \{U_s : s \in S\}$ of the space X and construct a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that $\mathcal{U}_0 = \mathcal{U}$ and $\mathcal{U}_{n+1}$ is a star refinement of the cover $\mathcal{U}_n$ for each $n \in \omega$. For each $s \in S$ and $i \in \mathbb{N}$ let $U_{s,i} = \bigcup\{V \in \tau(X) : \mathrm{St}(V, \mathcal{U}_i) \subset U_s\}$. Since every $\mathcal{U}_i$ is a star refinement of $\mathcal{U}$, the family $\{U_{s,i} : s \in S\}$ is an open refinement of $\mathcal{U}$ for each $i \in \mathbb{N}$. It is also clear that $U_{s,i} \subset U_{s,i+1}$ for any $s \in S$ and $i \in \mathbb{N}$. We will use the following important property: $(*)$ If $x \in U_{s,i}$ and $y \notin U_{s,i+1}$ then no $U \in \mathcal{U}_{i+1}$ can contain both points x and y.

Indeed, if $x \in U \in \mathcal{U}_{i+1}$ then take any $V \in \mathcal{U}_i$ such that $\mathrm{St}(U, \mathcal{U}_{i+1}) \subset V$. Then $x \in V$ and hence $V \subset \mathrm{St}(x, \mathcal{U}_i) \subset U_s$. As a consequence $\mathrm{St}(U, \mathcal{U}_{i+1}) \subset U_s$ and therefore $U \subset U_{s,i+1}$.

Take any well order $<$ on S and let $V_{s,i} = U_{s,i} \backslash \overline{\bigcup\{U_{t,i+1} : t < s\}}$ for every $s \in S$ and $i \in \mathbb{N}$. The family $\mathcal{B}_i = \{V_{i,s} : s \in S\}$ is discrete for each $i \in \mathbb{N}$. To see this, take any $x \in X$ and any $U \in \mathcal{U}_{i+1}$ with $x \in U$. If $t < s$ then $V_{s,i} \subset X \setminus U_{t,i+1}$ and $(*)$ implies that U cannot intersect both sets $V_{t,i} \subset U_{t,i}$ and $V_{s,i} \subset X \setminus U_{t,i+1}$. Therefore U can intersect at most one element of $\mathcal{B}_i$ so $\mathcal{B}_i$ is discrete. To conclude the proof it suffices to show that $\mathcal{B} = \bigcup\{\mathcal{B}_i : I \in \mathbb{N}\}$ is a cover of X. Take any $x \in X$ and let $s(x)$ to be the minimal $s \in S$ such that $x \in U_{s,i}$ for some $i \in \mathbb{N}$. The existence of $s(x)$ follows from the fact that $\{U_{s,i} : s \in S\}$ is a cover of X for all $i \in \mathbb{N}$. Since $x \notin U_{s,i+2}$ for all $s < s(x)$, it follows from $(*)$ that $\mathrm{St}(x, \mathcal{U}_{i+2}) \cap (\bigcup\{U_{s,i+1} : s < s(x)\}) = \emptyset$ which implies $x \in V_{s(x),i}$ and the implication (xi) ⇒ (v) is proved.

Since the implication (v) ⇒ (iv) is obvious and (iv) ⇔ (i) we proved that the properties (i)–(v), (x) and (xi) are equivalent. Every locally finite family is closure-preserving (Fact 2 of S.221), so we have (i) ⇒ (vi). The implication (vi) ⇒ (vii) is evident. Note that if $\mathcal{F}$ is a closure-preserving family then the family $\{\overline{A} : A \in \mathcal{F}\}$ is also closure-preserving. Now assume that we have (vii) and $\mathcal{U}$ is an open cover of the space X. For each $x \in X$, take $U \in \mathcal{U}$ with $x \in U$. By regularity of X there is $U_x \in \tau(x, X)$ such that $\overline{U}_x \subset U$. If $\mathcal{F}$ is a closure-preserving refinement of the cover $\{U_x : x \in X\}$ then $\mathcal{G} = \{\overline{A} : A \in \mathcal{F}\}$ is a closure-preserving closed refinement of $\mathcal{U}$ and hence (vii) ⇒ (viii).

We will now prove that (ix) ⇒ (vii). It is easy to verify that the following statement holds:
$(**)$ If $\mathcal{C}$ is a closure-preserving family of closed subsets of X and F is closed in X then the family $\mathcal{C}_F = \{C \cap F : C \in \mathcal{C}\}$ is also closure-preserving.

Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, take $U \in \mathcal{U}$ with $x \in U$. By regularity of X there is $U_x \in \tau(x, X)$ such that $\overline{U}_x \subset U$. Take an open refinement $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ of the family $\{U_x : x \in X\}$ such that $\mathcal{B}_n$ is closure-preserving for each $n \in \omega$. For any $n \in \omega$ let $U' = \overline{U} \backslash (\bigcup\{\bigcup \mathcal{B}_i : i < n\})$ for every $U \in \mathcal{B}_n$. Let $\mathcal{B}'_n = \{U' : U \in \mathcal{B}_n\}$ and $\mathcal{B}' = \bigcup\{\mathcal{B}'_n : n \in \omega\}$. We claim that $\mathcal{B}'$ is a closure-preserving refinement of $\mathcal{U}$. To see this, observe first that $(**)$ implies that $\mathcal{B}'_n$ is closure-preserving because so is $\{\overline{U} : U \in_n \mathcal{B}_n\}$. Now take any $x \in X$ and let $n = \min\{i \in \omega : x \in \bigcup \mathcal{B}_i\}$. There is $U \in \mathcal{B}_n$ with $x \in U$. It is clear that $x \in U' \in \mathcal{B}'$ and hence $\mathcal{B}'$ is a refinement of $\mathcal{U}$. To see that $\mathcal{B}'$ is closure-preserving take any $\mathcal{C} \subset \mathcal{B}'$

and any $x \in \overline{\bigcup \mathcal{C}}$. If $x \in U \in \mathcal{B}_n$ then U is an open neighbourhood of x with $U \cap (\bigcup \mathcal{B}'_m) = \emptyset$ for any $m > n$. As a consequence $x \in \overline{C_0 \cup \cdots \cup C_n}$ where $C_i = \bigcup(\mathcal{C} \cap \mathcal{B}'_i)$ for each $i \leqslant n$. Therefore there is $j \leqslant n$ with $x \in \overline{C}_j = \bigcup \overline{(\mathcal{C} \cap \mathcal{B}'_j)}$. Since the family $\mathcal{B}'_j$ is closure-preserving, there is $W \in \mathcal{C} \cap \mathcal{B}'_j$ such that $x \in \overline{W}$. The point x having been chosen arbitrarily, we proved that $\mathcal{B}'$ is closure-preserving and hence (ix) $\Rightarrow$ (vii).

Since (v)$\Rightarrow$(ix)$\Rightarrow$(vii) and (v)$\Rightarrow$(vi)$\Rightarrow$(vii)$\Rightarrow$(viii), we will finish our solution proving that (viii)$\Rightarrow$(v).

Claim 1. If (viii) holds for X then X is normal.

Proof. Assume that F, G are closed disjoint subsets of X. Find a closure-preserving closed refinement $\mathcal{F}$ for the open cover $\mathcal{U} = \{X \setminus F, X \setminus G\}$ of the space X. Let $U = X \setminus (\bigcup\{P \in \mathcal{F} : P \cap F = \emptyset\})$ and $V = X \setminus (\bigcup\{P \in \mathcal{F} : P \cap G = \emptyset\})$. It is clear that $F \subset U$ and $G \subset V$ so it suffices to prove that $U \cap V = \emptyset$. By definition of U, if $x \in U$ then $P \cap F \neq \emptyset$ for every $P \in \mathcal{F}$ with $x \in P$. Analogously, if $x \in V$ then $P \cap G \neq \emptyset$ for any $P \in \mathcal{F}$ with $x \in P$. Thus, if $x \in U \cap V$ then there is $P \in \mathcal{F}$ such that $P \cap F \neq \emptyset \neq P \cap G$ which is a contradiction with the fact that each $P \in \mathcal{F}$ lies in $X \setminus F$ or in $X \setminus G$. Claim 1 is proved.

Claim 2. Suppose that $\{U_s : s \in S\} \subset \tau(X)$ and $X = \bigcup\{U_s : s \in S\}$. Then there exist closed sets F_s, $s \in S$ such that $F_s \subset U_s$ for all $s \in S$ and $\bigcup\{F_s : s \in S\} = X$.

Proof. Take any closure-preserving closed refinement $\mathcal{F}$ of the cover $\{U_s : s \in S\}$ of the space X. For each $P \in \mathcal{F}$ fix $s = s(P) \in S$ such that $P \subset U_s$ and let $F_s = \bigcup\{P \in \mathcal{F} : s(P) = s\}$ for each $s \in S$. Each F_s is closed because $\mathcal{F}$ is closure-preserving. It is obvious that $F_s \subset U_s$ for each $s \in S$ and $\bigcup\{F_s : s \in S\} = \bigcup \mathcal{F} = X$ so Claim 2 is proved.

Now take any open cover $\mathcal{U} = \{U_s : s \in S\}$ of the space X. We will consider the set S to be well ordered by $<$. We are in position to apply Claim 2 to find a closed cover $\mathcal{F}_1 = \{F_{s,1} : s \in S\}$ of the space X such that $F_{s,1} \subset U_s$ for each $s \in S$. Suppose that we have closed covers $\mathcal{F}_1, \cdots, \mathcal{F}_n$ of the space X such that

(1) $\mathcal{F}_i = \{F_{s,i} : s \in S\}$.
(2) $F_{s,i+1} \subset U_s \setminus (\bigcup\{F_{t,i} : t < s\}$ for all $s \in S$ and $i < n$.

Let $V_s = U_s \setminus (\bigcup\{F_{t,n} : t < s\})$ for each index $s \in S$. Then, the family $\{V_s : s \in S\}$ is an open cover of the space X. Indeed, for any point $x \in X$, let $s(x)$ be the minimal of the elements $s \in S$ such that $x \in U_s$. It follows from $x \notin \bigcup\{U_s : s < s(x)\} \supset \bigcup\{F_{s,n} : s < s(x)\}$ that $x \in V_{s(x)}$. Apply Claim 2 once more to find a closed cover $\mathcal{F}_{n+1} = \{F_{s,n+1} : s \in S\}$ of the space X such that $F_{s,n+1} \subset V_s$ for each $s \in S$. It is clear that the closed covers $\mathcal{F}_1, \ldots, \mathcal{F}_n, \mathcal{F}_{n+1}$ satisfy (1) and (2) so the inductive construction goes on. Once we have the sequence $\{\mathcal{F}_i : i \in \mathbb{N}\}$, let $W_{s,i} = X \setminus (\bigcup\{F_{t,i} : t \neq s\})$ for each $s \in S$ and $i \in \mathbb{N}$. Note first that the family $\mathcal{W}_i = \{W_{s,i} : s \in S\}$ is disjoint for each $i \in \mathbb{N}$ because $W_{s,i} \subset F_{s,i}$ for all $s \in S$. We claim that the family $\mathcal{W} = \bigcup\{\mathcal{W}_i : i \in \mathbb{N}\}$ is a refinement of $\mathcal{U}$. Of course, we must only prove that

$\bigcup\mathcal{W} = X$. So take any $x \in X$ and denote by t the minimal among all $s \in S$ such that $x \in F_{s,i}$ for some $i \in \mathbb{N}$. Take any $j \in \mathbb{N}$ with $x \in F_{t,j}$.

Observe that (2) implies that $x \notin F_{s,j+1}$ for all $s > t$. On the other hand $x \notin F_{s,j+1}$ for any $s < t$ by the choice of t. Since $\mathcal{F}_{j+1}$ is a cover of X, we have $x \in V_{t,j+1}$ which proves that $\mathcal{W}$ is a cover of X. If $O_i = \bigcup\mathcal{W}_i$ for each $i \in \mathbb{N}$ then $\{O_i : i \in \mathbb{N}\}$ is an open cover of X. Apply Claim 2 once more to find closed $P_i \subset O_i$ such that $\bigcup\{P_i : i \in \mathbb{N}\} = X$. The space X is normal by Claim 1 so there exist $G_i \in \tau(P_i, X)$ such that $\overline{G}_i \subset O_i$ for all $i \in \mathbb{N}$. Let $\mathcal{V}_i = \{W_{s,i} \cap G_i : s \in S\}$. Since $P_i \subset \bigcup \mathcal{V}_i$ for each $i \in \mathbb{N}$, the family $\mathcal{V} = \bigcup\{\mathcal{V}_i : i \in \mathbb{N}\}$ is a cover of X. It is clear that $\mathcal{V}$ is a refinement of $\mathcal{U}$ so it suffices to show that $\mathcal{V}_i$ is discrete for each i.

Take any $x \in X$. If $x \notin O_i$ then $X \backslash \overline{G}_i$ is a neighbourhood of x which meets no elements of $\mathcal{V}_i$. Now, if $x \in O_i$ then $x \in W_{s,i}$ for some $s \in S$ and hence $W_{s,i}$ is a neighbourhood of x which meets only one element of $\mathcal{V}_i$. Thus $\mathcal{V}$ is a σ-discrete refinement of $\mathcal{U}$. We proved that (viii) $\Rightarrow$ (v) and hence our solution is complete.

S.231. *Prove that any paracompact space is collectionwise normal. In particular, every metrizable space is collectionwise normal.*

Solution. Suppose that X is a paracompact space and take any discrete family $\mathcal{F} = \{F_s : s \in S\}$ of closed subsets of X. For any $x \in X$ fix an open $V_x \ni x$ which meets at most one of the elements of $\mathcal{F}$. Let $\mathcal{B}$ be a closure-preserving closed refinement of the open cover $\{V_x : x \in X\}$ (see Problem 230(viii)). It is clear that any $B \in \mathcal{B}$ intersects at most one element of $\mathcal{F}$. If $U_s = X\backslash(\bigcup\{B \in \mathcal{B} : B \cap F_s = \emptyset\})$ then U_s is open and $F_s \subset U_s$ for any $s \in S$.

The family $\mathcal{U} = \{U_s : s \in S\}$ is disjoint. Indeed, if $s \neq t$ and $x \in U_s \cap U_t$ then take any $B \in \mathcal{B}$ with $x \in B$. It is immediate from the definition of U_s and U_t that $B \cap F_s \neq \emptyset \neq B \cap F_t$ which is a contradiction. This proves, in particular, that X is normal and hence we can choose $W \in \tau(X)$ such that $F \subset W \subset \overline{W} \subset \bigcup\mathcal{U}$, where $F = \bigcup\{F_s : s \in S\}$.

Now if $W_s = U_s \cap W$ then $F_s \subset W_s \in \tau(X)$ for each $s \in S$ so it suffices to show that the family $\mathcal{W} = \{W_s : s \in S\}$ is discrete. Given $x \in X$ suppose that $x \notin \bigcup\mathcal{U}$. Then $X \backslash \overline{W}$ is a neighbourhood of X which does not intersect any element of $\mathcal{W}$. If $x \in \mathcal{U}$ then $x \in U_s$ for some $s \in S$ and hence $U_s \in \tau(x, X)$ intersects at most one element of $\mathcal{W}$. Hence $\mathcal{W}$ is discrete and we proved collectionwise normality of each paracompact space. To finish our solution, observe that every metrizable space is paracompact (Problem 218) and hence collectionwise normal.

S.232. *Give an example of a space which is collectionwise normal but not paracompact.*

Solution. The underlying set of our space X will be the set ω_1 of all countable ordinals, i.e., $X = \{\alpha : \alpha < \omega_1\}$. Given $\alpha, \beta < \omega_1$ we will need the intervals $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $[\alpha, \beta) = \{\gamma : \alpha \leqslant \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leqslant \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leqslant \gamma \leqslant \beta\}$. Note that some of the defined intervals can be empty. For example, $(\alpha, \beta) = \emptyset$ if $\alpha \geqslant \beta$. The topology of X is generated by the family $\mathcal{B} = \{0\} \cup \{(\alpha, \beta) :$

$\alpha < \beta < \omega_1\}$ as a base. This generation makes sense because any non-empty intersection of two elements of $\mathcal{B}$ belongs to $\mathcal{B}$ and $\bigcup \mathcal{B} = X$.

The space X is Hausdorff. To see this, take distinct $\alpha, \beta \in X$. We may assume that $\alpha < \beta$. If $\alpha = 0$ then the open sets $U = \{0\}$ and $V = (0, \beta + 1)$ separate the points α and β. If $\alpha > 0$ then the open sets $U = (0, \alpha + 1)$ and $V = (\alpha, \beta + 1)$ separate the points α and β.

Note that $(\beta, \alpha + 1) = (\beta, \alpha]$ for all $\beta < \alpha < \omega_1$. As a consequence, the family $\mathcal{B}_\alpha = \{(\beta, \alpha] : \beta < \alpha\}$ is a local base at α for any $\alpha > 0$. It is easy to see that every element of $\mathcal{B}_\alpha$ is *clopen* ($\equiv$ closed-and-open) subset of X. Since $\{0\}$ is a clopen local base at 0, the space X has a base which consists of clopen sets. We will call such spaces *zero-dimensional*.

Fact 1. Any zero-dimensional T_0-space Y is Tychonoff.

Proof. Take distinct $x, y \in Y$. There exists an open set U such that $U \cap \{x, y\}$ consists of exactly one point. Suppose, for example that $U \cap \{x, y\} = \{x\}$. Then $x \in U$ and there is a clopen V such that $x \in V \subset U$. Then V and $W = Y \backslash V$ are disjoint open neighbourhoods of the points x and y, respectively. The case when $U \cap \{x, y\} = \{y\}$ is considered in the same way so we proved that Y is Hausdorff and, in particular, Y is a T_1-space.

Now, assume that $x \in Y$ and F is a closed set such that $x \notin F$. Since $U = Y \backslash F \in \tau(x, Y)$, we can find a clopen set W with $x \in W \subset U$. If $f(z) = 1$ for $z \in W$ and $f(z) = 0$ for all $z \in Y \backslash W$ then $f : Y \to [0, 1]$ is a continuous function, $f(x) = 1$ and $f(F) \subset \{0\}$ so Y is Tychonoff and Fact 1 is proved.

The space X is zero-dimensional, so it is Tychonoff by Fact 1.

Fact 2. The subspace $[0, \alpha]$ is compact for any $\alpha < \omega_1$.

Proof. It is clear that $[0, \alpha]$ is countable and hence Lindelöf. Thus it is sufficient to prove that $[0, \alpha]$ is countably compact (see Problem 138). Assume the contrary. Then there is a closed discrete infinite $D \subset [0, \alpha]$. Let $\alpha_0 = \min\{\beta : \beta \leqslant \alpha$ and $[0, \beta] \cap D$ is infinite$\}$. Since $D = D \cap [0, \alpha]$ is infinite, the ordinal α_0 is infinite and well defined. Take any $U \in \tau(\alpha_0, X)$. There exists $\beta < \alpha_0$ such that $(\beta, \alpha_0] \subset U$. If $(\beta, \alpha_0] \cap D$ is finite then $[0, \beta] \cap D$ is infinite while $\beta < \alpha_0$, a contradiction. Hence $U \cap D \supset (\beta, \alpha_0] \cap D$ is infinite for any $U \in \tau(\alpha_0, X)$ which contradicts the fact that D is closed and discrete. Fact 2 is proved.

Fact 3. The space X is normal.

Proof. Let F and G be disjoint closed subsets of X. If one of them is empty then the proof is trivial so we assume that $F \neq \emptyset \neq G$. We claim that one of the sets F, G is countable. To see this, assume that $|F| = |G| = \omega_1$ and take $\alpha_0 \in F$. Since G is uncountable, there is $\beta_0 \in G$ such that $\beta_0 > \alpha_0$. Suppose that we constructed ordinals $\{\alpha_i, \beta_i, i \leqslant n\}$ such that $\alpha_i \in F, \beta_i \in G$ for all $i \leqslant n$ and $\alpha_0 < \beta_0 < \alpha_1 < \cdots < \alpha_n < \beta_n$.

The set F being uncountable we can find $\alpha_{n+1} \in F$ with $\alpha_{n+1} > \beta_n$. By the same property of G there is $\beta_{n+1} \in G$ with $\beta_{n+1} > \alpha_{n+1}$ and the inductive construction can go on. Once we have the sequences $\{\alpha_n : n \in \omega\} \subset F$ and $\{\beta_n : n \in \omega\} \subset G$, let $\gamma = \min\{\delta : \alpha_n < \delta$ for all $n \in \omega\}$. Note that γ is well defined because the set $A = \bigcup\{[0, \alpha_n] : n \in \omega\} = \bigcup\{[0, \beta_n] : n \in \omega\}$ is countable and hence any $\delta \in X \backslash A$ is greater that all α_n's. Since $\beta_n < \alpha_{n+1} < \gamma$ for all $n \in \omega$, we have $\beta_n < \gamma$ for all $n \in \omega$.

Observe that, for any $\delta < \gamma$ there exists $m \in \omega$ such that $\alpha_m \geqslant \delta$ and hence $\alpha_n \in (\delta, \gamma]$ for all $n > m$. Since $\beta_n > \alpha_n > \delta$ for all $n > m$, we have $\beta_n \in (\delta, \gamma]$ for all $n > m$ which proves that $\alpha_n \to \gamma$ and $\beta_n \to \gamma$. The sets F and G being closed we have $\gamma \in F \cap G$, a contradiction. Hence one of our sets, say F, is countable. Consequently, the set $A_F = \bigcup\{[0, \alpha] : \alpha \in F\}$ is countable so $F \subset [0, \beta]$ for any $\beta \in X \backslash A_F$. The set $U' = [0, \beta]$ is compact by Fact 2 and open because $[0, \beta] = [0, \beta + 1)$. The set $G' = G \cap U'$ is closed and disjoint from F is the compact space U'. Therefore there are open sets U, V' (in U' and hence in X) such that $F \subset U, G' \subset V'$ and $U \cap V' = \emptyset$. Then $V = (X \backslash [0, \beta]) \cup V'$ is open in X, contains G and $U \cap V = \emptyset$ which yields normality of X. Fact 3 is proved.

Fact 4. The space X is countably compact.

Proof. If D is a countably infinite closed discrete subspace of X then there is $\alpha < \omega_1$ such that $D \subset [0, \alpha]$. The space $[0, \alpha]$ being compact by Fact 2, the set D cannot be closed and discrete in $[0, \alpha]$; this contradiction proves Fact 4.

Fact 5. Any countably compact normal space Y is collectionwise normal.

Proof. Let $\mathcal{F}$ be a discrete family of non-empty closed sets of Y. If $\mathcal{F}$ is infinite, then, choosing $x_F \in F$ for any $F \in \mathcal{F}$ we obtain an infinite closed discrete $D = \{x_F : F \in \mathcal{F}\} \subset X$ which contradicts the countable compactness of X. Hence $\mathcal{F}$ is finite, say $\mathcal{F} = \{F_0, \ldots, F_n\}$. Letting $f(x) = i$ for any $x \in F_i$, we obtain a continuous function $f : F = \bigcup\mathcal{F} \to \mathbb{R}$. By normality of X there is $g \in C(X)$ such that $g|F = f$ (Problem 032). Consider the sets $U_i = g^{-1}\left(\left(i - \frac{1}{3}, i + \frac{1}{3}\right)\right)$ for each $i \leqslant n$. It is immediate that $F_i \subset U_i$ for each $i \leqslant n$ and the family $\{\overline{U}_0, \ldots, \overline{U}_n\}$ is disjoint. It is easy to check that the family $\{U_0, \ldots, U_n\}$ is discrete and Fact 5 is proved.

Returning to our solution, we can conclude that X is collectionwise normal by Facts 3–5, so we only have to show that X is not paracompact.

Assume that X is paracompact. Then any open cover of X has a locally finite refinement which has to be finite by Fact 4 and Problem 136. Therefore X is compact. However, the family $\{[0, \alpha) : \alpha < \omega_1\}$ is an open cover of X which has no finite subcover; this contradiction shows that our solution is complete.

S.233. *Let* $N = \{(x, y) \in \mathbb{R}^2 : y \geqslant 0\}$. *Given* $z = (a, b)$ *such that* $b > 0$, *let* $\mathcal{B}_z = \{U_n(z) : n \in \mathbb{N}, n > \frac{1}{b}\}$ *where* $U_n(z) = \{(x, y) \in N : (x - a)^2 + (y - b)^2 < \frac{1}{n^2}\}$ *for each* $n \in \mathbb{N}, n > \frac{1}{b}$. *If* $z = (a, 0)$ *then* $\mathcal{B}_z = \{U_n(z) : n \in \mathbb{N}\}$ *where* $U_n(z) = \{z\} \cup \{(x, y) : (x - a)^2 + (y - \frac{1}{n})^2 < \frac{1}{n^2}\}$. *Show that*

(*i*) *The families* $\{\mathcal{B}_z : z \in N\}$ *generate a topology* τ *on* N *as local bases. The resulting space* $N = (N, \tau)$ *is called the Niemytzki plane.*
(*ii*) N *is a separable Tychonoff space with* $iw(N) = \omega$.

(iii) $\text{ext}(N) = \mathfrak{c}$ *and hence N is not normal.*
(iv) *N is a locally metrizable non-metrizable space.*

Solution. The underlying set of N is contained in $\mathbb{R}^2$ so we have the metric $d = \rho_2$ on N introduced in Problem 205. This metric does not generate the topology of N but comes in handy for quite a few useful considerations.

(i) Let $L = \{(x, 0) : x \in \mathbb{R}\} \subset N$ and $P = N \backslash L$. The property (LB1) is evident for any $\mathcal{B}_z$ (see Problem 007). For each $z \in N$ there is $m \in \mathbb{N}$ such that $\mathcal{B}_z = \{U_n(z) : n \in \mathbb{N}, n \geqslant m\}$ where $U_{n+1}(z) \subset U_n(z)$ for all $n \geqslant m$. Thus $U \cap V \in \mathcal{B}_z$ for any $U, V \in \mathcal{B}_z$ and hence (LB2) is also fulfilled. Now assume that $z \in U \in \mathcal{B}_t$. If $z = t$ then there is nothing to prove. If $z \neq t$ then $z \notin L$ because $(\bigcup \mathcal{B}_t) \cap L \subset \{t\}$ for any $t \in N$. Therefore, $z = (x, y)$ where $y > 0$. By definition of $\mathcal{B}_t$ the set $U \backslash \{t\}$ is a circle if $t \in L$ and a circle without center otherwise. In both cases the set $U \backslash \{t\} \ni z$ is open in $\mathbb{R}^2$ (see Problems 201 and 205) and hence there is $\varepsilon > 0$ such that $B_d(z, \varepsilon) \subset U \backslash \{t\}$. Then, for any $n > \frac{1}{\varepsilon}$ we have $U_n(z) \subset B_d(z, \varepsilon) \subset U$ so the property (LB3) holds as well. Finally, apply Problem 007 to conclude that the family $\{\mathcal{B}_z : z \in N\}$ generates a topology on N for which $\mathcal{B}_z$ is a local base at z for every point $z \in N$.

(ii) Let us check first that N is Hausdorff and hence T_1. Observe that, for any $z \in N$, we have $d(z, z') < \frac{2}{n}$ for any $z' \in U_n(z)$. Indeed, if $z \in P$ then z is the center of the circle $U_n(z)$ so $d(z, z') < \frac{1}{n}$. If $z \in L$ then for the center $\mathbf{c}$ of the circle $U_n(z) \backslash \{z\}$ we have $d(z, z') \leqslant d(z, c) + d(c, z') < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$. Now take any $z, t \in N$ with $z \neq t$ and find $n \in \mathbb{N}$ such that $n > \frac{4}{d(z,t)}$. We claim that $U_n(z) \cap U_n(t) = \emptyset$. Indeed, by our observation, if $z' \in U_n(z) \cap U_n(t)$ then $d(z, z') < \frac{2}{n}$ and $d(t, z') < \frac{2}{n}$ which implies $d(z, t) \leqslant d(z, z') + d(z', t) < \frac{4}{n} < d(z, t)$ which is a contradiction.

We will denote by N', L' and P' the respective sets N, L and P with the topology induced from $\mathbb{R}^2$. Let us prove that the map $\text{id} : N \to N'$ defined by $\text{id}(z) = z$, is continuous. Given any $z \in N$ and any $U \in \tau(z, N')$ there is $\varepsilon > 0$ such that $B_d(z, \varepsilon) \subset U$. Take any $n \in \mathbb{N}$ with $n > \frac{2}{\varepsilon}$. If $t \in U_n(z)$ then $d(t, z) < \frac{2}{n} < \varepsilon$ by the observation before. Therefore $t = \text{id}(t) \in U$ for any $t \in U_n(z)$ whence $\text{id}(U_n(z)) \subset U$, i.e., the map id is continuous at the point z. This immediately implies

Fact 1. Any map continuous on N' (at a point $z \in N'$) is continuous on N (at the same point $z \in N$, respectively).

Fact 2. The map id is a condensation of N onto a second countable space N' and hence $iw(N) \leqslant \omega$.

Proof. We only have to prove that $w(N') \leqslant \omega$. But this is immediate from the fact that $N' \subset \mathbb{R}^2$ and Problem 209.

Now it is easy to prove complete regularity at the points of P. Take any $z \in P$ and any closed $F \subset N$ with $z \notin F$. There is $n \in \mathbb{N}$ such that $U_n(z) \cap F = \emptyset$. The set $U_n(z)$ is open in N' being the ball of radius $\frac{1}{n}$ centered at z. Since N' is completely regular (Problem 202), there is a continuous function $f : N' \to [0, 1]$ such that $f(z) = 1$ and $f(N' \backslash U_n(z)) \subset \{0\}$. The function $f : N \to [0, 1]$ is also continuous by Fact 1 and we have $f(z) = 1$ and $f(F) \subset \{0\}$ so complete regularity is verified at all points of P.

Now take any $z = (x, 0) \in L$. Given a closed $F \subset N$ with $z \notin F$, there exists $n \in \mathbb{N}$ such that $U_n(z) \cap F = \emptyset$. Consider the set $C = \{(a, b) \in \mathbb{R}^2 : (a - x)^2 + (b - \frac{1}{n})^2 \leqslant \frac{1}{n^2}\} \backslash \{z\}$ endowed with the topology induced from N'. In other words, C is a closed circle centered at $(x, \frac{1}{n})$ of radius $\frac{1}{n}$ without the point z. We will need the subspaces $E = \{(a, b) \in \mathbb{R}^2 : (a - x)^2 + (b - \frac{1}{n})^2 = \frac{1}{n^2}\} \backslash \{z\}$ and $D = \{(a, b) \in \mathbb{R}^2 : (a - x)^2 + (b - \frac{1}{2n})^2 \leqslant \frac{1}{4n^2}\} \backslash \{z\}$ of the space C. It is clear that E and D are closed and disjoint in C. The space C is normal being second countable (see Problems 209 and 231) and hence there is a continuous function $g : C \to [0, 1]$ such that $g(E) \subset \{0\}$ and $g(D) \subset \{1\}$ (Problem 031). Define a function $f : N \to [0, 1]$ by $f(z) = 1$, $f(t) = g(t)$ for all $t \in C$ and $f(t) = 0$ for all $t \in N \backslash U_n(z)$. Since $F \subset N \backslash U_n(z)$, we have $f(F) \subset \{0\}$. Thus, to finish the proof of the Tychonoff property of N, we must only show that f is continuous.

The function f is continuous at z because $D \cup \{z\}$ contains z in its interior and $f(D \cup \{z\}) = \{1\}$. Take any $t \in N \backslash (C \cup \{z\})$. The set $C \cup \{z\}$ is a closed circle of radius $\frac{1}{n}$ centered at $w = \left(x, \frac{1}{n}\right)$ so $d(t, w) = r > \frac{1}{n}$. If $s = r - \frac{1}{n}$ then the set $B_d(w, s)$ is open in N' and does not meet $C \cup \{z\}$. This shows that the set $C \cup \{z\}$ is closed in N' and hence in N. The set $W = N \backslash (C \cup \{z\})$ is open in N and $f(W) = \{0\}$. Therefore f is continuous at any point of W.

Note that the set P is open in N because for any $t = (a, b) \in P$ we have $U_n(t) \subset P$ for every $U_n(t) \in \mathcal{B}_z$. Therefore continuity of f at every point of $t \in P$ is equivalent to continuity of $f|P$ at the point t. By Fact 1 it suffices to prove continuity of $f|P'$ at every point $t \in C$. If $t \in U_n(z) \backslash \{z\}$ then f is continuous at t because $W = U_n(z) \backslash \{z\}$ is an open subset of P' and $f|W = g|W$ and the function g is continuous. Finally, if $t \in E$ then, by continuity of g, for any $\varepsilon > 0$ there is $V' \in \tau(C)$ such that $t \in V'$ and $g(V') \subset [0, \varepsilon)$. Take any $V \in \tau(P')$ such that $V \cap C = V'$ and observe that $f(V \backslash C) = \{0\}$ and therefore $f(V) \subset [0, \varepsilon)$ so f is continuous at the point t. This proves that N is Tychonoff.

To finish the proof of (ii) we must show that N is separable. Since P is dense in N, it suffices to prove that P is separable. Note that P is homeomorphic to P'; the same map $\mathrm{id} : P \to P'$ is a homeomorphism. To see this we must only show that $\mathrm{id} : P' \to P$ is continuous. This follows from the fact that the family $\mathcal{B} = \bigcup\{\mathcal{B}_t : t \in P\}$ is a base in P and $\mathrm{id}^{-1}(U) = U$ is an open circle which is open in P'. Applying Problem 009 (ii) we conclude that $\mathrm{id} : P' \to P$ is continuous and hence P is homeomorphic to P'. Since $P' \subset \mathbb{R}^2$, we have $d(P') \leqslant \omega(P') \leqslant \omega$ (see Problems 209 and 156(i)). Thus P is separable and hence so is N.

(iii) We already saw that L is closed. Since $U_1(z) \cap L = \{z\}$ for each $z \in L$, the subspace L is discrete. Since $|L| = \mathfrak{c}$, we have $\mathrm{ext}(N) = \mathfrak{c}$. Applying Problem 164, we can conclude that N is not normal.

(iv) Since N is not normal, it cannot be metrizable (Problem 231). To see that N is locally metrizable, take any $z \in N$. If $z \in P$ then any $W = U_n(z) \in \mathcal{B}_z$ is an open neighbourhood of z and lies in a metrizable space P'. Thus W is metrizable and hence N is locally metrizable at z. Now take any $z \in L$. The space P is separable and metrizable so we can fix a countable base $\mathcal{C}$ in P. It is clear that the family $\mathcal{C} \cup \mathcal{B}_z$ is a base in the space $P \cup \{z\}$ and therefore $P \cup \{z\}$ is a second countable (and hence metrizable) neighbourhood of z. This proves that N is locally metrizable and our solution is complete.

S.234. *Prove that any paracompact locally metrizable space is metrizable.*

Solution. Let X be a paracompact locally metrizable space. For each $x \in X$ fix a metrizable $U_x \in \tau(x, X)$ and find a locally finite closed refinement $\mathcal{F} = \{F_s : s \in S\}$ of the open cover $\{U_x : x \in X\}$ of the space X. It is evident that F_s is metrizable for each $s \in S$.

Fact 1. Suppose that Y_t is a metrizable space for each index $t \in T$. Then the space $Y = \bigoplus \{Y_t : t \in T\}$ is metrizable (see Problem 113 for the definition of the discrete union).

Proof. For each $t \in T$ fix a base $\mathcal{B}_t$ in the space Y_t such that $\mathcal{B}_t = \bigcup \{\mathcal{B}_t^n : n \in \omega\}$ and each $\mathcal{B}_t^n$ is a discrete family (see Problem 221). We will identify each Y_t with the respective clopen summand of Y. Then the family $\mathcal{B}_n = \bigcup \{\mathcal{B}_t^n : t \in T\}$ is discrete for each $n \in \omega$ and $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ is a base in Y. Applying Problem 221 once more we can conclude that Y is metrizable. Fact 1 is proved.

For each $s \in S$ let $i_s : F_s \to X$ be the respective identity map, i.e., $i_s(x) = x$ for each $s \in S$. The space $F = \bigoplus \{F_s : s \in S\}$ is metrizable by Fact 1. We also identify each F_s with the respective clopen subspace of F. Given $x \in F$ take $s \in S$ with $x \in F_s$ and let $f(x) = i_s(x)$. The resulting map $f : F \to X$ is perfect. Indeed, the family $\mathcal{F}$ is a cover of X which implies that f is onto. Since $\mathcal{F}$ is point-finite, every $f^{-1}(x)$ is finite and hence compact. Given a closed $P \subset X$ we have $f^{-1}(P) = \bigcup\{P \cap F_s : s \in S\}$ (here each F_s is considered to be the respective subspace of F). The last set is closed in F because each $P \cap F_s$ is closed and the family $\{P \cap F_s : s \in S\}$ is discrete in F. This shows that the map f is continuous.

To finally see that f is closed, take any closed $Q \subset F$. Then $Q \cap F_s$ is closed for each $s \in S$ and hence $\bigcup\{Q \cap F_s : s \in S\}$ is closed in X (here each F_s is considered to be the respective subspace of X) because the family $\{Q \cap F_s : s \in S\}$ is locally finite and hence closure-preserving (Fact 2 of S.221). Since $f(Q) = \bigcup\{Q \cap F_s : s \in S\}$, we proved that the map f is closed and hence perfect. A perfect image of a metrizable space is metrizable (Problem 226) so our solution is complete.

S.235. *Let N be the Niemytzki plane. Prove that* $\text{ext}(C_p(N)) = \mathfrak{c}$. *Deduce from this fact that $C_p(N)$ is not normal.*

Solution. For any $t \in [0, 1]$, we will define a (discontinuous) function $f_t : \mathbb{R} \to \mathbb{R}$. First let $f_t(x) = 0$ for any $x \in (-\infty, -1) \cup [-t, 0) \cup [t, +\infty)$. If $x \in [-1, -t)$ then $f_t(x) = -\frac{1}{t+x}$ and $f_t(x) = \frac{1}{t-x}$ for all $x \in [0, t)$.

Fact 1. The set $D = \{f_t : t \in [0, 1]\}$ is closed and discrete in $\mathbb{R}^{\mathbb{R}}$.

Proof. Observe that $f_t(t) = f_t(-t) = 0$ for all $t \in [0, 1]$. Given $s > t$ we have $f_s(t) = \frac{1}{s-t} \geqslant \frac{1}{s} \geqslant 1$. If $s < t$ then $f_s(-t) = -\frac{1}{s-t} = \frac{1}{t-s} \geqslant \frac{1}{t} \geqslant 1$. As a consequence, the set $V_t = \{f \in \mathbb{R}^{\mathbb{R}} : \{f(t), f(-t)\} \subset (-\frac{1}{2}, \frac{1}{2})\}$ is open in $\mathbb{R}^{\mathbb{R}}$ and $V_t \cap D = \{f_t\}$ which shows that D is discrete.

Let h be the function equal to zero at all points of $\mathbb{R}$. Note that $h \in V_1$ and $V_1 \cap D = \{f_1\} \neq h$. This shows that $h \notin \overline{D}$. Suppose now that $g \in \overline{D} \backslash D$ and $g \neq h$. It is evident that $g(x) \geqslant 0$ for all $x \in \mathbb{R}$.

Claim. Suppose that $x_0 \in [0, 1)$, $g(x_0) > 0$ and $g(x_0) = f_t(x_0)$ for some $t \in [0, 1]$. Then $g(x) = f_t(x)$ for any $x \in [0, 1)$. Analogously, if $x_0 \in [-1, 0)$, $g(x_0) > 0$ and $g(x_0) = f_t(x_0)$ for some $t \in [0, 1]$, then $g(x) = f_t(x)$ for any $x \in [-1, 0)$.

Proof of the Claim. Let $x_0 \in [0, 1)$. Then $g(x_0) = f_t(x_0) = \frac{1}{t-x_0}$ and $x_0 < t$. Take any $x \in [0, t)$ for which $g(x) \neq f_t(x)$ and consider $\varepsilon = |g(x) - f_t(x)\ | > 0$. If $\delta > 0$ and $|f_s(x_0) - f_t(x_0)| = |\frac{1}{s-x_0} - \frac{1}{t-x_0}| < \delta$ then $|t - s| < \delta|(s - x_0)(t - x_0)| \leqslant \delta$. If δ is sufficiently small and $|\ t - s\ | < \delta$ then $|\ (s - x)(t - x)| \geqslant A > 0$, where A is a constant which does not depend on δ. For such δ we have $|f_s(x) - f_t(x)| = \frac{|t-s|}{|(s-x)(t-x)|} < \frac{\delta}{A}$. This proves that for a sufficiently small $\delta > 0$ we have $|f_s(x) - f_t(x)| < \frac{\varepsilon}{2}$ for any $s \in (t - \delta, t + \delta)$.

Observe that $g \in \overline{D}$ implies that, for the found $\delta > 0$, there is an $s \in [0, 1]$ such that $|f_s(x_0) - g(x_0)| = |f_s(x_0) - f_t(x_0)| < \delta$ and $|f_s(x) - g(x)| < \frac{\varepsilon}{2}$. We saw already that $|s - t| < \delta$ and hence $|f_s(x) - f_t(x)| < \frac{\varepsilon}{2}$. Therefore $\varepsilon = |g(x) - f_t(x)| \leqslant |g(x) - f_s(x)| + |f_s(x) - f_t(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ which is a contradiction proving that $g(x) = f_t(x)$ for all $x \in [0, t)$. We must also show that $g(x) = 0$ for all $x \geqslant t$. Suppose not. Since $f_s(x) = 0$ or $f_s(x) = \frac{1}{s-x} \geqslant \frac{1}{1-x}$ for each $s \in [0, 1]$, we have $g(x) \geqslant \frac{1}{1-x} = f_1(x)$. Thus there exists $s \in [0, 1]$ with $f_s(x) = g(x)$. Evidently, $s \neq t$. The first part of the proof of this claim shows that $f_s(y) = g(y)$ for all $y \in [0, s)$ and hence $g(0) = f_s(0) = \frac{1}{s} = f_t(0) = \frac{1}{t}$ whence $s = t$, a contradiction. The case $x_0 \in [0, 1)$ is settled and the proof is analogous for $x_0 \in [-1, 0)$. The claim is proved.

Returning to the proof of Fact 1 observe that $g(x_0) > 0$ for some $x_0 \in [-1, 1)$. Suppose first that $x_0 \geqslant 0$. We already saw that $f_t(x_0) = 0$ or $f_t(x_0) \geqslant \frac{1}{1-x_0}$ for each $t \in [0, 1]$ which means $g(x_0) \geqslant \frac{1}{1-x_0}$. This implies that $g(x_0) = f_t(x_0)$ for some $t \in [0, 1]$. Our claim shows that $g(x) = f_t(x)$ for each $x \geqslant 0$ and, in particular, $g(t) = 0$. If $g(-t) = 0$ then $g \in V_t$ and $V_t \cap D$ has only one point which contradicts $g \in \overline{D} \backslash D$. Therefore $g(-t) > 0$ and there is $s \neq t$ with $g(-t) = f_s(-t)$. Applying again our claim we conclude that $g(x) = f_s(x)$ for any $x < 0$.

If $x_0 < 0$ the same reasoning shows that there are distinct $s, t \in [0, 1]$ such that $g(x) = f_s(x)$ for all $x < 0$ and $g(x) = f_t(x)$ for every $x \geqslant 0$. Now observe that $g \in \overline{D}$ implies that for any $\varepsilon > 0$ there exists $\omega \in [0, 1]$ with $|f_w(-1) - g(-1)| < \varepsilon$ and $|f_w(0) - g(0)| < \varepsilon$. In other words $|\frac{1}{1-w} - \frac{1}{1-s}| < \varepsilon$ and $|\frac{1}{t} - \frac{1}{w}| < \varepsilon$. However, these inequalities cannot both be fulfilled for sufficiently small ε because they imply $|w - s| < \varepsilon$ and $|w - t| < \varepsilon$ which gives a contradiction for $\varepsilon < \frac{|s-t|}{3}$. Fact 1 is proved.

Now let N be the Niemytzki plane. We will use the notation from Problem 233. Let $L = \{(x, 0) : x \in \mathbb{R}\} \subset N$ and $P = N \backslash L$. We will denote by N', L' and P' the respective sets N, L and P with the topology induced from $\mathbb{R}^2$.

Fact 2. For every number $t \in [0, 1]$ there exists a function $g_t \in C_p(N)$ such that $f_t = g_t|L$. The functions f_t are the ones from Fact 1 and we identify the sets L and $\mathbb{R}$ considering $(x, 0) \in L$ and $x \in \mathbb{R}$ to be the same point.

Proof. Given any $z = (x, 0) \in L$ and $n \in \mathbb{N}$, consider the set $C_n(z) = \{(a, b) \in \mathbb{R}^2 : (a-x)^2 + (b-\frac{1}{n})^2 \leqslant \frac{1}{n^2}\} \backslash \{z\}$. In other words, $C_n(z)$ is a closed circle centered at the point $\left(x, \frac{1}{n}\right)$ of radius $\frac{1}{n}$ without the point z. Fix any $t \in [0, 1]$ and pick $n \in \mathbb{N}$ such that the sets $C_n(-1)$, $C_n(-t)$, $C_n(0)$ and $C_n(t)$ are disjoint. Denote by P_t the set $\{-1, -t, 0, t\} \subset L$. The set $F = (L' \backslash P_t) \cup C_n(-1) \cup C_n(-t) \cup C_n(0) \cup C_n(t)$ is closed in the normal space $N' \backslash P_t$. Let $\varphi_t(z) = f_t(z)$ if $z \in L$ and $\varphi_t(z) = 0$ for all $z \in C_n(-1) \cup C_n(-t) \cup C_n(0) \cup C_n(t)$. It is easy to see that φ_t is a continuous function on F. By normality of $N' \backslash P_t$ there exists $\Phi_t \in C(N' \backslash P_t)$ with $\Phi_t | F = \varphi_t$. Finally, let $g_t(z) = \Phi_t(z)$ for any $z \in N' \backslash P_t$ and $g_t(z) = 0$ if $z \in P_t$. The function g_t is as promised.

Indeed, it is evident that $g_t | L = f_t$ so we only have to check that g_t is continuous on N. Now g_t is continuous at every point of $N' \backslash P_t$ and hence at every point of $N \backslash P_t$ (Fact 1 of S.233). But g_t is also continuous (in N!) at all points of P_t because every $z \in P_t$ has a neighbourhood on which g_t is identically zero. This finishes the proof of Fact 2.

Now it is easy to finish our solution. The restriction map $\pi_L : C_p(N) \to \mathbb{R}^L$ is continuous and $\pi_L(C_p(N)) \supset D = \{f_t : t \in [0, 1]\}$ by Fact 2. The set D is closed and discrete in $\pi_L(C_p(N))$ by Fact 1 and $\pi_L(H) = D$ where $H = \{g_t : t \in [0, 1]\}$. It is immediate from the equality $\pi_L(H) = D$ that H is closed and discrete in $C_p(N)$. This proves that $\text{ext}(C_p(N)) = \boldsymbol{c}$. Since $iw(N) = \omega$ (Problem 233(ii)), the space $C_p(N)$ is separable (Problem 174). Now apply Problem 164 to conclude that $C_p(N)$ is not normal. Our solution is complete.

S.236. *Let (X, d) be a metric space. Say that a family $\mathcal{F}$ of subsets of X has elements of arbitrarily small diameter if, for any $\varepsilon > 0$, there is $F \in \mathcal{F}$ such that $diam(F) < \varepsilon$. Prove that the following properties are equivalent:*

(*i*) *(X, d) is complete;*
(*ii*) *for every decreasing sequence $F_1 \supset F_2 \supset \ldots$ of closed non-empty subsets of X such that $diam(F_i) \to 0$ when $i \to \infty$, we have $\bigcap\{F_i : i \in \mathbb{N}\} \neq \emptyset$.*
(*iii*) *for any centered family $\mathcal{F}$ of closed subsets of X which has elements of arbitrarily small diameter, we have $\bigcap \mathcal{F} \neq \emptyset$.*

Solution. Suppose that (X, d) is complete and $\{F_i : i \in \mathbb{N}\}$ is a decreasing sequence of non-empty closed sets with $\text{diam}(F_i) \to 0$. Take any $x_i \in F_i$ for all $i \in \mathbb{N}$. We claim that the sequence $\{x_i\}$ is fundamental. Indeed, if $\varepsilon > 0$ then find $m \in \mathbb{N}$ such that $\text{diam}(F_m) < \varepsilon$ and take any $n, k \geqslant m$. We have $x_n \in F_n \subset F_m$ and $x_k \in F_k \subset F_m$ which implies $d(x_n, x_k) \leqslant \text{diam}(F_m) < \varepsilon$. As a consequence, there is $x \in X$ such that $x_i \to x$. For any $n \in \mathbb{N}$ we have $\{x_i : i \geqslant n\} \subset F_n$ and therefore $x \in \overline{F}_n = F_n$. Thus $x \in \bigcap\{F_n : n \in \mathbb{N}\}$ and (i)$\Rightarrow$(ii) is proved.

Assume that (ii) holds and take any centered family $\mathcal{F}$ as in (iii). For each $n \in \mathbb{N}$ there is $P_n \in \mathcal{F}$ such that diam $(P_n) < \frac{1}{n}$. For each $n \in \mathbb{N}$ consider the set $F_n = P_1 \cap \cdots \cap P_n$. It is clear that $\{F_n\}$ is a decreasing family of non-empty closed subsets of X and $\text{diam}(F_n) \leqslant$ diam $(P_n) < \frac{1}{n}$. Therefore, $\text{diam}(F_n) \to 0$ and hence (ii) is applicable: there exists $x \in X$ with $x \in \bigcap\{F_n : n \in \mathbb{N}\} = \bigcap\{P_n : n \in \mathbb{N}\}$. We claim that $x \in \bigcap \mathcal{F}$. Indeed, if not then $x \notin P$ for some $P \in \mathcal{F}$. Take any $\varepsilon > 0$ with $B(x, \varepsilon)$

$\cap\ P = \emptyset$ and any $n > \frac{1}{\varepsilon}$. We have $x \in F_n \subset P_n$ and for any $y \in P_n$, we have $d(y, x) \leqslant$ diam $(P_n) < \frac{1}{n} < \varepsilon$ whence $y \in B(x, \varepsilon)$. This shows that $P_n \subset B(x, \varepsilon)$ and therefore $P_n \cap P = \emptyset$, a contradiction with the fact that $\mathcal{F}$ is centered. This contradiction shows that $x \in \bigcap \mathcal{F}$ and (ii) $\Rightarrow$ (iii) is proved.

Observe that any sequence $\{F_i\}$ as in (ii) is a centered family as in (iii). This shows that (iii)$\Rightarrow$(ii).

Fact 1. Let A be any subset of a metric space (Y, ρ). Then $\mathrm{diam}_\rho(A) = \mathrm{diam}_\rho(\overline{A})$.

Proof. It is absolutely evident that a larger set has a greater diameter which implies $s = \mathrm{diam}_\rho(A) \leqslant \mathrm{diam}_\rho(\overline{A})$. Now, if $\varepsilon > 0$ and $x, y \in \overline{A}$ then there are $a, b \in A$ with $\rho(x, a) < \frac{\varepsilon}{2}$ and $\rho(y, b) < \frac{\varepsilon}{2}$. Therefore

$$\rho(x, y) \leqslant \rho(x, a) + \rho(a, b) + \rho(b, y) < \mathrm{diam}_\rho(A) + \varepsilon.$$

Since x, y were chosen arbitrarily, we have $\mathrm{diam}_\rho(\overline{A}) \leqslant \mathrm{diam}_\rho(A) + \varepsilon$. The number $\varepsilon > 0$ being arbitrary we have $\mathrm{diam}_\rho(\overline{A}) \leqslant \mathrm{diam}_\rho(A)$ and Fact 1 is proved.

Returning to our solution, assume that the property (ii) holds and take any fundamental sequence $\{x_n : n \in \omega\} \subset X$. Let $F_n = \overline{\{x_k : k \geqslant n\}}$ for all $n \in \omega$. It is clear that $\{F_n\}$ is a decreasing sequence of non-empty closed sets. Given $\varepsilon > 0$ there is $m \in \omega$ such that $d(x_n, x_k) < \varepsilon$ for each $n, k \geqslant m$. This means exactly that diam $(\{x_n : n \geqslant m\}) \leqslant \varepsilon$ and hence $\mathrm{diam}(F_m) < \varepsilon$ by Fact 1. Of course, $\mathrm{diam}(F_n) \leqslant$ diam $(F_m) \leqslant \varepsilon$ for each $n \geqslant m$ and this shows that $\mathrm{diam}(F_n) \to 0$.

Applying (ii) we conclude that there exists $x \in \bigcap\{F_n : n \in \omega\}$. To show that $x_n \to x$, take any $\varepsilon > 0$. There exists $m \in \omega$ with $\mathrm{diam}(F_m) < \varepsilon$. Thus for any $n \geqslant m$ we have $x \in F_n \subset F_m$ and $x_n \in F_n \subset F_m$ whence $d(x, x_n) \leqslant \mathrm{diam}(F_m) < \varepsilon$. We proved that $x_n \to x$ so (ii)$\Rightarrow$(i) holds and our solution is complete.

S.237. *Show that every metric space X is isometric to a dense subspace of a complete metric space $\tilde{X}$, which is called the completion of X.*

Solution. We will need the following fact.

Fact 1. Let (X, d) be a complete metric space. Then for any closed $F \subset X$ the metric space (F, d_F) is complete where $d_F = d|(F \times F)$.

Proof. If $S = \{x_n\} \subset F$ is a fundamental sequence then S is fundamental considered as a sequence in X. Therefore $x_n \to x$ for some $x \in X$. It is clear that $x \in \overline{F} = F$ and therefore $\{x_n\}$ converges to $x \in F$ considered as a sequence in F. Fact 1 is proved.

Fact 2. Let X be an arbitrary space. Given any functions $f, g \in C^*(X)$ let $\rho(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. Then ρ is a complete metric on $C^*(X)$.

Proof. There exists $K \in \mathbb{R}$ such that $|f(x)| \leqslant K$ and $|g(x)| \leqslant K$ for all $x \in X$. Therefore $|f(x) - g(x)| \leqslant |f(x)| + |g(x)| \leqslant 2K$ and hence $\rho(f, g)$ is well defined. Let us check that ρ is a metric on $C^*(X)$. If $f = g$ then $f(x) - g(x) = 0$ for all $x \in X$ and hence $\rho(f, g) = 0$. If $\rho(f, g) = 0$ then $f(x) - g(x) = 0$ for all $x \in X$ which implies $f = g$ so (MS1) holds for ρ. The axiom of symmetry holds for ρ because $|f(x) - g(x)| = |g(x) - f(x)|$ for all $x \in X$. Finally, if $f, g, h \in C^*(X)$ then, for any $x \in X$, we have

$|f(x) - h(x)| \leqslant |f(x) - g(x)| + |g(x) - h(x)| \leqslant \rho(f, g) + \rho(g, h)$. Since $\rho(f, g) + \rho(g, h)$ does not depend on x, we can pass to the supremum in the last inequality obtaining $\rho(f, h) \leqslant \rho(f, g) + \rho(g, h)$ i.e., the triangle inequality also holds for ρ. Hence ρ is a metric on $C^*(X)$ and we only have to verify that ρ is complete.

Let $\{f_n : n \in \omega\}$ be a Cauchy sequence in $(C^*(X), \rho)$. Given an arbitrary $x \in X$ and $\varepsilon > 0$ there is $m \in \omega$ such that $\rho(f_n, f_k) < \varepsilon$ for all $n, k \geqslant m$. Therefore $|f_n(x) - f_k(x)| \leqslant \rho(f_n, f_k) < \varepsilon$ and hence the numeric sequence $\{f_n(x)\}$ is fundamental in $\mathbb{R}$ with the usual metric. Since $\mathbb{R}$ is complete (Problem 205), the sequence $\{f_n(x)\}$ converges to a number we will call $f(x)$. To finish the proof of Fact 2 it suffices to show that f is a limit of the sequence $\{f_n\}$ in the space $(C^*(X), \rho)$.

Given $\varepsilon > 0$ there exists $m \in \omega$ such that $\rho(f_n, f_k) < \frac{\varepsilon}{2}$ for all $n, k \geqslant m$. As a consequence, we have $|f_n(x) - f_k(x)| \leqslant \rho(f_n, f_k) < \frac{\varepsilon}{2}$ for all $x \in X$. Taking the limit of the sequence $\{f_k(x)\}$ when $k \to \infty$ we obtain $|f_n(x) - f(x)| \leqslant \frac{\varepsilon}{2} < \varepsilon$ for any $n \geqslant m$ and $x \in X$. This proves that the sequence $\{f_n\}$ converges uniformly to f and hence f is continuous (Problem 029).

By uniform convergence of $\{f_n\}$ there is $m \in \omega$ such that $|f_m(x) - f(x)| < 1$ for all $x \in X$. The function f_m is bounded so there is $K \in \mathbb{R}$ such that $|f_m(x)| < K$ for all $x \in X$. Therefore $|f(x)| \leqslant |f_m(x)| + 1 < K + 1$ for every $x \in X$ which proves that f is also a bounded function, i.e., $f \in C^*(X)$.

Let U be an open set in $(C^*(X), \tau(\rho))$ with $f \in U$. There is $\varepsilon > 0$ such that $B_\rho(f, \varepsilon) \subset U$. Since $f_n \rightrightarrows f$, there is $m \in \omega$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $x \in X$ and $n \geqslant m$. It is immediate from the definition of ρ that $\rho(f_n, f) \leqslant \frac{\varepsilon}{2} < \varepsilon$ and hence $f_n \in B_\rho(f, \varepsilon) \subset U$ for all $n \geqslant m$. Hence $f_n \to f$ in the space $(C^*(X), \rho)$ and our proof of Fact 2 is complete.

Returning to our solution, take any metric space (X, d) and fix a point $a \in X$. For any $x \in X$ let $\varphi(x)(z) = d(z, x) - d(z, a)$ for all $z \in X$. This gives us a function $\varphi(x) : X \to \mathbb{R}$. Since d is continuous (Problem 202), the map $\varphi(x)$ is continuous for each $x \in X$. By triangle inequality, we have $d(a, z) \leqslant d(a, x) + d(x, z)$ and therefore $\varphi(x)(z) = d(z, x) - d(z, a) \geqslant - d(a, x)$ for any $z \in Z$. Analogously, $d(x, z) \leqslant d(x, a) + d(a, z)$ which shows that $\varphi(x)(z) = d(x, z) - d(z, a) \leqslant d(a, x)$ for all $z \in X$. We proved that $-d(a, x) \leqslant \varphi(x)(z) \leqslant d(a, x)$ for all $z \in X$, i.e., $\varphi(x)$ is a bounded function. Consequently $\varphi : X \to C^*(X)$. We will prove that φ is an isometry of (X, d) onto $(Y, \rho Y)$, where $Y = \varphi(X)$ and $\rho_Y = \rho|(Y \times Y)$. Here ρ is the metric on $C^*(X)$ introduced in Fact 2.

It suffices to show that $\rho(\varphi(x), \varphi(y)) = d(x, y)$ for any $x, y \in X$. Given any $z \in X$ we have

$|\varphi(x)(z) - \varphi(y)(z)| = |d(z, x) - d(z, a) - d(y, z) + d(z, a)| = |d(z, x) - d(z, y)| \leqslant d(x, y)$, and therefore we have $\rho(\varphi(x), \varphi(y)) \leqslant d(x, y)$. On the other hand, the equality $|\varphi(x)(y) - \varphi(y)(y)| = d(x, y)$ implies that $d(x, y) \leqslant \rho(\varphi(x), \varphi(y))$. Thus ρ is an isometry and the space Y is a dense subspace of the space $\tilde{X} = \overline{Y}$ (the closure is taken in $(C^*(X), \tau(\rho))$), which is complete by Facts 1 and 2. Since (X, d) is isometric to (Y, ρ_Y), our solution is finished.

S.238. *Let A be a dense subset of a metric space (X, d). Suppose that (Y, ρ) is a complete metric space, $B \subset Y$ and $\varphi : A \to B$ is an isometry. Prove that there exists an isometry $\phi : X \to Y' \subset Y$ such that $\phi|A = \varphi$.*

Solution. Let us formulate the following easy fact for further references.

Fact 1. Suppose that X is a space, $\{x_n\} \subset X$ and $x_n \to x$. If $f : X \to Y$ is a continuous map then $f(x_n) \to f(x)$.

Proof. An easy exercise.

Take any $x \in X$. Since $x \in \overline{A}$, there is a sequence $\{a_n(x)\} \subset A$ with $a_n(x) \to x$ (Problem 210). The sequence $\{\varphi(a_n(x))\}$ has to be fundamental because it is isometric to the convergent sequence $\{a_n(x)\}$ and any convergent sequence is fundamental. The space (Y, ρ) is complete and hence there is $y \in Y$ such that $\varphi(a_n(x)) \to y$. Letting $\varphi(x) = y$ we obtain a function $\phi : X \to Y$. Let us show that ϕ is as promised.

(1) $\phi|A = \varphi$. For any $x \in A$ we have $a_n(x) \to x$ and hence $\varphi(a_n(x)) \to \varphi(x)$ by continuity of φ and Fact 1. We also have $\varphi(a_n(x)) \to \phi(x)$ by definition of $\phi(x)$. It is easy to see that a convergent sequence in a Hausdorff space has only one limit, so $\phi(x) = \varphi(x)$ and hence $\phi|A = \varphi$.
(2) $\rho(\phi(x), \phi(y)) = d(x, y)$ for any $x, y \in X$. Let $z_n = (a_n(x), a_n(y)) \in X \times X$ for all $n \in \omega$. It is clear that $z_n \to z = (x, y)$ so $d(a_n(x), a_n(y)) \to d(x, y)$ by continuity of d (Problem 202) and Fact 1. Now, $\rho(\varphi(a_n(x)), \varphi(a_n(y))) \to \rho(\phi(x), \phi(y))$ by continuity of ρ and Fact 1. Noting that $\rho(\varphi(a_n(x)), \varphi(a_n(y))) = d(a_n(x), a_n(y)) \to d(x, y)$ we convince ourselves that $\rho(\phi(x), \phi(y)) = d(x, y)$.

Letting $Y'; = \phi(X) \subset Y$ we have the promised isometry $\phi : X \to Y'$, so our solution is complete.

S.239. *Let (X, d) and (Y, ρ) be complete metric spaces. Suppose that A is dense in X, and B is dense in Y. Prove that any isometry between A and B (with the metrics induced from X and Y, respectively) can be extended to an isometry between (X, d)* and *(Y, ρ). In particular, the completion $\tilde{X}$ of a metric space X is unique in the sense that if Z is another completion of X then there is an isometry between $\tilde{X}$ and Z which is the identity restricted to the respective copies of X.*

Solution. Let $f : A \to B$ be an isometry. It is evident that f is a bijection and $g = f^{-1} : B \to A$ is also an isometry. Apply Problem 238 to obtain an isometry $F : X \to Y' \subset Y$ such that $F|A = f$. It suffices to prove that $Y' = Y$ so take any $y \in Y$. Since $y \in \overline{B}$, there is a sequence $\{b_n : n \in \omega\} \subset B$ with $b_n \to y$. If we let $a_n = g(b_n)$ for all $n \in \omega$ then the sequence $\{a_n\}$ is fundamental being isometric to the convergent (and hence fundamental) sequence $\{b_n\}$. The space (X, d) being complete there is $x \in X$ with $a_n \to x$. Being an isometry the map F is continuous which implies $F(a_n) \to F(x)$ and $b_n = f(a_n) = F(a_n) \to y$ whence $F(x) = y$. The point y was chosen arbitrarily so F is an isometry between (X, d) and (Y, ρ).

To finish our proof suppose that $i : X \to M$ and $j : X \to L$ are isometries, the spaces M and L are complete (with their respective metrics) and the images $i(X)$ and $j(X)$ are dense in M and L, respectively. The map $f = j \circ i^{-1} : i(X) \to j(X)$ is an isometry, so we can apply what we proved before to the sets $A = i(X)$ and $B = j(X)$ to obtain an isometry $F : M \to L$ such that $F|i(x) = f$. If we identify X with $i(X)$ and $j(X)$ then F is an isometry between M and L such that $F|X$ is an identity.

S.240. *Given metric spaces (X, d) and (Y, ρ), call a map $f : X \to Y$ a contraction if there is $k \in (0, 1)$ such that $\rho(f(x), f(y)) \leqslant k \cdot d(x, y)$ for any $x, y \in X$. Prove that any contraction is a uniformly continuous map.*

Solution. Given $\varepsilon > 0$ let $\delta = \frac{\varepsilon}{k}$. If $x, y \in X$ are arbitrary points with $d(x, y) < \delta$ then $\rho(f(x), f(y)) \leqslant k \cdot d(x, y) < k \cdot \delta = \varepsilon$ and we are done.

S.241. *Let (X, d) be a complete metric space. Prove that if $f : X \to X$ is a contraction, then it has a unique fixed point, i.e., there is a unique $x \in X$ such that $f(x) = x$.*

Solution. Let $k \in (0, 1)$ be the coefficient of contraction for the function f, i.e., $d(f(a), f(b)) \leqslant k \cdot d(a, b)$ for all $a, b \in X$. Fix any point $x_0 \in X$ and consider the sequence $x_1 = f(x_0)$, $x_2 = f(x_1)$, ..., $x_{n+1} = f(x_n)$ and so on. The sequence $\{x_n : n \in \omega\}$ being constructed, let us establish that

(∗) $d(x_n, x_{n+1}) \leqslant k^n \cdot d(x_0, x_1)$ for all $n \in \mathbb{N}$.

We will prove the property (∗) by induction on $n \in \mathbb{N}$. Note that we have $d(x_1, x_2) = d(f(x_0), f(x_1)) \leqslant k \cdot d(x_0, x_1)$ which proves (∗) for $n = 1$. Now, assume that we proved the inequality (∗) for $n = m$. Then $d(x_{m+1}, x_{m+2} = d(f(x_m), f(x_{m+1})) \leqslant k \cdot d(x_m, x_{m+1}) \leqslant k \cdot k^m \cdot d(x_0, x_1) = k^{m+1} \cdot d(x_0, x_1)$ and (∗) is proved for $n = m + 1$ and hence it holds for all $n \in \mathbb{N}$.

Applying (∗) we obtain the following property:

$$(**) \quad \begin{aligned} d(x_m, x_{m+p}) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{m+p-1}, x_{m+p}) \\ &\leqslant (k^m + k^{m+1} + \cdots + k^{m+p-1}) \cdot d(x_0, x_1) \\ &= k^m \cdot \tfrac{1-k^p}{1-k} \cdot d(x_0, x_1) \leqslant \tfrac{k^m}{1-k} \cdot d(x_0, x_1), \end{aligned}$$

for any $m, p \in \mathbb{N}$. We are now in position to prove that the sequence $\{x_n\}$ is fundamental. Given any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\frac{k^n}{1-k} \cdot d(x_0, x_1) < \varepsilon$ for all $n \geqslant m$. For any $n, k \geqslant m$ such that $n < k$ apply (∗∗) to $m = n$ and $p = k - n$ to conclude that $d(x_n, x_k) \leqslant \frac{k^n}{1-k} \cdot d(x_0, x_1) < \varepsilon$. Thus the sequence $\{x_n\}$ is fundamental. Since (X, d) is complete, there is $x \in X$ such that $x_n \to x$. The map f being continuous, we have $x_{n+1} = f(x_n) \to f(x)$ whence $x = f(x)$, i.e., x is a fixed point of the function f.

Finally, to see that x is a unique fixed point, suppose that $f(y) = y$ for some $y \in X$. Then $d(x, y) = d(f(x), f(y)) \leqslant k \cdot d(x, y)$ which immediately implies $d(x, y) = 0$, i.e., $x = y$ so our solution is complete.

S.242. *Let (X, d) be a compact metric space. Prove that, for any metric space (Y, ρ) and any continuous $f : X \to Y$, the map f is uniformly continuous.*

Solution. Take any $\varepsilon > 0$ and consider the open cover $\mathcal{U} = \{B_\rho(y, \frac{\varepsilon}{2}) : y \in Y\}$ of the space Y. The family $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X. For each $x \in X$ fix $\delta(x) > 0$ such that $B_d(x, 2\delta(x)) \subset V$ for some $V \in \mathcal{V}$. There exist $x_1, \ldots, x_n \in X$ such that $X = B_d(x_1, \delta(x_1)) \cup \cdots \cup B_d(x_n, \delta(x_n))$. The number $\delta = \min\{\delta(x_1), \ldots, \delta(x_n)\}$ is as required. Indeed, suppose that $d(a, b) < \delta$. There exists $i \leqslant n$ such that $a \in B_d(x_i, \delta(x_i))$. Then $d(x_i, b) \leqslant d(x_i, a) + d(a, b) < \delta(x_i) + \delta \leqslant 2\delta(x_i)$ and therefore $a, b \in B_d(x_i, 2\delta(x_i)) \subset V$ for some $V \in \mathcal{V}$. Take $U = B_\rho(y, \frac{\varepsilon}{2}) \in \mathcal{U}$ such that $V = f^{-1}(U)$ and observe that $f(a), f(b) \in B_\rho(y, \frac{\varepsilon}{2})$ implies $\rho(f(a), f(b)) \leqslant \rho(f(a), y) + \rho(y, f(b)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and our solution is complete.

S.243. *Let (X, d) be a metric space such that, for any metric space (Y, ρ), any continuous map $f : X \to Y$ is uniformly continuous. Must X be a compact space?*

Solution. No, X need not be compact. Consider $X = D(\omega)$ with the metric d defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise. Then d generates the discrete topology of X (Problem 204) and X is not compact. Now take any metric space (Y, ρ), any map $f : X \to Y$ and any $\varepsilon > 0$. If $\delta = \frac{1}{2}$ and $d(x, y) < \delta$ then $x = y$ and therefore $\rho(f(x), f(y)) = 0 < \varepsilon$ which shows that f is uniformly continuous.

S.244. *Let (X, d) be a compact metric space. Prove that, for any open cover $\mathcal{U}$ of the space X, there is a number $\delta = \delta(\mathcal{U}) > 0$ such that for each $A \subset X$ with $\mathrm{diam}_d(A) < \delta$ there exists a set $U \in \mathcal{U}$ such that $A \subset U$. The number $\delta(\mathcal{U})$ is called the Lebesgue number of the cover $\mathcal{U}$.*

Solution. For any $x \in X$ fix $\delta(x) > 0$ such that $B(x, 2\delta(x)) \subset U$ for some $U \in \mathcal{U}$. Find $x_1, \ldots, x_n \in X$ such that $X = B(x_1, \delta(x_1)) \cup \cdots \cup B(x_n, \delta(x_n))$ and consider $\delta = \min\{\delta(x_1), \ldots, \delta(x_n)\} > 0$. The number $\delta = \delta(\mathcal{U})$ is as required. Indeed, take any set $A \subset X$ with $\mathrm{diam}_d(A) < \delta$. If $A = \emptyset$ then there is nothing to prove. If not, fix a point $x \in A$; there is $i \leqslant n$ such that $x \in B(x_i, \delta(x_i))$; by our choice of $\delta(x_i)$ there is $U \in \mathcal{U}$ such that $B(x_i, 2\delta(x_i)) \subset U$. We have $d(x_i, y) \leqslant d(x_i, x) + d(x, y) < \delta(x_i) + \delta \leqslant 2\delta(x_i)$ and hence $y \in B(x_i, 2\delta(x_i)) \subset U$ for any $y \in A$ so $A \subset U$ and we are done.

S.245. *Let (X, d) and (Y, ρ) be metric spaces. A set $\mathcal{F} \subset C(X, Y)$ is called equicontinuous at a point $x \in X$ if, for any $\varepsilon > 0$, there is $\delta > 0$ such that $f(B_d(x, \delta)) \subset B_\rho(f(x), \varepsilon)$ for every $f \in \mathcal{F}$. The set $\mathcal{F}$ is called an equicontinuous family if it is equicontinuous at every $x \in X$. Prove that every finite $\mathcal{F} \subset C(X, Y)$ is equicontinuous.*

Solution. Let $\mathcal{F} = \{f_1, \ldots, f_n\}$. Given a point $x \in X$ and $\varepsilon > 0$ we can find, for each $i \leqslant n$, a positive δ_i such that $f_i(B_d(x, \delta_i)) \subset B_\rho(f_i(x), \varepsilon)$. To show that $\delta = \min\{\delta_i : i \leqslant n\}$ is as promised take an arbitrary $i \leqslant n$ and observe that $f_i(B_d(x, \delta)) \subset f_i(B_d(x, \delta_i)) \subset B_\rho(f_i(x), \varepsilon)$.

S.246. *Given metric spaces (X, d) and (Y, ρ), a set $\mathcal{F} \subset C(X, Y)$ is called a uniformly equicontinuous family if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x, y \in X$ with $d(x, y) < \delta$, we have $\rho(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$. Prove that*

(*i*) *Every subset of a uniformly equicontinuous set is uniformly equicontinuous;*
(*ii*) *If $\mathcal{F}$ is uniformly equicontinuous then every $f \in \mathcal{F}$ is uniformly continuous;*
(*iii*) *A finite set of maps $\mathcal{F}$ is uniformly equicontinuous if and only if each $f \in \mathcal{F}$ is uniformly continuous.*

Solution. (i) If $\mathcal{G} \subset \mathcal{F}$ and $\varepsilon > 0$ then there is $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$ and, in particular, for all $f \in \mathcal{G}$. Thus $\mathcal{G}$ is uniformly equicontinuous.

(ii) Fix $f_0 \in \mathcal{F}$ and $\varepsilon > 0$. There exists a number $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$. In particular, $d(x, y) < \delta$ implies $\rho(f_0(x), f_0(y)) < \varepsilon$ and hence f_0 is uniformly continuous.

(iii) If a set $\mathcal{F}$ is uniformly equicontinuous then each $f \in \mathcal{F}$ is uniformly continuous by (ii). Now suppose that $\mathcal{F} = \{f_1, \dots, f_n\}$ and each f_i is uniformly continuous. Fix $\varepsilon > 0$ and find, for each $i \leqslant n$, a positive δ_i such that $x, y \in X$ and $d(x, y) < \delta_i$ implies $\rho(f_i(x), f_i(y)) < \varepsilon$. The number $\delta = \min\{\delta_i : i \leqslant n\} > 0$ is as needed because if $d(x, y) < \delta$ then, for any $i \leqslant n$, we have $d(x, y) < \delta_i$ and therefore $\rho(f_i(x), f_i(y)) < \varepsilon$. This proves that $\mathcal{F}$ is uniformly equicontinuous.

S.247. *Let (X, d) be a compact metric space. Given a metric space (Y, ρ) and an equicontinuous family $\mathcal{F} \subset C(X, Y)$, prove that $\mathcal{F}$ is uniformly equicontinuous.*

Solution. Fix any $\varepsilon > 0$ and find, for any $x \in X$, a positive $\delta(x)$ such that $f(B_d(x, \delta(x))) \subset B_\rho(f(x), \frac{\varepsilon}{2})$ for any $f \in \mathcal{F}$. The family $\mathcal{U} = \{B_d(x, \delta(x)) : x \in X\}$ is an open cover of the compact space X. Apply Problem 244 to find $\delta > 0$ such that $d(x, y) < \delta$ implies $\{x, y\} \subset U$ for some $U \in \mathcal{U}$. We claim that δ witnesses the uniform equicontinuity of $\mathcal{F}$. Indeed, take any $f \in \mathcal{F}$. If $a, b \in X$ and $d(a, b) < \delta$ then there is $U = B(x, \delta(x)) \in \mathcal{U}$ such that $\{a, b\} \subset U$. By definition of $\mathcal{U}$ we have $\{f(a), f(b)\} \subset f(U) \subset B_\rho(f(x), \frac{\varepsilon}{2})$ and therefore $\rho(f(a), f(b)) \leqslant \rho(f(a), f(x)) + \rho(f(x), f(b)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and our solution is over.

S.248. *Suppose that X is a space and (Y, ρ) is a (complete) metric space. For any functions $f, g \in C^*(X, Y)$ let $\sigma(f, g) = \sup\{\rho(f(x), g(x)) : x \in X\}$. Show that σ is a (complete) metric on $C^*(X, Y)$. It is called the metric of uniform convergence.*

Solution. Given any $f, g \in C^*(X, Y)$ the sets $f(X)$ and $g(X)$ are bounded in (Y, ρ). Consequently, there exist $y, z \in Y$ and $r, s > 0$ such that $f(X) \subset B(y, r)$ and $g(X) \subset B(z, s)$. This means $\rho(f(x), g(x)) \leqslant \rho(f(x), y) + \rho(y, z) + \rho(z, g(x)) < K = r + s + \rho(y, z)$. Since K does not depend on $x \in X$, we have $\rho(f(x), g(x)) < K$ and therefore $\sigma(f, g) \leqslant K < \infty$, i.e., $\sigma(f, g)$ is well defined for all $f, g \in C^*(X, Y)$.

Let us check that σ is a metric on $C^*(X, Y)$. If $f = g$ then $\rho(f(x), g(x)) = 0$ for all $x \in X$ and hence $\sigma(f, g) = 0$. If $\sigma(f, g) = 0$ then $\rho(f(x), g(x)) = 0$ for all $x \in X$ which implies $f(x) = g(x)$ for all $x \in X$ and hence $f = g$. This proves that (MS1) holds for σ. The axiom of symmetry holds for σ because $\rho(f(x), g(x)) = \rho(g(x), f(x))$ for all $x \in X$. Finally, if $f, g, h \in C^*(X, Y)$ then, for any $x \in X$, we have $\rho(f(x), h(x)) \leqslant \rho(f(x), g(x)) + \rho(g(x), h(x)) \leqslant \sigma(f, g) + \sigma(g, h)$. Since $\sigma(f, g) + \sigma(g, h)$ does not depend on

x, we can pass to the supremum in the last inequality obtaining $\sigma(f, h) \leqslant \sigma(f, g) + \sigma(g, h)$ i.e., the triangle inequality also holds for σ and hence σ is a metric on $C^*(X, Y)$.

Let (Z, δ) be a metric space. If T is an arbitrary space and $h_n : T \to Z$ is a function for each $n \in \omega$, say that the sequence $\{h_n\}$ *converges uniformly* to a function $h : T \to Z$ if, for any $\varepsilon > 0$ there is $m \in \omega$ such that $\delta(h_n(x), h(x)) < \varepsilon$ for all $x \in T$ and $n \geqslant m$. This will also be denoted by $h_n \rightrightarrows h$.

Fact 1. Suppose that T is a space and (Z, δ) is a metric space. Assume that $h_n \in C(T, Z)$ of each $n \in \omega$ and $h_n \rightrightarrows h$ for some $h : T \to Z$. Then the function h is continuous.

Proof. We will show that h is continuous at every point $t \in T$. Take any $\varepsilon > 0$ and find $m \in \omega$ such that $\delta(h_n(y), h(y)) < \frac{\varepsilon}{3}$ for all $n \geqslant m$ and any $y \in T$. The function h_m is continuous at t so there exists $U \in \tau(t, T)$ such that $h_m(U) \subset B_\delta(h_m(t), \frac{\varepsilon}{3})$. To prove that $h(U) \subset B_\delta(h(t), \varepsilon)$ take any $u \in U$. We have the inequalities $\delta(h(t), h(u)) \leqslant \delta(h(t), \quad h_m(t)) + \delta(h_m(t), h_m(u)) + \delta(h_m(u), h(u)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ which shows that $h(U) \subset B_\delta(h(t), \varepsilon)$ and therefore h is continuous at t. The point t was chosen arbitrarily so h is continuous and Fact 1 is proved.

Now suppose that the metric space (Y, ρ) is complete. Let $\{f_n : n \in \omega\}$ be a Cauchy sequence in $(C^*(X, Y), \sigma)$. Given an arbitrary $x \in X$ and $\varepsilon > 0$ there is $m \in \omega$ such that $\sigma(f_n, f_k) < \varepsilon$ for all $n, k \geqslant m$. Therefore $\rho(f_n(x), f_k(x)) \leqslant \sigma(f_n, f_k) < \varepsilon$ and hence the sequence $\{f_n(x)\} \subset Y$ is fundamental in (Y, ρ). Since (Y, ρ) is complete, the sequence $\{f_n(x)\}$ converges to some $y = f(x) \in Y$. To finish our solution it suffices to show that $f \in C^*(X, Y)$ and f is a limit of the sequence $\{f_n\}$ in the space $(C^*(X, Y), \sigma)$.

Given $\varepsilon > 0$ there exists $m \in \omega$ such that $\sigma(f_n, f_k) < \frac{\varepsilon}{2}$ for all $n, k \geqslant m$. As a consequence, we have $\rho(f_n(x), f_k(x)) \leqslant \sigma(f_n, f_k) < \frac{\varepsilon}{2}$ for all $x \in X$. Taking the limit of the sequence $\{f_k(x)\}$ when $k \to \infty$ and applying continuity of the metric ρ (Problem 202), we obtain $\rho(f_n(x), f(x)) \leqslant \frac{\varepsilon}{2} < \varepsilon$ for any $n \geqslant m$ and $x \in X$. Therefore, the sequence $\{f_n\}$ converges uniformly to the function f and hence f is continuous by Fact 1.

By uniform convergence of the sequence $\{f_n\}$ there exists a number $m \in \omega$ such that $\rho(f_m(x), f(x)) < 1$ for all $x \in X$. The function f_m is bounded so there is a number $K \in \mathbb{R}$ such that $f_m(X) \subset B_\rho(a, K)$ for some $a \in X$. Therefore, we have the inequalities $\rho(a, f(x)) \leqslant \rho(a, f_m(x)) + \rho(f_m(x), f(x)) < K + 1$ for every $x \in X$ which proves that $f(X) \subset B_\rho(a, K+1)$, i.e., f is also a bounded function, i.e., $f \in C^*(X, Y)$.

Let U be an open set in $(C^*(X, Y), \tau(\sigma))$ with $f \in U$. There is $\varepsilon > 0$ such that $B_\sigma(f, \varepsilon) \subset U$. Since $f_n \rightrightarrows f$, there is $m \in \omega$ such that $\rho(f_n(x), f(x)) < \frac{\varepsilon}{2}$ for all $x \in X$ and $n \geqslant m$. It is immediate from the definition of σ that $\sigma(f_n, f) \leqslant \frac{\varepsilon}{2} < \varepsilon$ and hence $f_n \in B_\sigma(f, \varepsilon) \subset U$ for all $n \geqslant m$. Hence $f_n \to f$ in the space $(C^*(X, Y), \sigma)$ and our solution is complete.

S.249. *Let (X, d) be a totally bounded metric space. Suppose that (Y, ρ) is a metric space and $\mathcal{F} \subset C(X, Y)$ has the following properties:*

(1) *The family $\mathcal{F}$ is uniformly equicontinuous.*
(2) *For any $x \in X$, the set $\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\}$ is totally bounded in (Y, ρ).*

Prove that $\mathcal{F} \subset C^*(X, Y)$ and the family $\mathcal{F}$ is totally bounded in $C^*(X, Y)$. Here $C^*(X,Y)$ is considered with the metric σ of uniform convergence.

Solution. It turns out that being a totally bounded set can be expressed as an internal property.

Fact 1. Let (Z, ρ) be a metric space. A set $A \subset Z$ is totally bounded in Z if and only if the metric space (A, ρ_A) is totally bounded. Here ρ_A is the metric induced from Z, i.e., $\rho A = \rho|(A \times A)$.

Proof. Given any $z \in Z$ and $r > 0$, we denote by $B(z, r)$ the ball centered at z of radius r in Z, i.e., $B(z, r) = \{y \in Z : \rho(y, z) < r\}$. If $z \in A$ then the set $B_A(z, r) = \{y \in A : \rho_A(y, z) < r\}$ is the respective ball in the metric space (A, ρ_A). It is immediate that $B_A(z, r) = B(z, r) \cap A$ for any $z \in A$ and $r > 0$.

Assume that A is totally bounded in Z. Given $\varepsilon > 0$ there exists a finite set $F \subset Z$ such that $A \subset \bigcup\{B(z, \frac{\varepsilon}{2}) : z \in F\}$. Let $\{z_1, \ldots, z_n\}$ be an enumeration of all $z \in F$ such that $B(z, \frac{\varepsilon}{2}) \cap A \neq \emptyset$. For each $i \leqslant n$ pick $a_i \in B(z, \frac{\varepsilon}{2}) \cap A$. The set $B = \{a_i : i \leqslant n\} \subset A$ is finite and $\bigcup\{B_A(a_i, \varepsilon) : i \leqslant n\} = A$. Indeed, if $a \in A$ then there is $z \in F$ with $a \in B(z, \frac{\varepsilon}{2})$. In particular, $B(z, \frac{\varepsilon}{2}) \cap A \neq \emptyset$ and hence $z = z_i$ for some $i \leqslant n$. We have $\rho_A(a_i, a) = \rho(a_i, a) \leqslant \rho(a_i, z_i) + \rho(z_i, a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This implies $a \in B_A(a_i, \varepsilon)$ and proves that (A, ρ_A) is totally bounded.

Assume now that (A, ρ_A) is totally bounded. Given any $\varepsilon > 0$ there exists a finite set $B \subset A$ such that $\bigcup\{B_A(a, \varepsilon) : a \in B\} = A$. Then $\bigcup\{B(a, \varepsilon) : a \in B\} \supset \bigcup\{B_A(a, \varepsilon) : a \in B\} = A$ and hence the same set B witnesses the fact that A is totally bounded in Z. Fact 1 is proved.

Let us first prove that $\mathcal{F} \subset C^*(X, Y)$, i.e., that each $f \in \mathcal{F}$ is bounded. The function f is uniformly continuous by Problem 246(ii) so there is $\delta > 0$ such that $a, b \in X$ and $d(a, b) < \delta$ implies $\rho(f(a), f(b)) < 1$. Since the space (X, d) is totally bounded, there is a finite $F \subset X$ such that $X = \bigcup\{B_d(x, \delta) : x \in F\}$. Let $K = \max\{\rho(f(x), f(y)) : x, y \in F\} + 2$ and choose any $a \in F$. We claim that $f(X) \subset B_\rho(z, r)$ where $z = f(a)$ and $r = K + 2$. Indeed, take any $x \in X$. There is $y \in F$ such that $d(x, y) < \delta$ and hence $\rho(f(x), f(y)) < 1$. Therefore $\rho(z, f(x)) \leqslant \rho(z, f(y)) + \rho(f(y), f(x)) < K + 1 < r$ and hence $f(X) \subset B_\rho(z, r)$ which proves that f is bounded.

Denote by X' set X endowed with the discrete topology. Then any function $f : X' \to Y$ is continuous so $C^*(X, Y) \subset C^*(X', Y)$. If σ' is the metric of uniform convergence on $C^*(X', Y)$ then σ' induces the metric σ of uniform convergence on the space $C^*(X, Y)$. Therefore, the metrics σ and σ' induce the same metric $\aleph$ on the set $\mathcal{F}$. Fact 1 says that total boundedness of $\mathcal{F}$ in $C^*(X, Y)$ is equivalent to total boundedness of $(\mathcal{F}, \aleph)$ which in turn is equivalent to total boundedness of $\mathcal{F}$ in $C^*(X', Y)$. Thus it suffices to prove that $\mathcal{F}$ is totally bounded in $C^*(X', Y)$.

Take any $\varepsilon > 0$. There exists $\delta > 0$ such that $a, b \in X$ and $d(a, b) < \delta$ implies $\rho(f(a), f(b)) < \frac{\varepsilon}{3}$ for any $f \in \mathcal{F}$. The space (X, d) being totally bounded there is a finite set $A = \{a_1, \ldots, a_n\} \subset X$ such that $\bigcup\left\{B_d\left(x, \frac{\delta}{2}\right) : x \in A\right\} = X$.

Let $U_i = B_d\big(a_i, \frac{\delta}{2}\big)$ for each $i \leqslant n$. We will need the sets $P_1 = U_1, P_2 = U_2 \backslash U_1, \ldots,$ $P_n = U_n \backslash (U_1 \cup \cdot \cup U_{n-1})$. It is clear that the family $\mathcal{H} = \{H_i : i \leqslant n\}$ is disjoint and $\bigcup \mathcal{H} = X$. Observe that, for any $i \leqslant n$ and any $a, b \in H_i \subset B_d\,(a_i, \delta/2)$, we have $d(a, b) < \delta$ because $d(a, b) \leqslant d(a, a_i) + d(a_i, b)\; < \frac{\delta}{2} + \frac{\delta}{2} = \delta$. Recalling the way δ was chosen we obtain

(∗) given $i \leqslant n$, we have $\rho(f(x), f(y)) < \frac{\varepsilon}{3}$ for any $x, y \in H_i$ and $f \in \mathcal{F}$.

Without loss of generality we may assume that $H_i \neq \emptyset$ for each $i \leqslant n$ because we can throw away the empty elements of $\mathcal{H}$ and enumerate the non-empty ones obtaining a family of non-empty disjoint sets which still has (∗) and whose union is the space X. Pick $x_i \in H_i$ for all $i \leqslant n$. The set $P_i = \mathcal{F}(x_i)$ is totally bounded in (Y, ρ) by (2) and hence the set $P = \bigcup\{P_i : i \leqslant n\}$ is totally bounded in (Y, ρ) (it is an easy exercise to show that a finite union of totally bounded sets is totally bounded). Therefore we can find a finite $T \subset Y$ such that

(∗∗) $\bigcup\big\{B_\rho\big(t, \frac{\varepsilon}{3}\big) : t \in T\big\} \supset P$.

Let $E = \{g : X' \to Y : g(X') \subset T$ and g is constant on H_i for every $i \leqslant n\}$. It is easy to see that E is finite. Since $g(X')$ is finite, the function g is bounded for each $g \in E$. Thus E is a finite subset of $C^*(X', Y)$. Now take any $f \in \mathcal{F}$. For every $i \leqslant n$ we have $f(x_i) \in P$ and hence we can apply (∗∗) to find $t_i \in T$ such that $\rho(t_i, f(x_i)) < \frac{\varepsilon}{3}$. Now let $g(x) = t_i$ for all $x \in H_i$. Having done this for all $i \leqslant n$ we obtain a function $g \in E$.

Given any point $x \in X$ there exists $i \leqslant n$ with $x \in H_i$. We have $\rho(f(x), g(x)) \leqslant \rho(f(x), f(x_i)) + \rho(f(x_i), g(x)) = \rho(f(x), f(x_i)) + \rho(f(x_i), g(x_i)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$. The last inequality takes place because $\rho(f(x), f(x_i)) < \frac{\varepsilon}{3}$ by (∗) and $\rho(f(x_i), g(x_i)) = \rho(f(x_i), t_i)) < \frac{\varepsilon}{3}$ by the choice of t_i. We proved that $\rho(f(x), g(x)) < \frac{2\varepsilon}{3}$ for all $x \in X$ and hence $\sigma'(f, g) \leqslant \frac{2\varepsilon}{3} < \varepsilon$. Therefore $f \in B_{\sigma'}(g, \varepsilon)$ and, the function $f \in \mathcal{F}$ having been chosen arbitrarily, we have $\bigcup\{B_{\sigma'}(g, \varepsilon) : g \in E\} \supset \mathcal{F}$. This proves that $\mathcal{F}$ is totally bounded in $C^*(X', Y)$ and hence in $C^*(X', Y)$ so our solution is complete.

S.250. *Let X be a compact (not necessarily metrizable) space. Given a metric space (Y, ρ), prove that any continuous map $f : X \to Y$ is bounded, i.e., we have the equality $C(X, Y) = C^*(X, Y)$.*

Solution. Fix any $y \in Y$. The family $\mathcal{U} = \{B_\rho(y, n) : n \in \mathbb{N}\}$ is an open cover of the compact space $f(X) \subset Y$. If $\gamma = \{B_\rho(y, n_1), \ldots, B_\rho(y, n_k)\}$ is a finite subcover of $f(X)$ then, for $n = \max\{n_1, \ldots, n_k\}$ we have $f(X) \subset \bigcup \gamma \subset B_\rho(y, n)$ and hence f is bounded.

S.251. *Let (X, d) be a compact metric space. Given a metric space (Y, ρ) and a family $\mathcal{F} \subset C^*(X, Y)$, prove that $\mathcal{F}$ is totally bounded if and only if $\mathcal{F}$ is equicontinuous and $\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\}$ is totally bounded in (Y, ρ) for any $x \in X$. Here $C^*(X, Y)$ is assumed to be endowed with the metric σ of uniform convergence.*

Solution. Given $\varepsilon > 0$ the open cover $\{B_d(x, \varepsilon) : x \in X\}$ of the compact space X has a finite subcover. Taking the centers of the balls of this subcover, we convince ourselves that (X, d) is totally bounded. If $\mathcal{F}$ is equicontinuous then it is uniformly equicontinuous by Problem 247. If, additionally, $\mathcal{F}(x)$ is totally bounded for every $x \in X$ then Problem 249 can be applied to conclude that $\mathcal{F}$ is totally bounded.

Now assume that $\mathcal{F}$ is totally bounded. Given $x \in X$ and $\varepsilon > 0$, find a finite $P \subset \mathcal{F}$ such that $\bigcup\{B_\sigma(g, \varepsilon) : g \in P\} \supset \mathcal{F}$. The set $P(x) = \{g(x) : x \in P\} \subset \mathcal{F}(x)$ is finite. If $y \in \mathcal{F}(x)$ then $y = f(x)$ for some $f \in \mathcal{F}$. Pick any $g \in P$ with $\sigma(f, g) < \varepsilon$. Then $\rho(y, g(x)) = \rho(f(x), g(x)) \leqslant \sigma(f, g) < \varepsilon$ which shows that $\cup\{B_\rho(z, \varepsilon) : z \in P(x)\} \supset \mathcal{F}(x)$ and hence $\mathcal{F}(x)$ is totally bounded.

To prove that $\mathcal{F}$ is uniformly equicontinuous, take any $\varepsilon > 0$ and fix a finite $Q \subset \mathcal{F}$ such that $\bigcup\{B_\sigma(g, \frac{\varepsilon}{3}) : g \in Q\} \supset \mathcal{F}$. Every $f \in \mathcal{F}$ is uniformly continuous because X is compact. Apply Problem 246(iii) to conclude that Q is uniformly equicontinuous and hence there exists $\delta > 0$ such that $a, b \in X$ and $d(a, b) < \delta$ implies $\rho(g(a), g(b)) < \frac{\varepsilon}{3}$ for any $g \in Q$. We claim that δ witnesses the uniform equicontinuity of $\mathcal{F}$. Indeed, take any $a, b \in X$ with $d(a, b) < \delta$. If $f \in \mathcal{F}$ then there exists $g \in Q$ such that $\sigma(f, g) < \frac{\varepsilon}{3}$. As a consequence, $\rho(f(a), f(b)) \leqslant \rho(f(a), g(a)) + \rho(g(a), g(b)) + \rho(g(b), f(b)) \leqslant \sigma(f, g) + \frac{\varepsilon}{3} + \sigma(f, g) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. This proves uniform equicontinuity of $\mathcal{F}$ and hence our solution is complete.

S.252. *Given a compact metric space (X, d), suppose that (Y, ρ) is a complete metric space. Prove that a set $\mathcal{F} \subset C^*(X, Y)$ is compact if and only if it is closed, equicontinuous and $\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\}$ is compact for any $x \in X$. Here $C^*(X, Y)$ is considered to be endowed with the metric σ of uniform convergence.*

Solution. Assume that $\mathcal{F}$ is compact. Then it is closed in $C^*(X, Y)$ by Problem 121. Since any compact subset of any metric space is totally bounded, we can apply Problem 251 to conclude that $\mathcal{F}$ is equicontinuous. Now fix any $x \in X$. Consider the map $\psi : C^*(X, Y) \to Y$ defined by $\psi(f) = f(x)$ for any $f \in C^*(X, Y)$. To see that this map is continuous, fix any function $f \in C^*(X, Y)$ and $\varepsilon > 0$. If $g \in U = B_\sigma(f, \varepsilon)$ then $\rho(\psi(g), \psi(f)) = \rho(g(x), f(x)) \leqslant \sigma(f, g) < \varepsilon$ which shows that $\psi(U) \subset B_\rho(\psi(f), \varepsilon)$ and hence ψ is continuous at the point f. Observe that $\mathcal{F}(x) = \psi(\mathcal{F})$ and therefore the set $\mathcal{F}(x)$ is compact being a continuous image of a compact space $\mathcal{F}$. This finishes the proof of necessity.

To prove sufficiency, suppose that $\mathcal{F}$ is closed, equicontinuous and $\mathcal{F}(x)$ is compact (and hence totally bounded) for any $x \in X$. Applying Problem 251 we can see that $\mathcal{F}$ is totally bounded in $C^*(X, Y)$. The space $(C^*(X, Y), \sigma)$ is complete by Problem 248. As a consequence, $(\mathcal{F}, \sigma_{\mathcal{F}})$ is complete by Fact 1 from S.237. Here $\sigma_{\mathcal{F}} = \sigma|(\mathcal{F} \times \mathcal{F})$. Any complete totally bounded metric space is compact by Problem 212(iv) so $\mathcal{F}$ is compact and our solution is complete.

S.253. *Let (X, d) and (Y, ρ) be compact metric spaces. Show that a set $\mathcal{F} \subset C(X, Y) = C^*(X, Y)$ is compact if and only if it is closed and equicontinuous. Here $C^*(X, Y)$ is assumed to be endowed with the metric σ of uniform convergence.*

Solution. It is easy to see that every compact metric space is complete. Therefore (Y, ρ) is complete and Problem 252 is applicable to conclude that $\mathcal{F}$ is closed and equicontinuous if it is compact.

Now suppose that $\mathcal{F}$ is closed and equicontinuous. To show that it is compact, observe that $\mathcal{F}(x) = \{f(x) : x \in \mathcal{F}\}$ is totally bounded being a subset of a totally bounded space (Y, ρ). This makes it possible to apply 251 to conclude that $\mathcal{F}$ is totally bounded in $C^*(X, Y)$. Since $\mathcal{F}$ is closed in $C^*(X, Y)$, it has to be complete by Problem 248 and Fact 1 from S.237. Now we can apply Problem 212(iv) again to conclude that $\mathcal{F}$ is compact.

S.254. *Let (X, d) be a compact metric space. Prove that a set $\mathcal{F} \subset C^*(X)$ is compact if and only if it is closed, bounded and equicontinuous. Here the space $C^*(X) = C(X)$ is assumed to be endowed with the metric σ of uniform convergence.*

Solution. Since $\mathbb{R}$ is a complete metric space (Problem 205), we can apply Problem 252 to see that $\mathcal{F}$ is closed and equicontinuous if its is compact. It is an easy exercise to prove that any compact subspace of any metric space is bounded so necessity is established.

Now suppose that $\mathcal{F} \subset C^*(X)$ is closed, bounded and equicontinuous. Fix $x \in X$. Since $\mathcal{F}$ is bounded, we have $\text{diam}(\mathcal{F}) = K < \infty$. Fix $f \in \mathcal{F}$ and take any $g \in \mathcal{F}$. Then $|g(x) - f(x)| \leqslant \sigma(f, g) \leqslant K$ and therefore $g(x) \in [f(x) - K, f(x) + K]$, i.e., $\mathcal{F}(x) \subset [f(x) - K, f(x) + K]$. This shows that $\mathcal{F}(x)$ is bounded in $\mathbb{R}$. Every bounded subset of $\mathbb{R}$ has a compact closure which implies that $\mathcal{F}(x)$ is totally bounded in $\mathbb{R}$. Thus we can apply Problem 251 to assert that $\mathcal{F}$ is totally bounded in $C^*(X)$. Since $C^*(X)$ is complete (Problem 248), so is $\mathcal{F}$. Every totally bounded complete metric space is compact by Problem 212(iv), so $\mathcal{F}$ is compact and our solution is finished.

S.255. *Given a space X_t for each $t \in T$, let $X = \bigcup\{X_t \times \{t\} : t \in T\}$. Define the map $q_t : X_t \times \{t\} \to X_t$ by the formula $q_t(x, t) = x$ for all $t \in T$ and $x \in X_t$. If $U \subset X$, let $U \in \tau$ if $q_t(U \cap (X_t \times \{t\}))$ is open in X_t for all $t \in T$. The family τ is a topology on X (see Problem 113); the space (X, τ) is called the discrete (or free) union of the spaces X_t and we also denote (X, τ) by $\bigoplus \{X_t : t \in T\}$. Suppose that $X = \bigoplus \{X_t : t \in T\}$, where $|X_t| \leqslant \omega$ for each $t \in T$. Prove that $C_p(X)$ is homeomorphic to a product of metric spaces. Give an example of a space Y such that $C_p(Y)$ is homeomorphic to a product of metric spaces but Y cannot be represented as a discrete union of countable spaces.*

Solution. Apply Problem 114 to conclude that $C_p(X)$ is homeomorphic to the space $\prod \{C_p(X_t) : t \in T\}$. Since X_t is countable, the space $C_p(X_t)$ is metrizable (see Problem 210) for each $t \in T$ so $C_p(X)$ is homeomorphic to a product of metrizable spaces.

To construct our space Y we will need the set $N_\alpha = \{y_n^\alpha : n \in \omega\}$ where $y_n^\alpha = (n, \alpha)$ for each $n \in \omega$ and $\alpha < \omega_1$. Let $F = \bigcup\{N_\alpha : \alpha < \omega_1\} \cup \{0\}$. All points of $\bigcup\{N_\alpha : \alpha < \omega_1\}$ are isolated in F and $U \ni 0$, $U \subset F$ is open in F if and only if $N_\alpha \backslash U$ is finite for all $\alpha < \omega_1$. We leave to the reader the trivial verification that F is a Tychonoff space.

Let $D = \{d_\alpha : \alpha < \omega_1\}$ be a discrete space of cardinality ω_1. Then $Y = F \bigoplus D$ is our promised space. Observe that if a space Z is a discrete union of countable spaces then every point of Z has a countable neighbourhood. Since any $U \in \tau(0, Y)$ is an uncountable set, the space Y cannot be represented as a discrete union of countable spaces. So to finish our solution, we will establish that $C_p(Y)$ is a product of metric spaces.

Given $\alpha < \omega_1$, let $S_\alpha = \{x_n^\alpha : n \in \omega\} \cup \{x_\alpha\}$ be a faithfully enumerated sequence converging to the point x_α for each $\alpha < \omega_1$. Take any new point p and let $S = \bigoplus \{S_\alpha : \alpha < \omega_1\} \bigoplus \{p\}$. It is evident that S is a discrete union of countable spaces so $C_p(S)$ is homeomorphic to a product of metric spaces. Thus it is sufficient to show that $C_p(Y)$ is homeomorphic to $C_p(S)$.

First let $i(0) = p$, $i(d_\alpha) = x_\alpha$ and $i(y_n^\alpha) = x_n^\alpha$ for any $\alpha < \omega_1$ and $n \in \omega$. It is evident that $i : Y \to S$ is a (discontinuous!) bijection. Given $f \in C_p(S)$, define a function $g = \varphi(f) \in C_p(Y)$ as follows: $g(0) = f(p)$ and $g(y_n^\alpha) = f(x_n^\alpha) - f(x_\alpha) + f(p)$ for all $\alpha < \omega_1$ and $n \in \omega$. Besides, we let $g(d_\alpha) = f(x_\alpha)$ for all $\alpha < \omega_1$. It is immediate that g is indeed a continuous function on Y. To see that the mapping $\varphi : C_p(S) \to C_p(Y)$ is continuous, let us check its continuity at an arbitrary point $f_0 \in C_p(S)$. So, if $g_0 = \varphi(f_0)$ and $W \in \tau(g_0, C_p(Y))$ then there exist a finite set $A \subset Y$ and $\varepsilon > 0$ such that $O(g_0, A, \varepsilon) = \{g \in C_p(Y) : |g(y) - g_0(y)| < \varepsilon \text{ for all } y \in A\} \subset W$. If we make A larger, the set $O(g_0, A, \varepsilon)$ becomes smaller, so we can assume, without loss of generality, that $0 \in A$ and $d_\alpha \in A$ whenever $y_n^\alpha \in A$ for some $\alpha < \omega_1$ and $n \in \omega$.

The set $U = \{f \in C_p(S) : |f(x) - f_0(x)| < \frac{\varepsilon}{3} \text{ for any } x \in i(A)\}$ is an open neighbourhood of f_0 in $C_p(S)$. If $f \in U$, $g = \varphi(f)$ and $y \in A$ then there are three possibilities:

(a) $y = 0$. Then $i(y) = p$ and hence $|g(y) - g_0(y)| = |f(p) - f_0(p)| < \frac{\varepsilon}{3} < \varepsilon$.
(b) $y = d_\alpha$ for some $\alpha < \omega_1$. Then $|g(y) - g_0(y)| = |f(x_\alpha) - f_0(x_\alpha)| < \frac{\varepsilon}{3} < \varepsilon$.
(c) $y = y_n^\alpha$ for some $\alpha < \omega_1$ and $n \in \omega$. Then $|g(y) - g_0(y)| = |f(x_n^\alpha) - f(x_\alpha) + f(p) - f_0(x_n^\alpha) + f_0(x_\alpha) - f_0(p)| \leqslant |f(x_n^\alpha) - f_0(x_n^\alpha)| + |f(x_\alpha) - f_0(x_\alpha)| + |f(p) - f_0(p)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$,
so in all cases $g = \varphi(f) \in O(g_0, A, \varepsilon) \subset W$ whence $\varphi(U) \subset O(g_0, A, \varepsilon) \subset W$ and therefore φ is continuous at the point f_0.
Given an arbitrary $g \in C_p(Y)$, define a function $f = \delta(g) \in C_p(S)$ as follows: $f(p) = g(0)$, $f(x_\alpha) = g(d_\alpha)$ and $f(x_n^\alpha) = g(y_n^\alpha) - g(0) + g(d_\alpha)$ for all $\alpha < \omega_1$ and $n \in \omega$. It is immediate that $f : S \to \mathbb{R}$ is indeed a continuous function and δ is the inverse map of φ. Let us show that $\delta : C_p(Y) \to C_p(S)$ is continuous at an arbitrary point $g_0 \in C_p(Y)$. Take any open neighbourhood U of the function $f_0 = \delta(g_0)$ in the space $C_p(S)$. There exists a finite $B \subset S$ and $\varepsilon > 0$ such that $W(f_0, B, \varepsilon) = \{f \in C_p(S) : |f(x) - f_0(x)| < \varepsilon \text{ for all } x \in B\} \subset U$. If we make B larger, the set $W(f_0, B, \varepsilon)$ becomes smaller, so we can assume, without loss of generality, that $p \in B$ and $x_\alpha \in B$ whenever $x_n^\alpha \in B$ for some $\alpha < \omega_1$ and $n \in \omega$. The set $V = O(g_0, i^{-1}(B), \frac{\varepsilon}{3})$ is an open neighbourhood of g_0 and $\delta(V) \subset W(f_0, B, \varepsilon) \subset U$. Indeed, take any $g \in V$ and let $f = \delta(g)$. Given $x \in B$, we have three possibilities:
(d) $x = p$. Then $|f(x) - f_0(x)| = |g(0) - g_0(0)| < \frac{\varepsilon}{3} < \varepsilon$.
(e) $x = x_\alpha$ for some $\alpha < \omega_1$. Then $|f(x) - f_0(x)| = |g(d_\alpha) - g_0(d_\alpha)| < \frac{\varepsilon}{3} < \varepsilon$.

(f) $x = x_n^\alpha$ for some $\alpha < \omega_1$ and $n \in \omega$. Then $|f(x) - f_0(x)| = |g(y_n^\alpha) + g(d_\alpha) - g(0) - g_0(y_n^\alpha) - g_0(d_\alpha) + g_0(0)| \leqslant |g(y_n^\alpha) - g_0(y_n^\alpha)| + |g(d_\alpha) - g_0(d_\alpha)| + |g(0) - g_0(0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$,
so in all cases $f = \delta(g) \in W(f_0, B, \varepsilon) \subset U$ whence $\delta(V) \subset W(f_0, B, \varepsilon) \subset U$ and therefore δ is continuous at the point g_0. This proves that φ is a homeomorphism and finishes our solution.

S.256. *Suppose that $C_p(X)$ is homeomorphic to a product of metrizable spaces. Prove that, if X is Lindelöf or pseudocompact, then it is countable.*

Solution. Suppose that $C_p(X) = \prod\{M_t : t \in T\}$, where M_t is a metrizable space for each $t \in T$. If $\pi_t : C_p(X) \to M_t$ is the natural projection then we have $M_t = \pi_t(C_p(X))$ and hence $c(M_t) \leqslant c(C_p(X)) = \omega$ (see Problems 111 and 157(i)). Since the space M_t is metrizable, its weight has to be countable (Problem 214) and hence $C_p(X)$ is homeomorphic to a product of second countable spaces. Note that we can assume that every M_t has at least two points because a product does not change if we throw away its one-point factors.

Suppose that X is uncountable. Then $w(C_p(X)) > \omega$ (see Problem 169) and hence T has to be uncountable too (see Problem 209, 207 and 210). Choose a set $S \subset T$ with $|S| = \omega_1$ and fix distinct points $a_t, b_t \in M_t$ for each $t \in T$. The space $D = \prod\{\{a_t, b_t\} : t \in S\} \times \prod\{\{a_t\} : t \in T\backslash S\}$ is a subspace of the product $\prod\{M_t : t \in T\} = C_p(X)$ and has one-point factors for all indices in $T\backslash S$. As a consequence, D is a subspace of $C_p(X)$ homeomorphic to $\{0, 1\}^{\omega_1}$ so we will identify D with $\{0, 1\}^{\omega_1}$.

Let $\mathcal{B}_0 = \{\{0\}\}$; given $\alpha \leqslant \omega_1, \alpha > 0$ let $\mathcal{B}_\alpha = \{(\beta, \alpha] : \beta < \alpha\}$, where $(\beta, \alpha] = \{\gamma \in \omega_1 : \beta < \gamma \leqslant \alpha\}$.

Fact 1. The families $\{\mathcal{B}_\alpha : \alpha \leqslant \omega_1\}$ generate a topology τ on $\omega_1 + 1$ as local bases; this topology is Tychonoff and compact.

Proof. We leave it to the reader to check the properties (LB1)–(LB3) from Problem 007; let us prove that the space $(\omega_1 + 1, \tau)$ is Tychonoff. Observe first that the set $(\leftarrow, \alpha] = \{\beta < \omega_1 : \beta \leqslant \alpha\}$ is an open neighbourhood of α for any $\alpha \in \omega_1$. The next observation is that any interval $I = (\beta, \alpha]$ is a clopen($\equiv$closed an open) subset of $\omega_1 + 1$. Indeed, $I \in \mathcal{B}_\alpha$ so $I \in \tau$. To see that I is closed note that $(\leftarrow, \gamma] \cap I = \emptyset$ if $\gamma \leqslant \beta$ and $(\alpha, \gamma] \cap I = \emptyset$ if $\alpha < \gamma$. Thus every $\gamma \in (\omega_1 + 1)\backslash I$ has an open neighbourhood lying in $(\omega_1 + 1)\backslash I$ whence I is closed.

Now, if $\alpha \in \omega_1$ and F is a closed set, $\alpha \notin F$ then there is $\beta < \alpha$ such that $(\beta, \alpha] \cap F = \emptyset$. Since the set $W = (\beta, \alpha]$ is clopen, the function f equal to 1 on W and to zero on $(\omega_1 + 1)\backslash W$, is continuous. Since $f(\alpha) = 1$ and $f(F) \subset \{0\}$, the function f witnesses the Tychonoff property of $\omega_1 + 1$.

To finally see that $\omega_1 + 1$ is compact, take any open cover $\mathcal{U}$ of $\omega_1 + 1$. Let $\beta_0 = \omega_1$ and pick any $U_0 \in \mathcal{U}$ with $\beta_0 \in U_0$. There is $\beta_1 < \beta_0$ such that $(\beta_1, \beta_0] \subset U_0$. If we have chosen $U_0, \ldots, U_n \in \mathcal{U}$ and $\beta_0 > \beta_1 > \cdots > \beta_{n+1}$ such that $\beta_i \in U_i$ and $(\beta_{i+1}, \beta_i] \subset U_i$ for all $i \leqslant n$, choose $U_{n+1} \in \mathcal{U}$ with $\beta_{n+1} \in U_{n+1}$ and $\beta_{n+2} < \beta_{n+1}$

such that $(\beta_{n+2},\ \beta_{n+1}] \subset U_{n+1}$. This inductive construction cannot go on for all $n \in \omega$ because the set $\omega_1 + 1$ is well ordered, so $\beta_{n+1} = 0$ for some $n \in \omega$ which means that the finite family $\{U_0, \dots, U_n\} \subset \mathcal{U}$ covers all points of ω_1 except, maybe, 0. Therefore $\mathcal{U}$ has a finite subcover which shows that $\omega_1 + 1$ is compact. Fact 1 is proved.

Fact 2. Let $r : \omega_1 \to \omega_1$ be a function such that $r(\alpha) < \alpha$ for any $\alpha < \omega_1$. Then there is $\beta < \omega_1$ such that $r^{-1}(\beta)$ is uncountable.

Proof. Assume that $|r^{-1}(\alpha)| \leqslant \omega$ for any $\alpha < \omega_1$ and let $A_0 = \{0\}$. If we have a countable set $A_n \subset \omega_1$ let $A_{n+1} = \bigcup\{r^{-1}(\alpha) : \alpha \leqslant \sup(A_n)\}$. It is easy to see that the set $A = \bigcup\{A_n : n \in \omega\} \neq \emptyset$ is countable and $r^{-1}(\alpha) \subset A$ for any $\alpha \in A$. Observe that $\beta = \sup A \notin A$ for otherwise there is $\gamma > \beta$ with $r(\gamma) = \beta$ so $\gamma \in r^{-1}(\beta) \subset A$, a contradiction with $\gamma > \beta = \sup A$.

Since $\alpha = r(\beta) < \beta$, there is $\gamma \in A$ such that $\alpha < \gamma$. If $\gamma \in A_n$ for some $n \in \omega$ then $r^{-1}(\alpha) \subset A_{n+1} \subset A$ which is a contradiction with $\beta \in r^{-1}(\alpha)\backslash A$. Fact 2 is proved.

Fact 3. There is no Lindelöf subspace of the space $C_p(\omega_1 + 1)$ which separates the points of $\omega_1 + 1$.

Proof. Let $L \subset C_p(\omega_1 + 1)$ be a Lindelöf subspace which separates the points of $\omega_1 + 1$. Since the map $f \to (-f)$ is a homeomorphism of $C_p(\omega_1 + 1)$ onto itself, the sets $-L = \{-f : f \in L\}$ and $L \cup (-L)$ are also Lindelöf (it is an easy exercise that a union of two (or even countably many) Lindelöf spaces is a Lindelöf space). This shows that we can assume that $(-f) \in L$ for any $f \in L$.

For each $\alpha < \omega_1$ fix rational numbers s_α, t_α and a function $f_\alpha \in L$ such that $f_\alpha(\alpha) < s_\alpha < t_\alpha < f_\alpha(\omega_1)$ or $f_\alpha(\alpha) > s_\alpha > t_\alpha > f_\alpha(\omega_1)$. However, if we have the second inequality then, for the function $(-f_\alpha) \in L$, we have the first one. Therefore we can assume that $f_\alpha(\alpha) < s_\alpha < t_\alpha < f_\alpha(\omega_1)$ for all $\alpha < \omega_1$. Since each f_α is continuous, there exists $\beta_\alpha < \alpha$ such that $f_\alpha(\gamma) < s_\alpha$ for each $\gamma \in (\beta_\alpha, \alpha]$.

The map $r : \omega_1 \to \omega_1$ defined by $r(\alpha) = \beta_\alpha$ satisfies the hypothesis of Fact 2 so there is $\beta < \omega_1$ and an uncountable $R \subset \omega_1$ such that $\beta_\alpha = \beta$ for all $\alpha \in R$. Passing to a smaller uncountable subset of R if necessary, we can assume that there are $s, t \in \mathbb{Q}$ such that $s_\alpha = s$ and $t_\alpha = t$ for all $\alpha \in R$.

Given $f \in L$ we let $E = \{f_\alpha : \alpha \in R\}$ and $O_f = L\backslash\overline{E}$ if $f \notin \overline{E}$. Then O_f is an open neighbourhood of f in L such that $O_f \cap E = \emptyset$. If $f \in \overline{E}$ then $f(\omega_1) \geqslant t$ because $g(\omega_1) > t$ for all $g \in E$. Choose any $s' \in (s, t)$ and observe that, by continuity of f, there is $\gamma > \beta$ such that $f(\gamma) > s' > s$. The set $O_f = \{g \in L : g(\gamma) > s'\}$ is an open neighbourhood of f in L. If $\alpha > \gamma$ then $\gamma \in (\beta, \alpha] = (\beta_\alpha, \alpha]$ which implies, by the choice of β_α, that $f_\alpha(\gamma) < s < s'$ whence $f_\alpha \notin O_f$. As a consequence, $O_f \cap E \subset \{f_\alpha : \alpha \leqslant \gamma\}$ and therefore $O_f \cap E$ is a countable set.

The family $\mathcal{U} = \{O_f : f \in L\}$ is an open cover of the Lindelöf space L such that every $U \in \mathcal{U}$ intersects only countably many elements of E. If $\mathcal{U}'$ is a countable subcover of $\mathcal{U}$ then the uncountable set E is contained in $\bigcup \mathcal{U}'$ while $\mathcal{U}'$ is countable and every element of $\mathcal{U}'$ contains only countably many elements of E. This contradiction finishes the proof of Fact 3.

Fact 4. The space $\omega_1 + 1$ embeds in $D = \{0, 1\}^{\omega_1}$.

Proof. Let $f_\alpha(\beta) = 0$ if $\beta \leqslant \alpha$ and $f_\alpha(\beta) = 1$ for all $\beta > \alpha$. Then the map $f_\alpha : (\omega_1 + 1) \to \{0, 1\}$ is continuous for each $\alpha < \omega_1$. Let $\pi_\alpha : \{0, 1\}^{\omega_1} \to \{0, 1\}$ be the natural projection onto the αth factor.

Given ordinals $\alpha \leqslant \omega_1$ and $\beta < \omega_1$ let $\varphi(\alpha)(\beta) = f_\alpha(\beta)$; this defines a point $\varphi(\alpha) \in \{0, 1\}^{\omega_1}$. The formula $\alpha \to \varphi(\alpha)$ defines a map $\varphi : (\omega_1 + 1) \to \{0, 1\}^{\omega_1}$ which is continuous because $\pi_\alpha \circ \varphi = f_\alpha|\omega_1$ is a continuous map for each $\alpha < \omega_1$ (see Problem 102). If $\alpha < \beta \leqslant \omega_1$ then $f_\alpha(\alpha) = 0 \neq 1 = f_\alpha(\beta)$ which shows that $\varphi(\alpha) \neq \varphi(\beta)$. Thus the map $\varphi : (\omega_1 + 1) \to \varphi(\omega_1 + 1)$ is a condensation; the space $\omega_1 + 1$ being compact (Fact 1), the map φ is a homeomorphism (Problem 123) so Fact 4 is proved.

Fact 5. No Lindelöf subspace of $C_p(D)$ separates the points of $D = \{0, 1\}^{\omega_1}$.

Proof. Assume that some Lindelöf $L \subset C_p(D)$ separates the points of D. By Fact 4, there exists a subspace $W \subset D$ which is homeomorphic to $\omega_1 + 1$. Recall that the restriction map $\pi_W : C_p(D) \to C_p(W)$ is defined by $\pi_W(f) = f|W$ for any $f \in C_p(D)$. The map π_W is continuous (Problem 152) so $\pi_W(L)$ is a Lindelöf subspace of $C_p(W)$. It is immediate that $\pi_W(L)$ separates the points of $W = \omega_1 + 1$ which contradicts Fact 3 and proves Fact 5.

Returning to our proof, for any $x \in X$, let $\psi_x(f) = f(x)$ for any $f \in D$ (we now consider D to be a subspace of $C_p(X)$). The map $\psi : X \to C_p(D)$ defined by $\psi(x) = \psi_x$ for all $x \in X$, is continuous (Problem 166); it is an easy exercise to see that the set $Y = \psi(X) \subset C_p(D)$ separates the points of D.

If X is Lindelöf then Y is also a Lindelöf subspace of $C_p(D)$ which separates the points of D, a contradiction with Fact 5.

Now, assume that X is pseudocompact. Since the space D is separable (Problem 108), the space $C_p(D)$ and hence Y, condenses onto a second countable space (Problem 173). This condensation is a homeomorphism if restricted to Y because Y is pseudocompact (Problem 140). Thus Y is a second countable (and hence Lindelöf) subspace of $C_p(D)$ which separates the points of D; this contradiction with Fact 5 settles the case of a pseudocompact X and makes our solution complete.

S.257. *Let X be any space. Prove that, for any compact space Y and any continuous map $\varphi : X \to Y$, there exists a continuous map $\Phi : \beta X \to Y$ such that $\Phi|X = f$.*

Solution. By Problem 126, there is a set B such that Y embeds in $\mathbb{I}^B$ and hence we can assume that $Y \subset \mathbb{I}^B$. For the set $A = C(X, \mathbb{I})$ we can identify X with the subset $\tilde{X} = \{\beta_x : x \in X\} \subset \mathbb{I}^A$ Z, where $\beta_x(f) = f(x)$ for any $x \in X$ and $f \in A$. By definition, βX is the closure of $\tilde{X}$ in $\mathbb{I}^A$, so we consider that $X \subset \beta\ X = \overline{X} \subset \mathbb{I}^A$. Given a coordinate $b \in B$, denote by $p_b : \mathbb{I}^B \to \mathbb{I}$ is the natural projection onto the bth factor.

Analogously, the map $q_f : \mathbb{I}^A \to \mathbb{I}$ is the natural projection to the fth factor. Observe that $q_f|X = f$ for any $f \in A = C(X, \mathbb{I})$.

For any $b \in B$, the map $p_b \circ \varphi$ belongs to $C(X, \mathbb{I}) = A$ so fix $f_b \in A$ with $p_b \circ \varphi = f_b$. It is clear that $q_{fb}|X = f_b$ and therefore $q_{fb}|\beta X : \beta X \to \mathbb{I}$ is an extension of the map f_b to βX. For any $x \in \beta X$ let $\Phi(x)(b) = q_{fb}(x) \in \mathbb{I}$; this defines a point $\Phi(x) \in \mathbb{I}^B$ so we have a map $\Phi : \beta X \to \mathbb{I}^B$. We claim that the map Φ is continuous, $\Phi|X = \varphi$ and $\Phi(\beta X) \subset Y$, i.e., $\Phi : \beta X \to Y$ is a continuous extension of the map φ.

The map Φ is continuous because $p_b \circ \Phi = q_{fb}$ is continuous for any $b \in B$ (see Problem 102). If $x \in X$ then $\Phi(x)(b) = q_{f_b}(x) = f_b(x) = p_b \circ \varphi(x) = \varphi(x)(b)$ for every $b \in B$; this shows that $\Phi|X = \varphi$. Finally, X is dense in βX implies that $\varphi(X)$ is dense in $\Phi(\beta X)$ so $\Phi(\beta X) \subset \overline{\varphi(X)} \subset \overline{Y} = Y$ (the closure is taken in $\mathbb{I}^B$ and the last equality holds because Y is compact and hence closed in $\mathbb{I}^B$). We proved that the map $\Phi : \beta X \to Y$ is an extension of φ so our solution is complete.

S.258. *Let cX be a compact extension of a space X. Prove that the following properties are equivalent*:

(i) *For any compact space Y and any continuous map $f : X \to Y$ there exists a continuous map $F : cX \to Y$ such that $F \mid X = f$.*
(ii) *For any compact extension bX of the space X there exists a continuous map $\pi : cX \to bX$ such that $\pi(x) = x$ for all $x \in X$.*
(iii) *There is a homeomorphism $\varphi : cX \to \beta X$ such that $\varphi(x) = x$ for any $x \in X$.*

Solution. Take $Y = bX$ and $f : X \to Y$ defined by $f(x) = x$ for any $x \in X$. If $F : cX \to Y$ is the extension of f whose existence is guaranteed by (i), then $\pi = F$ satisfies (ii) so (i) $\Rightarrow$ (ii) is established.

(ii) $\Rightarrow$ (iii). Fix a continuous map $\pi : cX \Rightarrow \beta X$ such that $\pi(x) = x$ for any $x \in X$. It suffices to prove that $\varphi = \pi$ is the required homeomorphism. Observe first that $\pi(cX)$ is a compact subset of βX which contains X. Therefore $\beta X = \overline{X} \subset \pi(cX)$ so $\pi(cX) = \beta X$. Since every condensation of a compact space is a homeomorphism (Problem 123), it suffices to show that π is an injection.

The map $f : X \to cX$ defined by $f(x) = x$ for all $x \in X$, has a continuous extension $F : \beta X \to cX$ by Problem 257. Suppose that z, t are distinct points of cX with $y = \pi(z) = \pi(t)$. Take sets $U \in \tau(z, cX)$ and $V \in \tau(t, cX)$ such that $\overline{U} \cap \overline{V} = \emptyset$. The sets $U_1 = U \cap X$ and $V_1 = V \cap X$ are open subsets of X for which $\overline{U}_1 = \overline{U}$ and $\overline{V}_1 = \overline{V}$ so $\overline{U}_1 \cap \overline{V}_1 = \emptyset$ (the bar denotes the closure in cX). Now $z \in \overline{U}_1$ implies $y = \pi(z) \in \mathrm{cl}(\pi(U_1)) = \mathrm{cl}(U_1)$. Here the set $\mathrm{cl}(U_1)$ is the closure of U_1 in βX. Analogously, $y \in \mathrm{cl}(V_1)$; since the map F is continuous, we must have $F(y) \in \overline{F(U_1)} \cap \overline{F(V_1)} = \overline{U}_1 \cap \overline{V}_1 = \emptyset$, this contradiction shows that π is a bijection and hence a homeomorphism.

(iii) $\Rightarrow$ (i). Let $f : X \to Y$ be a continuous map of X to a compact space Y. By Problem 257 there exists a continuous $F_1 : \beta X \to Y$ such that $F_1|X = f$. Then $F = F_1 \circ \varphi$ maps cX continuously into Y and if $x \in X$ then $F(x) = F_1(\varphi(x)) = F_1(x) = f(x)$ and therefore $F|X = f$.

S.259. *Prove that the following conditions are equivalent for any space X*:

(i) *X is Čech-complete.*
(ii) *X is a G_δ-set in some compact extension of X.*
(iii) *X is a G_δ-set in any compact extension of X.*
(iv) *X is a G_δ-set in any extension of X.*

Solution. Since βX is a compact extension of X, and X is a G_δ-set in βX, we have (i) $\Rightarrow$ (ii).

Fact 1. Given a compact extension cX of a space X, let $f : \beta X \to cX$ be the unique continuous map such that $f(x) = x$ for any $x \in X$ (see Problem 258(ii)). Then $f(\beta X \backslash X) \subset cX \backslash X$ and therefore $f^{-1}(cX \backslash X) = \beta X \backslash X$ and $f(\beta X \backslash X) = cX \backslash X$.

Proof. Assume that there exists a point $y \in \beta X \backslash X$ such that $f(y) = x \in X$. Fix any sets $U \in \tau(y, \beta X)$ and $V \in \tau(x, \beta X)$ such that $\overline{U} \cap \overline{V} = \emptyset$. Since f is continuous, there is $W \in \tau(y, \beta X)$ such that $W \subset U$ and $f(\overline{W}) \subset V$. If $W_1 = W \cap X$ then $\overline{W}_1 = \overline{W}$ while $W_1 = f(W_1) \subset f(\overline{W}_1) = f(\overline{W}) \subset U$. Since $W_1 \subset X$, it follows from $W_1 \subset U$ that $W_1 \subset U \cap X = U_1$. The set W_1 is non-empty, which implies $\emptyset \neq W_1 = W_1 \cap U_1 \subset \overline{W}_1 \cap \overline{U}_1 = \overline{W} \cap \overline{U} \subset \overline{U} \cap \overline{V} = \emptyset$; this contradiction proves that $f(\beta X \backslash X) \subset cX \backslash X$. An immediate consequence of this inclusion is that $\beta X \backslash X \subset f^{-1}(cX \backslash X)$. Since it is evident that $f^{-1}(cX \backslash X) \subset \beta X \backslash X$, we obtain the equality $f^{-1}(cX \backslash X) = \beta X \backslash X$. Note that $f(\beta X)$ is a closed subset of cX which contains X; this implies $f(\beta X) = cX$ and $f(\beta X \backslash X) = cX \backslash X$ so Fact 1 is proved.

Fact 2. Suppose that $f : Y \to Z$ is a perfect map and $K \subset Z$ is compact. Then $L = f^{-1}(K)$ is a compact subspace of Y.

Proof. Observe first that the subspace L is closed in Y which easily implies that the map $f : L \to K$ is also perfect. Thus we can forget about Y and Z and consider f to be a perfect map from L onto K.

Given an open cover $\mathcal{U}$ of the space L, let $\mathcal{V}$ be the family of all finite unions of the elements from $\mathcal{U}$. For every $y \in K$ the set $f^{-1}(y)$ is compact so there is $V_y \in \mathcal{V}$ with $f^{-1}(y) \subset V_y$. By Fact 1 from S.226 there exists $U_y \in \tau(y, Y)$ such that $f^{-1}(U_y) \subset V_y$. The space Y being compact, there are $y_1, \ldots, y_n \in Y)$ such that $U_{y_1} \cup \cdots \cup U_{y_n} = Y$ and hence $V_{y_1} \cup \cdots \cup V_{y_n} = X$. Since each V_{y_i} can be covered by a finite subfamily of $\mathcal{U}$, there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that $\bigcup \mathcal{U}' = X$ and hence we proved the compactness of L and Fact 2.

Suppose now that cX is a compact extension of X in which X is a G_δ-set. Then $cX \backslash X = \bigcup \{F_n : n \in \omega\}$ where each F_n is closed in cX and hence compact. Take the continuous map $f : \beta X \to cX$ such that $f(x) = x$ for any $x \in X$. Since f is perfect map (see Problems 120 and 122), the set $G_n = f^{-1}(F_n)$ is compact for each $n \in \omega$ by Fact 2. Since $\bigcup \{G_n : n \in \omega\} = \beta X \backslash X$ by Fact 1, the set $\beta X \backslash X$ is a countable union of closed subsets of βX (see Problem 121). Of course, this implies that X is a G_δ-set in βX so we proved that (ii) $\Rightarrow$ (i).

To see that (i) $\Rightarrow$ (iii) take any compact extension bX of the space X and a continuous map $f : \beta X \to bX$ such that $f(x) = x$ for any $x \in X$. If X is Čech-complete then $\beta X \backslash X = \bigcup\{G_n : n \in \omega\}$ where each G_n is closed in βX and hence compact. Letting $F_n = f(G_n)$ we have $\bigcup\{F_n : n \in \omega\} = bX \backslash X$ by Fact 1 so $bX \backslash X$ is a countable union of compact (and hence closed) subsets of bX. This implies that X is a G_δ-set in bX which proves that (i) $\Rightarrow$ (iii).

Since it is evident that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i), we must only show that (iii) $\Rightarrow$ (iv). Let Y be an extension of X. Then X is dense in Y and Y is dense in βY whence X is dense in βY, i.e., βY is a compact extension of X. The set X is G_δ in βY by (iii) and hence X is a G_δ-set in a smaller space Y which proves the implication (iii) $\Rightarrow$ (iv) and finishes our solution.

S.260. *Prove that*

(i) *Any closed subspace of a Čech-complete space is Čech-complete.*
(ii) *Any G_δ-subspace of a Čech-complete space is Čech-complete. In particular, every open subspace of a Čech-complete space is Čech-complete.*

Solution. (i) Let X be a Čech-complete space. If F is closed in X then the space $G = \mathrm{cl}_{\beta X}(F)$ is a compact extension of F. Fix a family $\gamma = \{U_n : n \in \omega\} \subset \tau(\beta X)$ such that $\bigcap\gamma = X$. If $V_n = U_n \cap G$ then $\mu = \{V_n : n \in \omega\}$ is a family of open subsets of G such that $F = \bigcap\mu$ because $\bigcap\mu = \left(\bigcap\gamma\right) \cap G = X \cap G = F$. Now apply Problem 259(ii) to conclude that F is Čech-complete.

(ii) If the space X is Čech-complete and P is a G_δ-subset of X then the subspace $F = \overline{P}$ is Čech-complete by (i). Take a family $\gamma = \{U_n : n \in \omega\} \subset \tau(X)$ such that $\bigcap\gamma = P$ and let $V_n = U_n \cap F$ for all $n \in \omega$. The space $Y = \mathrm{cl}_{\beta X}(F)$ is a compact extension of both F and P. The space F being Čech-complete there exists a family $\mu = \{W_n : n \in \omega\} \subset \tau(Y)$ such that $\bigcap\mu = F$. Given $n \in \omega$, the set V_n is open in F so there exists a set $O_n \in \tau(Y)$ such that $O_n \cap F = V_n$. The family $\nu = \mu \cup \{O_n : n \in \omega\} \subset \tau(Y)$ is countable and $\bigcap\nu = P$. Indeed,

$$\bigcap\nu = \left(\bigcap\mu\right) \cap \left(\bigcap\{O_n : n \in \omega\}\right) = F \cap \left(\bigcap\{O_n \cap F : n \in \omega\}\right)$$
$$= F \cap \left(\bigcap\{V_n : n \in \omega\}\right) = \bigcap\{V_n : n \in \omega\} = P,$$

and hence P is a G_δ-set in Y. Now apply Problem 259(ii) to finish our solution.

S.261. *Prove that any perfect image as well as any perfect preimage of a Čech-complete space is Čech-complete.*

Solution. We will first prove some general statements about closed and perfect maps.

Fact 1. Let $f : X \to Y$ be a closed map. Then, for any subspace $A \subset Y$, the map $f_A = f \mid f^{-1}(A) : f^{-1}(A) \to A$ is closed.

Proof. It is clear that f_A is continuous and onto. If F is a closed subset of $f^{-1}(A)$ then $f_A(F) = f(F) = f(\overline{F} \cap f^{-1}(A)) = f(\overline{F}) \cap A$ is a closed subset of A so Fact 1 is proved.

Fact 2. Let $f : X \to Y$ be a perfect map. Then, for any subspace $A \subset Y$, the map $f_A = f \mid f^{-1}(A) : f^{-1}(A) \to A$ is perfect.

Proof. The map f_A is closed by Fact 1. Besides, $f_A^{-1}(a) = f^{-1}(a)$ is a compact set for any $a \in A$ because f is perfect, Thus the map f_A is perfect and Fact 2 is proved.

Given a continuous map $f : X \to Y$, it can be considered as a map from X to βY so there exists a continuous map $\tilde{f} : \beta X \to \beta Y$ such that $\tilde{f}|X = f$.

Fact 3. A map $f : X \to Y$ is perfect if and only if $\tilde{f}(\beta X \backslash X) = \beta Y \backslash Y$.

Proof. Suppose that f is perfect. Since $\tilde{f}(\beta X) = \beta Y$, it suffices to show that $\tilde{f}\,(\beta X \backslash X) \subset \beta Y \backslash Y$. Assume that $\tilde{f}\,(x) = y \in Y$ for some $x \in \beta X \backslash X$. The set $K = f^{-1}(y) \subset X$ is compact so there exist $U, V \in \tau(\beta X)$ such that $x \in U$, $K \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$. Let $U_1 = U \cap X$ and $V_1 = V \cap X$. Apply Fact 1 of S.226 to find $W \in \tau(y, Y)$ such that $f^{-1}(W) \subset V_1$. We have $f(U_1) \cap W = \emptyset$ which implies $y \notin \text{cl}_Y(f(U_1))$. However, $\overline{U}_1 = \overline{U}$ (the bar denotes the closure in the space βX) and hence $x \in \overline{U}_1$. Since the map $\tilde{f}$ is continuous, we have

$$y = \tilde{f}(x) \in \text{cl}_{\beta Y}(\tilde{f}(U_1)) = \text{cl}_{\beta Y}(f(U_1)),$$

and therefore $y \in \text{cl}_{\beta Y}(f(U_1)) \cap Y = \text{cl}_Y(f(U_1))$; this contradiction shows that $\tilde{f}\,(\beta X \backslash X) \subset \beta Y \backslash Y$ and necessity is established.

Now, if $\tilde{f}\,(\beta X \backslash X) = \beta Y \backslash Y$ then $\beta X \backslash X = (\tilde{f})^{-1}(\beta Y \backslash Y)$ and therefore we have $Y = (\tilde{f})^{-1}(X)$. The map $\tilde{f}$ is perfect (Problem 122) so we can apply Fact 2 to conclude that $f = \tilde{f}\,|X : X \to Y$ is also perfect. Fact 3 is proved.

Returning to our solution, assume that $f : X \to Y$ is a perfect map. If X is Čech-complete then X is a G_δ-set in βX and therefore $\beta X \backslash X = \bigcup\{K_n : n \in \omega\}$ where each K_n is closed and hence compact subset of βX. By Fact 3, we have $\tilde{f}(\beta X \backslash X) = \bigcup\{\tilde{f}\,(K_n) : n \in \omega\} = \beta Y \backslash Y$. Since each $\tilde{f}\,(K_n)$ is compact, the space $\beta Y \backslash Y$ is a countable union of closed subsets of βY whence Y is a G_δ-set in βY, i.e., Y is Čech-complete.

Assume now that Y is Čech-complete and hence $\beta Y \backslash Y = \bigcup\{L_n : n \in \omega\}$ where each L_n is closed in βY. It follows from Fact 3 that we have the equalities $(\tilde{f})^{-1}(\beta Y \backslash Y) = \beta X \backslash X = \bigcup\{(\tilde{f})^{-1}(L_n) : n \in \omega\}$ where each $(\tilde{f})^{-1}(L_n)$ is closed in βX by continuity of f. Thus the space $\beta X \backslash X$ is a countable union of closed subsets of βX whence X is a G_δ-set in βX, i.e., X is Čech-complete.

S.262. *Prove that any discrete union as well as any countable product of Čech-complete spaces is a Čech-complete space.*

Solution. Assume that X_t is Čech-complete for any $t \in T$. It is immediate that $X = \bigoplus \{X_t : t \in T\}$ is a dense subspace of the space $Y = \bigoplus \{\beta X_t : t \in T\} \subset \beta Y$. Since Y is dense in βY, the space βY is a compact extension of the space X.

The space Y is open in βY; to see this take any $y \in Y$ and $t \in T$ such that $y \in \beta X_t$. The set $U = \beta X_t$ is an open neighbourhood of y in Y. If $W \in \tau(\beta Y)$ and $W \cap Y = U$ then the compact set U is dense in W and hence $U = W$. This shows that Y is a neighbourhood of every $y \in Y$ and hence Y is open in βY. Since each βX_t is open in Y, any open subset of βX_t is open in Y and hence in βY.

By Čech-completeness of X_t there is a family $\{U_n^t : n \in \omega\} \subset \tau(\beta X_t)$ such that $X_t = \bigcap \{U_n^t : n \in \omega\}$ for each $t \in T$. The set $U_n = \bigcup \{U_n^t : t \in T\}$ is open in βY for each $n \in \omega$ and $\bigcap\{U_n : n \in \omega\} = \bigoplus \{X_t : t \in T\}$, i.e., the space X is a G_δ-set in the compact extension βY of the space X. Now apply Problem 259(ii) to conclude that X is Čech-complete.

To settle the case of a countable product, suppose that X_n is Čech-complete for any $n \in \omega$. The space $Y = \prod \{\beta X_n : n \in \omega\}$ is a compact extension of the space $X = \prod \{X_n : n \in \omega\}$. For each $n \in \omega$ fix a family $\{F_m^n : m \in \omega\}$ of compact subsets of βX_n such that $\bigcup \{F_m^n : m \in \omega\} = \beta X_n \backslash X_n$ for each $n \in \omega$; this is possible because each X_n is Čech-complete.

Let $\pi_n : Y \to \beta X_n$ be the natural projection of Y onto its nth factor. The set $G_m^n = \pi_n^{-1}(F_m^n)$ is compact for all $m, n \in \omega$ and it is easy to see that $Y \backslash X = \bigcup \{G_m^n : n, m \in \omega\}$. Thus X is a G_δ-set in its compact extension Y, so we can apply Problem 259(ii) to conclude that X is Čech-complete and finish our solution.

S.263. *Let X be a Čech-complete space. Given a compact $K \subset X$, prove that there exists a compact $L \subset X$ such that $K \subset L$ and $\chi(L, X) = \omega$. In particular, any point of X is contained in a compact set of countable character in X.*

Solution. Fix a family $\{O_n : n \in \omega\} \subset \tau(\beta X)$ such that $X = \bigcap\{O_n : n \in \omega\}$. Using normality of the compact space βX it is easy to construct by induction a family $\{U_n : n \in \omega\} \subset \tau(\beta X)$ with the following properties:

(1) $K \subset U_n \subset \overline{U}_n \subset O_n$ for each $n \in \omega$;
(2) $\overline{U}_{n+1} \subset U_n$ for al $n \in \omega$.

The set $L = \bigcap \{U_n : n \in \omega\} = \bigcap \{\overline{U}_n : n \in \omega\}$ is compact; it is contained in X because $L \subset U_n \subset O_n$ for every natural n. It is evident that $K \subset L$. Now take any $U \in \tau(L, \beta X)$. The set $F = \beta X \backslash U$ is compact and hence so is the set $F_n = \overline{U}_n \cap F$ for any $n \in \omega$. We have $F_{n+1} \subset F_n$ and $G = \bigcap\{F_n : n \in \omega\} = \emptyset$ because G is contained in $L \cap F = \emptyset$. The compactness of F implies $F_n = \emptyset$ for some n and hence $U_n \subset U$. This proves that the family $\{U_n : n \in \omega\}$ is an outer base of L in βX. It is straightforward to verify that the family $\{U_n \cap X : n \in \omega\}$ is an outer base of L in X and hence $\chi(L, X) = \omega$.

S.264. *Let X be any non-empty space. Suppose that Y and Z are dense Čech-complete subspaces of X. Prove that $Y \cap Z \neq \emptyset$.*

Solution. Both spaces Y and Z are dense in X and hence in βX, so βX is a compact extension of both Y and Z. Apply Čech-completeness of Y and Z to find countable families γ and μ of open subsets of βX such that $\bigcap\gamma = Y$ and $\bigcap\mu = Z$. The family $\nu = \gamma \cup \mu$ is countable and $\bigcap\nu = (\bigcap\gamma) \cap (\bigcap\mu) = Y \cap Z = \emptyset$. Take some faithful enumeration $\{O_n : n \in \omega\}$ of the family ν. Take any $x_0 \in O_0$ and fix some $U_0 \in \tau(x_0, \beta X)$ such that $\overline{U}_0 \subset O_0$. Suppose that we have constructed non-empty sets $U_0, \ldots, U_n \in \tau(\beta X)$ with the following properties:

(1) $\overline{U}_{i+1} \subset U_i$ for each $i \leqslant n - 1$;
(2) $U_i \subset O_i$ for all $i \leqslant n$.

To construct U_{n+1} observe that $U_n \cap O_{n+1} \neq \emptyset$ because the set O_{n+1} is dense in βX. Take any $x_{n+1} \in U_n \cap O_{n+1}$ and find $U_{n+1} \in \tau(x_{n+1}, \beta X)$ such that $\overline{U}_{n+1} \subset U_n \cap O_{n+1}$. It is clear that the properties (1) and (2) hold for the sets $U_0, \dots, U_n, U_{n+1}$ and hence our inductive construction goes on.

Once we have the sequence $\{U_n : n \in \omega\}$ observe that the family $\{\overline{U}_n : n \in \omega\}$ of compact subsets of βX, is decreasing so $\bigcap \{U_n : n \in \omega\} = \bigcap \{\overline{U}_n : n \in \omega\} \neq \emptyset$ by compactness of βX. On the other hand $\bigcap\{U_n : n \in \omega\} \subset \bigcap\{O_n : n \in \omega\} = \emptyset$; this contradiction proves that $Y \cap Z \neq \emptyset$.

S.265. *Prove that the following are equivalent for any space X*:

(i) *The space $C_p(X)$ is Čech-complete.*
(ii) *$C_p(X)$ has a dense Čech-complete subspace.*
(iii) *X is countable and discrete.*

Solution. It is evident that (i) $\Rightarrow$ (ii). If X is countable and discrete then $C_p(X)$ is homeomorphic to $\mathbb{R}^\omega$. Observe that $\mathbb{R}$ is homeomorphic to the interval $(-1, 1)$ which is open (and hence G_δ) in its compact extension $\mathbb{I}$. This shows that $\mathbb{R}$ is Čech-complete. Now apply Problem 262 to conclude that $C_p(X) = \mathbb{R}^\omega$ is Čech-complete. This proves (iii) $\Rightarrow$ (i).

Fact 1. A space X is discrete if and only if $C_p(X) = \mathbb{R}^X$.

Proof. If X is discrete then any $f \in C_p(X)$ is continuous on X so $C_p(X)$ coincides with $\mathbb{R}^X$. Now, if $C_p(X) = \mathbb{R}^X$ then, for any $A \subset X$, the function f equal to 1 on A and to zero on $X \backslash A$, is continuous whence $A = f^{-1}((0, +\infty))$ is an open subset of X. It turns out that all subsets of X are open so it is discrete and Fact 1 is proved.

To show that (ii) $\Rightarrow$ (iii), take any dense Čech-complete subspace $D \subset C_p(X)$. Let us prove first that $C_p(X) = \mathbb{R}^X$. If not, pick any $f \in \mathbb{R}^X \backslash C_p(X)$ and consider the space $Y = f + C_p(X) = \{f + g : g \in C_p(X)\} \subset \mathbb{R}^X$. Since the map $\varphi : \mathbb{R}^X \to \mathbb{R}^X$ defined by $\varphi(h) = h + f$, is a homeomorphism (recall that $\mathbb{R}^X = C_p(Z)$ where Z is the set X with the discrete topology, and apply Problem 079), the space $D_1 = f + D$ is a dense Čech-complete subspace of $\mathbb{R}^X$. It is easy to see that $Y \cap C_p(X) = \emptyset$ and hence D and D_1 are dense disjoint Čech-complete subspaces of $\mathbb{R}^X$ which contradicts Problem 264 and proves that $C_p(X) = \mathbb{R}^X$. Now apply Fact 1 to conclude that X is discrete.

Fact 2. Let Y be a dense subspace of a space X. Then $\chi(F, Y) = \chi(F, X)$ for any compact $F \subset Y$. In particular this is true if F is a point of Y.

Proof. It is evident that $\chi(F, Y) \leqslant \chi(F, X)$. To prove the inverse inequality, take any outer base $\mathcal{B}$ of the set F in Y. For any $U \in \mathcal{B}$ choose $O_U \in \tau(X)$ with $O_U \cap Y = U$. It suffices to show that $\mathcal{C} = \{O_U : U \in \mathcal{B}\}$ is an outer base of F in X. Let $W \in \tau(F, X)$. For any $x \in F$ fix $V_x \in \tau(x, X)$ for which $\overline{V}_x \subset W$ and use compactness of F to find $x_1, \dots, x_n \in F$ such that $V = V_{x_1} \cup \dots \cup V_{x_n} \supset F$; observe also that $\overline{V} \subset W$. Since $\mathcal{B}$ is an outer base of F in Y, there exists $U \in \mathcal{B}$ such that $U \subset V \cap Y$. As a consequence, $\overline{O_U} = \overline{U}$ (the bar denotes the closure in X) and we have $F \subset U \subset O_U \subset \overline{O_U} = \overline{U} \subset \overline{V} \subset W$ and Fact 2 is proved.

To finish our solution, apply Problem 263 to find a compact $K \subset D$ with $\chi(K, D) = \omega$. It follows from Fact 2 that $\chi(K, C_p(X)) = \chi(K, D) = \omega$. Therefore X is countable by Problem 170 and our solution is complete.

S.266. *Considering $C_p(X)$ as a subspace of $\mathbb{R}^X$, assume that it is a G_δ-subset of $\mathbb{R}^X$. Prove that X is discrete (and hence $C_p(X) = \mathbb{R}^X$).*

Solution. Suppose that $C_p(X)$ is a G_δ-subset of $\mathbb{R}^X$. If X is non-discrete, fix a function $f \in \mathbb{R}^X \backslash C_p(X)$. Since the map $\varphi : \mathbb{R}^X \to \mathbb{R}^X$ defined by $\varphi(h) = h + f$, is a homeomorphism (recall that $\mathbb{R}^X = C_p(Z)$ where Z is the set X with the discrete topology, and apply Problem 079), the space $Y = \varphi(C_p(X))$ is a dense G_δ-subspace of $\mathbb{R}^X$ disjoint from $C_p(X)$. Fix countable families $\mu, \nu \subset \tau(\mathbb{R}^X)$ such that $C_p(X) = \bigcap \mu$ and $Y = \bigcap \nu$. If $\{O_n : n \in \omega\}$ is an enumeration of the family $\mu \cup \nu$ then every O_n is dense in $\mathbb{R}^X$ and $\bigcap\{O_n : n \in \omega\} = \left(\bigcap \mu\right) \cap \left(\bigcap \nu\right) = Y \cap C_p(X) = \emptyset$.

Take any $f_0 \in O_0$ and find a finite $A_0 \subset X$ and $\varepsilon_0 \in (0, 1)$ such that $O(f_0, A_0, \varepsilon_0) = \{g \in \mathbb{R}^X : |g(x) - f_0(x)| < \varepsilon_0 \text{ for all } x \in A_0\} \subset O_0$. Assume that we constructed finite sets $A_0, \ldots, A_n \subset X$, positive numbers $\varepsilon_0, \ldots, \varepsilon_n$ and functions $f_0, \ldots, f_n \in \mathbb{R}^X$ with the following properties:

(1) $\varepsilon_0 > \cdots > \varepsilon_n$ and $\varepsilon_i < \frac{1}{i+1}$ for all $i \leqslant n$.
(2) $A_0 \subset \cdots \subset A_n$.
(3) $[f_{i+1}(x) - \varepsilon_{i+1}, f_{i+1}(x) + \varepsilon_{i+1}] \subset (f_i(x) - \varepsilon_i, f_i(x) + \varepsilon_i)$ for each $i < n$ and $x \in A_i$.
(4) $O(f_i, A_i, \varepsilon_i) \subset O_i$ for each $i \leqslant n$.

Since the set O_{n+1} is dense in $\mathbb{R}^X$, we can choose $f_{n+1} \in O_{n+1} \cap O(f_n, A_n, \varepsilon_n)$. The set O_{n+1} is open in $\mathbb{R}^X$ and $f_{n+1}(x) \in (f_n(x) - \varepsilon_n, f_n(x) + \varepsilon_n)$ for each $x \in A_n$ so there exists a number $\varepsilon_{n+1} \in \left(0, \frac{1}{n+2}\right)$ and a finite set $A_{n+1} \subset X$ such that $A_n \subset A_{n+1}$, $O(f_{n+1}, A_{n+1}, \varepsilon_{n+1}) \subset O_{n+1}$ and

$$[f_{n+1}(x) - \varepsilon_{n+1}, f_{n+1}(x) + \varepsilon_{n+1}] \subset (f_n(x) - \varepsilon_n, f_n(x) + \varepsilon_n)$$

for all $x \in A_n$. It is clear that the properties (1)–(4) also hold for the sets $A_0, \ldots, A_{n+1}$, numbers $\varepsilon_0, \ldots, \varepsilon_{n+1}$ and functions $f_0, \ldots, f_{n+1} \in \mathbb{R}^X$ so the inductive construction goes on.

Given a point $x \in A = \bigcup\{A_n : n \in \omega\}$, take the minimal $n \in \omega$ such that $x \in A_i$ for all $i \geqslant n$. The sets $I_k = [f_k(x) - \varepsilon_k, f_k(x) + \varepsilon_k]$, $k \geqslant n$ being compact, there exists a point $h(x) \in \bigcap\{I_k : k \geqslant n\}$. Letting $h(x) = 0$ for all $x \in X \backslash A$, we obtain a function $h \in \mathbb{R}^X$. Given a number $n \in \omega$ and a point $x \in A_n$, the property (3) implies $h(x) \in \bigcap\{I_k : k \geqslant n\} = \bigcap\{(f_k(x) - \varepsilon_k, f_k(x) + \varepsilon_k) : k \geqslant n\}$ whence $h(x) \in (f_n(x) - \varepsilon_n, f_n(x) + \varepsilon_n)$ and therefore $h \in O(f_n, A_n, \varepsilon_n)$. We chose the number n arbitrarily so $h \in \bigcap\{O(f_k, A_k, \varepsilon_k) : k \in \omega\} \subset \bigcap\{O_k : k \in \omega\} = \emptyset$ which is a contradiction. Thus $C_p(X) = \mathbb{R}^X$ and we can apply Fact 1 of S.265 to see that our solution is complete.

S.267. *Considering $C_p(X)$ as a subspace of $\mathbb{R}^X$, assume that it is an F_σ-subset of $\mathbb{R}^X$. Prove that X is discrete (and hence $C_p(X) = \mathbb{R}^X$).*

Solution. Fix a family $\{F_n : n \in \omega\}$ of closed subsets of the space $\mathbb{R}^X$ such that $C_p(X) = \bigcup\{F_n : n \in \omega\}$. The set $C_n = C_p(X, [-n, n])$ is closed in $C_p(X)$ for every $n \in \omega$ and therefore $G_{n,k} = F_n \cap C_k$ is closed in $\mathbb{R}^X$ for any $n, k \in \omega$. If $\mathcal{G} = \{G_{n,k} : n, k \in \omega\}$ then $\mathcal{G}$ is a countable family of closed subsets of $\mathbb{R}^X$ such that $\bigcup \mathcal{G} = C_p^*(X)$.

Given $f, g \in C_p^*(X)$, let $\rho(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. Then ρ is a complete metric on $C^*(X)$ (Problem 248). Any set closed in $C_p^*(X)$ is also closed in $(C^*(X), \rho) = C_u^*(X)$ (see Problems 086 and 211) so the elements of G are closed in $C_u^*(X)$ as well.

Suppose first that $\text{Int}(G) = \emptyset$ for all $G \in \mathcal{G}$ (the interior is taken in $C_u^*(X)$). Then, for any $G \in \mathcal{G}$, we have the property

$(*)$ For any non-empty $U \in \tau(\rho)$ and any $\varepsilon > 0$ there exists a non-empty $V \in \tau(\rho)$ such that $\overline{V} \subset U \setminus G$ and $\text{diam}_\rho(V) < \varepsilon$.

To find such a set V, take any function $f \in U \setminus G$, find a number $r \in (0, \varepsilon)$ such that $B_\rho(f, r) = \{g \in C^*(X) : \rho(f, g) < r\} \subset U \setminus G$ and let $V = B_\rho(f, \frac{r}{2})$.

Take an enumeration $\{G_n : n \in \omega\}$ of the family $\mathcal{G}$ and use $(*)$ to construct a sequence $\{U_n : n \in \omega\} \subset \tau(\rho)$ such that

(1) $\overline{U}_{n+1} \subset U_n$ for each $n \in \omega$.
(2) $\text{diam}_\rho(U_n) \to 0$.
(3) $U_n \cap G_n = \emptyset$ for any $n \in \omega$.

If $F_n = \overline{U}_n$ for any $n \in \omega$, observe that (1) and (2) imply $\text{diam}_\rho(F_n) \to 0$. Since $(C^*(X), \rho)$ is a complete metric space, we have can apply Problem 236 to conclude that $F = \bigcap\{F_n : n \in \omega\} = \bigcap\{U_n : n \in \omega\} \neq \emptyset$. If $x \in F$ then $x \notin G_n$ for any $n \in \omega$ by (3). This contradiction with $\bigcup \mathcal{G} = C_p^*(X)$ proves that $\text{Int}(G) \neq \emptyset$ for some $G \in \mathcal{G}$. As a consequence, we can find $f \in G$ and $\varepsilon > 0$ such that $P_\varepsilon(f) = \{g \in C^*(X) : |g(x) - f(x)| \leqslant \varepsilon\} \subset G$. This implies that the set $P_\varepsilon(f)$ is closed in $\mathbb{R}^X$ and hence the set $P_\varepsilon(f) - f = \{g - f : g \in P_\varepsilon(f)\} = C(X, [-\varepsilon, \varepsilon])$ is closed in $\mathbb{R}^X$. It follows easily from the Tychonoff property of X that $C(X, [-\varepsilon, \varepsilon])$ is dense in $[-\varepsilon, \varepsilon]^X$ so $C(X, [-\varepsilon, \varepsilon]) = [-\varepsilon, \varepsilon]^X$, i.e., any function $h : X \to [-\varepsilon, \varepsilon]$ is continuous. We leave it as a simple exercise to the reader to prove that this condition implies discreteness of X so apply Fact 1 of S.265 to see that our solution is complete.

S.268. *Let X be a space. Given a sequence $\Gamma = \{\gamma_n : n \in \omega\}$ of open covers of X, say that a filter $\mathcal{F}$ is dominated by Γ if, for any $n \in \omega$, there is $F_n \in \mathcal{F}$ with $F_n \subset U_n$ for some $U_n \in \gamma_n$. The sequence Γ is called complete if for any filter $\mathcal{F}$ in X, dominated by Γ, we have $\bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$. Prove that a space X is Čech-complete if and only if there is a complete sequence of open covers of X.*

Solution. To prove necessity, suppose that a space X is Čech-complete and fix a family $\mathcal{U} = \{O_n : n \in \omega\} \subset \tau(\beta X)$ such that $X = \bigcap \mathcal{U}$. Given a number $n \in \omega$, let $\gamma_n = \{V \in \tau(X) : \text{cl}_{\beta X}(V) \subset U_n\}$. If $n \in \omega$ then, for any $x \in X$ we can take $W \in \tau(x, \beta X)$

such that $\mathrm{cl}_{\beta X}(W) \subset O_n$. Then $W' = W \cap X \in \gamma_n$ and $x \in W$. This shows that γ_n is an open cover of X for each $n \in \omega$.

To show that the sequence $\Gamma = \{\gamma_n : n \in \omega\}$ is complete, take any filter $\mathcal{F}$ dominated by Γ. Since the space βX is compact, there is $x \in \bigcap\{\mathrm{cl}_{\beta X}(F) : F \in \mathcal{F}\}$, so it suffices to show that $x \in X$. Take any $n \in \omega$ and an element $F \in \mathcal{F}$ such that $F \subset U \in \gamma_n$ for some $U \in \gamma_n$. We have $\mathrm{cl}_{\beta X}(F) \subset \mathrm{cl}_{\beta X}(U) \subset O_n$ by definition of γ_n which implies $x \in \mathrm{cl}_{\beta X}(F) \subset O_n$. The number n was chosen arbitrarily, so $x \in \bigcap \{O_n : n \in \omega\} = X$ and necessity is proved.

To establish sufficiency, fix a complete sequence $\Gamma = \{\gamma_n : n \in \omega\}$ of open covers of X. Let $\mu_n = \{U \in \tau(\beta X) : U \cap X \in \gamma_n\}$ and $O_n = \bigcup \mu_n$ for each $n \in \omega$. It suffices to show that $X = \bigcap\{O_n : n \in \omega\}$. Observe first that $X \subset \bigcup \gamma_n \subset \bigcup \mu_n$ for each $n \in \omega$ so $X \subset \bigcap\{O_n : n \in \omega\}$. To prove the reverse inclusion, take any $x \in \bigcap\{O_n : n \in \omega\}$ and let $\mathcal{F} = \{P \subset X : U \cap X \subset P$ for some $U \in \tau(x, \beta X)\}$. We leave to the reader the simple verification that $\mathcal{F}$ is a filter on X. Take any $n \in \omega$ and $U \in \mu_n$ with $x \in U$. Then $V = U \cap X \in \mathcal{F}$ and $V \subset V \in \gamma_n$ by the definition of μ_n. This shows that $\mathcal{F}$ is dominated by Γ and therefore we can fix $y \in \bigcap \{\overline{F} : F \in \mathcal{F}\}$ (the bar denotes the closure in X). If $y \neq x$ then take any set $W \in \tau(x, \beta X)$ such that $y \in \mathrm{cl}_{\beta X}(W)$ and observe that $W' = W \cap X$ belongs to $\mathcal{F}$ and $y \notin \overline{W}'$ this contradiction shows that $x \in X$ and hence $X = \bigcap\{O_n : n \in \omega\}$, i.e., X is Čech-complete.

S.269. *Prove that a metrizable space is Čech-complete if and only if it is metrizable by a complete metric.*

Solution. To prove sufficiency, fix a complete metric space (X, ρ) and consider the family $\gamma_n = \{U \in \tau(X) : \mathrm{diam}(U) < \frac{1}{n}\}$. It is clear that γ_n is an open cover of X; let us prove that the sequence $\Gamma = \{\gamma_n : n \in \omega\}$ is complete. Take any filter $\mathcal{F}$ dominated by Γ and, for each $n \in \omega$, find $F_n \in \mathcal{F}$ and $U_n \in \gamma_n$ such that $F_n \subset U_n$. If we let $G_n = \overline{F_0 \cap \dots \cap F_n}$ for each $n \in \omega$ then the sequence $\{G_n : n \in \omega\}$ has the following properties:

(1) $G_0 \supset G_1 \supset \cdots \subset G_n \supset \cdots$; this is evident.
(2) $\mathrm{diam}(G_n) < \frac{1}{n}$; this is true because $G_n \subset \overline{F}_n \subset \overline{U}_n$ and hence we have diam $(G_n) \leqslant$ diam $(\overline{U}_n) = \mathrm{diam}(U_n) < \frac{1}{n}$.

Thus $\mathrm{diam}(G_n) \to 0$ for a decreasing sequence $\{G_n : n \in \omega\}$ of closed non-empty subsets of a complete metric space (X, ρ). Apply Problem 236 to conclude that $\bigcap\{G_n : n \in \omega\} \neq \emptyset$ and take any $x \in \bigcap\{G_n : n \in \omega\}$; it suffices to show that $x \in \bigcap\{\overline{F} : F \in \mathcal{F}\}$. Observe first that $x \in G_n \subset \overline{F}_n$ so $x \in \overline{F}_n$ for all $n \in \omega$. If there exists $F \in \mathcal{F}$ such that $x \notin \overline{F}$, take $n \in \omega$ such that $B_\rho\left(x, \frac{2}{n}\right)$ is disjoint from F; since $\mathcal{F}$ is a filter, we can find a point $y \in F \cap F_n$. The point y lying outside of the ball $B_\rho\left(x, \frac{2}{n}\right)$, we have $\rho(x, y) \geqslant \frac{2}{n}$. On the other hand, $x, y \in \overline{F}_n$ so $\rho(x, y) \leqslant \mathrm{diam}(F_n) < \frac{1}{n}$ which is a contradiction showing that $x \in \bigcap \{\overline{F} : F \in \mathcal{F}\}$. This contradiction shows that the sequence Γ is complete so X is Čech-complete by Problem 268 and sufficiency is proved.

Fact 1. Let (X, d) be a metric space. Suppose that P is a G_δ-subset of X. Then P embeds in $X \times \mathbb{R}^\omega$ as a closed subspace.

Proof. Let $\psi : P \to X$ be the embedding, i.e., $\psi(x) = x$ for all $x \in P$. Represent $X\backslash P$ as $\bigcup\{F_i : i \in \mathbb{N}\}$, where F_i is closed in X for all $i \in \mathbb{N}$; given $x \in P$, let $\varphi_i(x) = \frac{1}{d(x,F_i)}$ where $d(x, F_i) = \inf\{d(x, y);\ y \in F_i\}$ for each $i \in \mathbb{N}$. Observe that $d(x, F_i) > 0$ for all $x \in P$ and apply Fact 1 of S.212 to see that $\varphi_i : P \to \mathbb{R}$ is a continuous map for all $i \in \mathbb{N}$. Let $X_0 = X$ and $X_n = \mathbb{R}$ for all $n \in \mathbb{N}$; it is clear that $Z = \prod\{X_i : i \in \omega\}$ is homeomorphic to $X \times \mathbb{R}^\omega$. Let $\pi_i : Z \to X_i$ be the natural projection for all $i \in \omega$. Now define a map $\varphi : P \to Z$ by the formulas $\varphi(x)(0) = \psi(x)$ and $\varphi(x)(n) = \varphi_n(x)$ for all $x \in P$ and $n \in \mathbb{N}$. The map $\varphi : P \to Z$ is continuous because $\pi_i \circ \varphi = \varphi_i$ if $i \in \mathbb{N}$ and $\pi_0 \circ \varphi = \psi$ which shows that $\pi_i \circ \varphi : P \to X_i$ is a continuous map for all $i \in \omega$ (see Problem 102). It suffices to establish that $\varphi : P \to P' = \varphi(P)$ is a homeomorphism and P' is closed in Z. The map φ is an injection because so is $\psi = \pi_0 \circ \varphi$. Since $\varphi^{-1} = \psi^{-1} \circ \pi_0$, we can see that φ^{-1} is continuous so φ is an embedding.

To show that P' is closed in Z, it suffices to establish that any $z \in Z\backslash P'$ has a neighbourhood contained in $Z\backslash P'$. Consider first the case $z_0 = z(0) \in P$. Since $z \notin P'$, there is $i \in \mathbb{N}$ such that $z(i) \neq \varphi_i(z_0)$. Find disjoint $U \in \tau(z(i), \mathbb{R})$ and $V \in \tau(\varphi_i(z_0), \mathbb{R})$ and apply continuity of φ_i to fix $W \in \tau(z_0, X)$ such that $\varphi_i(W \cap P) \subset V$. The set $O = \pi_0^{-1}(W) \cap \pi_i^{-1}(U)$ is open in Z and $z \in O \subset X\backslash P'$. Indeed, if $y \in P' \cap O$ then $y_0 = y(0) \in W \cap P$ and hence $y(i) = \varphi_i(y) \in V$ which shows that $y(i) \notin U$, i.e., $y \notin O$. This contradiction shows that $O \cap P' = \emptyset$.

Now, if $z_0 = z(0) \in X\backslash P$ then $z_0 \in F_i$ for some $i \in \mathbb{N}$. Take any $r > 0$ with $\frac{1}{r} > z(i) + 1$; let $U = B(z_0, r)$ and $V = (-\infty, z(i)+1)$. The set $O = \pi_0^{-1}(U) \cap \pi_i^{-1}(V)$ is open in Z and $z \in O \subset X\backslash P'$. Indeed, if $y \in P' \cap O$ then $y_0 = y(0) \in U \cap P$ and hence $\varphi_i(y_0) > \frac{1}{r} > z(i) + 1$. On the other hand $\pi_i(y) = y(i) \in V$ and hence $y(i) < z(i) + 1$. This contradiction shows that $O \subset X\backslash P'$ and finishes the proof of Fact 1.

Now it is easy to finish the proof of necessity. Suppose that a metrizable space X is Čech-complete. Then X is embeddable into a completely metrizable space Y as a dense subspace (see Problem 237). Since Y is an extension of X, the set X is G_δ in Y (Problem 259) so we can apply Fact 1 to conclude that X embeds in $Y \times \mathbb{R}^\omega$ as a closed subspace. The space $Y \times \mathbb{R}^\omega$ is completely metrizable by Problem 208; any closed subset of a completely metrizable space is completely metrizable so the space X is completely metrizable. This concludes the proof of necessity and our solution.

S.270. *Prove that, for any Čech-complete space X, we have $w(X) = nw(X)$. In particular, any Čech-complete space with a countable network is second countable.*

Solution. Fix a network $\mathcal{N}$ in the space X with $|\mathcal{N}| = \kappa = nw(X)$ and a countable family $\mathcal{F}$ of closed subsets of βX such that $\bigcup\mathcal{F} = \beta X\backslash X$. The family $\mathcal{G} = \{\mathrm{cl}_{\beta X}(N) : N \in \mathcal{N}\} \cup \mathcal{F}$ also has cardinality κ and the following property:

$(*)$ For any $x \in X$ and $y \in \beta X\backslash\{x\}$ there exist $F, G \in \mathcal{G}$ such that $x \in F$, $y \in G$ and $F \cap G = \emptyset$.

To prove $(*)$, suppose first that $y \in X$ and find $U, V \in \tau(\beta X)$ such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. There are $M, N \in \mathcal{N}$ such that $x \in M \subset U \cap X$ and $y \in N \subset V \cap X$.

Then $\overline{M}, \overline{N} \in \mathcal{G}, x \in \overline{M}, y \in \overline{N}$ and $\overline{M} \cap \overline{N} \subset \overline{U} \cap \overline{V} = \emptyset$, i.e., $(*)$ holds for such x and y. Now, if $y \in \beta X \backslash X$ then there is $F \in \mathcal{F}$ such that $y \in F$. Take $W \in \tau(x, \beta X)$ such that $\overline{W} \cap F = \emptyset$. Since $\mathcal{N}$ is a network in X, there exists $N \in \mathcal{N}$ such that $x \in N \subset W \cap X$. As a consequence, $\overline{N} \in \mathcal{G}$, $F \in \mathcal{G}$, $x \in \overline{N}$, $y \in F$ and $\overline{N} \cap F \subset \overline{W} \cap F = \emptyset$ so $(*)$ is proved.

For any disjoint pair $p = (F, G)$ of elements of $\mathcal{G}$, fix $U(p), V(p) \in \tau(\beta X)$ such that $F \subset U(p)$, $G \subset V(p)$ and $U(p) \cap V(p) = \emptyset$. It is possible to do this because every compact space is normal (Problem 124). The family $\mathcal{U} = \{U(p), V(p) : p$ is a disjoint pair of the elements of $\mathcal{G}\}$ has cardinality $\leqslant \kappa$ so the family $\mathcal{W}$ of all finite intersections of the elements of $\mathcal{U}$ also has cardinality $\leqslant \kappa$. To finish our solution it is sufficient to show that the family $\mathcal{W}' = \{W \cap X : W \in \mathcal{W}\}$ is a base in X.

Fix any $x \in X$ and any $O' \in \tau(x, X)$; take $O \in \tau(\beta X)$ such that $O \cap X = O'$. For any $y \in E = \beta X \backslash O$ find $F_y, G_y \in \mathcal{G}$ such that $x \in F_y$, $y \in G_y$ and $F_y \cap G_y = \emptyset$ (see $(*)$). The pair $p = (F_y, G_y)$ is disjoint so the sets $U_y = U(p)$ and $V_y = V(p)$ are disjoint. The family $\{V_y : y \in E\}$ is an open cover of the compact set E so we can find $y_1, \dots, y_n \in E$ such that $E \subset V_{y_1} \cup \dots \cup V_{y_n}$. The set $W = U_{y_1} \cap \dots \cap U_{y_n}$ belongs to $\mathcal{W}$ and $W \cap F = \emptyset$. Therefore, $W' = W \cap X \in \mathcal{W}'$ and $x \in W' \subset O'$ which proves that $\mathcal{W}'$ is a base in X and finishes our solution.

S.271. *Let X be a Lindelöf Čech-complete space. Prove that X^ω is a Lindelöf space.*

Solution. Assume that we are given a map $f_t : X_t \to Y_t$ for each $t \in T$. Given a point $x \in X = \prod \{X_t : t \in T\}$, let $f(x)(t) = f_t(x(t)) \in Y_t$ for all $t \in T$. Then $f(x) \in Y = \prod\{Y_t : t \in T\}$ and hence we have a map $f : X \to Y$ which is called *the product of the maps* $\{f_t : t \in T\}$. This product is usually denoted by $\prod_{t\in T} f_t$.

Fact 1. Any product of continuous maps is a continuous map.

Proof. Assume that we are given a continuous map $f_t : X_t \to Y_t$ for each $t \in T$. Let $X = \prod\{X_t : t \in T\}$ and $Y = \prod\{Y_t : t \in T\}$. The maps $p_t : X \to X_t$ and $q_t : Y \to Y_t$ are the respective natural projections. To see that $f = \prod_{t\in T} f_t : X \to Y$ is continuous, observe that $q_t \circ f = f_t \circ p_t$ is a continuous map for any $t \in T$ so f is continuous by Problem 102. Fact 1 is proved.

Fact 2. A continuous onto map $g : Y \to Z$ is closed if and only if for any $z \in Z$ and any $O \in \tau(g^{-1}(z), Y)$, there exists $O' \in \tau(z, Z)$ such that $g^{-1}(O') \subset O$.

Proof. Necessity was proved in Fact 1 of S.226. To prove sufficiency, take any closed set $F \subset Y$. It suffices to show that any point $z \in Z \backslash g(F)$ has a neighbourhood which is contained in $Z \backslash g(F)$. Since $z \notin g(F)$, we have $g^{-1}(z) \cap F = \emptyset$, i.e., $O = Y \backslash F \in \tau(g^{-1}(z), Y)$. Our hypothesis says that there is $O' \in \tau(z, Z)$ with $g^{-1}(O') \subset O$ and, in particular, $g^{-1}(O') \cap F = \emptyset$ whence $O' \subset Z \backslash g(F)$ so Fact 2 is proved.

Fact 3. (The Wallace theorem) Assume that K_t is a compact subspace of a space X_t for all $t \in T$; let $X = \prod\{X_t : t \in T\}$ and $K = \prod\{K_t : t \in T\}$. Then, for every set $W \in \tau(K, X)$ we can choose sets $U_t \in \tau(K_t, X_t)$, $t \in T$ such that $U_t \neq X_t$ only for finitely many t and $\prod\{U_t : t \in T\} \subset W$.

Proof. We first consider the product of two spaces, i.e., $T = \{1, 2\}$. Fix a point $x \in K_1$ and for any point $y \in K_2$ choose sets $U(y) \in \tau(x, X_1)$ and $V(y) \in \tau(y, X_2)$ such that $U(y) \times V(y) \subset W$. By compactness of K_2, we can find points $y_1, \dots, y_n$ such that $\bigcup\{U(y_i) \times V(y_i) : i \leqslant n\} \supset \{x\} \times K_2$. Now, if $G(x) = \bigcap\{U(y_i) : i \leqslant n\}$ and $H(x) = \bigcup\{V(y_i) : i \leqslant n\}$ then $G(x) \in \tau(x, X_1), H(x) \in \tau(K_2, X_2)$ and $G(x) \times H(x) \subset W$.

Apply compactness of K_1 to find $x_1, \dots, x_k$ such that $G(x_1) \cup \dots \cup G(x_k) \supset K_1$. Now if $U_1 = \bigcup\{G(x_i) : i \leqslant n\}$ and $U_2 = \bigcap\{H(x_i) : i \leqslant n\}$ then $U_i \in \tau(K_i, X_i)$ for $i = 1, 2$ and $U_1 \times U_2 \subset W$, i.e., our Fact is proved for the products of two factors.

Assume now that our Fact is true for all products of $\leqslant n - 1$ factors, $n \geqslant 3$ and take $T = \{1, \dots, n\}$. Representing $K = K_1 \times \dots \times K_n$ as $K_1 \times (K_2 \times \dots \times K_n)$, we have a product of two compact subsets in the product of two spaces: X_1 and $X_2 \times \dots \times X_n$. Applying our Fact for the mentioned product of two factors, we can obtain sets $U_1 \in \tau(K_1, X_1)$ and $U_2' \in \tau(K_2, \times \dots \times K_n, X_2 \times \dots \times X_n)$ such that $U_1 \times U_2' \subset W$. By the inductive hypothesis there exist $U_2 \in \tau(K_2, X_2), \dots, U_n \in \tau(K_n, X_n)$ such that $U_2 \times \dots \times U_n \subset U_2'$. One readily sees that the sets $U_1, \dots, U_n$ are as promised so our Fact is proved for all finite products.

Finally, we consider the case of an arbitrary T. If $O_t \in \tau(X_t)$ for all $t \in T$, the set $\prod_{t \in T} O_t$ is called *standard* if $O_t \neq X_t$ only for finitely many t. The standard sets are open in X and form a base in X (see Problem 101). If $O = \prod_{t \in T} O_t$ is a standard set then $\text{supp}(O) = \{t \in T : O_t \neq X_t\}$.

For every point $x \in K$ find a standard set U_x such that $x \in U_x \subset W$. The set K being compact its open cover $\{U_x : x \in K\}$ has a finite subcover, i.e., there exist $x_1, \dots, x_k \in K$ with $K \subset U_{x1} \cup \cdot \cup U_{xk} \subset W$. If $T_0 = \bigcup\{\text{supp}(U_{x_i}) : i \leqslant n\}$ then, for every $i \leqslant n$, we have the equality $U_{x_i} = \prod_{t \in T} O_t^i$ where $O_t^i = X_t$ for all $t \in T \setminus T_0$. If $W_1 = \bigcup\{\prod_{t \in T_0} O_t^i : i \leqslant n\}, W_2 = \prod_{t \in T \setminus T_0} X_t, L_1 = \prod_{t \in T_0} K_t$ and $L_2 = \prod_{t \in T \setminus T_0} K_t$ then $L_1 \subset W_1$, $L_2 \subset W_2$ and $K = L_1 \times L_2 \subset W_1 \times W_2 \subset W$. Since our Fact is proved for finite products, for each $t \in T_0$ we can choose $U_t \in \tau(K_t, X_t)$ such that $\prod_{t \in T_0} U_t \subset W_1$. Taking $U_t = X_t$ for all $t \in T \setminus T_0$ we obtain the family $\{U_t : t \in T\}$ with the promised properties. Fact 3 is proved.

Fact 4. Any product of perfect maps is a perfect map.

Proof. Given a perfect map $f_t : X_t \to Y_t$ for each $t \in T$, let $X = \prod\{X_t : t \in T\}$ and $Y = \prod\{Y_t : t \in T\}$. To prove that the mapping $f = \prod_{t \in T} f_t : X \to Y$ is perfect, fix any $y \in Y$ and observe that the space $f^{-1}(y) = \prod\{f_t^{-1}(y(t)) : t \in T\}$ is compact being a product of compact spaces (Problem 125). To show that f is closed, take any $O \in \tau(f^{-1}(y), X)$. There is a finite $T_0 \subset T$ and sets $O_t \in \tau(f_t^{-1}(y(t)), X_t)$, $t \in T$ such that $O_t = X_t$ for all $t \in T \setminus T_0$ and $\prod_{t \in T} O_t \subset O$ (Fact 3). The map f_t being closed, for every $t \in T_0$, there exists $V_t \in \tau(y(t), Y_t)$ such that $f_t^{-1}(V_t) \subset O_t$ (see Fact 2). If $V_t = Y_t$ for all $t \in T \setminus T_0$ and $V = \prod\{V_t : t \in T\}$ then it is clear that $V \in \tau(y, Y)$ and $f^{-1}(V) = \prod\{f_t^{-1}(V_t) : t \in T\} \subset \prod\{O_t : t \in T\} \subset O$; apply Fact 2 again to see that f is closed concluding the proof of Fact 4.

Fact 5. If $f : Y \to Z$ is a perfect map and $l(Z) \leqslant \kappa$ then $l(Y) \leqslant \kappa$. In particular, any perfect preimage of a Lindelöf space is Lindelöf.

Proof. Given an open cover $\mathcal{U}$ of the space Y, let $\mathcal{V}$ be the family of all finite unions of the elements from $\mathcal{U}$. For every $z \in Z$ the set $f^{-1}(z)$ is compact so there is $V_z \in \mathcal{V}$ with $f^{-1}(z) \subset V_z$. By Fact 2 there exists $U_z \in \tau(z, Z)$ such that $f^{-1}(U_z) \subset V_z$. It follows from $l(Z) \leqslant \kappa$ that there is $A \subset Z$ such that $|A| \leqslant \kappa$ and $\cup\{U_z : z \in A\} = Z$ and hence $\bigcup\{V_z : z \in A\} = Y$. Since each V_z can be covered by a finite subfamily of $\mathcal{U}$, there is a family $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| \leqslant k$ and $\bigcup \mathcal{U}' = Y$ so we have $l(Y) \leqslant \kappa$ and hence Fact 5 is proved.

Fact 6. If X_n is a σ-compact space for all $n \in \omega$ then the space $P = \prod\{X_n : n \in \omega\}$ is Lindelöf.

Proof. Every X_n is a closed subspace of the space $Y = \bigoplus\{X_n : n \in \omega\}$ (see the definition and basic properties of discrete unions in Problem 113) and hence P is homeomorphic to a closed subspace of Y^ω. This shows that, without loss of generality, we can consider that all factors of the product P are the same, i.e., $X_n = X$ for all $n \in \omega$. By our assumption, $X = \bigcup\{K_i : i \in \omega\}$ where each K_i is a non-empty compact set. If $Z = \bigoplus\{K_i : i \in \omega\}$ then Z maps continuously onto X: this is an easy exercise for the reader. Therefore, the space Z^ω maps continuously onto X^ω (Fact 1) so it suffices to prove the Lindelöf property of Z^ω. Let D be the set ω with the discrete topology. Letting $f(x) = i$ for any $i \in \omega$ and $x \in L_i = K_i \times \{i\}$, we obtain a map $f : Z \to D$ which is perfect. Indeed, the inverse image of any subset of D is a union of some L_i's which are open in Y. Thus f is continuous and $f^{-1}(i) = L_i$ is a compact set for any $i \in D$. Finally, f is closed because any subset of D is closed.

Now if $Z_i = Z$, $D_i = D$ and $f_i = f$ for all $i \in \omega$ then $h = \prod_{i\in\omega} f_i : Z^\omega \to D^\omega$ is a perfect map by Fact 4. The space D^ω is second countable (S.135, Observation four) and hence Lindelöf (S.140, Observation one) so we can apply Fact 5 to conclude that Z^ω is also Lindelöf. Fact 6 is proved.

Fact 7. Given a space Y and $X_t \subset Y$ for each $t \in T$ the space $X = \bigcap\{X_t : t \in T\}$ embeds in $\prod\{X_t : t \in T\}$ as a closed subspace.

Proof. Let $p_t : \prod\{X_t : t \in T\} \to X_t$ be the natural projection. We define a map $f_t : X \to X_t$ by the formula $f_t(x) = x$ for any $x \in X$ and $t \in T$. Clearly, f_t is a homeomorphic embedding and the formula $f(x)(t) = f_t(x)$ defines a continuous mapping $f : X \to \prod \{X_t : t \in T\}$ because $p_t \circ f = f_t$ for all $t \in T$. In fact, the mapping $f : X \to Z = f(X)$ is a homeomorphism because $f_t^{-1} \circ p_t$ is its continuous inverse for each $t \in T$. To see that Z is closed in $P = \prod\{X_t : t \in T\}$, take any $y \in P \setminus Z$. Then $y(t) \neq y(s)$ for some $t, s \in T$ and we can fix sets $U \in \tau(y(t), Y)$, $V \in \tau(y(s), Y)$ such that $U \cap V = \emptyset$. If $U' = U \cap X_t$ and $V' = V \cap X_s$ then the set $O = \{z \in P : z(t) \in U'$ and $z(s) \in V'\}$ is an open neighbourhood of y in P and $O \cap Z = \emptyset$. Hence Z is closed in P and Fact 7 is proved.

To finish our solution, take a Lindelöf Čech-complete space X and fix a family $\{U_n : n \in \omega\} \subset \tau(\beta X)$ such that $\bigcap\{U_n : n \in \omega\} = X$. For any $n \in \omega$ and any $x \in X$ find $V_x^n \in \tau(x, \beta X)$ such that $\mathrm{cl}_{\beta X}(V_x^n) \subset U_n$. The open cover $\gamma_n = \{V_x^n : x \in X\}$ of the

Lindelöf space X has a countable subcover μ_n. If $P_n = \bigcup\{\text{cl}_{\beta X}(U) : U \in \mu_n\}$ then P_n is a σ-compact space and $X \subset P_n \subset O_n$ for any $n \in \omega$. As a consequence, $X = \bigcap\{P_n : n \in \omega\}$. The space X embeds as a closed subspace in the space $P = \prod\{P_n : n \in \omega\}$ by Fact 7 and hence X^ω embeds as a closed subspace in the space P^ω. Since the space P^ω can be represented as a countable product of σ-compact spaces, it is also Lindelöf by Fact 6. Since every closed subspace of a Lindelöf space is a Lindelöf space, the space X^ω is Lindelöf and hence our solution is complete.

S.272. *Prove that the Sorgenfrey line is not Čech-complete. Recall that the Sorgenfrey line is the space* $(\mathbb{R}, \tau_s)$, *where* τ_s *is the topology generated by the family* $\{[a, b) : a, b \in \mathbb{R}, a < b\}$ *as a base.*

Solution. We know that the Sorgenfrey line S is a Lindelöf space while $S \times S$ is not Lindelöf (Problem 165) so S cannot be Čech-complete by Problem 271.

S.273. *Prove that a second countable space is Čech-complete if and only if it embeds into* $\mathbb{R}^\omega$ *as a closed subspace.*

Solution. The space $\mathbb{R}^\omega$ is Čech-complete by Problems 205, 269 and 262. Since any closed subspace of a Čech-complete space is Čech-complete (see Problem 260), we have sufficiency.

Now, if X a second countable Čech-complete space, it can be embedded in $\mathbb{R}^\omega$ as a G_δ-subspace. Indeed, there is $Y \subset \mathbb{R}^\omega$ homeomorphic to X by Problem 209. Since Y is Čech-complete, it is a G_δ-set in $\overline{Y}$ (Problem 259(iv)). The set $\overline{Y}$ being a G_δ-subspace of $\mathbb{R}^\omega$, the set Y is also G_δ in $\mathbb{R}^\omega$. Now apply Fact 1 of S.269 to conclude that X embeds as a closed subspace in $\mathbb{R}^\omega \times \mathbb{R}^\omega = \mathbb{R}^\omega$ finishing the proof of necessity.

S.274. *Prove that*

(i) *Any Čech-complete space has the Baire property.*
(ii) *Any pseudocompact space has the Baire property.*

Solution. (i) Since any open subset of a Čech-complete space is Čech-complete (Problem 260(ii)), it suffices to show that any Čech-complete space is of second category in itself (we omit the simple proof of the fact that an open set of a space is of second category in that space if and only if it is of second category in itself).

Assuming the contrary we can find a Čech-complete space X and a family $\{F_n : n \in \omega\}$ of nowhere dense subspaces of X such that $X = \bigcup\{F_n : n \in \omega\}$. Fix a family $\mathcal{U} = \{U_n : n \in \omega\} \subset \tau(\beta X)$ such that $X = \bigcap \mathcal{U}$. Since the set $G_0 = \text{cl}_{\beta X}(F_0)$ is nowhere dense in βX, we can take a point $x_0 \in U_0 \backslash G_0$ and a set $W_0 \in \tau(x_0, \beta X)$ such that $\text{cl}_{\beta X}(W_0) \subset U_0$ and $\text{cl}_{\beta X}(W_0) \cap F_0 = \emptyset$. If we have a non-empty $W_n \in \tau(\beta X)$ then we can take a point $x_{n+1} \in (U_n \cap W_n)\backslash\text{cl}_{\beta X}(F_{n+1})$ (because $\text{cl}_{\beta X}(F_{n+1})$ is no where dense in βX) and a set $W_{n+1} \in \tau(x_{n+1}, \beta X)$ such that $\text{cl}_{\beta X}(W_{n+1}) \subset (U_n \cap W_n)\backslash\ \text{cl}_{\beta X}(F_{n+1})$.

Having constructed the sequence $\{W_n : n \in \omega\}$, observe that the compact set $F = \bigcap\{W_n : n \in \omega\} = \bigcap\{\text{cl}_{\beta X}(W_n) : n \in \omega\}$ is non-empty because the sequence

$\{\mathrm{cl}_{\beta X}(W_n) : n \in \omega\}$ is decreasing and consists of compact sets. Observe also that $W_n \subset U_n$ for all $n \in \omega$ whence $F \subset X$. Finally, $F \cap (\bigcup\{F_n : n \in \omega\}) = \emptyset$ because $W_n \cap F_n = \emptyset$ for all $n \in \omega$. As a consequence, $X \neq \bigcup\{F_n : n \in \omega\}$; this contradiction finishes the proof of (i).

(ii) Take any pseudocompact space X and any $U \in \tau^*(X)$. If U is of first category then there is a family $\mathcal{P} = \{P_n : n \in \omega\}$ of nowhere dense subsets of U such that $U = \bigcup\mathcal{P}$. Take any $x_0 \in U \backslash \overline{P}_0$ and find $U_0 \in \tau(x_0, X)$ such that $\overline{U}_0 \subset U\backslash\overline{P}_0$. if we have a non-empty $U_n \in \tau(X)$, take any $x_{n+1} \in U_n\backslash\overline{P}_{n+1}$ and $U_{n+1} \in \tau(x_{n+1}, X)$ such that $\overline{U}_{n+1} \subset U_n\backslash\overline{P}_{n+1}$. This construction gives us a sequence $U_0 \supset U_1 \supset \dots$ of open subsets of X such that $\overline{U}_0 \subset U$ and $\overline{U}_n \cap P_n = \emptyset$ for all $n \in \omega$. Since X is pseudocompact, we have $F = \bigcap\{\overline{U}_i : i \in \omega\} \neq \emptyset$ (Problem 136); if $u \in F$ then $u \in U\backslash\big(\bigcup\{P_n : n \in \omega\}\big)$; this contradiction finishes the proof of (ii) and completes our solution.

S.275. *Let X be a Baire space. Prove that any extension of X as well as any open subspace of X is a Baire space. Show that a closed subspace of a Baire space is not necessarily a Baire space.*

Solution. Assume that X is a dense subset of a space Y. Given a non-empty $U \in \tau(Y)$ suppose that $U = \bigcup\{F_n : n \in \omega\}$ where F_n is nowhere dense in Y. It is easy to see that $U' = U \cap X = \bigcup\{F'_n : n \in \omega\}$ where $F'_n = F_n \cap X$ is nowhere dense in X. Thus $U' \in \tau^*(X)$ is a union of countably many nowhere dense subsets of X which is a contradiction with the Baire property of X.

If U is an open subset of a Baire space X and $V \in \tau(U)$ then V is also open in X so it is of second category in X. It is immediate that V is also of second category in U so we proved that any open subset of a Baire space is a Baire space.

To give the promised example, consider the space $Z = (\mathbb{R} \times (\mathbb{R} \setminus\{0\})) \cup (\mathbb{Q} \times \{0\})$ with the topology induced from $\mathbb{R}^2$. It is easy to see that $F = \mathbb{Q} \times \{0\}$ is a closed subspace of Z homeomorphic to $\mathbb{Q}$. However, $\mathbb{Q}$ is a countable union of its one-point sets each one of which is nowhere dense in $\mathbb{Q}$. Therefore $\mathbb{Q}$ is of first category in itself.

To prove that Z is a Baire space note that the set $W = \mathbb{R} \times (\mathbb{R}\setminus\{0\})$ is dense in Z. Clearly, W is an open set of the Čech-complete space $\mathbb{R}^2$ (see Problems 205 and 269). Therefore W is Čech-complete (Problem 260) and hence Baire (Problem 274). Since Z is an extension of W, it is a Baire space so our solution is complete.

S.276. *Prove that a dense G_δ-subspace of a Baire space is a Baire space. As a consequence, $\mathbb{Q}$ is not a G_δ-subset of $\mathbb{R}$.*

Solution. Take an arbitrary Baire space X and a dense subspace $Y \subset X$ such that $Y = \bigcap\{U_n : n \in \omega\}$ for some family $\{U_n : n \in \omega\} \subset \tau(X)$. It is evident that the set $F_n = X \setminus U_n$ is nowhere dense in X for any $n \in \omega$. If $W \in \tau^*(Y)$ fix $W' \in \tau(X)$ with $W' \cap Y = W$. Suppose that W is of first category in Y and take a family $\{P_n : n \in \omega\}$ of nowhere dense subsets of Y such that $W = \bigcup\{P_n : n \in \omega\}$. Then each P_n is also nowhere dense in X and $W' = \bigcup\{F_n \cap W' : n \in \omega\} \cup \{P_n : n \in \omega\}$ which shows that the set $W' \in \tau^*(X)$ is represented as a countable union of nowhere dense subsets of X which is a contradiction with the Baire property of X.

To finish our solution, observe that $\mathbb{Q}$ is not a Baire space because it is of first category in itself. The space $\mathbb{R}$ is Čech-complete and hence Baire, so $\mathbb{Q}$ cannot be a G_δ-subset of $\mathbb{R}$.

S.277. *Prove that an open image of a Baire space is a Baire space.*

Solution. Suppose that X is a Baire space and take any open continuous map $f : X \to Y$. Observe that if $P \subset Y$ is nowhere dense in Y then $f^{-1}(P)$ is nowhere dense in X. Indeed, if $U \in \tau^*(X)$ and $U \subset \text{cl}_X(f^{-1}(P))$ then $f(U) \in \tau^*(Y)$ and $f(U) \subset \text{cl}_Y(P)$ which is a contradiction.

An evident consequence is that $f^{-1}(N)$ is of first category in X whenever N is of first category in Y. Thus, if $U \in \tau^*(Y)$ is of first category in Y then $f^{-1}(U) \in \tau^*(X)$ is of first category in X which contradicts the Baire property of X. Therefore Y is a Baire space.

S.278. *Prove that $C_p(X)$ is a Baire space if and only it is of second category in itself. Give an example of a non-Baire space Y which is of second category in itself.*

Solution. If $C_p(X)$ is a Baire space then every open subset of $C_p(X)$ and, in particular, the whole $C_p(X)$ is of second category in $C_p(X)$. Now suppose that $C_p(X)$ is of second category in itself and some $U \in \tau^*(C_p(X))$ is of first category. Take any $u_0 \in U$ and consider a maximal disjoint family γ of open subsets of $C_p(X)$ each one of which is homeomorphic to a non-empty open subset of U. We claim that $\bigcup\gamma$ is dense in X. Indeed, if $V = C_p(X)\backslash\overline{\bigcup\gamma} \neq \emptyset$ then take any $v_0 \in V$ and observe that the set $W = U + (v_0 - u_0) = \{u + (v_0 - u_0) : u \in U\}$ is a neighbourhood of v_0 because the map $f \to f + (v_0 - u_0)$ is a homeomorphism of $C_p(X)$ onto itself (Problem 079). As a consequence, $W \cap V$ is a non-empty open subset of V homeomorphic to an open subset of U and hence the family $\gamma \cup \{W \cap V\}$ is still disjoint and consists of sets which are homeomorphic to open subsets of U; this contradiction with maximality of γ proves that $\overline{\bigcup\gamma} = C_p(X)$ and hence the set $F = C_p(X)\backslash(\bigcup\gamma)$ is nowhere dense in $C_p(X)$.

Every element $W \in \gamma$ is of first category being homeomorphic to an open subset of U; fix a family $\mathcal{F}_W = \{F_n^W : n \in \mathbb{N}\}$ of nowhere dense subsets of W such that $\bigcup\mathcal{F}_W = W$. Letting $F_0 = F$ and $F_n = \bigcup\{F_W^n : W \in \gamma\}$ for all $n \in \mathbb{N}$, we obtain a family $\mathcal{F} = \{F_n : n \in \omega\}$ of nowhere dense subsets of $C_p(X)$ with $\bigcup\mathcal{F} = C_p(X)$, a contradiction with the fact that $C_p(X)$ is of second category. Thus, second category and Baire property are equivalent in spaces $C_p(X)$.

To give a promised example, consider the space $Y = \mathbb{R} \oplus \mathbb{Q}$ (see the definition and basic properties of discrete unions in Problem 113). The space Y is of second category because $\mathbb{R}$ is an open subset of Y which is of second category; the space Y is not Baire because $\mathbb{Q}$ is an open subset of Y which is of first category.

S.279. *Suppose that X is an infinite set and ξ is a free ultrafilter on X (i.e., ξ is an ultrafilter on X and $\bigcap\xi = \emptyset$). Denote by X_ξ the set $X \cup \{\xi\}$ with the topology*

$\tau_\xi = \{A : A \subset X\} \cup \{B : \xi \in B$ and $X \setminus B \notin \xi\}$. Show that τ_ξ is indeed a topology on X_ξ such that ξ is the unique non-isolated point of X_ξ. Prove that $C_p(X_\xi)$ is a Baire space.

Solution. Since $\emptyset = X \setminus X_\xi \notin \xi$ and $\xi \in X_\xi$, we have $X_\xi \in \tau_\xi$. The empty set belongs to τ_ξ because it is a subset of X. If $U, V \in \tau_\xi$ and $\xi \notin U \cap V$ then $U \cap V \subset X$ and hence $U \cap V \in \tau_\xi$. Now, if $\xi \in U \cap V$ then $X \setminus U \notin \xi$ and $X \setminus V \notin \xi$ and therefore $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \notin \xi$ because ξ is an ultrafilter. Thus $U \cap V \in \tau_\xi$. If $\gamma \subset \tau_\xi$ and $\bigcup\gamma \subset X$ then $\bigcup\gamma \in \tau_\xi$. If $\xi \in U$ for some $U \in \gamma$ then $U \cap X \in \xi$ and therefore $(\bigcup\gamma) \cap X \in \xi$ whence $X \setminus (\bigcup\gamma) \notin \xi$, i.e., $\bigcup\gamma \in \tau_\xi$ and we proved that τ_ξ is a topology on X_ξ.

To prove that $C_p(X_\xi)$ is a Baire space we need the following notion. Say that a map σ is *a strategy on X_ξ* if it has the following properties:

(1) The domain of σ is the family $\mathcal{D} = \{(S_0, S_1, \dots, S_{2n}) : n \in \omega$ and $\{S_0, \dots, S_{2n}\}$ is a disjoint family of finite subsets of $X_\xi\}$.
(2) Given any $(S_0, \dots, S_{2n}) \in \mathcal{D}$, its image $\sigma(S_0, \dots, S_{2n})$ is a finite subset of X_ξ disjoint from $S_0 \cup \dots \cup S_{2n}$.

Call a sequence $\{S_i : i \in \omega\}$ of disjoint finite subsets of X *a σ-play (or a play in which the strategy σ is applied)* if $S_{2n+1} = \sigma(S_0, \dots, S_{2n})$ for all $n \in \omega$. A sequence $(S_0, S_1, \dots, S_{2n}) \in \mathcal{D}$ is called *a partial σ-play* if $S_{2i+1} = \sigma(S_0, \dots, S_{2i})$ for each $i = 0, \dots, n-1$. In particular, every finite set $S_0 \subset X_\xi$ is a partial σ-play.

The strategy σ is called *winning* if, for any σ-play $\{S_i : i \in \omega\}$, the point ξ is an accumulation point of the set $\cup\{S_{2i+1} : i \in \omega\}$.

Fact 1. If $C_p(X_\xi)$ is not a Baire space then there exists a winning strategy σ on the space X_ξ.

Proof. Suppose that $C_p(X)$ is not Baire. Then it is of first category (Problem 278) and hence we can find a sequence $S = \{O_n : n \in \omega\}$ of dense open subsets in $C_p(X)$ such that $\bigcap S = \emptyset$. Observe first that to get a winning strategy σ, we only need to check its values on partial σ-plays so we will only define it on partial σ-plays considering that $\sigma(S_0, \dots, S_{2n}) = \emptyset$ for the rest of the sequences $(S_0, \dots, S_{2n}) \in \mathcal{D}$.

Given a finite set $A \subset X_\xi$, a function $f : A \to \mathbb{R}$ and $\varepsilon > 0$ we will need the set $M(f, A, \varepsilon) = \{g \in C_p(X_\xi) : |g(x) - f(x)| < \varepsilon$ for all $x \in A\}$. It is evident that the sets $M(f, A, \varepsilon)$ form a local base at f in the space $C_p(X_\xi)$. Given any finite $S_0 \subset X_\xi$, let $\varepsilon_0 = 1$ and $f_0(x) = 0$ for all $x \in S_0$. The open set $U_0 = M(f_0, S_0, \varepsilon_0)$ has to intersect the dense open set O_0 and therefore there exist a finite $T_1 \supset S_0$, $\varepsilon_1 > 0$ and a function $f_1 : T_1 \to \mathbb{R}$ such that

(1) $\varepsilon_1 < 2^{-1}$ and $\xi \in T_1$.
(2) $M(f_1, T_1, 2\varepsilon_1) \subset U_0 \cap O_0$.

We let $\sigma(S_0) = T_1 \setminus S_0$. Suppose that, for some $n > 0$, we have a partial σ-play $S_0, \dots, S_{2n}$ for which we have chosen functions $f_0, \dots, f_{2n-1}$ and positive numbers $\varepsilon_0, \dots, \varepsilon_{2n-1}$ with the following properties:

(3) $f_k : T_k = S_0 \cup \ldots \cup S_k \to \mathbb{R}$ for each $k \leqslant 2n - 1$.
(4) $\varepsilon_k < 2^{-k}$ for each $k \leqslant 2n - 1$.
(5) $M(f_{2k+1}, T_{2k+1}, 2\varepsilon_{2k+1}) \subset M(f_{2k}, T_{2k}, \varepsilon_{2k}) \cap O_k$ for each $k = 0, \ldots, n - 1$.
(6) $f_{2k}|T_{2k-1} = f_{2k-1}$ and $f_{2k}(x) = f_{2k-1}(\xi)$ for all $x \in S_{2k}$, $k = 1, \ldots, n - 1$.

We are going to define a function $f_{2n} : T_{2n} = S_0 \cup \cdots \cup S_{2n} \to \mathbb{R}$ by the formulas $f_{2n}(x) = f_{2n-1}(x)$ for all $x \in T_{2n-1}$ and $f_{2n}(x) = f_{2n-1}(\xi)$ if $x \in S_{2n}$. If we let $\varepsilon_{2n} = (1/2)\varepsilon_{2n-1}$ then the property (6) holds also for $k = n$. The open set $M(f_{2n}, T_{2n}, \varepsilon_{2n})$ must intersect the open dense set O_n so we can find a finite $T_{2n+1} \supset T_{2n}$, a function $f_{2n+1} : T_{2n+1} \to \mathbb{R}$ and $\varepsilon_{2n+1} \in (0, 2^{-2n-1})$ such that we have the inclusion $M(f_{2n+1}, T_{2n+1}, 2\varepsilon_{2n+1}) \subset M(f_{2n}, T_{2n}, \varepsilon_{2n}) \cap O_n$. To finish our inductive construction let $\sigma(S_0, \ldots, S_{2n}) = T_{2n+1} \setminus T_{2n}$.

Suppose that we have a σ-play $\{S_i : i \in \omega\}$, the respective functions $\{f_i : i \in \omega\}$ and the sequence $\{\varepsilon_i : i \in \omega\}$. If $x \in S = \bigcup\{S_i : i \in \omega\}$ then $x \in S_n$ for some $n \in \omega$ and therefore $x \in T_k$ for all $k \geqslant n$. Observe that (5) and (6) imply that, for any $k \geqslant n$, we have $|f_{k+1}(x) - f_k(x)| < \varepsilon_k < 2^{-k}$ so it is a standard exercise to show that $\{f_k(x) : k \in \omega\}$ is a Cauchy sequence and hence it converges to some number $f(x) \in \mathbb{R}$. The properties (4), (5) and (6) imply that the sequence $\mathcal{V} = \{[f_k(x) - \varepsilon_k, f_k(x) + \varepsilon_k] : k \geqslant n\}$ is decreasing and the diameters of its elements tend to zero. This, together with the convergence $f_k(x) \to f(x)$ implies that $\{f(x)\} = \bigcap \mathcal{V}$

Assume that the function $f : S \to \mathbb{R}$ is continuous. The property (1) implies that $\xi \in S$ so the set S is closed in X_ξ. Any space with a unique non-isolated point is normal (Claim 2 of S.018) so X_ξ is a normal space. Apply Problem 032 to find a function $g \in C_p(X)$ such that $g|S = f$. Given any number $k \in \omega$ we can apply the observations of the previous paragraph to conclude that $g(x) = f(x)$ and $f(x) \in [f_{2k+1}(x) - \varepsilon_{2k+1}, f_{2k+1}(x) + \varepsilon_{2k+1}] \subset (f_{2k+1}(x) - 2\varepsilon_{2k+1}, f_{2k+1}(x) + 2\varepsilon_{2k+1})$ for every $x \in T_{2k+1}$ so $g \in M(f_{2k+1}, T_{2k+1}, 2\varepsilon_{2k+1}) \subset O_k$ and therefore $g \in O_k$. The number $k \in \omega$ was taken arbitrarily so this proves, that $g \in \bigcap\{O_k : k \in \omega\} = \emptyset$, a contradiction which shows that the function f cannot be continuous.

The space S has only one non-isolated point ξ so f is not continuous at ξ. Observe that, for any point $x \in S_{2k}$, we have $|f(x) - f_{2k}(x)| \leqslant \varepsilon_{2k} < 2^{-2k}$ and $|f_{2k-1}(\xi) - f(\xi)| \leqslant \varepsilon_{2k-1} < 2^{-2k+1}$ and hence we can apply the second equality of (6) to conclude that $|f(x) - f(\xi)| \leqslant |f(x) - f_{2k}(x)| + |f_{2k-1}(\xi) - f(\xi)| < 2^{-2k+2}$ for any $x \in S_{2k}$. An easy consequence is that the function f is continuous on the set $\bigcup\{S_{2k} : k \in \omega\} \cup \{\xi\}$. Thus f is discontinuous on the set $S' = \bigcup\{S_{2k+1} : k \in \omega\}$ and hence ξ has to be an accumulation point of the set S' which finally shows that σ is a winning strategy on X_ξ so Fact 1 is proved.

Fact 2. The space X_ξ has the following "moving off" property: for any sequence $\{\mathcal{F}_n : n \in \omega\}$ of infinite disjoint families of finite subsets of X_ξ, we can choose an element $F_n \in \mathcal{F}_n$ for every $n \in \omega$ in such a way that $\bigcup\{F_n : n \in \omega\}$ has no accumulation points in X_ξ.

Proof. Choose distinct sets $A_0, B_0 \in \mathcal{F}_0$ arbitrarily. Suppose that we have chosen sets $A_i, B_i \in \mathcal{F}_i$ for all $i \leqslant n$ so that the family $\{A_i, B_i : i \leqslant n\}$ is disjoint. Since the set $S_n = \left(\bigcup_{i \leqslant n} A_i\right) \cup (\bigcup_{i \leqslant n} B_i)$ is finite, there exist distinct $A_{n+1}, B_{n+1} \in \mathcal{F}_{n+1}$ such that

$A_{n+1} \cap S_n = B_{n+1} \cap S_n = \emptyset$ so the inductive choice goes on. Once we have the sequences $\{A_i : i \in \omega\}$ and $\{B_i : i \in \omega\}$, observe that their unions A and B are disjoint so one of the sets A, B has no accumulation points in X_ξ. If, for example, A has no accumulation points in X_ξ, we let $F_n = A_n$ for all $n \in \omega$ finishing the proof of Fact 2.

If $C_p(X_\xi)$ is not a Baire space then there exists a winning strategy σ on the space X_ξ by Fact 1. Let $T(0) = \emptyset$ and, if the sets $T(0), \ldots, T(n-1)$ are defined, let $T(n) = \sigma(T(0) \cup \cdot \cup T(n-1))$. Suppose that $k \geqslant 1$ and the set $T(i_1, \ldots, i_k)$ is defined for every k-tuple $(i_1, \ldots, i_k)$ of elements of ω.

Let $T(i_1, \ldots, i_k, 0) = \emptyset$ and, if the set $T(i_1, \ldots, i_k, j)$ is defined for all numbers $j = 0, \ldots, n-1$, consider the set $T(i_1, \ldots, i_k, n) = \sigma(S_0, S_1, S_2, \ldots, S_{2k-1}, T(i_1, \ldots, i_k, 0) \cup \cdots \cup T(i_1, \ldots, i_k, n-1))$, where the sets S_0, S_1, S_2, $\ldots$, S_{2k-1} are defined as follows:

$$S_0 = T(0) \cup \cdots \cup T(i_1 - 1), S_1 = T(i_1), S_2 = T(i_1, 0) \cup \cdots \cup T(i_1, i_2 - 1),$$
$$S_3 = T(i_1, i_2), \ldots, S_{2j-1} = T(i_1, \ldots, i_j)$$

and $S_{2j} = T(i_1, \ldots, i_j, 0) \cup \cdots \cup T(i_1, \ldots, i_j, i_{j+1} - 1)$ for all $j \leqslant k$.

Once we have the sets $T(i_1, \ldots, i_k)$ for all k-tuples $(i_1, \ldots, i_k)$, we can define the families $\mathcal{F} = \{T(j) : j \in \mathbb{N}\}$ and $\mathcal{F}(i_1, \ldots, i_k) = \{T(i_1, \ldots, i_k, j) : j \in \mathbb{N}\}$ for every k-tuple $(i_1, \ldots, i_k)$ of elements of $\mathbb{N}$. Observe that the collections $\mathcal{F}$, $\mathcal{F}(i_1, \ldots, i_k)$ are infinite, disjoint and consist of finite sets. Fact 2 makes it possible to choose one element from each of these collections in such a way that the union of the chosen sets is closed and discrete.

To be more specific, let the chosen sets be $T(m) \in \mathcal{F}$ and, for each k-tuple $(i_1, \ldots, i_k)$, let $T(i_1, \ldots, i_k, m(i_1, \ldots, i_k))$ be the chosen member of $\mathcal{F}(i_1, \ldots, i_k)$. We will need a sequence $\{j_k : k \in \mathbb{N}\}$ where

$$j_1 = m, j_2 = m(j_1), \ldots, j_{n+1} = m(j_1, \ldots, j_n), \cdots$$

Consider the sets

$$S_0 = T(0) \cup \cdots \cup T(j_1 - 1), S_1 = T(j_1), S_2 = T(j_1, 0) \cup \cdots \cup T(j_1, j_2 - 1),$$
$$S_3 = T(j_1, j_2), \ldots, S_{2n-1} = T(j_1, \ldots, j_n),$$

and $S_{2n} = T(j_1, \ldots, j_n, 0) \cup \cdots \cup T(j_1, \ldots, j_n, j_{n+1} - 1)$ for all $n \in \mathbb{N}$.

It is clear that the sequence $\{S_i : i \in \omega\}$ is a σ-play so ξ must be an accumulation point for the set $\bigcup\{S_{2i+1} : i \in \omega\} = \bigcup\{T(j_1, \ldots, j_k) : k \in \mathbb{N}\}$. However, the set $\bigcup\{T(j_1, \ldots, j_k) : k \in \mathbb{N}\}$ has to be closed and discrete because it was chosen to witness the "moving off" property of the families $\mathcal{F}$, $\mathcal{F}(i_1, \ldots, i_k)$. This contradiction shows that $C_p(X_\xi)$ is a Baire space and finishes our solution.

S.280. *Show that $C_p(X)$ is a Baire space if and only if $\pi_A(C_p(X))$ is a Baire space for any countable $A \subset X$. Here $\pi_A : C_p(X) \to C_p(A)$ is the restriction map defined by $\pi_A(f) = f \mid A$ for every $f \in C_p(X)$.*

Solution. To prove necessity, fix any countable $A \subset X$. Given a finite $P \subset X$, a function $f \in C_p(X)$ and $\varepsilon > 0$, we let $O(f, P, \varepsilon) = \{g \in C_p(X) : |g(x) - f(x)| < \varepsilon$ for all $x \in P\}$. It is evident that the sets $O(f, P, \varepsilon)$ form a local base at f in $C_p(X)$. Denote by $C(A \mid X)$ the subspace $\pi_A(C_p(X))$ of the space $C_p(A)$. Given $h \in C(A \mid X)$, a finite $B \subset A$ and $\varepsilon > 0$, let $M(h, B, \varepsilon) = \{g \in C(A \mid X) : |g(x) - h(x)| < \varepsilon$ for all $x \in B\}$. All possible sets $M(h, B, \varepsilon)$, evidently, constitute a local base at h in the space $C(A|X)$.

Claim. Given a finite set $P \subset X$, a function $f \in C_p(X)$ and $\varepsilon > 0$, the set $\pi_A(O(f, P, \varepsilon))$ is dense in the set $M(\pi_A(f), P \cap A, \varepsilon)$. Besides, if $P \subset A$ then $\pi_A(O(f, P, \varepsilon)) = M(\pi_A(f), P, \varepsilon)$ and, in particular, the set $\pi_A(O(f, P, \varepsilon))$ is open in $C(A \mid X)$.

Proof of the claim. It is immediate that $\pi_A(O(f, P, \varepsilon)) \subset M(\pi_A(f), P \cap A, \varepsilon)$; for $g = \pi_A(f)$ take any $h \in M(g, P \cap A, \varepsilon)$. It suffices to prove that any basic neighbourhood of h in $C(A \mid X)$ intersects the set $\pi_A(O(f, P, \varepsilon))$. So take any finite $B \subset A$ and $\delta > 0$. Applying Problem 034 we can find $f_1 \in C_p(X)$ such that $f_1(x) = h(x)$ for all $x \in B$ and $f_1(x) = f(x)$ for all $x \in P \setminus B$. It is straightforward to check that $f_1 \in O(f, P, \varepsilon)$ and $\pi_A(f_1) \in M(h, B, \delta) \cap \pi_A(O(f, P, \varepsilon))$ whence $h \in \overline{\pi_A(O(f, P, \varepsilon))}$ (the closure is taken in $C(A \mid X)$) and the first part of our claim is proved.

Now, if P is contained in A then $\pi_A(O(f, P, \varepsilon)) \subset M(\pi_A(f), P, \varepsilon)$ because, for any $x \in P$, we have $|\pi_A(g)(x) - \pi_A(f)(x)| = |f(x) - g(x)| < \varepsilon$ for any $g \in O(f, P, \varepsilon)$. If, on the other hand, $h \in M(\pi_A(f), P, \varepsilon)$ then there is $h' \in C_p(X)$ with $\pi_A(h') = h$; for this h' we have $|h'(x) - f(x)| < \varepsilon$ for all $x \in P$ so $h' \in O(f, P, \varepsilon)$ and hence $h = \pi_A(h') \in \pi_A(O(f, P, \varepsilon))$ so $\pi_A(O(f, P, \varepsilon)) \supset M(\pi_A(f), P, \varepsilon)$ and the proof of our claim is complete.

Note that the Baire property of a space Z is equivalent to the fact that the intersection of any countable family of dense open subsets of Z is dense in Z. To show that $C(A|X)$ is a Baire space, take any family $\{U_n : n \in \omega\}$ of open dense subsets of $C(A|X)$. The set $V_n = \pi_A^{-1}(U_n)$ is an open dense subset of $C_p(X)$ for all $n \in \omega$. The openness of V_n is clear so let prove that V_n is dense in $C_p(X)$. Take any function $f \in C_p(X)$, any finite $P \subset X$ and $\varepsilon > 0$. If $O(f, P, \varepsilon) \cap V_n = \emptyset$ then $\pi_A(O(f, P, \varepsilon)) \cap U_n = \emptyset$. Since U_n is dense in the space $C(A|X)$, the set $W = U_n \cap M(\pi_A(f), P \cap A, \varepsilon)$ is non-empty; the set $\pi_A(O(f, P, \varepsilon))$ being dense in $M(\pi_A(f), P \cap A, \varepsilon)$ by our Claim, we have $\pi_A(O(f, P, \varepsilon)) \cap W \neq \emptyset$ and hence $\pi_A(O(f, P, \varepsilon)) \cap U_n \neq \emptyset$ which is a contradiction showing that V_n is dense in $C_p(X)$. The space $C_p(X)$ being Baire, the set $D = \bigcap\{V_n : n \in \omega\}$ is dense in $C_p(X)$. As a consequence the set $\pi_A(D)$ is dense in $C(A|X)$ and contained in $\bigcap\{U_n : n \in \omega\}$. Thus $\bigcap\{U_n : n \in \omega\}$ is dense in $C(A|X)$ so necessity is proved.

To establish sufficiency, suppose that $C(A|X)$ is Baire for every countable $A \subset X$. If $C_p(X)$ is not Baire then it is of first category in itself (278) so we can fix a sequence $\{W_n : n \in \omega\}$ of open dense subsets of $C_p(X)$ such that $\bigcap\{W_n : n \in \omega\} = \emptyset$. Let $\mathcal{B} = \{O(f, P, \varepsilon) : f \in C_p(X), P$ is a finite subset of X and $\varepsilon > 0\}$. It is clear that $\mathcal{B}$ is a base in $C_p(X)$; if $U = O(f, P, \varepsilon) \in \mathcal{B}$ then $\text{supp}(U) = P$.

For any number $n \in \omega$ consider a maximal disjoint family $\gamma_n \subset \mathcal{B}$ such that $V_n = \bigcup \gamma_n \subset W_n$. The set $A = \bigcup\{\text{supp}(U) : U \in \gamma_n, n \in \omega\}$ is countable and hence $C(A|X)$ is a Baire space. The set V_n is dense in $C_p(X)$ and hence $G_n = \pi_A(V_n)$ is dense in $C(A|X)$. For any $U \in \gamma_n$ we have $\text{supp}(U) \subset A$ so the second part of our claim is applicable to conclude that $\pi_A(U)$ is open in $C(A|X)$ for any $U \in \gamma_n$. As a consequence, the set $G_n = \bigcup\{\pi_A(U) : U \in \gamma_n\}$ is open in $C(A|X)$ which implies, together with the Baire property of $C(A|X)$, that $\bigcap\{G_n : n \in \omega\} \neq \emptyset$. Pick any $h \in C_p(X)$ such that $\pi_A(h) \in \bigcap\{G_n : n \in \omega\}$; given $n \in \omega$ there is $U = O(f, P, \varepsilon) \in \gamma_n$ such that $\pi_A(h) \in \pi_A(U)$. Since $P = \text{supp}(U) \subset A$, we have $\pi_A(U) = M(\pi_A(f), P, \varepsilon)$ by the second part of the claim. This implies $|\pi_A(h)(x) - \pi_A(f)(x)| = |h(x) - f(x)| < \varepsilon$ for every point $x \in P$ and therefore $h \in O(f, P, \varepsilon) \subset V_n \subset W_n$. The number $n \in \omega$ was chosen arbitrarily so we proved that $f \in \bigcap\{U_n : n \in \omega\} = \emptyset$ which is a contradiction. Hence $C_p(X)$ is a Baire space and our solution is finished.

S.281. *Prove that a countable product of second countable Baire spaces is a Baire space.*

Solution. Call a subset A of a space Z *residual* if $Z \backslash A$ is of first category.

Fact 1. Let Z be an arbitrary space. Suppose that T is a second countable space and $\pi : Z \times T \to Z$ is the natural projection. If we have a countable family $\mathcal{U}$ of open dense subsets of $Z \times T$ then the set $A = \{z \in Z : \pi^{-1}(z) \cap U$ is dense in $\pi^{-1}(z)$ for any $U \in \mathcal{U}\}$ is residual in Z.

Proof. Fix a countable base $\mathcal{B} = \{B_n : n \in \omega\} \subset \tau^*(T)$ of the space T; given any $U \in \mathcal{U}$ and $n \in \omega$, let $R(n, U) = \{z \in Z : (\{z\} \times B_n) \cap U = \emptyset\}$. Since the set $(Z \times T)\backslash U$ is closed in $Z \times T$, it is easy to see that $R(n, U)$ is closed in Z. If $W = \text{Int}(R(n, U)) \neq \emptyset$ then $W \times B_n$ is a non-empty open set contained in $(Z \times T)\backslash U$ which contradicts the density of U. Therefore, the set $R(n, U)$ is nowhere dense in Z. Observe finally that $Z \backslash A = \bigcup\{R(n, U) : U \in \mathcal{U}, n \in \omega\}$ and hence the set A is residual. Fact 1 is proved.

If we have second countable Baire spaces $\{Y_n : n \in \omega\}$ such that the space $Y = \prod\{Y_n : n \in \omega\}$ is not Baire then some set $O \in \tau^*(Y)$ must be of first category in the space Y. There exist $n \in \omega$ and $U_i \in \tau^*(Y_i)$, $i = 0, \ldots, n$ such that $U_0 \times \cdots \times U_n \times \prod\{Y_i : i > n\} \subset O$ and hence the family $\{U_0, \ldots, U_n, Y_{n+1}, \ldots\}$ consists of Baire spaces (Problem 275) whose product is of first category. This shows that if we prove that every countable product of second countable Baire spaces if of second category then every such product is a Baire space.

We first prove that the product of two second countable Baire spaces is a Baire space. By the observation of the previous paragraph it suffices to rule out the possibility of existence of two second countable Baire spaces whose product is of first category.

Take any second countable Baire spaces X and Y such that $X \times Y$ is of first category; fix a countable family $\mathcal{U}$ of open dense subsets of $X \times Y$ with $\bigcap \mathcal{U} = \emptyset$. Given $x \in X$, let $Y_x = \{(x, y) : y \in Y\}$. It is clear that Y_x is homeomorphic to Y for each $x \in X$; furthermore, $Y_x = \pi^{-1}(x)$ where $\pi : X \times Y \to X$ is the natural projection. Note that, in any Baire space, any residual set is non-empty; this makes it possible to apply Fact 1 to conclude that there is a point $x \in X$ such that $U \cap Y_x$ is dense in Y_x for all $U \in \mathcal{U}$. The family $\mathcal{U}_x = \{U \cap Y_x : U \in \mathcal{U}\}$ consists of open dense subsets of a Baire space Y_x and hence $\bigcap \mathcal{U}_x \neq \emptyset$. Thus $\bigcap \mathcal{U} \supset \bigcap \mathcal{U}_x \neq \emptyset$; this contradiction proves that $X \times Y$ is of second category and hence every product of two second countable Baire spaces is a Baire space.

A trivial induction shows that any finite product of second countable Baire spaces is a Baire space so let us consider the general case. Again, we must only prove that a countable product of second countable Baire spaces cannot be of first category. Suppose that $\{X_n : n \in \omega\}$ is a sequence of second countable Baire spaces such that the space $X = \prod\{X_n : n \in \omega\}$ is of first category. Let $Y_n = \prod\{X_i : i > n\}$ for all $n \in \omega$. Denote by $\pi_n : X \to X_0 \times \cdots \times X_n$ the projection defined by $\pi_n(x) = x|\{0, \ldots, n\}$ for all $n \in \omega$. If $m, n \in \omega$ and $n \leqslant m$, let $p^m_n : \prod_{i \leqslant m} X_i \to \prod_{i \leqslant n} X_i$ be the analogous projection, i.e., $\pi^m_n(x) = x|\{0, \ldots, n\}$ for every $x \in X_0 \times \cdots \times X_m$. Fix a sequence $\{O_n : n \in \omega\}$ of open dense sets in X such that $\bigcap\{O_n : n \in \omega\} = \emptyset$.

We are going to construct a sequence $k_0 < k_1 < \cdots < k_n < \ldots$ of elements of ω and points $w_n \in X_0 \times \cdots \times X_{kn}$, $n \in \omega$ with the following properties:

(1) If $n < m$ then w_m extends w_n, i.e., $w_m |\{0, \ldots, k_n\} = w_n$.
(2) For each $n \in \omega$, if $x \in X$ and $\pi_{k_n}(x) = w_n$ then $x \in U_n$.
(3) $(\{w_n\} \times Y_{k_n}) \cap O_i$ is dense in $\{w_n\} \times Y_{k_n}$ for each $i > n$.

Since O_0 is a non-empty open subset of the space X, there exists $k_0 \in \omega$ and sets $W^0_i \in \tau^*(X_i), i \leqslant k_0$ such that $U_0 = \prod_{i \leqslant k_0} W^0_i \times \prod_{i > k_0} X_i \subset O_0$. The set $W^0 = \prod_{i \leqslant k_0} W^0_i$ is a Baire space being a finite product of second countable Baire spaces. We can apply Fact 1 to the product $U_0 = W^0 \times Y_{k_0}$ and to the dense open sets $\{O_i \cap U_0 : i > 0\}$ of the product $W^0 \times Y_{k_0}$ to conclude that there is a point $w_0 \in W^0$ such that $(\{w_0\} \times Y_{k_0}) \cap O_i$ is dense in $\{w_0\} \times Y_{k_0}$ for each $i > 0$. It is clear that properties (2) and (3) hold for k_0 and w_0. Suppose that we have natural numbers $k_0 < \cdots < k_n$ and point $w_0, \ldots, w_n$ with the properties (1)–(3).

Apply property (3) to see that the intersection $O_{n+1} \cap (\{w_n\} \times Y_{k_n})$ is an open non-empty subset of the product $\{w_n\} \times Y_{k_n}$; therefore we can find a natural number $k_{n+1} > k_n$ and sets $W^{n+1}_i \in \tau^*(X_i), i = k_n + 1, \ldots, k_{n+1}$ such that $\{w_n\} \times \prod_{k_n < i \leqslant k_{n+1}} W^{n+1}_i \times Y_{k_{n+1}} \subset O_{n+1}$. If $W^{n+1} = \{w_n\} \times \prod_{k_n < i \leqslant k_{n+1}} W^{n+1}_i$ then $O_i \cap (W^{n+1} \times Y_{k_{n+1}})$ is a dense subset of $W^{n+1} \times Y_{k_{n+1}}$ for all $i > n + 1$ by (3). Since W^{n+1} is a Baire space, we can apply Fact 1 to conclude that there exists $w_{n+1} \in W^{n+1}$ such that $(\{w_{n+1}\} \times Y_{k_{n+1}}) \cap O_i$ is dense in $\{w_{n+1}\} \times Y_{k_{n+1}}$ for each $i > n + 1$. Therefore (3) is true for $w_0, \ldots, w_{n+1}$ and $k_0, \ldots, k_{n+1}$. Since $p^{k_{n+1}}_{k_n}(w_{n+1}) = w_n$, the property (1) also holds for $w_0, \ldots w_{n+1}$. The property (2) is true for w_{n+1} because $\{w_{n+1}\} \times Y_{k_{n+1}} \subset O_{n+1}$. As a consequence, our inductive construction can be carried out for all $n \in \omega$ and therefore there exist sequences $\{w_n : n \in \omega\}$ and $\{k_n : n \in \omega\}$ with the properties (1)–(3). The property (1) shows that there exists $y \in X$ such that $\pi_n(y) = w_n$ for each $n \in \omega$. The property (2) implies $y \in O_n$ for all numbers $n \in \omega$ and hence $y \in \bigcap\{O_n : n \in \omega\} = \emptyset$ which is a contradiction. Therefore X is a Baire space and our solution is complete.

S.282. *Prove that, for every Baire space X, we have $p(X) = c(X)$.*

Solution. It is evident that $c(X) \leqslant p(X)$ for any space X. Now assume that X is a Baire space, $c(X) = \kappa$ and there exists a point-finite family $\gamma \subset \tau^*(X)$ with $|\gamma| = \kappa^+$. Given $x \in X$ and $n \in \omega$, say that $\text{ord}(x) \leqslant n$ if the point x belongs to $\leqslant n$ elements of γ. Observe that the set $F_n = \{x \in X : \text{ord}(x) \leqslant n\}$ is closed in the space X. Indeed, if $x \in X \backslash F_n$ then there are $n + 1$ distinct sets $U_1, \ldots, U_{n+1} \in \gamma$ with $x \in U_i$ for all $i \leqslant n + 1$. Then $U = \bigcap\{U_i : i \leqslant n + 1\}$ is an open neighbourhood of x which does not meet F_n.

The family γ being point-finite, we have $X = \bigcup\{F_n : n \in \omega\}$. If $U \in \tau^*(X)$ then $F_n \cap U$ cannot have empty interior for all $n \in \omega$ because then the set U would be of first category which is impossible because of the Baire property of X. If n is the minimal number for which $U \cap \text{Int}(F_n) \neq \emptyset$ then $U \cap \text{Int}(F_n \backslash F_{n-1}) \neq \emptyset$.

This shows that the set $W = \bigcup\{\text{Int}(F_n \backslash F_{n-1}) : n \in \mathbb{N}\}$ is dense in X. As a consequence, $W \cap U \neq \emptyset$ for every $U \in \gamma$ so there exists $n \in \omega$ such that the family $\mu = \{U \in \gamma\text{: } U \cap \text{Int}(F_n \backslash F_{n-1}) \neq \emptyset\}$ has cardinality κ^+.

Every point of $V = \text{Int}(F_n \backslash F_{n-1})$ belongs to exactly n elements of μ. For any $x \in V$, let $\mu_x = \{U \in \mu : x \in U\}$ and $V_x = (\bigcap \mu_x) \cap V$. The family $\mathcal{V} = \{V_x : x \in V\}$ is disjoint in the sense that, for any $x, y \in V$, we have $V_x = V_y$ or $V_x \cap V_y = \emptyset$. To see this, observe that $z \in V_x \cap U$ and $U \in \mu$ implies $U \in \mu_x$ because otherwise the point z belongs to more than n elements of μ. Therefore $V_x \cap V_y \neq \emptyset$ implies every $U \in \mu_y$ intersects V_x so $\mu_y \subset \mu_x$. Since the situation is symmetric, we also have $\mu_x \subset \mu_y$ whence $\mu_x = \mu_y$ and therefore $V_x = V_y$. Since every V_x intersects only finitely many elements of μ, there must be κ^+ distinct (and hence disjoint) elements of $\mathcal{V}$ which is a contradiction with $c(X) \leqslant \kappa$. This proves that $p(X) \leqslant c(X)$ and therefore $p(X) = c(X)$.

S.283. *Prove that, if $C_p(X_t)$ is a Baire space for all $t \in T$, then the product $\prod\{C_p(X_t) : t \in T\}$ is a Baire space.*

Solution. Given a space Z and $B \subset Z$ let $\pi_B : C_p(Z) \to C_p(B)$ be the restriction map, i.e., $\pi_B(f) = f|B$ for any $f \in C_p(Z)$. The map π_B is continuous for any $B \subset Z$ (Problem 152). Let $C(B|Z)$ be the set $\pi_B(C_p(Z))$ with the topology induced from $C_p(B)$.

Observe that the space $\prod\{C_p(X_t) : t \in T\}$ is homeomorphic to the space $C_p(X)$ where $X = \bigoplus \{X_t : t \in T\}$ (see Problems 113 and 114), so it suffices to prove that $C_p(X)$ is a Baire space. We will consider X_t to be a clopen subspace of X. Given a countable $A \subset X$, the set $T_0 = \{t \in T : A \cap X_t \neq \emptyset\}$ is countable and $A = \bigcup\{A_t : t \in T_0\}$ where $A_t = A \cap X_t$ for all $t \in T_0$. Observe that, for any $h \in C(X_t)$ there is a function $h_1 \in C(X)$ with $h_1|X_t = h$; an immediate consequence is that $C(A_t|X) = C(A_t|X_t)$ for any $t \in T_0$. For an arbitrary function $f \in C(A|X)$ take any $f_1 \in C_p(X)$ with $\pi_A(f_1) = f$ and let $\varphi(f)(t) = \pi_{A_t}(f_1)$ for each $t \in T_0$. It is evident that $\varphi(f) \in \prod \{C(A_t|X) : t \in T_0\} = \prod\{C(A_t|X_t) : t \in T_0\}$ for each $f \in C(A|X)$ so we have a map $\varphi : C(A|X) \to \prod\{C(A_t|X_t) : t \in T_0\}$.

To check that φ is continuous, consider the map $q_t \circ \varphi$ where $q_t : \prod\{C(A_t|X_t) : t \in T_0\} \to C(A_t|X_t)$ is the natural projection. It is easy to see that $q_t \circ \varphi$ coincides with the restriction map from $C(A|X)$ to $C(A_t|X) = C(A_t|X_t)$

for every $t \in T_0$; the mentioned restriction map being continuous, this proves continuity of φ. If $f, g \in C(A|X)$ and $f \neq g$ then $f(x) \neq g(x)$ for some $x \in A_t$, $t \in T_0$. Then $\varphi(f)(t) \neq \varphi(g)(t)$ which proves that $\varphi(f) \neq \varphi(g)$, i.e., φ is an injection.

If $f_t : X_t \to \mathbb{R}$ is a continuous function for each $t \in T_0$ then there exists $f \in C(X)$ such that $f|X_t = f_t$ for all $t \in T_0$: one has to define $f(x)$ to be $f_t(x)$ if $x \in X_t$ for some index $t \in T_0$ and $f(x) = 0$ for the rest of $x \in X$. This fact easily implies that φ is an onto map. The map φ^{-1} is also continuous; to see that recall that φ^{-1} maps $\prod\{C(A_t|X_t) : t \in T_0\}$ to the product $\mathbb{R}^A$ so it suffices to verify that $e_a \circ \varphi^{-1}$ is continuous for each $a \in A$. Here $e_a(f) = f(a)$ for each $f \in \mathbb{R}^A$. Take $t \in T$ with $a \in X_t$ and observe that $e_a \circ \varphi^{-1}$ coincides with the continuous map $d_a \circ q_t$, where $d_a(f) = f(a)$ for all $f \in C(A_t|X_t)$.

This proves that φ is a homeomorphism and hence $C(A|X)$ is homeomorphic to $P = \prod\{C(A_t|X_t) : t \in T_0\}$. Each $C(A_t|X_t)$ is second countable and Baire (see Problems 209 and 280); applying Problem 281 we convince ourselves that P is also a Baire space. It turns out that $C(A|X)$ is a Baire space for each countable $A \subset X$ so we can apply Problem 280 again to conclude that $C_p(X)$ is a Baire space.

S.284. *Given a (not necessarily metric!) space X, call a subset $A \subset X$ bounded if, for any $f \in C(X)$, the set $f(A)$ is bounded in $\mathbb{R}$. Prove that if $C_p(X)$ is a Baire space then every bounded subset of X is finite. In particular, every pseudocompact subspace of X is finite. As a consequence, if X is a metrizable space such that $C_p(X)$ is has the Baire property, then X is discrete.*

Solution. Assume that A is an infinite unbounded subset of X. Then $C_p(X) = \bigcup\{C_n: n \in \mathbb{N}\}$ where $C_n = \{f \in C_p(X) : f(A) \subset [-n, n]\}$ for each $n \in \mathbb{N}$. It is easy to see that each C_n is a closed subset of $C_p(X)$. If $\text{Int}(C_n) \neq \emptyset$ then there is a finite $K \subset X$, $\varepsilon > 0$ and $f : K \to \mathbb{R}$ such that $O(f, K, \varepsilon) = \{g \in C_p(X) : |g(x) - f(x)| < \varepsilon \text{ for all } x \in K\} \subset C_n$. Take any $a \in A \setminus K$ and a function $g \in C_p(X)$ such that $g|K = f|K$ and $g(a) = n + 1$ (see Problem 034). It is evident that $g \in O(f, K, \varepsilon)\setminus C_n$ which is a contradiction. As a consequence, each C_n is nowhere dense and hence $C_p(X)$ is of first category which contradicts the Baire property of $C_p(X)$. Thus, every bounded subset of X is finite.

If P is a pseudocompact subspace of X then $f|P$ is continuous on P for any $f \in C(X)$. Therefore $f(P)$ is bounded in $\mathbb{R}$ which shows that every pseudocompact subset of X is bounded in X. Thus, if $C_p(X)$ is Baire, then all pseudocompact subspaces of X are finite. If X is a metric space then it has no non-trivial convergent sequences because every such sequence is infinite and compact. Each metrizable space is first countable and hence sequential (Problem 210), so if A is a non-closed subset of X then there is a sequence $(a_n)_{n\in\omega} \subset A$ such that $a_n \to x \notin A$. It is clear that $\{a_n\}_{n\in\omega} \cup \{x\}$ is a non-trivial convergent sequence which is a contradiction. Thus every subset of X is closed and hence X is discrete.

S.285. *Prove that there exist spaces X such that $C_p(X)$ is not a Baire space while all bounded subsets of X are finite.*

Solution. Given a family $\mathcal{A} = \{A_t : t \in T\}$ of subsets of a space Y, say that $\mathcal{A}$ *has a discrete open expansion* if there exists a discrete $\mathcal{U} = \{U_t : t \in T\} \subset \tau(Y)$ such that

$A_t \subset U_t$ for each $t \in T$. Given a finite $P \subset Y$, a function $f \in C_p(Y)$ and $\varepsilon > 0$, we let $O(f, P, \varepsilon) = \{g \in C_p(Y) : |g(x) - f(x)| < \varepsilon$ for all $x \in P\}$. Consider the family $\mathcal{B} = \{O(f, P, \varepsilon) : f \in C_p(Y), P$ is a finite subset of Y and $\varepsilon > 0\}$. It is clear that $\mathcal{B}$ is a base in $C_p(Y)$.

Fact 1. Suppose that $C_p(Y)$ is a Baire space. Then Y has the following "moving off" property: for any sequence $\{\mathcal{F}_n : n \in \omega\}$ of infinite disjoint families of non-empty finite subsets of Y, we can choose an element $F_n \in \mathcal{F}_n$ for every $n \in \omega$ in such a way that $\bigcup\{F_n : n \in \omega\}$ has an open discrete expansion.

Proof. Given $U \in \tau(\mathbb{R})$ and a finite $P \subset Y$, let $[P, U] = \{f \in C_p(Y) : f(P) \subset U\}$. It is clear that $[P, U]$ is open in $C_p(Y)$ for any $U \in \tau(\mathbb{R})$ and finite $P \subset Y$. Observe that if $U_n = (2n, 2n + 1)$ for each $n \in \omega$ then the family $\{U_n : n \in \omega\} \subset \tau^*(\mathbb{R})$ is discrete. It is an easy exercise to see that, for any function $f \in C_p(Y)$, the family $\gamma = \{f^{-1}(U_n) : n \in \omega\}$ is discrete in Y (while many elements of γ can be empty).

The set $O_n = \bigcup\{[F, U_n] : F \in \mathcal{F}_n\}$ is open in $C_p(Y)$ for each $n \in \omega$; we claim that it is also dense in $C_p(Y)$. Indeed, if $W = O(f, P, \varepsilon) \in \mathcal{B}$ there exists $F \in \mathcal{F}_n$ such that $F \cap P = \emptyset$. Apply Problem 034 to find a function $h \in C_p(Y)$ such that $h|P = f|P$ and $h(x) = 2n + \frac{1}{2}$ for all $x \in F$. It is evident that $h \in W \bigcap O_n$; since the set $W \in \mathcal{B}$ was chosen arbitrarily, we have $O_n \cap W \neq \emptyset$ for any $W \in \mathcal{B}$, i.e., O_n is dense in $C_p(Y)$. The space $C_p(Y)$ being Baire there exists a function $f \in \cap\{O_n : n \in \omega\}$. For each $n \in \omega$, there is $F_n \in \mathcal{F}_n$ such that $f \in [F_n, U_n]$ and therefore the family $\{f^{-1}(U_n) : n \in \omega\}$ is a discrete open expansion of the family $\{F_n : n \in \omega\}$. Fact 1 is proved.

The underlying set of our promised space X is $\mathbb{Q} \cup \{z\}$ where $z = \sqrt{2}$. All points of $\mathbb{Q}$ are isolated and a set $U \ni z$ is open in X if and only if $\mathbb{Q}\backslash U$ is nowhere dense in $\mathbb{Q}$. It is easy to see that X is a Tychonoff space and, given $A \subset \mathbb{Q}$, we have $z \in \mathrm{cl}_X(A)$ if and only if the closure of A in $\mathbb{R}$ contains some non-empty open subset of $\mathbb{R}$. Apply Claim 2 of S.018 to conclude that X is also normal. Thus, if $D = \{d_n : n \in \omega\}$ is a closed discrete subset of X then there exists $f \in C(X)$ such that $f(d_n) = n$ for all $n \in \omega$. As a consequence, no subset of X, which contains an infinite closed discrete subset, is bounded.

The next observation is that, for every infinite $A \subset X$ there is infinite $B \subset A$ such that B is closed and discrete in X. To see this, observe that if A is closed and discrete in $\mathbb{Q}$ then we can take $B = A$. If not then A contains a non-trivial convergent sequence B (in the space $\mathbb{R}$). It is clear that $B \subset A$ is nowhere dense in $\mathbb{Q}$ and hence B is closed and discrete in X. This shows that every bounded subset of X is finite.

To show that $C_p(X)$ is not Baire, we will prove that X does not have the "moving off" property. Given $\varepsilon > 0$, call a subset $B \subset [0, 1] \cap \mathbb{Q}$ *an ε-net* if, for any $x \in [0, 1]$ there is $b \in B$ such that $|b - x| < \varepsilon$. Given $n \in \omega$ it is easy to construct a family $\mathcal{F}_n$ of infinitely many disjoint finite 2^{-n}-nets in $[0, 1]$. If X has the "moving off" property then it is possible to choose $F_n \in \mathcal{F}_n$ for each $n \in \omega$ so that the family $\{F_n : n \in \omega\}$ has an open discrete expansion in X. This means, in particular, that $z \notin \overline{\bigcup\{F_n : n \in \omega\}}$ while $\bigcup\{F_n : n \in \omega\}$ is dense in the open set $(0, 1)$ (in the topology of $\mathbb{R}$) and hence $z \in \overline{\bigcup\{F_n : n \in \omega\}}$; this contradiction shows that X does not have the "moving off" property and therefore $C_p(X)$ is not Baire by Fact 1. Our solution is complete.

S.286. *Prove that if $C_p(X)$ is a Baire space then $C_p(X, \mathbb{I})$ is also Baire. Give an example of a space X such that $C_p(X, \mathbb{I})$ is a Baire space but $C_p(X)$ does not have the Baire property.*

Solution. Let $w : \mathbb{R} \to (-1, 1)$ be a homeomorphism (Problem 025). Given $f \in C_p(X)$, let $\varphi(f) = w \circ f$; then the map $\varphi : C_p(X) \to C_p(X, (-1, 1))$ is continuous as well as its inverse defined by the formula $\varphi^{-1}(g) = w^{-1} \circ g$ for any function $g \in C_p(X, (-1, 1))$ (Problem 091). Thus $C_p(X)$ is homeomorphic to $C_p(X, (-1, 1))$ which is dense in $C_p(X, \mathbb{I})$. If $C_p(X)$ is Baire then $C_p(X, \mathbb{I})$ contains a dense Baire subspace $C_p(X, (-1, 1))$ so we can apply Problem 275 to conclude that $C_p(X, \mathbb{I})$ is also a Baire space.

To construct a promised example, we will first need several facts.

Fact 1. Fix an arbitrary set M. Given any $B \subset M$, denote by $\pi_B : \mathbb{I}^M \to \mathbb{I}^B$ the projection defined by $\pi_B(x) = x|B$ for any $x \in \mathbb{I}^M$. A dense subspace $X \subset \mathbb{I}^M$ is pseudocompact if and only if $\pi_B(X) = \mathbb{I}^B$ for every countable $B \subset M$.

Proof. If X is pseudocompact and B is a countable subset of M then $\pi_B(X)$ is a second countable pseudocompact space because π_B is a continuous map (Problem 107). Any second countable pseudocompact space is compact (Problem 138) so $\pi_B(X)$ is a compact dense subspace of $\mathbb{I}^B$. Hence $\pi_B(X) = \mathbb{I}^B$ and we proved necessity.

The family $\mathcal{B} = \{\prod_{t\in M} U_t : U_t \in \tau(\mathbb{I})$ for all t, and the set $\{t \in M : U_t \neq \mathbb{I}\}$ is finite$\}$ is a base for the space $\mathbb{I}^M$ (Problem 101). Given any set $U = \prod_{t\in M} U_t \in \mathcal{B}$, let $\text{supp}(U) = \{t \in M : U_t \neq \mathbb{I}\}$.

Suppose now that $\pi_B(X) = \mathbb{I}^B$ for any countable $B \subset M$. If X is not pseudocompact then there exists a discrete family $\mathcal{O} = \{O_n : n \in \omega\} \subset \tau^*(X)$. There exists a family $\{U_n : n \in \omega\} \subset \mathcal{B}$ such that $U_n \cap X \subset O_n$ for all $n \in \omega$. Let $B = \bigcup\{\text{supp}(U_n) : n \in \omega\}$. It is easy to see that $\pi_B^{-1}(\pi_B(U_n)) = U_n$ for all $n \in \omega$. The map π_B is open (Problem 107) so $V_n = \pi_B(U_n)$ is open in $\mathbb{I}^B$ for all n.

The space $\mathbb{I}^B$ is compact so the family $\mathcal{V} = \{V_n : n \in \omega\}$ cannot be discrete in $\mathbb{I}^B$; fix a point $y \in \mathbb{I}^B$ whose every neighbourhood intersects infinitely many elements of $\mathcal{V}$. Take any $x \in X$ with $\pi_B(x) = y$ and any open neighbourhood W of the point x in X. We claim that W intersects infinitely many elements of $\mathcal{O}$.

To prove this, take any $G = \prod_{t\in M} G_t \in \mathcal{B}$ with $x \in G$ and $G \cap X \subset W$. The set $\pi_B(G)$ is an open neighbourhood of y in $\mathbb{I}^B$ and hence it intersects infinitely many elements of the family $\mathcal{V}$; thus it suffices to show that if $\pi_B(G) \cap V_n \neq \emptyset$ then $W \cap O_n \neq \emptyset$. Take any $z \in \pi_B(G) \cap V_n$ and fix $g \in G$ with $\pi_B(g) = z$. The set $C = B \cup \text{supp}(G)$ is countable so there exists $z_1 \in X$ such that with $\pi_C(z_1) = h = \pi_C(g)$. Since $\pi_B^{-1}(z) \subset U_n$ and $\pi_B(z_1) = \pi_B(g) = z$, we have $z_1 \in U_n \cap X \subset O_n$. By definition of G we have $\pi_C^{-1}(h) \subset G$ and therefore $z_1 \in G$. As a consequence $z_1 \in (G \cap X) \cap O_n \subset W \cap O_n$, so $W \cap O_n \neq \emptyset$ whenever $\pi_B(G) \cap V_n \neq \emptyset$. The set $W \in \tau(x, X)$ was chosen arbitrarily, so we proved that $\mathcal{O}$ is not locally finite at x; this contradiction proves that X is pseudocompact so Fact 1 is proved.

Fact 2. Let us identify the discrete space $D(\omega)$ with ω. Suppose that K is a compact extension of ω such that $\mathrm{cl}_K(A) \cap \mathrm{cl}_K(B) = \emptyset$ for any $A, B \subset \omega$ with $A \cap B = \emptyset$. Then there exists a homeomorphism $f : \beta\omega \to K$ such that $f(n) = n$ for any $n \in \omega$. In particular, any map from ω to a compact space can be extended continuously over K.

Proof. There exists a continuous map $f : \beta\omega \to K$ such that $f(n) = n$ for any $n \in \omega$ (Problem 257). Since $\omega \subset f(\beta\omega)$, the compact set $f(\beta\omega)$ is dense in K and hence $f(\beta\omega) = K$. Given $x, y \in \beta\omega, x \neq y$ pick $U \in \tau(x, \beta\omega)$, $V \in \tau(y, \beta\omega)$ such that $\overline{U} \cap \overline{V} = \emptyset$ (the bar denotes the closure in $\beta\omega$). We have $\overline{U \cap \omega} = \overline{U}$ and $\overline{V \cap \omega} = \overline{V}$ so $f(U \cap \omega) = U \cap \omega = U_1$ is dense in $f(U)$ and $f(V \cap \omega) = V \cap \omega = V_1$ is dense in $f(V)$. Since $U_1 \cap V_1 = \emptyset$, the mentioned property of K implies $\mathrm{cl}_K(U_1) \cap \mathrm{cl}_K(V_1) = \emptyset$. Since $f(x) \in \mathrm{cl}_K(U_1)$ and $f(y) \in \mathrm{cl}_K(V_1)$, we proved that $f(x) \neq f(y)$, i.e., the map f is a condensation and hence a homeomorphism (Problem 123). Fact 2 is proved.

Fact 3. If N is a set and $|N| = \kappa$, then there exists an enumeration $\{n_\alpha : \alpha < \kappa\}$ of the set N such that each $n \in N$ occurs κ-many times in this enumeration, i.e., the set $\{\alpha < \kappa : n_\alpha = n\}$ has cardinality κ for any $n \in N$.

Proof. Let $f : \kappa \to N$ be any onto map. Since $|\kappa \times \kappa| = \kappa$, there exists an onto map $g : \kappa \to \kappa \times \kappa$. Besides, $\pi : \kappa \times \kappa \to \kappa$ is the projection onto the first factor. Finally, let $n_\alpha = f(\pi(g(\alpha)))$ for each $\alpha < \kappa$. It is immediate that the enumeration $\{n_\alpha : \alpha < \kappa\}$ is as promised. Fact 3 is proved.

Given any $\alpha < \mathfrak{c}$, we denote by $\pi_\alpha : \mathbb{I}^{\mathfrak{c}} \to \mathbb{I}$ the natural projection onto the α-th factor. Consider the set $G = \{x \in \mathbb{I}^{\mathfrak{c}} : |\{\alpha \in \mathfrak{c} : x(\alpha) \neq 0\}| \leqslant \omega\} \subset \mathbb{I}^{\mathfrak{c}}$. Since $|\mathbb{I}| = \mathfrak{c}$, we have $|\mathbb{I}^B| = \mathfrak{c}^\omega = \mathfrak{c}$ for any countable set B. Therefore, for any countable $B \subset \mathfrak{c}$ the set $G_B = \{x \in \mathbb{I}^{\mathfrak{c}} : x(\alpha) = 0 \text{ for all } \alpha \in \mathfrak{c}\backslash B\}$ has cardinality $\mathfrak{c}$. Since $G = \bigcup\{G_B : B \text{ is a countable subset of } \mathfrak{c}\}$, we have $|G| = \mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$. Apply Fact 3 to fix an enumeration $\{g_\alpha : \alpha < \mathfrak{c}\}$ of the set G in which every $g \in G$ occurs $\mathfrak{c}$ times. The family $\varepsilon = \{B \subset \mathfrak{c} : B \text{ is countable}\}$ also has cardinality continuum so we can use Fact 3 again to choose an enumeration $\{A_\alpha : \alpha < \mathfrak{c}\}$ of the family ε such that every $A \in \varepsilon$ occurs $\mathfrak{c}$ times in this enumeration.

For any $\alpha < \mathfrak{c}$ define a point $x_\alpha \in \mathbb{I}^{\mathfrak{c}}$ vas follows: $x_\alpha(\beta) = g_\alpha(\beta)$ for each $\beta \leqslant \alpha$; if $\beta > \alpha$ and $\alpha \in A_\beta$ then $x_\alpha(\beta) = 1$; if $\beta > \alpha$ and $\alpha \notin A_\beta$ then $x_\alpha(\beta) = 0$. Then $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ is our promised space.

Take any countable $B \subset \mathfrak{c}$ and any $g \in \mathbb{I}^B$. If $h(\beta) = g(\beta)$ for any $\beta \in B$ and $h(\beta) = 0$ for all $\beta \in \mathfrak{c}\backslash B$ then $h \in G$ and $\pi_B(h) = g$. There exists $\alpha > \sup(B)$ such that $g_\alpha = h$; then $\pi_B(x_\alpha) = \pi_B(h) = g$. This proves that $\pi_B(X) = \mathbb{I}^B$ for each countable $B \subset \mathfrak{c}$ so X is pseudocompact by Fact 1.

Take any countable set $P \subset X$; there exits a countable $B \subset \mathfrak{c}$ such that $P = \{x_\alpha : \alpha \in B\}$. If $\gamma \notin B$ then find $\beta > \sup(B \cup \{\gamma\})$ such that $A_\beta = B$ and observe that $\gamma \notin B = A_\beta$ and therefore $\pi_\beta(x_\alpha) = x_\alpha(\beta) = 1$ for any $\alpha \in B = A_\beta$ while $\pi_\beta(x_\gamma) = x_\gamma(\beta) = 0$. Since the map $\pi_\beta : X \to \mathbb{I}$ is continuous, it is impossible that $x_\gamma \in \overline{\{x_\alpha : \alpha \in B\}} = \overline{P}$ so the set P is closed in X. Thus every countable set is closed in X. Besides, every countable subset $P \subset X$ is also discrete because every subset of P is also countable and hence closed in P.

Now fix any countable $P \subset X$ and denote by K the closure of P in $\mathbb{I}^{\mathfrak{c}}$. Then K is a compact extension of the countable discrete space P. Suppose that $R, S \subset P$ and $R \cap S = \emptyset$. Fix countable $B, D \subset \mathfrak{c}$ such that $R = \{x_\alpha : \alpha \in B\}$ and $S = \{x_\alpha : \alpha \in D\}$. Then $B \cap D = \emptyset$ find $\beta > \sup(B \cup D)$ such that $A_\beta = B$. The same verification as in the previous paragraph shows that $\pi_\beta(R) = \{1\}$ and $\pi_\beta(S) = \{0\}$. Therefore $[R] \cap [S] \subset \pi_\beta^{-1}(0) \cap \pi_\beta^{-1}(1) = \emptyset$ (the brackets denote the closure in $\mathbb{I}^{\mathfrak{c}}$). As a consequence $\mathrm{cl}_K(R) \cap \mathrm{cl}_K(S) = \emptyset$ for any disjoint $R, S \subset P$ so we can apply Fact 2 to conclude that K is homeomorphic to $\beta\omega = \beta P$. Thus, if $f : P \to \mathbb{R}$ is a bounded function, it is continuous because P is discrete. Besides, there is $n \in \omega$ such that $f : P \to [-n, n]$ because f is bounded. Since $K = \beta P$, there exists a continuous $h : K \to [-n, n]$ such that $h|P = f$ (Problem 257). The space $\mathbb{I}^{\mathfrak{c}}$ is compact and hence normal, so there is a continuous $g_1 : \mathbb{I}^{\mathfrak{c}} \to \mathbb{R}$ such that $g_1 |K = h$. Therefore $g = g_1|X$ is continuous and $g|P = f$. Thus, we proved the following fact.

Fact 4. There exists an infinite dense pseudocompact $X \subset \mathbb{I}^{\mathfrak{c}}$ with the following properties:

(1) Every countable subspace of X is closed and discrete.
(2) For every countable $B \subset X$ and every $f : B \to \mathbb{I}$ there exists $g \in C(X, \mathbb{I})$ such that $g|B = f$.

Now it is easy to finish our solution. Consider the space $C_p(X, \mathbb{I}) \subset \mathbb{I}^X$ Z. If $B \subset X$ is countable and $f \in \mathbb{I}^B$ then there exists a continuous $g \in C_p(X, \mathbb{I})$ with $g|B = f$. This shows that $\pi_B(C_p(X, \mathbb{I})) = \mathbb{I}^B$ for every countable $B \subset X$. Therefore Fact 1 is applicable to $C_p(X, \mathbb{I})$ and hence $C_p(X, \mathbb{I})$ is pseudocompact. Any pseudocompact space is Baire (Problem 274) so $C_p(X, \mathbb{I})$ is a Baire space. However, $C_p(X)$ is not Baire because X is an infinite pseudocompact space (Problem 284) so our solution is complete.

S.287. *Prove that if $C_p(X, \mathbb{I})$ has a dense Čech-complete subspace then X is discrete.*

Solution. Let us prove that X has to be discrete if $C_p(X, \mathbb{I})$ has a dense Čech-complete subspace D. Since $\mathbb{I}^X$ is a compact extension of D, the set D has to be G_δ in $\mathbb{I}^X$. Fix a family $\{O_n : n \in \omega\} \subset \tau(\mathbb{I}^X)$ such that $\bigcap\{O_n : n \in \omega\} = D$. The family $\mathcal{B} = \{\prod_{x \in X} U_x : U_x \in \tau(\mathbb{I})$ for all x, and the set $\{x \in X : U_x \neq \mathbb{I}\}$ is finite$\}$ is a base for the space $\mathbb{I}^X$ (101). Given $U = \prod_{x \in X} U_x \in \mathcal{B}$, let $\mathrm{supp}(U) = \{x \in X : U_x \neq \mathbb{I}\}$. If $A \subset X$ then $\pi_A : \mathbb{I}^X \to \mathbb{I}^A$ is the restriction map defined by the formula $\pi_A(f) = f|A$ for any $f \in \mathbb{I}^X$. A set $H \subset \mathbb{I}^X$ is called *A-saturated* if $\pi_A^{-1}(\pi_A(H)) = H$. It is straightforward that any union and any intersection of A-saturated sets is an A-saturated set.

Let γ_n be a maximal disjoint subfamily of $\mathcal{B}$ such that $\bigcup\gamma_n \subset O_n$. Since $c(\mathbb{I}^X) = \omega$ (Problem 109), the family γ_n is countable for each $n \in \omega$ and hence the set $A = \bigcup\{\mathrm{supp}(U) : U \in \gamma_n, n \in \omega\}$ is countable. If $V_n = \bigcup\gamma_n$ then $V_n \subset O_n$ is a dense open subset of O_n (by maximality of γ_n) and hence of $\mathbb{I}^X$. Since the compact space $\mathbb{I}^X$ has the Baire property (Problem 274), the set $E = \bigcap\{V_n : n \in \omega\}$ is dense in $\mathbb{I}^X$. The space E is also Čech-complete because it is a G_δ-set in its compact extension $\mathbb{I}^X$. Another important observation is that $E \subset C_p(X, \mathbb{I})$ because $V_n \subset O_n$ for each $n \in \omega$.

Observe that if $U \in \mathcal{B}$ and $\text{supp}(U) \subset B$ then U is B-saturated; therefore every element of γ_n is A-saturated and hence $V_n = \bigcup \gamma_n$ is also A-saturated for all $n \in \omega$. This, in turn, implies that E is A-saturated. The set $U_n = \pi_A(V_n)$ is dense and open in $\mathbb{I}^A$ for any $n \in \omega$ (107) and hence $L = \bigcap\{U_n : n \in \omega\}$ is a dense Čech-complete subspace of $\mathbb{I}^A$; besides, $L \subset C_p(A, \mathbb{I})$ because $L = \pi_A(E)$ and $E \subset C_p(X, \mathbb{I})$. Fix any $h \in L$; then $\pi_A^{-1}(h) \subset E \subset C_p(X, \mathbb{I})$. This means that any $f : X \to \mathbb{I}$ with $f|A = h$, is continuous on X. Given any $g : X \setminus A \to \mathbb{I}$ there exists $f : X \to \mathbb{I}$ such that $f|(X \setminus A) = g$ and $f|A = h$; this proves that $C_p(X \setminus A, \mathbb{I}) = \mathbb{I}^{X \setminus A}$ and therefore $X \setminus A$ is a discrete subspace of X. If $y \in \overline{A} \setminus A$ then let $f_1(x) = 1$ for each $x \in X \setminus A$ and $f_1(x) = h(x)$ for all $x \in A$. Analogously, let $f_2(x) = 0$ for each $x \in X \setminus A$ and $f_1(x) = h(x)$ for all $x \in A$. Then $f_1, f_2 \in C(X)$ and hence $f = f_1 - f_2$ is also continuous on X. However, $f(A) = \{0\}$ and $f(y) = 1$ which is impossible by $y \in \overline{A}$; therefore we proved that A is closed in X.

Now, if $y \in \overline{X \setminus A} \cap A$ then take any $f : X \to \mathbb{I}$ for which $f|A = h$ and $f|(X \setminus A)$ is a constant distinct from $h(y)$. It is clear that f cannot be continuous on X; this contradiction shows that $X \setminus A$ is also closed in X and hence both sets A and $X \setminus A$ are open.

For any $a \in A$ let $W_a = \{f \in C_p(A, \mathbb{I}) : f(a) \in (-1, 1)\}$. It is evident that W_a is a dense open set of $C_p(A, \mathbb{I})$ and therefore $M = \bigcap\{W_a : a \in A\} \cap L$ is a dense Čech-complete subspace of $C_p(A, \mathbb{I})$. It is clear that M is contained in $C_p(A, (-1, 1))$ which is homeomorphic to $C_p(A)$ (see the first paragraph of S.286). As a consequence $C_p(A)$ has a dense Čech-complete subspace so A is also discrete (Problem 265) and hence X is discrete so our solution is complete.

S.288. *Prove that the following are equivalent for any normal space X:*

(i) *X is countably paracompact.*
(ii) *$X \times K$ is normal for any metrizable compact K.*
(iii) *$X \times \mathbb{I}$ is normal.*
(iv) *$X \times A(\omega)$ is normal.*

Solution. The following characterization of countable paracompactness can often be useful.

Fact 1. The following conditions are equivalent for any space X:

(a) X is countably paracompact.
(b) For any countable open cover $\{U_i : i \in \omega\}$ of the space X there exists a locally finite open cover $\{V_i : \in \omega\}$ of X such that $V_i \subset U_i$ for every $i \in \omega$.
(c) For any increasing sequence $W_0 \subset W_1 \subset \ldots$ of open subsets of X satisfying $\bigcup_{i \in \omega} W_i = X$ there exists a sequence $\{F_i : i \in \omega\}$ of closed subsets of X such that $F_i \subset W_i$ for all $i \in \omega$ and $\bigcup_{i \in \omega} \text{Int}(F_i) = X$.

(d) For any decreasing sequence $G_0 \supset G_1 \supset \dots$ of closed subsets of X satisfying $\bigcap_{i\in\omega} G_i = \emptyset$ there exists a sequence $\{O_i : i \in \omega\}$ of open subsets of X such that $G_i \subset O_i$ for all $i \in \omega$ and $\bigcap_{i\in\omega} \overline{O}_i = \emptyset$.

Proof. To show that (a) $\Rightarrow$ (b) take any locally finite refinement $\mathcal{U}$ of the cover $\{U_i : i \in \omega\}$, choose for every $V \in \mathcal{U}$ a natural number $i(V)$ such that $V \subset U_{i(V)}$ and let $V_i = \bigcup\{V \in \mathcal{U} : i(V) = i\}$ for each $i \in \omega$.

As to (b) $\Rightarrow$ (c), take a locally finite open cover $\{V_i : i \in \omega\}$ of the space X such that $V_i \subset W_i$ for all $i \in \omega$. The sets $F_i = X\backslash(\bigcup\{V_j : j > i\})$ are closed and $F_i \subset \bigcup\{V_j : j \leqslant i\} \subset \bigcup\{W_j : j \leqslant i\} = W_i$ for each $i \in \omega$ so we have $F_i \subset W_i$ for all $i \in \omega$. The family $\{V_i : i \in \omega\}$ being locally finite, every $x \in X$ has a neighbourhood O_x which does not intersect the set $\bigcup\{V_j : j > i\}$ for some i. Therefore $x \in O_x \subset F_i$ so $x \in \mathrm{Int}(F_i)$ which proves that $\bigcup_{i\in\omega} \mathrm{Int}(F_i) = X$.
(c) $\Rightarrow$ (d). If $W_i = X\backslash G_i$ for all $i \in \omega$ then the sequence $\{W_i : i \in \omega\}$ is increasing and $\bigcup\{W_i : i \in \omega\} = X$. Find a sequence $\{F_i : i \in \omega\}$ of closed subsets of X such that $F_i \subset W_i$ and $\bigcup_{i\,\in\,\omega} \mathrm{Int}(F_i) = X$; if $O_i = X\backslash F_i$ for all $i \in \omega$ then the family $\{O_i : i \in \omega\}$ is as required.
(d) $\Rightarrow$ (a). Take an arbitrary open cover $\gamma = \{U_i : i \in \omega\}$ of the space X. The family $G_i = X \setminus \bigcup\{U_j : j \leq i\}$ form a decreasing family, which, consists of closed sets and $\bigcap_{i\in\omega} G_i = \emptyset$. Choose a sequence $\{O_i : i \in \omega\} \subset \tau(X)$ such that $G_i \subset O_i$ for all $i \in \omega$ and $\bigcap_{i\in\omega} \overline{O}_i = \emptyset$. The set $V_i = U_i \cap \left(\bigcap\{O_j : j < i\}\right)$ is open for all $i \in \omega$; the family $\mathcal{U} = \{V_i : i \in \omega\}$ is a cover of X because $V_i \supset U_i\backslash\left(\bigcup\{U_j : j < i\}\right)$ for all $i \in \omega$. Now if $x \in X$ then $x \notin \overline{O}_n$ for some $n \in \omega$ and hence $W = X\backslash\overline{O}_n$ is a neighbourhood of x which does not intersect any V_i if $i > n$. Thus the family $\mathcal{U}$ is a locally finite refinement of γ. Fact 1 is proved.

Fact 2. The following conditions are equivalent for any normal space X.

(a) X is countably paracompact.
(b) For any decreasing sequence $F_0 \supset F_1 \supset \dots$ of closed subsets of X satisfying $\bigcap_{i\in\omega} F_i = \emptyset$ there exists a sequence $\{W_i : i \in \omega\}$ of open subsets of X such that $F_i \subset W_i$ for all $i \in \omega$ and $\bigcap_{i\in\omega} W_i = \emptyset$.
(c) For any countable open cover $\{U_i : i \in \omega\}$ of the space X there exists a locally finite open cover $\{V_i : \in \omega\}$ of X such that $\overline{V}_i \subset U_i$ for every $i \in \omega$.
(d) For any countable open cover $\{U_i : i \in \omega\}$ of the space X there exists a closed cover $\{G_i : i \in \omega\}$ of X such that $G_i \subset U_i$ for every $i \in \omega$.

Proof. The implication (a) $\Rightarrow$ (b) is an immediate consequence of Fact 1(d). To prove (b) $\Rightarrow$ (a) suppose that we have a decreasing sequence $F_0 \supset F_1 \supset \dots$ of closed subsets of X satisfying $\bigcap_{i\in\omega} F_i = \emptyset$ and take a sequence $\{W_i' : i \in \omega\}$ of open subsets of X such that $F_i \subset W_i'$ for all $i \in \omega$ and $\bigcap_{i\in\omega} W_i' = \emptyset$. By normality of X there exists $W_i \in \tau(F_i, X)$ such that $\overline{W}_i \subset W_i'$. It is clear that we have $\bigcap_{i\in\omega} \overline{W}_i = \emptyset$ so Fact 1 (d) is applicable to conclude that X is countably paracompact. Thus (a)$\Leftrightarrow$(b).

(a) $\Rightarrow$ (c). Given a countable open cover $\{U_i : i \in \omega\}$ of the space X, apply Fact 1(b) to find an open locally finite cover $\{V_i' : i \in \omega\}$ of X such that $V_i' \subset U_i$ for each $i \in \omega$. The set $F_0 = X \backslash (\bigcup\{V_i' : i > 0\}$ is closed in X and $F_0 \subset V_0$. Applying normality of X we can find an open set V_0 with $F_0 \subset V_0 \subset \overline{V}_0 \subset V_0'$. Suppose that we have open sets V_i, $i < n$ such that $\overline{V}_i \subset V_i'$ for each $i < n$ and the family $\{V_0, \ldots, V_{n-1}, V_n', V_{n+1}', \ldots\}$ is a cover of X. Then $F_n = X \backslash \big((\bigcup_{i<n} V_i) \cup (\bigcup_{i>n} V_i') \big)$ is a closed set contained in V_n'. By normality of X there is $V_n \in \tau(F_n, X)$ such that $\overline{V}_n \subset V_n'$ so the inductive process can be continued and we can construct the sequence $\mathcal{S} = \{V_i : i \in \omega\}$. It is clear that the family $\mathcal{S}$ is locally finite and $\overline{V}_i \subset V_i' \subset U_i$ for each $i \in \omega$. Observe also that, for any $x \in X$, there is $n \in \omega$ such that $x \notin \bigcup\{V_i' : i \geqslant n\}$ because the family $\{V_i' : i \in \omega\}$ is locally finite. Since $\{V_0, \ldots, V_{n-1}, V_n', V_{n+1}', \ldots\}$ is a cover of X, we have $x \in V_i$ for some $i < n$ so $\mathcal{S}$ is a cover of X and the implication (a) $\Rightarrow$ (c) is established.

The implication (c) $\Rightarrow$ (d) is obvious, so it suffices to prove that (d) $\Rightarrow$ (b). Take any decreasing sequence $F_0 \supset F_1 \supset \ldots$ of closed subsets of a space X satisfying $\bigcap_{i\in\omega} F_i = \emptyset$. If $U_i = X \backslash F_i$ for all $i \in \omega$ then the family $\{U_i : i \in \omega\}$ is an open cover of X so we can fix a sequence $\{G_i : i \in \omega\}$ of closed subsets of X such that $G_i \subset U_i$ for all $i \in \omega$ and $\bigcup\{G_i : i \in \omega\} = X$. Now, if $W_i = X \backslash G_i$ for all $i \in \omega$ then $F_i \subset W_i$ and $\bigcap_{i\in\omega} W_i = \emptyset$ so the proof of (d) $\Rightarrow$ (b) is complete. Fact 2 is proved.

Fact 3. For any space Z and any compact P the natural projection $\pi : Z \times P \to Z$ is a perfect map.

Proof. It is evident that π is continuous and onto. Given a point $z \in Z$, the space $\pi^{-1}(z) = \{z\} \times P$ is compact so the set $\pi^{-1}(z)$ is compact for any point $z \in Z$. If $U \in \tau(\pi^{-1}(z), Z \times P)$, then we can apply Fact 3 of S.271 to the product $\{z\} \times P$ to conclude that there is $V \in \tau(z, Z)$ and $W \in \tau(P, P)$ such that $V \times W \subset U$. Of course, $W = P$ so we have $\pi^{-1}(V) = V \times P \subset U$ and hence Fact 2 of S.271 is applicable to the map π to conclude that π is closed. Fact 3 is proved.

We are finally ready to establish the implication (i) $\Rightarrow$ (ii). Assume that a normal space X is countably paracompact and take a metrizable compact K. Let $\mathcal{B}$ be a countable base in K such that any finite union of elements of $\mathcal{B}$ belongs to $\mathcal{B}$. Denote by $p : X \times K \to X$ and $q : X \times K \to K$ the respective natural projections. Given any $M \subset X \times K$ and $x \in X$, let $M_x = q(M \cap p^{-1}(x))$.

Fix a pair A, B of disjoint closed subsets of the product $X \times K$ and define $O_U = \{x \in X : A_x \subset U \subset \overline{U} \subset K \backslash B_x\}$ for each $U \in \mathcal{B}$. Observe that the set

$$\begin{aligned} X \backslash O_U &= \{x \in X : A_x \cap (K \backslash U) \neq \emptyset\} \cup \{x \in X : B_x \cap \overline{U} \neq \emptyset\} \\ &= p(A \cap (X \times (K \backslash U))) \cup p(B \cap (X \times \overline{U})) \end{aligned}$$

is closed because the projection p is a closed map by Fact 3. Thus every O_U is an open set. Given any $x \in X$ the sets A_x and B_x are disjoint and compact so there exist $U_1, \ldots, U_n \in \mathcal{B}$ with $A_x \subset U = \bigcup_{i\leqslant n} U_i \subset \overline{U} \subset K \backslash B_x$. Since $U \in \mathcal{B}$ by the choice of $\mathcal{B}$, we have $x \in O_U$, i.e., the countable family $\{O_U : U \in \mathcal{B}\}$ is an open cover of X. Since X is countably paracompact, we can apply Fact 2(c) to conclude that there exists a locally finite open cover $\mathcal{W} = \{W_U : U \in \mathcal{B}\}$ of the space X such that $\overline{W_U} \subset O_U$ for any $U \in \mathcal{B}$. The set $O = \bigcup\{W_U \times U : U \in \mathcal{B}\}$ is open in $X \times K$ so it suffices to show that $A \subset O \subset \overline{O} \subset (X \times K)\backslash B$.

Given any $(x, y) \in A$ there is $U \in \mathcal{B}$ with $x \in W_U$; then $x \in O_U$ and hence $y \in A_x \subset U$ so $(x, y) \in W_U \times U \subset O$ which proves that $A \subset O$. Now assume that there is $(x, y) \in \overline{O} \cap B$; since the family $\mathcal{W}$ is locally finite, there is $V \in \tau(x, X)$ such that $V \cap W_U \neq \emptyset$ only for a finitely many U's. The neighbourhood $V \times K$ of the point (x, y) intersects only finitely many products $W_U \times U$, say $\{W_{U_i} \times U_i : i \leqslant n\}$. As a consequence, we convince ourselves that $(x, y) \in \bigcup\{\overline{W_{U_i} \times U_i} : i \leqslant n\}$ and therefore $(x, y) \in \overline{W_{U_i} \times U_i} = \overline{W_{U_i}} \times \overline{U_i}$ for some $i \leqslant n$. But then $x \in \overline{W_{U_i}} \subset O_{U_i}$ and $y \in \overline{U_i} \cap B_x$ which is a contradiction with the fact that $A_x \subset U_i \subset \overline{U_i} \subset K\backslash B_x$. The implication (i) $\Rightarrow$ (ii) is proved.

The implication (ii) $\Rightarrow$ (iii) is obvious and (iii) $\Rightarrow$ (iv) is true because $X \times A(\omega)$ is embeddable as a closed subspace into the space $X \times \mathbb{I}$ (see 018).

(iv) $\Rightarrow$ (i). We identify $A(\omega)$ with the subspace $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of the real line. Take any decreasing sequence $\mathcal{S} = \{F_i : i \in \mathbb{N}\}$ of closed subsets of X with $\bigcap \mathcal{S} = \emptyset$. The sets $F = \bigcup \left\{F_i \times \left\{\frac{1}{i}\right\} : i \in \mathbb{N}\right\}$ and $G = X \times \{0\}$ are disjoint and closed in the normal space $X \times A(\omega)$. If $U \in \tau(F, X \times A(\omega))$, $V \in \tau(G, X \times A(\omega))$ are disjoint then let $W_i = \{x \in X : (x, \frac{1}{i}) \in U\}$ for all $i \in \mathbb{N}$. It is obvious that W_i is an open subset of X and $F_i \subset W_i$ for all $i \in \mathbb{N}$. Assume that $x \in \bigcap\{W_i : i \in \mathbb{N}\}$; it is immediate that the sequence $\{(x, \frac{1}{n}) : n \in \mathbb{N}\} \subset U$ converges to $y = (x, 0) \in G$ whence $y \in \overline{U} \cap G \subset \overline{U} \cap V = \emptyset$, a contradiction. Thus, $\bigcap\{W_i : i \in \mathbb{N}\} = \emptyset$ and hence X is countably paracompact by Fact 2(b). This finishes the proof of the implication (iv) $\Rightarrow$ (i) and makes our solution complete.

S.289. *Prove that, if $C_p(X)$ is normal, then it is countably paracompact.*

Solution. Let us show first that some subspaces inherit normality.

Fact 1. Let Y be a normal space. Then any F_σ-subspace of Y is also normal.

Proof. Take any F_σ-set $P \subset Y$; fix a family $\{P_n : n \in \omega\}$ of closed subsets of Y with $P = \bigcup\{P_n : n \in \omega\}$. Take any $F \subset P$ which is closed in P and any $W \in \tau(F, P)$. Choose any $U \in \tau(Y)$ with $U \cap P = W$. The set $F_n = P_n \cap F$ is closed in Y so we can apply normality of the space Y to find $U_n \in \tau(F_n, Y)$ such that $\mathrm{cl}_Y(U_n) \subset U$. For the family $\{W_n = U_n \cap P : n \in \omega\}$, we have $F_n \subset W_n$ and $\mathrm{cl}_P(W_n) \subset W$ for each $n \in \omega$. Therefore $F \subset \bigcup\{W_n : n \in \omega\}$ and $\mathrm{cl}_P(W_n) \subset W$ for all $n \in \omega$. This shows that Fact 1 of S.221 can be applied to conclude that P is normal. Fact 1 is proved.

Fact 2. Suppose that $Y \times \mathbb{R}$ is normal for some space Y. Then $Y \times \mathbb{R}$ is countably paracompact.

Proof. The spaces Y and $Y \times \mathbb{I}$ are normal being each one a closed subspace of the normal space $Y \times \mathbb{R}$. Thus Y is countably paracompact (Problem 288). Apply Problem 288 again to observe that $Y \times (\mathbb{I} \times \mathbb{I})$ is also normal because $\mathbb{I} \times \mathbb{I}$ is a metrizable compact space. It is easy to see that $Y \times ((-1, 1) \times \mathbb{I})$ is an F_σ-subspace of $Y \times (\mathbb{I} \times \mathbb{I})$ so it is normal by Fact 1. It is evident that $Y \times ((-1, 1) \times \mathbb{I})$ is homeomorphic to $(Y \times \mathbb{R}) \times \mathbb{I}$ so Problem 288 can be applied once more to conclude that $Y \times \mathbb{R}$ is countably paracompact. Fact 2 is proved.

It is now easy to finish our solution. Take any point $x \in X$ and consider the set $Y = \{f \in C_p(X) : f(x) = 0\}$. Then $C_p(X)$ is homeomorphic to $Y \times \mathbb{R}$ (Problem 182) so Fact 2 can be applied to conclude that $C_p(X)$ is countably paracompact if it is normal.

S.290. *Prove that $C_p(X)$ is normal if and only if any F_σ-subset of $C_p(X)$ is countably paracompact.*

Solution. Suppose that $C_p(X)$ is normal and take any F_σ-set $P \subset C_p(X)$. Then P is also normal by Fact 1 of S.289. The space $C_p(X)$ is countably paracompact by Problem 289 and therefore $C_p(X) \times \mathbb{I}$ is normal (Problem 288). It is evident that $P \times \mathbb{I}$ is an F_σ-subset of $C_p(X) \times \mathbb{I}$ so it is normal by Fact 1 of S.289. Apply Problem 288 again to conclude that P is countably paracompact and finish the proof of necessity.

To prove sufficiency, suppose that every F_σ-subspace of $C_p(X)$ is countably paracompact. Denote by S the convergent sequence $\{0\} \cup \left\{\frac{1}{i} : i \in \mathbb{N}\right\}$ and let $S_n = \{0\} \cup \left\{\frac{1}{i} : i \geqslant n\right\}$ for any $n \in \mathbb{N}$. Take any $x \in X$ and let $Y = \{f \in C_p(X) : f(x) = 0\}$. Then $C_p(X)$ is homeomorphic to the space $Y \times \mathbb{R}$ (Problem 182). Let $p : Y \times S \to Y$ be the natural projection. Take any closed subset F of the space Y and any $U \in \tau(F, Y)$. The space $Z = (F \times \{0\}) \cup (\bigcup\{Y \times \left\{\frac{1}{i}\right\} : i \in \mathbb{N}\})$ is an F_σ-set in $C_p(X)$ so Z is countably paracompact. If $U_i = Y \times \left\{\frac{1}{i}\right\}$ for each $i \in \mathbb{N}$ and $U_0 = (U \times S) \cap Z$ then the family $\{U_i : i \in \omega\}$ is a countable open cover of the space Z. By countable paracompactness of Z, there is a locally finite family $\{V_i : i \in \omega\} \subset \tau(Z)$ such that $V_i \subset U_i$ for all $i \in \omega$ (Fact 1(b) of S.288). The set $W_i = Y \backslash \overline{p(V_i)}$ is open in Y for all $i \in \mathbb{N}$. Since V_i is an open subset of Y containing $(Y \backslash U) \times \left\{\frac{1}{i}\right\}$, the set $p(V_i)$ is an open subset of Y which contains $Y \backslash U$. This proves that $\overline{W_i} \subset U$ for all $i \in \mathbb{N}$.

Given a point $y \in F$, there is $G \in \tau(y, Y)$ such that $(G \times S_n) \cap Z$ intersects only finitely many of the sets V_i's. As a consequence, there is $j \in \mathbb{N}$ such that $(G \times S) \cap V_j = \emptyset$ and hence $y \notin \overline{p(V_j)}$, i.e., $y \in W_i$. This shows that, for an arbitrary closed $F \subset Y$ and any $U \in \tau(F, Y)$ we constructed a sequence $\{W_i : i \in \mathbb{N}\} \subset \tau(Y)$ such that $F \subset \bigcup\{W_i : i \in \mathbb{N}\}$ and $\overline{W_i} \subset U$ for all $i \in \mathbb{N}$. Now apply Fact 1 of S.221 to see that Y is normal. Since Y is also countably paracompact, the space $Y \times \mathbb{I}$ is also normal (Problem 288). Observing that $C_p(X) = Y \times \mathbb{R}$ is homeomorphic to an F_σ-subspace of $Y \times \mathbb{I}$ we conclude that $C_p(X)$ is normal (Fact 1 of S.289) so we proved sufficiency and hence our solution is complete.

S.291. *Suppose that $C_p(X)$ is normal and Y is closed in X. Prove that the space $\pi_Y(C_p(X)) = \{f|Y : f \in C_p(X)\} \subset C_p(Y)$ is also normal.*

Solution. Given a space Z and $A, B \subset Z$, say that A and B are *separated* if $\overline{A} \cap B = \emptyset = \overline{B} \cap A$. Call the sets A, B *open-separated* if there are open sets $U, V \subset Z$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. We will also say that the mentioned open sets U and V *separate* the sets A and B. It is easy to see that open-separated sets are separated. Given a product $M = \prod\{M_t : t \in T\}$ and $S \subset T$, the map $\pi_S : M \to M_S = \prod\{M_t : t \in S\}$ is the natural projection defined by $\pi_S(x) = x|S$ for all $x \in M$.

Fact 1. If Z is a hereditarily normal space then $A, B \subset Z$ are separated if and only if they are open-separated. In particular, this is true in metrizable spaces.

Proof. It suffices to prove that separated sets are open-separated, so let $A, B \subset Z$ be separated sets. If $F = \overline{A} \cap \overline{B}$ then $A \cup B \subset Z \backslash F$ and hence $\overline{A} \backslash F, \overline{B} \backslash F$ are disjoint closed subsets of the normal space $Z \backslash F$. If $U, V \in \tau(Z \backslash F)$ separate the sets $\overline{A} \backslash F$ and $\overline{B} \backslash F$ then they are open in Z and separate the sets A, B. To finish the proof observe that every metrizable space Z is hereditarily normal because every subspace of Z is metrizable and every metrizable space is normal. Fact 1 is proved.

Fact 2. Given subsets A, B of a space Z, suppose that $f : Z \to Y$ is a continuous map, the space Y is hereditarily normal and the sets $f(A), f(B)$ are separated in Y. Then they are open-separated in Z.

Proof. Use Fact 1 to find $U, V \in \tau(Y)$ which separate the sets $f(A)$ and $f(B)$. Then the open sets $f^{-1}(U)$ and $f^{-1}(V)$ separate the sets A and B. Fact 2 is proved.

Fact 3. Assume that M_t is a second countable space for all $t \in T$ and D is a dense subspace of the product $M = \prod\{M_t : t \in T\}$. Suppose also that A, B are arbitrary subsets of D. Then A and B are open-separated in the space D if and only if there exists a countable $S \subset T$ such that $\pi_S(A)$ and $\pi_S(B)$ are separated in the space $M_S = \prod\{M_t : t \in S\}$.

Proof. Assume that the sets $\pi_S(A)$ and $\pi_S(B)$ are separated in the space M_S for some countable $S \subset T$. Since M_S is second countable and the map $\pi_S|D : D \to M_S$ is continuous, Fact 2 can be applied to conclude that A and B are open-separated in D so we proved sufficiency.

Now suppose that we have $U' \in \tau(A, D)$, $V' \in \tau(B, D)$ such that $U' \cap V' = \emptyset$. Fix $U, V \in \tau(M)$ such that $U \cap D = U'$ and $V \cap D = V'$. Observe first that $U \cap V = \emptyset$ for if not, then $U \cap V$ is a non-empty open set which has to intersect the dense set D, so $\emptyset \neq U \cap V \cap D = U' \cap V' = \emptyset$ which is a contradiction.

Recall that the family $\mathcal{B} = \{\prod_{t \in T} W_t : W_t \in \tau(M_t)$ for all $t \in T$, and the set $\{t \in T : W_t \neq M_t\}$ is finite$\}$ is a base for the space M (see 101). Given any set $W = \prod_{t \in M} W_t \in \mathcal{B}$, let $\text{supp}(W) = \{t \in T : W_t \neq M_t\}$. Choose any maximal disjoint families γ_U, γ_V of elements of $\mathcal{B}$ such that $O_U = \bigcup \gamma_U \subset U$ and $O_V = \bigcup \gamma V \subset V$. The set $S = \bigcup\{\text{supp}(W) : W \in \gamma_U \cup \gamma_V\}$ is countable; let us prove that $\pi_S(A)$ and $\pi_S(B)$ are separated in the space M_S. It is easy to see that any two disjoint open sets are separated; the sets $\pi_S(U)$ and $\pi_S(V)$ are open in M_S (Problem 107) and $\pi_S(A) \subset \pi_S(U)$, $\pi_S(B) \subset \pi_S(V)$ so it suffices to prove that $\pi_S(U) \cap \pi_S(V) = \emptyset$.

Since $\text{supp}(W) \subset S$ for any $W \in \gamma_U$, we have $\pi_S^{-1}\pi_S(U) = U$ for any $U \in \gamma_U$; this easily implies $\pi_S^{-1}\pi_S(O_U) = O_U$. Analogously, $\pi_S^{-1}\pi_S(O_V) = O_V$. Besides, O_U is dense in U and O_V is dense in V which implies that $\pi_S(O_U)$ is an open dense subset of $\pi_S(U)$ and $\pi_S(O_V)$ is an open dense subset of $\pi_S(V)$. This shows that, if we have $\pi_S(U) \cap \pi_S(V) \neq \emptyset$ then $\pi_S(O_U) \cap \pi_S(O_V) \neq \emptyset$ and therefore $\emptyset \neq \pi_S^{-1}(\pi_S(O_U)) \cap \pi_S^{-1}(\pi_S(V)) = O_U \cap O_V \subset U \cap V = \emptyset$ which is a contradiction. Fact 3 is proved.

Returning to our solution, recall that the map $\pi_Y : C_p(X) \to C_p(Y)$ is defined by $\pi_Y(f) = f|Y$ for every $f \in C_p(X)$ (see Problem 152). Given any $Z \subset X$, denote the set $\pi_Z(C_p(X))$ by $C(Z|X)$; if we have a finite $K \subset Z$, a function $f \in C(Z|X)$ and $\varepsilon > 0$, we let $O_Z(f, K, \varepsilon) = \{g \in C(Z|X) : |g(x) - f(x)| < \varepsilon$ for all $x \in K\}$. It is clear that the family $\mathcal{C}(f, Z) = \{O_Z(f, K, \varepsilon) : K$ is a finite subset of Z and $\varepsilon > 0\}$ is a local base at f in $C(Z|X)$.

Take any disjoint $A, B \subset C(Y|X)$ which are closed in $C(Y|X)$. The sets $A' = \pi_Y^{-1}(A)$ and $B' = \pi_Y^{-1}(B)$ are closed in $C_p(X)$ and disjoint so they are open-separated in $C_p(X)$ because $C_p(X)$ is normal. Since $C_p(X)$ is dense in $\mathbb{R}^X$, we can apply Fact 3 to find a countable $Z \subset X$ such that the sets $\pi_Z(A')$ and $\pi_Z(B')$ are separated in $C(Z|X)$. The set $T = Z \cap Y$ is countable; we claim that the sets $\pi_T(A')$ and $\pi_T(B')$ are separated in $C(T|X)$.

If $\pi_T(A')$ and $\pi_T(B')$ are not separated in the space $C(T|X)$ assume, without loss of generality, that $f \in \pi_T(A') \cap \mathrm{cl}_{C(T|X)}(\pi_T(B'))$. Choose any $f_1 \in A'$ with $\pi_T(f_1) = f$ and take any finite $K \subset Z$ and $\varepsilon > 0$. There exists $g \in \pi_T(B')$ with $|g(x) - f(x)| < \varepsilon$ for all $x \in K \cap T = K \cap Y$. Fix $g_1 \in B'$ with $\pi_T(g_1) = g$; since Y is a closed set, there exists $h_1 \in C_p(X)$ such that $h_1|Y \equiv 0$ and $h_1(x) = f_1(x) - g_1(x)$ for all $x \in K \setminus Y$. Then $\pi_Y(h_1 + g_1) = \pi_Y(g_1) \in \pi_Y(B') = B$ and therefore $g_2 = h_1 + g_1 \in B'$. It is immediate that $\pi_Z(g_2) \in O_Z(\pi_Z(f_1), K, \varepsilon)$; the set $K \subset Z$ and $\varepsilon > 0$ being arbitrary, we proved that $\pi_Z(f_1) \in \mathrm{cl}_{C(Z|X)}(\pi_Z(B')) \cap \pi_Z(A')$ which is a contradiction with the fact that $\pi_Z(A')$ and $\pi_Z(B')$ are separated in $C(Z|X)$.

The last contradiction shows that the sets $\pi_T(A')$ and $\pi_T(B')$ are separated in $C(T|X)$; let $\pi_T^Y : C(Y|X) \to C(T|X)$ be the restriction map, i.e., $\pi_T^Y(f) = f|T$ for any $f \in C(Y \mid X)$. It is easy to see that $\pi_T(A') = \pi_T^Y(A)$ and $\pi_T(B') = \pi_T^Y(B)$ and therefore π_T^Y maps $C(Y|X)$ continuously into a second countable space $C(T|X)$ in such a way that the sets $\pi_T^Y(A)$ and $\pi_T^Y(B)$ are separated in $C(T|X)$. Therefore, Fact 2 is applicable to convince ourselves that A and B are open-separated in the space $C(Y|X)$. Since we proved that any disjoint closed sets $A, B \subset C(Y|X)$ are open-separated in $C(Y|X)$, our solution is complete.

S.292. *Prove that every perfectly normal space is hereditarily normal but not vice versa. Show that, for any space X, if $C_p(X)$ is hereditarily normal then it is perfectly normal.*

Solution. It is easy to see that in a perfectly normal space X every open set $O \subset X$ is an F_σ-set, so Fact 1 of S.289 is applicable to conclude that O is normal. Now take an arbitrary $Y \subset X$; if A, B are closed disjoint subsets of Y then $F = \overline{A} \cap \overline{B} \subset X \setminus Y$ (the bar denotes the closure in X) and hence $\overline{A} \setminus F, \overline{B} \setminus F$ are disjoint closed subsets of the normal space $O = X \setminus F$. Pick any disjoint sets $U' \in \tau(\overline{A} \setminus F, O), V' \in \tau(\overline{B} \setminus F, O)$ and note that the sets $U = U' \cap Y, V = V' \cap Y$ are open in Y, disjoint and contain A and B, respectively. As a consequence, Y is normal.

The space $A(\omega_1)$ is an example of a hereditarily normal space which is not perfectly normal. Indeed, any subspace $Y \subset A(\omega_1)$ is metrizable if $a \notin Y$; if $a \in Y$

then Y is compact so any $Y \subset A(\omega_1)$ is normal (see Problems 218, 231 and 124). However $A(\omega_1)$ is not perfect because the point a is not a G_δ-set in $A(\omega_1)$ (an easy proof is left to the reader).

Fact 1. Any perfectly normal space is countably paracompact.

Proof. Take a perfectly normal space X and any decreasing sequence $F_0 \supset F_1 \supset \ldots$ of closed subsets of X satisfying $\bigcap_{i\in\omega} F_i = \emptyset$. By perfect normality of X we can find a family $\mathcal{U}_i = \{U^i_n : n \in \omega\} \subset \tau(X)$ with $\bigcap \mathcal{U}_i = F_i$ for each $i \in \omega$. Let $W_i = \bigcap \{U^k_m : k, m \leqslant i\}$ for all $i \in \omega$. Then $F_i \subset W_i \in \tau(F_i, X)$ for all $i \in \omega$ and $\bigcap\{W_i : i \in \omega\} = \bigcap\{\bigcap \mathcal{U}_i : i \in \omega\} = \bigcap\{F_i : i \in \omega\} = \emptyset$; this shows that our normal space X satisfies condition (b) of Fact 2 of S.288 and hence X is countably paracompact. Fact 1 is proved.

Fact 2. If X is a space such that $X \times A(\omega)$ is hereditarily normal, then X is perfectly normal.

Proof. In this proof, we identify the space $A(\omega)$ with the usual convergent sequence $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. The map $\pi : X \times S \to X$ is the natural projection. The space X is normal because it embeds in the hereditarily normal space $X \times S$ so it suffices to show that every closed $F \subset X$ is a G_δ-set. If this is not true for some closed $F \subset X$ then consider the sets $A = (X \backslash F) \times \{0\}$ and $B = F \times (S \backslash \{0\})$. Then A and B are closed disjoint subspaces of the space $Y = (X \times S) \backslash (F \times \{0\})$; since Y is normal, we can fix $U \in \tau(A, Y)$ and $V \in \tau(B, Y)$ with $U \cap V = \emptyset$. It is clear that $U, V \in \tau(X \times S)$ so the set $W_n = \pi\big(V \cap \big(F \times \{\frac{1}{n}\}\big)\big)$ is open in X and contains F for each $n \in \mathbb{N}$. The set F is assumed not to be G_δ so there is $x \in (X \backslash F) \cap (\bigcap\{W_n : n \in \mathbb{N}\})$. As a consequence, $P = \{x\} \times (S \backslash \{0\}) \subset V$ and therefore $(x, 0) \in \overline{P} \cap A \subset \overline{V} \cap U = \emptyset$ which is a contradiction. Fact 2 is proved.

Fact 3. If X is a perfectly normal space then $X \times M$ is perfectly normal for any second countable space M.

Proof. In the first paragraph of our solution we proved that any perfectly normal space is hereditarily normal. It is evident that the property of being perfect is hereditary, i.e., if Z is perfect then any $Y \subset Z$ is also perfect. As a consequence, any subspace of a perfectly normal space is a perfectly normal space.

Our next observation is that it suffices to prove Fact 3 for any compact second countable space M. Indeed, M embeds into compact space $\mathbb{I}^\omega$ (Problem 209); if we prove that $X \times \mathbb{I}^\omega$ is perfectly normal then $X \times M$ is also perfectly normal being a subspace of $X \times \mathbb{I}^\omega$. Thus, we assume from now on that M is a second countable compact space.

The space X is countably paracompact by Fact 1; this implies that $X \times M$ is normal (Problem 288) so we must only prove that $X \times M$ is perfect. Take an arbitrary $U \in \tau(X \times M)$; fix a base $\mathcal{B} = \{O_i : i \in \omega\}$ in the space M and let $U_n = \{x \in X :$ there is $W \in \tau(x, X)$ such that $W \times O_n \subset U\}$. It is clear that U_n is an open subset of X and $U_n \times O_n \subset U$ for all $n \in \omega$. Given a point $(x, y) \in U$ there is $W \in \tau(x, X)$ and $n \in \omega$ such that $y \in O_n$ and $W \times O_n \subset U$. This shows that $W \subset U_n$ and hence

$\bigcup\{U_n \times O_n : n \in \omega\} = U$. The spaces X and M being perfect, the sets U_n and O_n are F_σ-sets in X and M respectively for all $n \in \omega$. This easily implies that $U_n \times O_n$ is an F_σ-sets in $X \times M$ for all $n \in \omega$. Another evident fact is that any countable union of F_σ-sets is an F_σ-set so U is an F_σ-set in $X \times M$ and Fact 3 is proved.

To finish our solution, suppose that $C_p(X)$ is hereditarily normal. Take any $x \in X$ and let $Y = \{f \in C_p(X) : f(x) = 0\}$. Then $C_p(X)$ is homeomorphic to $Y \times \mathbb{R}$ (182); the space $Y \times A(\omega)$ embeds in $Y \times \mathbb{R}$ so $Y \times A(\omega)$ is hereditarily normal. This, together with Fact 2, implies that Y is perfectly normal. Therefore $C_p(X) = Y \times \mathbb{R}$ is perfectly normal by Fact 3 and our solution is complete.

S.293. *Give an example of a space which is normal but not collectionwise normal.*

Solution. Given a space X and a set $M \subset X$, let $\tau_M = \{U \cup A : U \in \tau(X)$ and $A \subset X \backslash M\}$. It is easy to check that τ_M is a topology on X; denote by X_M the space (X, τ_M).

Fact 1. The space X_M is Tychonoff for any Tychonoff space X; all points of $X \backslash M$ are isolated in X_M and the topology induced on M from X_M coincides with the topology induced from X to M.

Proof. If $x \in X \backslash M$ then $\{x\} = \{x\} \cup \emptyset$ is open in X_M, so x is an isolated point of X_M. Thus the Tychonoff property is trivially true at all points of $X \backslash M$. Another easy fact is that $\tau(x, X)$ is a local base at x in X_M for any $x \in M$. Since $\tau(X) \subset \tau_M$, every $f \in C(X)$ is also continuous on X_M. If $x \in M$, F is closed in X_M and $x \notin F$ then there is $U \in \tau(x, X)$ with $U \subset X \backslash F$; the Tychonoff property of X implies existence of $f \in C(X, [0, 1])$ with $f(x) = 1$ and $f|(X \backslash U) \equiv 0$. Note that $f|F \equiv 0$ and, by our above observation, f is also continuous on X_M so X_M is a Tychonoff space. Finally, if W is open in M considered to be a subspace of X_M then there is a set $U \in \tau(X)$ and $A \subset X \backslash M$ such that $W = (U \cup A) \cap M = U \cap M$ and hence W is open in M considered to be a subspace of X. The inverse implication is obvious so Fact 1 is proved.

Fact 2. Suppose that a subspace M of a space X has the following property:

(∗) If A and B are closed subsets the space of M with $A \cap B = \emptyset$ then there exist sets $U \in \tau(A, X)$, $V \in \tau(B, X)$ such that $U \cap V = \emptyset$.

Then X_M is a normal space.

Proof. Take any closed disjoint sets F, G of the space X_M; then the disjoint sets $A = F \cap M$ and $B = G \cap M$ are closed the space in M by Fact 1 so there exist disjoint $U, V \in \tau(X)$ such that $A \subset U$ and $B \subset V$. It is evident that $U' = (U \backslash G) \cup (F \backslash M)$ and $V' = (V \backslash F) \cup (G \backslash M)$ are open in X_M, disjoint and contain the sets F and G respectively. Fact 2 is proved.

Consider the compact space $K = \beta(D(\omega_1))$; if $\omega(K) = k$ then K embeds in $\mathbb{I}^\kappa$ (Problem 209). Let $X = \mathbb{I}^\kappa$ and $M = D(\omega_1) \subset K \subset X$. We claim that the property (∗) holds for the subspace M of the space X. Given two disjoint $A, B \subset M$ define a function $f : M \to [0, 1]$ as follows: $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in M \backslash A$. Since M is discrete, the function f is continuous so there exists a continuous function $f_1 : K \to [0, 1]$ with $f_1|M = f$ (Problem 257). Since the compact space $\mathbb{I}^\kappa$ is normal and K is closed in $\mathbb{I}^\kappa$, there exists a continuous function $f_2 : \mathbb{I}^\kappa \to [0, 1]$ such that $f_2|K = f_1$ (Problem 031). It is clear that the sets $U = f_2^{-1}\left([0, \frac{1}{2})\right)$ and $V = f_2^{-1}\left((\frac{1}{2}, 1]\right)$ are open in X, disjoint and contain A and B respectively so (∗) holds for M. Therefore X_M is a normal space by Fact 2.

Since all points of $X \backslash M$ are isolated (Fact 1), the subspace M is closed in X_M; since M is a discrete subspace of X, it is also discrete in X_M (Fact 1), so M is a closed discrete subspace of X_M. If X_M is collectionwise normal then there exists a disjoint family $\{U_x : x \in M\} \subset \tau(X_M)$ such that $x \in U_x$ for each $x \in M$. For any $x \in M$ we have $U_x = V_x \cup P_x$ where $V_x \in \tau(X)$ and $P_x \subset X \backslash M$. As a consequence, $x \in V_x \in \tau(X)$ for each $x \in M$ and hence $\{V_x : x \in M\}$ is an uncountable disjoint family of non-empty open subsets of X which is a contradiction with the fact that $c(X) = \omega$ (Problem 109). Thus X_M is a normal space which is not collectionwise normal so our solution is complete.

S.294. *(Reznichenko's theorem). Call a set $P \subset \mathbb{I}^A$ convex if $tf + (1 - t)g \in P$ for any $f, g \in P$ and $t \in [0, 1]$. Let D be a dense convex subset of $\mathbb{I}^A$. Prove that, if D is normal then* $\text{ext}(D) = \omega$. *Deduce from this fact that any normal convex dense $D \subset \mathbb{I}^A$ is collectionwise normal.*

Solution. If, for some set B, we are given $H = \{h_\alpha : \alpha < \omega_1\} \subset \mathbb{I}^B$ and a number $t \in (0, 1)$, let $H(t) = \{th_\alpha + (1 - t)h_\beta : \alpha, \beta < \omega_1, \alpha \neq \beta\}$. If $P \subset \mathbb{I}^B$, K is a finite subset of B and $\varepsilon > 0$ then $O_P(f, K, \varepsilon) = \{g \in P : |g(x) - f(x)| < \varepsilon \text{ for all } x \in K\}$. It is clear that the family $\{O_P(f, K, \varepsilon) : K \text{ is a finite subset of } B \text{ and } \varepsilon > 0\}$ is a local base at f in the space P. Given any $C \subset B$ the map $\pi_C : \mathbb{I}^B \to \mathbb{I}^C$ is the natural projection defined by the formula $\pi_C(f) = f|C$ for any $f \in \mathbb{I}^B$.

Fact 1. Let P be a convex subset of $\mathbb{I}^A$ for some A. Assume that we have a set $H = \{h_\alpha : \alpha < \omega_1\} \subset P$ such that $\overline{H} \cap \overline{H(t)} = \emptyset$ for some $t \in (0, 1)$ (the bar denotes the closure in P). Then H is closed and discrete in P and $h_\alpha \neq h_\beta$ for all $\alpha, \beta < \omega_1$ with $\alpha \neq \beta$.

Proof. If $h_\alpha = h_\beta$ for some distinct $\alpha, \beta < \omega_1$ then $h_\alpha = th_\alpha + (1 - t)h_\beta \in H \cap H(t)$, a contradiction. Given $f \in P$ it is easy to check that $O_P(f, K, \varepsilon)$ is a convex set for any finite $K \subset A$ and $\varepsilon > 0$. Assume that f is an accumulation point of H; given an arbitrary $U \in \tau(f, P)$ there is a finite $K \subset A$ and $\varepsilon > 0$ such that $O_P(f, K, \varepsilon) \subset U$. Since f is an accumulation point of H, there exist distinct ordinals $\alpha, \beta < \omega_1$ with $h_\alpha, h_\beta \in O_P(f, K, \varepsilon)$; the set $O_P(f, K, \varepsilon)$ being convex, we have the equality $h = th_\alpha + (1 - t)h_\beta \in O_P(f, K, \varepsilon) \cap H(t)$ which shows that $f \in \overline{H} \cap \overline{H(t)}$, a contradiction. Thus, we proved that H has no accumulation points in P, i.e., H is closed and discrete in P so Fact 1 is proved.

Fact 2. Let P be a dense convex subset of $\mathbb{I}^A$ for some A; suppose that F is an uncountable discrete subspace of P. Then there exist $H = \{h_\alpha : \alpha < \omega_1\} \subset F$ and $t \in (0, 1)$ such that $H \cap \overline{H(t)} = \emptyset$ (the bar denotes the closure in P).

Proof. For each $h \in F$ choose a finite set $K_h \subset A$ and a number $\varepsilon_h \in (0, 1)$ such that $O_P(h, K_h, \varepsilon_h) \cap F = \{h\}$. It is possible to choose $H = \{h_\alpha : \alpha < \omega_1\} \subset F$ so that, for each $\alpha < \omega_1$, we have $\varepsilon_{h_\alpha} \geqslant \varepsilon$ where $\varepsilon > 0$ does not depend on α. We claim that

$t = \varepsilon/3$ is as promised, i.e., $H \cap \overline{H(t)} = \emptyset$. It suffices to show that the set $W_\alpha = O_P(h_\alpha, K_{h_\alpha}, \varepsilon^2/3)$ does not intersect $H(t)$ for all $\alpha < \omega_1$.

Take any distinct $\beta, \gamma < \omega_1$; if $\alpha \neq \gamma$ then $f_\gamma \notin O_P(h_\alpha, K_{h_\alpha}, \varepsilon)$ and hence there is $x \in K_{h_\alpha}$ such that $|h_\alpha(x) - h_\gamma(x)| \geqslant \varepsilon$. We have

$$\begin{aligned} |th_\beta(x) + (1-t)h_\gamma(x) - h_\alpha(x)| &= |t(h_\beta(x) - h_\gamma(x)) + (h_\gamma(x) - h_\alpha(x)| \\ &\geqslant |h_\alpha(x) - h_\gamma(x)| - t|h_\beta(x) - h_\gamma(x)| \\ &\geqslant \varepsilon - (\varepsilon/3)(|h_\beta(x)| + |h_\gamma(x)|) \\ &\geqslant \varepsilon - 2 \cdot (\varepsilon/3) = \varepsilon/3 > \varepsilon^2/3, \end{aligned}$$

which implies that $th_\beta + (1-t)h_\gamma \notin W_\alpha$. Now assume that $\gamma = \alpha$; then $\beta \neq \alpha$ and $h_\beta \notin \mathrm{O}_P(h_\alpha, K_{h_\alpha}, \varepsilon)$. Thus, there is $y \in K_{h_\alpha}$ for which $|h_\alpha(y) - h_\beta(y)| \geqslant \varepsilon$. We have

$$\begin{aligned} |th_\beta(y) + (1-t)h_\gamma(y) - h_\alpha(y)| &= |th_\beta(y) + (1-t)h_\alpha(y) - h_\alpha(y)| \\ &= |t(h_\beta(y) - h_\alpha(y))| \geqslant t \cdot \varepsilon = \varepsilon^2/3, \end{aligned}$$

and therefore $th_\beta + (1-t)h_\gamma \notin W_\alpha$. Fact 2 is proved.

Fact 3. Every normal space of countable extent is collectionwise normal.

Proof. Observe first that if $\mathcal{F}$ is a discrete family of closed subsets of an arbitrary normal space X then, for every $F \in \mathcal{F}$ there exists $U(F) \in \tau(F, X)$ such that the family $\mathcal{F}' = \{\overline{U(F)}\} \cup (\mathcal{F} \backslash \{F\})$ is also discrete. To see that this is true, use normality of the space X to find $U(F) \in \tau(F, X)$ such that $\overline{U(F)} \cap G = \emptyset$ where $G = \overline{\bigcup (\mathcal{F} \backslash \{F\})}$. Given any $x \in X$, fix $W \in \tau(x, X)$ which meets at most one element of $\mathcal{F}$. If $x \in \overline{U(F)}$ then $x \notin G$ so $W \backslash G$ is an open neighbourhood of x which can intersect no element of $\mathcal{F}'$ other than $\overline{U(F)}$. If $x \notin \overline{U(F)}$ then $W \backslash \overline{U(F)}$ is an open neighbourhood of x which intersects at most one element of $\mathcal{F}'$ and hence $\mathcal{F}'$ is discrete.

Now assume that X is a normal space with $\mathrm{ext}(X) = \omega$ and take a discrete family $\mathcal{F}$ of non-empty closed subsets of X. Choosing a point $x_F \in F$ for each $F \in \mathcal{F}$, we obtain a closed discrete subset $\{x_F : F \in \mathcal{F}\}$ of the space X. Since $\mathrm{ext}(X) = \omega$, the family $\mathcal{F}$ must be countable so let $\mathcal{F} = \{F_n : n \in \omega\}$. By the observation in the first paragraph there exists $U_0 \in \tau(F_0, X)$ such that the family $\mathcal{F}_0 = \{\overline{U}_0, F_1, F_2, \ldots\}$ is discrete. Assume that we have $U_i \in \tau(F_i, X)$ for each $i \leqslant n$ such that the family $\mathcal{F}_n = \{\overline{U}_0, \ldots, \overline{U}_n, F_{n+1}, F_{n+2}, \ldots\}$ is discrete. By the observation in the first paragraph there exists $U_{n+1} \in \tau(F_{n+1}, X)$ such that the family $\mathcal{F}_0 = \{\overline{U}_0, \ldots, \overline{U}_{n+1}, F_{n+2}, \ldots\}$ is discrete so our inductive process can be carried out giving us a disjoint family $\{U_n : n \in \omega\} \subset \tau(X)$ such that $F_n \subset U_n$ for any $n \in \omega$. If $F = \bigcup \mathcal{F}$ and $U = \bigcup \{U_n : n \in \omega\}$ then $U \in \tau(F, X)$ and hence there exists $W \in \tau(F, X)$ such that $F \subset W \subset \overline{W} \subset U$. We leave to the reader the trivial verification that the family $\{U_n \cap W : n \in \omega\}$ is discrete and $F_n \subset U_n \cap W$ for each $n \in \omega$ (see S.231 for the proof of an analogous fact). Fact 3 is proved.

Returning to our solution assume that F is an uncountable closed discrete subspace of D. Apply Fact 2 to find a set $H' = \{h'_\alpha : \alpha < \omega_1\} \subset F$ and $t \in (0, 1)$ such that $H' \cap \overline{H'(t)} = \emptyset$ (the bar denotes the closure in D). Since D is normal, there exist $U \in \tau(H', D)$ and $V \in \tau(\overline{H'(t)}, D)$ such that $U \cap V = \emptyset$. Apply Fact 3 of S.291 to find a countable $B \subset A$ such that the sets $P = \pi_B(H')$ and $Q = \pi_B(\overline{H'(t)})$ are separated in $I(B) = \mathbb{I}^B$, i.e., $\mathrm{cl}_{I(B)}(P) \cap Q = \emptyset = \mathrm{cl}_{I(B)}(Q) \cap P$. Every second countable space is perfectly normal (this is an easy exercise), so the closed set $R = \mathrm{cl}_{I(B)}(Q)$ is a G_δ-set in the second countable space $I(B)$. If we fix a sequence $\mathcal{O} = \{O_n : n \in \omega\} \subset \tau(I(B))$ such that $\bigcap \mathcal{O} = R$, then we have $\pi_B(H') \subset \bigcup \{I(B) \backslash O_n : n \in \omega\}$. Therefore, we can find an uncountable $H \subset H'$ such that $\pi_B(H) \cap O_n = \emptyset$ for some $n \in \omega$; it is immediate that $\mathrm{cl}_{I(B)}(\pi_B(H)) \cap O_n = \emptyset$. Choose some enumeration $\{h_\alpha : \alpha < \omega_1\}$ of the set H; since $\mathrm{cl}_{I(B)}(Q) \subset O_n$ and $H(t) \subset H'(t)$, we have $\mathrm{cl}_{I(B)}(\pi_B(H)) \cap \mathrm{cl}_{I(B)}(\pi_B(H(t))) = \emptyset$. If $f_\alpha = \pi_B(h_\alpha)$ for each $\alpha < \omega_1$ and $G = \{f_\alpha : \alpha < \omega_1\}$ then $G(t) = \pi_B(H(t))$ and therefore $\mathrm{cl}_{I(B)}(G) \cap \mathrm{cl}_{I(B)}(G(t)) = \emptyset$. Thus Fact 1 can be applied to conclude that G is an uncountable closed and discrete subspace of $I(B)$ which is a contradiction with the fact that $I(B)$ is second countable. This contradiction shows that we proved that $\mathrm{ext}(D) = \omega$ so we can apply Fact 3 to see that D is collectionwise normal and finish our solution.

S.295. *Prove that, if $C_p(X)$ is normal then* $\mathrm{ext}(C_p(X)) = \omega$. *Deduce from this fact that, if $C_p(X)$ is normal then it is collectionwise normal.*

Solution. The following statement is very easy but we formulate it as a Fact for further applications.

Fact 1. For any $a, b \in \mathbb{R}$ with $a < b$ and any space Z, the space $C_p(Z, (a, b))$ is homeomorphic to $C_p(Z)$.

Proof. Let $w : \mathbb{R} \to (a, b)$ be a homeomorphism (Problem 025). Given $f \in C_p(Z)$, let $\varphi(f) = w \circ f$; then the map $\varphi : C_p(Z) \to C_p(Z, (a, b))$ is continuous as well as its inverse defined by the formula $\varphi^{-1}(g) = w^{-1} \circ g$ for any $g \in C_p(Z, (a, b))$ (Problem 091). This shows that the map φ is a homeomorphism so Fact 1 is proved.

By Fact 1, the space $C_p(X)$ is homeomorphic to $C_p(X, (-1, 1))$ which is a dense convex (check it, please!) subset of $C_p(X, \mathbb{I})$ (see Problem 089). Since $C_p(X, \mathbb{I})$ is dense in $\mathbb{I}^X$ (this is an easy consequence of Problem 034), the space $C_p(X)$ is homeomorphic to a convex dense subset of $\mathbb{I}^X$. The space $C_p(X)$ being normal, we can apply Reznichenko's theorem (Problem 294) to see that $\mathrm{ext}(C_p(X)) = \omega$. Finally, apply Fact 3 of S.294 to conclude that $C_p(X)$ is collectionwise normal.

S.296. *Prove that, if $C_p(X, \mathbb{I})$ is normal then* $\mathrm{ext}(C_p(X, \mathbb{I})) = \omega$. *Deduce from this fact that, if $C_p(X, \mathbb{I})$ is normal then it is collectionwise normal.*

Solution. The space $C_p(X, \mathbb{I})$ is dense in $\mathbb{I}^X$ (this is an easy consequence of problem 034) and it is straightforward that it is also convex. Thus we can apply

Reznichenko's theorem (problem 294) to convince ourselves that $\text{ext}(C_p(X, \mathbb{I})) = \omega$. Finally, apply Fact 3 of S.294 to conclude that $C_p(X, \mathbb{I})$ is collectionwise normal.

S.297. *Give an example of a space X, for which* $\text{ext}(C_p(X)) = \omega$, *while* $C_p(X)$ *is not normal.*

Solution. The expression $Z \simeq T$ says that the spaces Z and T are homeomorphic. Give the set ω_1 the discrete topology and take any point $p \notin \omega_1$. Let $L = \omega_1 \cup \{p\}$ and $\tau(L) = \{A : A \subset \omega_1\} \cup \{B : p \in B \text{ and } L \setminus B \text{ is countable}\}$. It is easy to see that $\tau(L)$ is indeed a topology on L and that the unique non-isolated point of L is p. Observe that any G_δ-set H of L is open in L; this is evident if $H \subset \omega_1$. If $p \in H$ and $H = \bigcap_{n\in\omega} U_n$ where each U_n is open in L, then $L \setminus U_n$ is countable for each n and hence $L \setminus H = \bigcup\{L \setminus U_n : n \in \omega\}$ is countable, i.e., $H \in \tau(L)$. As a consequence, if $f \in C(L)$ then there exists $\beta < \omega_1$ such that $f(\alpha) = f(p)$ for each $\alpha \geqslant \beta$.

Let $a = 0$ and $a_n = 1/n$ for all $n \in \mathbb{N}$. Then $S = \{a\} \cup \{a_n : n \in \mathbb{N}\}$ is a convergent sequence with limit a. The space $Y = L \times S$ has the following properties:

(1) $W \times S$ is a clopen subspace of Y for each countable $W \subset \omega_1$.
(2) Given $f \in C(Y)$ there exists $\beta < \omega_1$ such that $f(\alpha, s) = f(p, s)$ for any $\alpha \geqslant \beta$ and $s \in S$.

The set $W \times S$ is open because W is open in L. Besides, $Y\setminus(W \times S) = (L \setminus W) \times S$ is also open in Y because $L \setminus W$ is open in L; this proves (1).

Given $s \in S$, the space $L \times \{s\}$ is homeomorphic to L so there is $\beta_s < \omega_1$ such that $f(\alpha, s) = f(p, s)$ for each $\alpha \geqslant \beta_s$. If $\beta = \sup\{\beta_s : s \in S\}$ then $\beta < \omega_1$ is as promised so (2) is also proved.

Consider the spaces $\Sigma = \{f \in C_p(Y) : f(p, s) = 0 \text{ for all } s \in S\}$ and $M = L \setminus \omega$; for $Z = M \times S$, let $\Sigma_Z = \{f \in C_p(Z) : f(p, s) = 0 \text{ for all } s \in S\}$. Let us prove that we have

(3) $\Sigma \simeq \Sigma_Z \times (C_p(S))^\omega$.

Observe first that $\{n\} \times S$ is clopen in $\omega \times S$ for each $n < \omega$ and therefore $\omega \times S = \bigoplus \{\{n\} \times S : n \in \omega\}$ (Problem 113). We have

(*) $C_p(\omega \times S) = C_p(\bigoplus\{\{n\} \times S : n \in \omega\}) = \prod\{C_p(\{n\} \times S) : n \in \omega\} = (C_p(S))^\omega$ by Problem 114 and the fact that $\{n\} \times S$ is homeomorphic to S for all $n \in \omega$. The property (1) implies $Y = (\omega \times S) \bigoplus Z$; define a function $\varphi : \Sigma \to C_p(\omega \times S) \times \Sigma_Z$ by the formula $\varphi(f) = (f|(\omega \times S), f|Z)$ for any $f \in \Sigma$. Since both restrictions are continuous maps (Problem 152), the map φ is also continuous. If $(g, h) \in C_p(\omega \times S) \times \Sigma_Z$ then define a map $f \in C_p(Y)$ by $f(t) = g(t)$ for all $t \in \omega \times S$ and $f(t) = h(t)$ for all $t \in Z$. It is straightforward that $f \in \Sigma$ and $\varphi(f) = (g, h)$, so we proved that φ is an onto map. If $f_1 \neq f_2$ then $f_1|(\omega \times S) \neq f_2|(\omega \times S)$ or $f_1|Z \neq f_2|Z$; this proves injectivity of φ.

To see that $\varphi^{-1} : C_p(\omega \times S) \times \Sigma_Z \to \Sigma$ is continuous, recall that φ^{-1} is a map into the product $\mathbb{R}^Y$ so it suffices to prove that $\pi_y \circ \varphi^{-1}$ is continuous for all $y \in Y$, where the projection $\pi_y : \mathbb{R}^Y \to \mathbb{R}$ is defined by $\pi_y(f) = f(y)$ for all $f \in \mathbb{R}^Y$. Let $p : C_p(\omega \times S) \times \Sigma_Z \to C_p(\omega \times S)$ and $q : C_p(\omega \times S) \times \Sigma_Z \to \Sigma_Z$ be the natural projections. Given $y \in \omega \times S$, the map $\pi_y^0 : \mathbb{R}^{\omega\times S} \to \mathbb{R}$ is the natural projection onto the yth factor; it is clear that $\pi_y^0(f) = f(y)$ for any $f \in \mathbb{R}^{\omega \times S}$. If $y \in Z$, then

$\pi_y^1 : \mathbb{R}^Z \to \mathbb{R}$ is also the natural projection onto the yth factor; it is also defined by the formula $\pi_y^1(f) = f(y)$ for any $f \in \mathbb{R}^Z$.

Now it is easy to prove that $\pi_y \circ \varphi^{-1}$ is continuous for any $y \in Y$: if $y \in \omega \times S$ then $\pi_y \circ \varphi^{-1} = \pi_y^0 \circ p$ or else if $y \in Z$ then $\pi_y \circ \varphi^{-1} = \pi_y^1 \circ q$ so in both cases $\pi_y \circ \varphi^{-1}$ is a continuous map and this proves that φ is a homeomorphism.

Finally, observe that we established that $\Sigma \simeq C_p(\omega \times S) \times \Sigma_Z \simeq \left(C_p(S)\right)^\omega \times \Sigma_Z$ by $(*)$ and, since the product is a commutative operation, the property (3) is proved.

Another important fact is the property

(4) $C_p(Y) \simeq \Sigma \times C_p(S)$.

Let $r : Y \to S$ be the natural projection. Define $r^* : C_p(S) \to C_p(Y)$ by the formula $r^*(f) = f \circ r$ for all $f \in C_p(S)$. The map r^* is continuous (Problem 163). Now if $f \in C_p(Y)$ then let $u(f)(s) = f(p, s)$ for all $s \in S$; it is evident that the map $u : C_p(Y) \to C_p(S)$ is continuous. Thus, if we define $\delta : C_p(Y) \to \Sigma \times C_p(S)$ by the formula $\delta(f) = (f - r^*(u(f)), u(f))$ for all $f \in C_p(Y)$ then δ is a continuous map. It is straightforward that $\delta^{-1}(h, g) = h + r^*(g)$ for each $(g, h) \in \Sigma \times C_p(S)$ so δ^{-1} is also continuous, i.e., δ is a homeomorphism and (4) is proved.

It is easy to deduce from the properties (3) and (4) that we have

(5) $C_p(Y) \simeq \Sigma$.

Indeed, $C_p(Y) \simeq \Sigma \times C_p(S) \simeq (\Sigma_Z \times (C_p(S))^\omega) \times C_p(S) \simeq \Sigma_Z \times (C_p(S))^\omega \simeq \Sigma$.

Our next step is to establish that $\text{ext}(\Sigma) = \omega$. Given $\alpha < \omega_1$, we define a mapping $r_\alpha : \Sigma \to \Sigma$ as follows: for any $x \in \Sigma$ and any $(\beta, s) \in \omega_1 \times S$, let $r_\alpha(f)(\beta, s) = f(\beta, s)$ if $\beta < \alpha$ and $r_\alpha(f)(\beta, s) = 0$ if $\beta \geqslant \alpha$; of course, $r_\alpha(f)(p, s) = 0$ for all $s \in S$. It is easy to see that the space $\Sigma_\alpha = r_\alpha(\Sigma)$ is second countable. Let $Y_\alpha = \alpha \times S \subset Y$ for each ordinal $\alpha < \omega_1$. Given a function $f \in \Sigma$, a finite set $K \subset Y$ and $\varepsilon > 0$, let $O(f, K, \varepsilon) = \{g \in \Sigma : |g(x) - f(x)| < \varepsilon \text{ for all } x \in K\}$. The family $\{O(f, K, \varepsilon) : K$ is a finite subset of Y and $\varepsilon > 0\}$ is a local base of the space Σ at the point f. For any $f \in \Sigma$, let $\text{supp}(f) = \{\alpha < \omega_1 : f(\alpha, s) \neq 0 \text{ for some } s \in S\}$. The set $\text{supp}(f)$ is countable for each $f \in \Sigma$ by (2).

Assume that F is a closed subset of Σ. Then

(6) For any $\lambda < \omega_1$ there is $\alpha \geqslant \lambda$, $\alpha < \omega_1$ such that $r_\alpha(F) \subset F$.

For $\alpha_0 = \lambda$ the space $r_{\alpha 0}(F) \subset \Sigma_{\alpha 0}$ is second countable; therefore there exists a countable $P_0 \subset F$ such that $r_{\alpha 0}(P_0)$ is dense in $r_{\alpha 0}(F)$. Suppose that we have defined ordinals $\alpha_0 \leqslant \cdots \leqslant \alpha_n < \omega_1$ and countable sets $P_0 \subset \cdots \subset P_n \subset F$; since the set P_n is countable, $\alpha_{n+1} = \max\{\alpha_n, \sup\{\sup(\text{supp}(f)) : f \in P_n\}\} < \omega_1$. Since $r_{\alpha_{n+1}}(F) \subset \Sigma_{\alpha_{n+1}}$ is second countable, there is countable $P_{n+1} \subset F$ such that $P_n \subset P_{n+1}$ and the set $\pi_{\alpha_{n+1}}(P_{n+1})$ is dense in $\pi_{\alpha_{n+1}}(F)$. The inductive construction having been carried out, we have the sequences $\{\alpha_n : n \in \omega\}$ and $\{P_n : n \in \omega\}$; let $\alpha = \sup\{\alpha_n : n \in \omega\}$, $P = \bigcup\{P_n : n \in \omega\}$ and $F' = F \cap \Sigma_\alpha$.

Note that $P \subset F'$ because $\text{supp}(f) \subset \alpha$ for any $f \in P$. It suffices to show that $r_\alpha(F) \subset F'$ so take any $f \in F$, any finite $K \subset Y$ and $\varepsilon > 0$. There is $n \in \omega$ such that $K' = K \cap Y_\alpha = K \cap Y_{\alpha_n}$; since $f' = r_{\alpha_n}(f) \in r_{\alpha_n}(F)$ and $r_{\alpha_n}(P_n)$ is dense in $r_{\alpha_n}(F)$, there is $g \in P_n$ such that $r_{\alpha_n}(g) \in O(f', K', \varepsilon)$. Observe that $g \in \Sigma_{\alpha_{n+1}} \subset \Sigma_\alpha$ so $r_\alpha(f)(x) = 0 = g(x)$ for any $x \in K \setminus K' \subset Y \setminus Y_\alpha$ and hence $g \in O(f, K, \varepsilon)$ which proves that $f \in \overline{P} \subset F'$ (the closure is taken in Σ and the last inclusion is true because F'

is closed in Σ). The function f being taken arbitrarily, we have $r_\alpha(F) \subset F' \subset F$ so (6) is proved.

Assume that D is a closed discrete subset of Σ. Then

(7) For any $\alpha < \omega_1$ the set $r_\alpha(D)$ is countable.

Let $D_\alpha = r_\alpha(D)$ for each $\alpha < \omega_1$. It is immediate that $r_\alpha(D_\beta) = D_\alpha$ for every $\beta \geqslant \alpha$. If D_α is uncountable for some $\alpha < \omega_1$ then apply (6) to find $\beta \geqslant \alpha$ such that $D_\beta \subset D \cap \Sigma_\beta$. The space Σ_β being second countable, we have $\text{ext}(\Sigma_\beta) = \omega$ and therefore $D \cap \Sigma_\beta$ is countable. Thus, the set $D_\beta \subset D \cap \Sigma_\beta$ is also countable and so is $D_\alpha = r_\alpha(D_\beta)$ which is a contradiction. Thus, D_α is countable for any $\alpha < \omega_1$ and (7) is proved.

Now suppose that E is an uncountable closed discrete subset of the space Σ. If $E_\alpha = r_\alpha(E)$ for all $\alpha < \omega_1$ then all sets E_α are countable by (7). Find $\beta_0 < \omega_1$ such that $E_{\beta 0} \subset E$; since E is uncountable, there is $h_0 \in E_{\beta 0} \cap E$ such that the set $Q_0 = \left(r_{\beta_0}^{-1}(h_0) \cap E\right) \backslash \{h_0\}$ is uncountable. Assume that we have chosen $h_0, \ldots, h_n \in E$, $\beta_0 < \cdots < \beta_n < \omega_1$ and $Q_0 \supset \cdots \supset Q_n$ with the following properties:

(a) The functions $h_0, \ldots, h_n$ are distinct and $\text{supp}(h_i) \subset \beta_i$ for all $i \leqslant n$.
(b) $r_{\beta_i}(h_j) = h_i$ for any $i, j \leqslant n$ with $i \leqslant j$.
(c) The set $Q_i = \left(r_{\beta_i}^{-1}(h_i) \cap E\right) \backslash \{h_0, \ldots, h_i\}$ is uncountable for each $i \leqslant n$.

Since $Q_n = \left(r_{\beta_n}^{-1}(h_n) \cap E\right) \backslash \{h_0, \ldots, h_n\}$ is an uncountable closed and discrete subset of the space Σ, we can apply (6) once again to find $\beta_{n+1} > \beta_n$ such that $r_{\beta_{n+1}}(Q_n) \subset Q_n$. The set $r_{\beta_{n+1}}(Q_n)$ being countable by the property (7), there exists $h_{n+1} \in r_{\beta n+1}(Q_n)$ such that $Q_{n+1} = \left(r_{\beta_{n+1}}^{-1}(h_{n+1}) \cap Q_n\right) \backslash \{h_0, \ldots, h_{n+1}\}$ is uncountable; since $h_{n+1} \in Q_n \cap \Sigma_{\beta_{n+1}}$, the function h_{n+1} is distinct from the previously chosen h_i's so properties (a) and (b) hold for all $i \leqslant n + 1$. Thus the inductive process goes on so and we can obtain the sequences $\{h_i : i \in \omega\}$ and $\{\beta_i : i \in \omega\}$ with the properties (a) and (b). Letting $\beta = \sup\{\beta_i : i \in \omega\}$ and applying (2) we can define $h \in \Sigma_\beta$ as follows: $h(x) = 0$ for all $x \in Y \backslash Y_\beta$; furthermore $h(x) = h_n(x)$ if $x \in Y_{\beta_n}$ for some $n \in \omega$; the property (2) makes this definition consistent. It is straightforward that the non-trivial sequence $\{h_n : n \in \omega\} \subset E$ converges to h which is a contradiction with the fact that E is closed and discrete. Thus, we finally proved that $\text{ext}(\Sigma) = \omega$ and hence $\text{ext}(C_p(Y)) = \omega$ by (5).

Now consider the space $X = Y \backslash \{p, a)\}$; we claim that X is C-embedded in Y. To prove this, take any $f \in C(X)$; then f is continuous on the subspace $L \times \{a_n\} \subset X$ for any $n \in \omega$. As a consequence there is $\beta_n < \omega_1$ such that $f(\alpha, a_n) = f(p, a_n)$ for all $\alpha \geqslant \beta_n$. If $\beta = \sup\{\beta_n : n \in \omega\}$ then $f(\alpha, s) = f(p, s)$ for any $s \in S \backslash \{a\}$ and any $\alpha \geqslant \beta$. Since f is continuous at any (α, a), we must have $f(\alpha, a) = \lim f(\alpha, a_n) = \lim f(p, a_n) = t$ for each $\alpha \geqslant \beta$. This shows that letting $f(p, a) = t$, we obtain a continuous extension of the function f to the space Y. As a consequence $\pi_X(C_p(Y)) = C_p(X)$ (see Problem 152) and hence $C_p(X)$ is a continuous image of the space $C_p(Y)$; this easily implies that $\text{ext}(C_p(X)) = \omega$.

To finish our solution, it suffices to rule out normality of $C_p(X)$. Suppose for a moment that $C_p(X)$ is normal. The set $F = \omega_1 \times \{a\}$ is a closed subset of X so

$C_F = \pi_F(C_p(X))$ has to be normal by 291. It is easy to see that $C_F = \{f \in \mathbb{R}^F :$ there is $\beta < \omega_1$ such that $f(\alpha, a) = f(\beta, a)$ for all $\alpha \geqslant \beta\}$.

Consider the sets $A = \{f \in C_F : f(F) \subset \mathbb{N}$ and $f^{-1}(i)$ has at most one element for each $i \in \mathbb{N}, i \neq 1\}$ and $B = \{f \in C_F : f(F) \subset \mathbb{N}$ and $f^{-1}(i)$ has at most one element for each $i \in \mathbb{N}, i \neq 2\}$. The sets A and B lie in $\mathbb{N}^F$ which is closed in $\mathbb{R}^F$, they are also closed in $\mathbb{N}^F$: this was established in Fact 2 of S.215 for $F = \omega_1$; the same proof can be applied if we identify (α, a) with α for all $\alpha < \omega_1$. Thus the sets A and B are closed in C_F; they are also disjoint: see the proof of Fact 2 of S.215 again to be convinced that this is true. Let us show that A and B cannot be separated by disjoint open sets in C_F, i.e., there exist no $U, V \in \tau(C_F)$ such that $F \subset U, G \subset V$ and $U \cap V = \emptyset$. Assume, on the contrary, that such U and V exist. Since C_F is a dense subspace of $\mathbb{R}^F$, we can apply Fact 3 of S.291 to conclude that there exists a countable $P = \{p_n : n \in \mathbb{N}\} \subset F$ such that $\pi_P(A) \cap \pi_P(B) = \emptyset$. Now let $f(p_n) = n$ for each $n \in \mathbb{N}$; define $h \in C_F$ as follows: $h|P = f$ and $h(x) = 1$ for all $x \in F \setminus P$. It is clear that $h \in A$ and $\pi_P(h) = f$. Now we can define a function $g \in C_F$ as follows: $g \setminus P = f$ and $g(x) = 2$ for all $x \in F \setminus P$. It is immediate that $g \in B$ and $\pi_P(g) = f$. As a consequence, $f \in \pi_P(A) \cap \pi_P(B)$; this contradiction shows that $C_p(X)$ cannot be normal so our solution is complete.

S.298. *Suppose that L is a subspace of a product $X = \prod\{X_t : t \in T\}$ and $l(L) \leqslant k$. Prove that, for any second countable space Y and any continuous map $f : L \to Y$, there exists $S \subset T$ with $|S| \leqslant k$ and a continuous map $h : p_S(L) \to Y$ such that $f = h \circ p_S$. Here $p_S : X \to X_S = \prod_{t \in S} X_t$ is the natural projection defined by $p_S(x) = x|S$ for any $x \in X$.*

Solution. Given an arbitrary product $Z = \prod\{Z_a : a \in A\}$, recall that a set $U = \prod_{a \in A} U_a$ is called *standard in Z* if $U_a \in \tau(Z_a)$ for all $a \in A$ and the set supp $(U) = \{a \in A : U_a \neq Z_a\}$ is finite. Standard sets form a base in Z (Problem 101). If we have a set $B \subset A$, denote by $q_B : Z \to Z_B = \prod\{Z_a : a \in B\}$ the projection defined by $q_B(z) = z|B$ for any $z \in Z$.

Fact 1. Let M be an arbitrary subset of a product $Z = \prod\{Z_a : a \in A\}$. Assume that U is a standard set of Z and $V = U \cap M$. If $B \subset A$ and $\text{supp}(U) \subset B$ then $q_B(V)$ is an open subset of $q_B(M)$ and $q_B(z) \in q_B(V)$ implies $z \in V$ for any $z \in M$.

Proof. Since U is standard in Z, we have $U = \prod_{a \in A} U_a$; it follows from supp(U) $\subset B$ that $U_a = Z_a$ for all $a \in A \setminus B$. The set $W = \prod_{a \in B} U_a$ is open in the space Z_B; let us prove that $W \cap q_B(M) = q_B(V)$. If $y \in W \cap q_B(M)$ then $y(a) \in U_a$ for all $a \in B$; if $z \in M$ and $q_B(z) = y$ then $z(a) \in U_a$ for all $a \in A$ because $z(a) = y(a) \in U_a$ if $a \in B$ and, if $a \in A \setminus B$ then $U_a = Z_a$ so $z(a) \in U_a$ anyway. As a consequence $z \in U \cap M = V$ whence $y = q_B(z) \in q_B(V)$; this shows that $W \cap q_B(M) \subset q_B(V)$. On the other hand, if $y \in q_B(V)$ then there is $z \in V$ with $q_B(z) = y$. Now, $V \subset U$ implies $z(a) \in U_a$ for all $a \in A$ and therefore $y(a) = q_B(z)(a) = (z|B)(a) = z(a) \in U_a$ for all $a \in B$ so $y \in W \cap q_B(M)$. This proves that $q_B(V) \subset W \cap q_B(M)$ so $W \cap q_B(M) = q_B(V)$ and hence the set $q_B(V)$ is open in $q_B(M)$.

Now assume that $y = q_B(z) \in q_B(V)$. Then $y \in W$ and hence $z(a) = y(a) \in U_a$ for all $a \in B$. Since $U_a = Z_a$ for all $a \in A \setminus B$, we have $z(a) \in U_a$ for all $a \in A$ and hence $z \in U$; as $z \in M$, we have $z \in U \cap M = V$ and Fact 1 is proved.

Returning to our solution say that $V \in \tau(L)$ is *L-standard* if there exists a standard $U \in \tau(X)$ such that $U \cap L = V$. It is evident that L-standard sets form a base in L. Fix a countable base $\mathcal{U} = \{O_n : n \in \omega\}$ in the space Y. It is easy to see that each O_n is an F_σ-set in Y so each $H_n = f^{-1}(O_n)$ is an F_σ-set in L.

The Lindelöf number is not increased by passing to closed subsets and countable unions so $l(H_n) \leqslant k$ for every $n \in \omega$. Since L-standard sets form a base in L, there exists a family $\{H^n_\alpha : \alpha < \kappa\}$ of L-standard sets such that $H_n = \bigcup\{H^n_\alpha : \alpha < \kappa\}$ for each $n \in \omega$. For all $n \in \omega$ and $\alpha < k$, take a standard set G^n_α such that $G^n_\alpha \cap L = H^n_\alpha$; the set $S = \bigcup\{\text{supp}(G^n_\alpha) : n \in \omega, \alpha < \kappa\}$ is what we are looking for.

It is trivial that $|S| \leqslant k$; let us check that we have the following property:

$(*)$ If $x, y \in L$ and $p_s(x) = p_s(y)$ then $f(x) = f(y)$.

Assume that $f(x) \neq f(y)$; then there exists a set $O_n \in \mathcal{U}$ such that $f(x) \in O_n$ and $f(y) \notin O_n$. This implies that $x \in H_n$ and $y \notin H_n$; if we take any $\alpha < k$ such that $x \in H^n_\alpha$ then $y \notin H^n_\alpha$. Observe that $U = G^n_\alpha$ is a standard set of X with $\text{supp}(U) \subset S$ and $U \cap L = V = H^n_\alpha$; since $p_S(y) = p_S(x) \in p_S(V)$, we can apply Fact 1 to conclude that $y \in V = H^n_\alpha$ which is a contradiction proving $(*)$.

Given any $y \in p_S(L)$, let $h(y) = f(x)$ where $x \in L$ is any point with $p_S(x) = y$; the property $(*)$ shows that this definition of $h : p_S(L) \to Y$ is consistent. It is immediate from the definition of h that $h \circ p_S = f$, so we must only prove that h is continuous. By Problem 009(ii) it suffices to show that $h^{-1}(O_n)$ is open in $p_S(L)$ for any $n \in \omega$. Observe that $h^{-1}(O_n) = p_S(H_n) = \bigcup\{p_S(H^n_\alpha) : \alpha < \kappa\}$; since $U = G^n_\alpha$ is a standard set of X with $\text{supp}(U) \subset S$ and $U \cap L = V = H^n_\alpha$, we can apply Fact 1 to conclude that $p_S(H^n_\alpha)$ is open in $p_S(L)$ for all $n \in \omega$ and $\alpha < k$. This shows that $p_s(H_n)$ is open in $p_S(L)$ being a union of open subsets of $p_S(L)$; hence $h^{-1}(O_n) = p_S(H_n)$ is also an open subset of $p_s(L)$ whence h is continuous so our solution is complete.

S.299. *Suppose that κ is an infinite cardinal and $nw(X_t) \leqslant \kappa$ for all $t \in T$. For the space $X = \prod\{X_t : t \in T\}$ and $T' \subset T$, the map $q_{T'} : X \to X_{T'} = \prod_{t \in T'} X_t$ is the natural projection defined by $q_{T'}(x) = x|T'$ for any $x \in X$. Suppose that D is a dense subspace of X and we are given a continuous onto map $f : D \to Y$ for some space Y. Let $Q = \{y \in Y : \chi(y, Y) \leqslant \kappa\}$ and $P = f^{-1}(Q)$. Prove that there exist a set $S \subset T$ with $|S| \leqslant \kappa$, a closed subset L of the space D, a closed subspace M of the space $q_S(D)$ and a continuous map $h : M \to Y$ with the following properties*:

(i) *$P \subset L$ and $q_S(L) = M$.*
(ii) *$f(x) = h(q_S(x))$ for every $x \in L$.*

Deduce from this fact that $nw(Q) \leqslant \kappa$. In particular, if $\chi(Y) \leqslant \kappa$ then we have $nw(Y) \leqslant \kappa$ and there exists a set $T' \subset T$ together with a continuous mapping $h : q_{T'}(D) \to Y$ such that $f = h \circ (q_{T'}|D)$.

Solution. Recall that a set $U = \prod_{t \in T} U_t$ is called *standard in X* if $U_t \in \tau(X_t)$ for all $t \in T$ and the set $\text{supp}(U) = \{t \in T : U_t \neq X_t\}$ is finite. Standard sets form a base in X

(Problem 101). If $E \subset X$ and $R \subset T$ then $E_R = q_R(E) \subset X_R = \prod_{t \in R} X_t$. Since the map q_R is continuous (Problem 107), the set D_R is dense in X_R for any $R \subset T$.

We call a space κ-*cosmic* if it has a network of cardinality at most κ.

Fact 1. If Z is a κ-cosmic space then $hd(Z) \leqslant \kappa$, i.e., $d(Y) \leqslant \kappa$ for any $Y \subset Z$.

Proof. Fix any $Y \subset Z$ and observe that any subspace of a κ-cosmic space must be κ-cosmic (Problem 159(ii)) so we can choose a network $S = \{S_\alpha : \alpha < \kappa\}$ in the space Y; we do not lose generality assuming that $S_\alpha \neq \emptyset$ for all ordinals $\alpha < \kappa$. If we take $z_\alpha \in S_\alpha$ for each $\alpha < \kappa$ then the set $\{z_\alpha : \alpha < \kappa\}$ is dense in Y and hence $d(Y) \leqslant \kappa$ for any $Y \subset Z$ so Fact 1 is proved.

Fact 2. Any product of at most κ-many κ-cosmic spaces is a κ-cosmic space and hence any subspace of such a product has density at most κ.

Proof. Let $Z = \prod\{Z_\alpha : \alpha < \kappa\}$ where $nw(Z_\alpha) \leqslant \kappa$ for all $\alpha < \kappa$. Fix a network S_α with $|S_\alpha| \leqslant \kappa$ in each Z_α; it is easy to check that the family $\mathcal{S} = \{\prod_{\alpha < \kappa} A_\alpha : A_\alpha \in \mathcal{S}_\alpha$ or $A_\alpha = Z_\alpha$ for each $\alpha < \kappa$ and the set $\{\alpha < \kappa : A_\alpha \neq Z_\alpha\}$ is finite$\}$ is a network in Z and $|S| \leqslant \kappa$ so Fact 2 is proved.

Returning to our solution choose a local base $\mathcal{B}_y$ of the space Y at each $y \in Q$ in such a way that $|\mathcal{B}_y| \leqslant \kappa$. By continuity of the function f, for each $x \in P$ there exists a family $\mathcal{U}_x$ of standard subsets of X such that $x \in \bigcap \mathcal{U}_x$ while $|\mathcal{U}_x| \leqslant \kappa$ and, for any $O \in \mathcal{B}_{f(x)}$, there exists a set $U \in \mathcal{U}_x$ such that $f(U \cap D) \subset O$. It is straightforward that the set $T_x = \bigcup\{\text{supp}(U) : U \in \mathcal{U}_x\}$ has cardinality at most κ for every $x \in P$. Another simple observation is that, for any $G \in \tau(f(x), Y)$, there is $O \in \mathcal{B}_{f(x)}$ with $O \subset G$. As a consequence, for any $G \in \tau(f(x), Y)$ there exists $U \in \mathcal{U}_x$ such that $f(U \cap D) \subset G$.

For an arbitrary $x_0 \in P$, let $A_0 = \{x_0\}$ and $S_0 = T_{x_0}$. If we have sets $A_n \subset P$ and $S_n \subset T$ such that $|A_n| \leqslant \kappa$ and $|S_n| \leqslant \kappa$, any subspace of the space $P_{S_n} = q_{S_n}(P)$ has density at most κ being a subspace of a κ-cosmic space X_{S_n} (see Fact 2) so it is possible to find a set $A_{n+1} \subset P$ with $A_{n+1} \supset A_n$ such that $|A_{n+1}| \leqslant \kappa$ and the set $q_{S_n}(A_{n+1})$ is dense in P_{S_n}. Letting $S_{n+1} = \bigcup\{T_x : x \in A_{n+1}\}$, we can follow this inductive procedure which gives us a sequence $\{A_n : n \in \omega\}$ of subsets of P and a sequence $\{S_n : n \in \omega\}$ of subsets of T such that $|A_n| \leqslant \kappa$ and $|S_n| \leqslant \kappa$ for all $n \in \omega$.

Once we have the sets $A = \bigcup\{A_n : n \in \omega\}$ and $S = \bigcup\{S_n : n \in \omega\}$, we can define the sets $M = [q_S(A)]$ (the brackets denote the closure in the space $q_S(D)$) and $L = q_S^{-1}(M) \cap D$. It is clear that L is closed in D and M is closed in $q_S(D)$. It is also immediate that $q_S(L) = M$.

Claim 1. The set $A_S = q_S(A)$ is dense in $P_S = q_S(P)$.

Proof. Take any $y \in P$ and any standard set $W = \prod\{W_t : t \in S\}$ of the space X_S such that $q_S(y) \in W$. Since $K = \text{supp}(W) = \{t \in S : W_t \neq X_t\}$ is finite, there is $n \in \omega$ such that $K \subset S_n$. Note that $q_{S_n}(y) \in P_{S_n}$ and $W' = \prod\{W_t : t \in S_n\}$ is an open neighbourhood of $q_{S_n}(y)$ in X_{S_n}. The set $q_{S_n}(A_{n+1})$ is dense in P_{S_n} so $W' \cap q_{S_n}(A_{n+1}) \neq \emptyset$ and hence there is $x \in A_{n+1}$ such that $x(t) \in W_t$ for all $t \in K$. We have $x(t) \in W_t$ for all $t \in S$ because $W_t = X_t$ for all $t \in S \setminus K$. Therefore we have $q_S(x) \in W \cap A_S$; the standard set

$W \ni q_S(y)$ of the product X_S has been chosen arbitrarily so $q_S(y)$ is in the closure of A_S. Since $y \in P$ has also been chosen arbitrarily, we proved that $q_S(A)$ is dense in P_S. Claim 1 is proved.

Claim 2. If $x, y \in L$ and $q_S(x) = q_S(y)$ then $f(x) = f(y)$.

Proof. If $f(x) \neq f(y)$ then there exist $O_x \in \tau(f(x), Y)$, $O_y \in \tau(f(y), Y)$ such that $\overline{O}_x \cap \overline{O}_y = \emptyset$. Fix any standard sets W_x and W_y of the space X such that we have $x \in W_x, y \in W_y, f(W_x \cap D) \subset O_x$ and $f(W_y \cap D) \subset O_y$. Next observe that $q_S(W_x)$ and $q_S(W_y)$ are open in X_S by Problem 107 and we have $q_S(x) = q_S(y) \in q_S(W_x) \cap q_S(W_y)$. Since $q_S(y) = q_S(x) \in M$ and the set A_S is dense in M, there is $z \in A$ such that $q_S(z) \in q_S(W_x) \cap q_S(W_y)$. The point $f(z)$ cannot belong to both sets $\overline{O}_x$ and $\overline{O}_y$; we assume without loss of generality that $f(z) \notin \overline{O}_x$. There is $O_z \in \mathcal{B}_{f(z)}$ such that $f(z) \in O_z \subset Y \backslash \overline{O}_x$; by the definition of $\mathcal{U}_z$, we can find $U_z \in \mathcal{U}_z$ such that $f(U_z \cap D) \subset O_z$. The set $q_S(U_z)$ is open in X_S (Problem 107) and $q_S(z) \in q_S(U_z) \cap q_S(W_x)$. We have $\text{supp}(U_z) \subset S$ so $q_S^{-1}(q_S(U_z)) = U_z$; an immediate consequence is that $U_z \cap W_x \neq \emptyset$. Recalling that D is dense in X we can see that $U_z \cap W_x \cap D \neq \emptyset$. Take any $w \in U_z \cap W_x \cap D$; then we hve $f(w) \in f(U_z \cap D) \subset O_z \subset Y \backslash \overline{O}_x$. However, $w \in W_x$ and hence $f(w) \in f(W_x \cap D) \subset O_x$; this contradiction concludes the proof of Claim 2.

Returning to our solution, observe that Claim 1 implies that $M = [A_S] = [P_S]$ and hence $P \subset L$ so (i) is holds for the sets M, L and S. For any $y \in M$, we let $h(y) = f(x)$ where $x \in L$ is any point with $q_S(x) = y$; Claim 2 shows that this definition of the mapping $h : M \to Y$ is consistent. It is immediate from the definition of h that $h(q_S(x)) = f(x)$ for any $x \in L$ so we must only prove that h is continuous. Take any $y \in M$ and fix any $x \in L$ with $q_S(x) = y$. Let $G \in \tau(h(y), Y)$; pick $O, H \in \tau(h(y), Y)$ such that $h(y) \in O \subset \overline{O} \subset H \subset \overline{H} \subset G$. Observe that $f(x) = h(y)$ and hence we have $O \in \tau(f(x), Y)$; the function f being continuous at the point x there is a standard set U of the space X such that $f(U \cap D) \subset O$. The set $V = q_S(U)$ is open in the space X_S (Problem 107) and $y \in V$.

Let us prove that $f(w) \in \overline{O}$ for any $w \in q_S^{-1}(V) \cap A$. If $f(w) \in Y \backslash \overline{O}$ then there is $O_w \in \mathcal{B}_{f(w)}$ with $O_w \subset Y \backslash \overline{O}$. Take any $W \in \mathcal{U}_w$ with $f(W \cap D) \subset O_w$. Since $\text{supp}(W) \subset S$, we have $q_S^{-1}(q_S(W)) = W$; recall that $q_S(w) \in q_S(W) \cap q_S(U)$ so $W \cap U \neq \emptyset$. The set D is dense in X so there is $d \in D \cap W \cap U$. Consequently, $f(d) \in f(W \cap D) \subset O_w \subset Y \backslash \overline{O}$. However, we also have $f(d) \in f(U \cap D) \subset O$ which is a contradiction.

Since $f(w) \in \overline{O} \subset H$ for each $w \in B = q_S^{-1}(V) \cap A$, by our construction of $\mathcal{U}_w$ there is $U_w \in \mathcal{U}_w$ such that $f(U_w \cap D) \subset H$ for each $w \in B$. Note also that $\text{supp}(U_w) \subset S$ implies that $q_S^{-1}(q_S(U_w)) = U_w$ for each $w \in B$. For the set $U(B) = \cup\{U_w : w \in B\}$, we also have $q_S^{-1}(q_S(U(B))) = U(B)$. Let us show that $q_S^{-1}(V) \cap L \subset \text{cl}_X(U(B))$. Indeed, otherwise $C = q_S^{-1}(V) \backslash \text{cl}_X(U(B))$ is an open subset of X with $C \cap L \neq \emptyset$. If $d \in C \cap L$ then $q_S(d) \in q_S(C)$; the set $q_S(C)$ is open in X_S and $q_S(C) \cap q_S(U(B)) = \emptyset$. Since $B \subset U(B)$, we have $q_S(B) = q_S(A) \cap V \subset q_S(U(B))$. The set $q_S(C)$ is an open neighbourhood of $q_S(d)$ and $q_S(d) \in [q_S(A)]$ whence $q_S(C) \cap q_S(A) \neq \emptyset$. Also note that $q_S(C) \subset V$ and therefore $\emptyset = q_S(U(B)) \cap$

$q_S(C) \supset (q_S(A) \cap V) \cap q_S(C) \supset q_S(A) \cap q_S(C) \neq \emptyset$. This contradiction shows that $q_S^{-1}(V) \cap L \subset \text{cl}_X(U(B))$.

Observe that $f(U(B) \cap D) = \bigcup\{f(U_w \cap D) : w \in B\} \subset H$, because we have $f(U_w \cap D) \subset H$ for every $w \in B$. Therefore

$$f(q_S^{-1}(V) \cap L) \subset f(\text{cl}_X(U(B)) \cap D) = f(\text{cl}_D(U(B) \cap D)) \subset \overline{H}.$$

It follows easily from Claim 2 that $h(V \cap M) = f(q_S^{-1}(V) \cap L)$ and hence we have $h(V \cap M) \subset \overline{H} \subset G$. Thus the set $V \cap M$ witnesses continuity of h at the point y.

Finally observe that the properties (i) and (ii) imply that $Q \subset h(M)$ and therefore $nw(Q) \leqslant nw(h(M)) \leqslant nw(M) \leqslant \kappa$ (see Problems 157(iii) and 159(ii)). If $\chi(Y) \leqslant \kappa$ then $Q = Y$ and $P = D$ so the properties (i) and (ii) hold for the sets $T' = S$, $M = q_S(D)$ and $L = D$, i.e., our solution is complete.

S.300. *Given a space Z and a second countable space Y show that, for any continuous map $p : C_p(Z) \to Y$, there is a countable set $A \subset Z$ and a continuous mapping $q : \pi_A(C_p(Z)) \to Y$ such that $p = q \circ \pi_A$. Here $\pi_A : C_p(Z) \to C_p(A)$ is the restriction map defined by the formula $\pi_A(f) = f|A$ for every $f \in C_p(Z)$.*

Solution. The space $C_p(Z)$ is a dense subspace of the product $\mathbb{R}^Z$ of second countable spaces (Problem 111) and, for any $B \subset Z$, the restriction map π_B coincides on $C_p(Z)$ with the natural projection of $p_B : \mathbb{R}^Z \to \mathbb{R}^B$ (see Problem 107). Thus Problem 299 is applicable to conclude that there exists a countable $A \subset Z$ and a continuous map $q : p_A(C_p(Z)) = \pi_A(C_p(Z)) \to Y$ such that $p = q \circ p_A = q \circ \pi_A$.

S.301. *Prove that, if $|X| > 1$, then the space $C_p(X)$ is not linearly ordered.*

Solution. We will see that many subspaces of $C_p(X)$ are homeomorphic to $\mathbb{I}$.

Fact 1. For any distinct $f, g \in C_p(X)$, the subset $[f, g] = \{tf + (1 - t)g : t \in [0, 1]\}$ is homeomorphic to the interval $[0, 1]$ and the mapping $\varphi : [0, 1] \to [f, g]$ defined by $\varphi(t) = tf + (1 - t)g$, is a homeomorphism.

Proof. For any $x \in X$, let $\pi_x(f) = f(x)$ for any $f \in \mathbb{R}^X$. It is clear that $\pi_x : \mathbb{R}^X \to \mathbb{R}$ is the natural projection of $\mathbb{R}^X$ onto the xth factor. Observe that the function $\pi_x \circ \varphi : [0, 1] \to [f(x), g(x)] \subset \mathbb{R}$ is continuous being a linear map; since $C_p(X)$ is a subspace of the product $\mathbb{R}^X$, this proves that φ is continuous (Problem 102).

If $t, s \in [0, 1]$, $t \neq s$ and $\varphi(t) = \varphi(s)$ then $tf + (1 - t)g = sf + (1 - s)g$; after trivial transformations we arrive at the equality $(t - s)f = (t - s)g$ which implies $f = g$, a contradiction. This shows that φ is a condensation and hence homeomorphism because the space $[0, 1]$ is compact. Fact 1 is proved.

Fact 2. If $|X| > 1$ then $[u, v]$ has empty interior in $C_p(X)$ for any $u, v \in C_p(X)$.

Proof. Note first that $u = v$ implies $[u, v] = u$; since $C_p(X)$ never has isolated points, the set $[u, v] = \{u\}$ has empty interior. Now fix distinct points $x, y \in X$ and distinct $u, v \in C_p(X)$ such that $U = \text{Int}([u, v]) \neq \emptyset$; take also $s \in [0, 1]$ with $\varphi(s) = su + (1 - s)v \in U$. Given any $h \in C_p(X)$, let $T_h(f) = f + h$ for all $f \in C_p(X)$. Then the map $T_h : C_p(X) \to C_p(X)$ is a homeomorphism for any $h \in C_p(X)$ (Problem 079). Observe that $T_h(tf + (1 - t)g) = tf + (1-t)g+h = t(f + h)+(1-t)(g+h)$ which shows that $T_h([u, v]) = [T_h(u), T_h(v)]$ for any $h \in C_p(X)$. Since T_h is a homeomorphism, the set $[T_h(u), T_h(v)]$ has non-empty interior for any $h \in C_p(X)$, in particular, for $h = -(su + (1 - s)v)$. If $u' = T_h(u)$ and $v' = T_h(v)$ then $f_0 \equiv 0$ belongs to the interior V of the set $[u', v']$. The functions u and v being distinct, we have $u' \neq v'$ so one of these functions, say u', is distinct from f_0 and hence $s \neq 1$. Observe that $su' + (1 - s)v' \equiv 0$ and hence $v' = \frac{s}{s-1}u'$.

Let us show that we have

(∗) For any $f \in C_p(X)$, there is $r \in \mathbb{R}$ such that $f = ru'$.

There exist $k \in \mathbb{N}$, $x_1, \dots, x_k \in X$ and $\varepsilon > 0$ such that $O(f_0, x_1, \dots, x_k, \varepsilon) \subset V$. Take any $n \in \mathbb{N}$ such that $\frac{1}{n}|f(x_i)| < \varepsilon$ for all $i \leqslant k$; then $\frac{1}{n}f \in V \subset [u', v']$ and therefore $\frac{1}{n}f = tu' + (1 - t)v'$ for some $t \in [0, 1]$. Thus $f = n(tu' + \frac{s}{s-1}(1 - t)u') = ru'$ where $r = nt + n(1 - t)\frac{s}{s-1}$. The property (∗) is proved.

There exist functions $f, g \in C_p(X)$ such that $f(x) = g(y) = 1$ and $f(y) = g(x) = 0$ (Problem 034). Apply (∗) to find $r, t \in \mathbb{R}$ such that $f = ru'$ and $g = tu'$; the numbers r and t are distinct from zero because $f \neq f_0$ and $g \neq f_0$. We have $u'(x) = \frac{1}{t}g(x) = 0$ and $u'(x) = \frac{1}{r}f(x) = \frac{1}{r} \neq 0$ which is a contradiction. Fact 2 is proved.

Fact 3. Suppose that G and H are non-empty open subsets of $I = [0, 1]$ such that $I = G \cup H$. Then $G \cap H \neq \emptyset$.

Proof. Suppose that $G \cap H = \emptyset$; without loss of generality we can assume that $0 \in G$. Consider the number $s = \inf(H) \in I$; if $s \in G$ then $[0, s) \subset G$ and, since G is open in I, there is $\varepsilon > 0$ such that $[s, s + \varepsilon) \subset G$. This proves that there are no points of H in $[0, s + \varepsilon)$, i.e., $\inf(H) \geqslant s + \varepsilon > s$, a contradiction. Therefore $s \in H$ and $s > 0$ because $0 \in G$. Since H is also open, there is $\delta > 0$ such that $(s - \delta, s] \subset H$ and hence $\inf(H) \leqslant s - \delta < s$, and this last contradiction shows that the sets G and H cannot be disjoint. Fact 3 is proved.

Now it is easy to finish our solution. Suppose that $\preceq$ is a linear order on $C_p(X)$ which generates the topology of $C_p(X)$. Given any distinct $f, g \in C_p(X)$, the set $J(f, g) = \{h \in C_p(X) : f \prec h \prec g\}$ is open in $C_p(X)$ by the definition of the order topology. Since $C_p(X)$ is an infinite space, we can choose distinct $u, v, w \in C_p(X)$ such that $u \prec w \prec v$; this shows that the open set $J(u, v)$ is non-empty. It is impossible that $J(u, v) \subset [u, v]$ by Fact 2 so we can fix $h \in J(u, v) \setminus [u, v]$. The sets $U = \{f \in C_p(X) : f \prec h\}$ and $V = \{f \in C_p(X) : h \prec f\}$ are open, disjoint and non-empty because $u \in U$ and $v \in V$. Furthermore, $U \cup V = C_p(X) \setminus \{h\}$ and hence $[u, v] \subset U \cup V$. As a consequence, $G \cup H = [u, v]$ where the sets $G = U \cap [u, v]$ and $H = V \cap [u, v]$ are open in $[u, v]$. We also have $G \neq \emptyset \neq H$ because $u \in G$ and $v \in H$. The interval $[u, v]$ is homeomorphic to $[0, 1]$ by Fact 1 so we have $G \cap H \neq \emptyset$ by

Fact 3. However, $G \cap H \subset U \cap V = \emptyset$ which is a contradiction. Our solution is complete.

S.302. *Prove that any linearly ordered space is collectionwise normal and T_1. Deduce from this fact that any linearly ordered space is Tychonoff.*

Solution. We will first need the following equivalent definition of collectionwise normality.

Fact 1. Suppose that, for any discrete family $\{F_s : s \in S\}$ of closed subsets of a space X, there exists a disjoint family $\{U_s : s \in S\}$ of open subsets of X such that $F_s \subset U_s$ for any $s \in S$. Then X is collectionwise normal.

Proof. A disjoint family of two closed subsets is discrete, so our hypothesis implies normality of the space X. Now take any discrete family $\{F_s : s \in S\}$ of closed subsets of the space X and find a disjoint family $\{U_s : s \in S\}$ of open subsets of X such that $F_s \subset U_s$ for each $s \in S$. The set $F = \cup\{F_s : s \in S\}$ is closed in X and $F \subset U = \cup\{U_s : s \in S\}$.

Since X is normal, we can choose $W \in \tau(X)$ such that $F \subset W \subset \overline{W} \subset U$. Now if $W_s = U_s \cap W$ then $F_s \subset W_s \in \tau(X)$ for each $s \in S$ so it suffices to show that the family $\mathcal{W} = \{W_s : s \in S\}$ is discrete. Given $x \in X$ suppose that $x \notin U$. Then $X \backslash \overline{W}$ is a neighbourhood of X which does not intersect any element of $\mathcal{W}$. If $x \in U$ then $x \in U_s$ for some $s \in S$ and hence $U_s \in \tau(x, X)$ intersects at most one element of $\mathcal{W}$. Hence $\mathcal{W}$ is discrete and we proved collectionwise normality of X and Fact 1.

Fact 2. Let $(L, \leqslant)$ be a linearly ordered space. If $D = \{x_s : s \in S\}$ is a discrete subspace of L then there exists a disjoint family $\{U_s : s \in S\}$ of open subsets of L such that $x_s \in U_s$ for any $s \in S$.

Proof. Let $\prec$ be any well order on the set L. For each $s \in S$ take $a_s, b_s \in L$ such that $a_s < x_s < b_s$ and $(a_s, b_s) \cap D = \{x_s\}$ for all $s \in S$. Of course, if x_s is the smallest element of L then only a point $b_s \in L$ is chosen so that $x < b_s$ and $(\leftarrow, b_s) \cap D = \{x_s\}$. If x_s is the largest element of L then only a point $a_s \in L$ is chosen so that $a_s < x$ and $(a_s, \rightarrow) \cap D = \{x_s\}$.

Let c_s be the $\prec$-minimal element of the set $[a_s, x_s] = \{x \in L : a_s \leqslant x < x_s\}$. Analogously, d_s is the $\prec$-minimal element of the set $(x_s, b_s) = \{x \in L : x_s < x \leqslant b_s\}$. Then $c_s < x_s < d_s$ and hence $U_s = (c_s, d_s)$ is an open neighbourhood of x_s for each $s \in S$. If x_s is the minimal element of L then $U_s = (\leftarrow, d_s)$ and $U_s = (c_s, \rightarrow)$ if x_s is the maximal element of L.

Fix any index $s \in S$; observe first that, if $x_t < x_s$ then $x_s \notin (a_t, b_t)$ implies $b_t < x_s$. Analogously, if $x_t > x_s$ then $a_t > x_s$. Now assume that $x_t < x_s$ and $(c_t, d_t) \cap (c_s, d_s) \neq \emptyset$. Since $d_t \leqslant b_t < x_s$, we have $d_t < x_s$ and $x_t < c_s$ so $c_s < d_t < x_s$. However, d_t is the $\prec$-minimal element of $(x_t, d_t]$ and therefore d_t is the $\prec$-minimal element of $[c_s, d_t]$. But c_s is the $\prec$-minimal element of $[c_s, x_s)$ and hence c_s is the $\prec$-minimal element of $[c_s, d_t]$. Since $c_s \neq d_t$, these two points cannot both be the minimal element of the same set $[c_s, d_t]$. The obtained contradiction proves that $(c_t, d_t) \cap (c_s, d_s) = \emptyset$. The proof for the case of $x_t > x_s$ is analogous so $U_s \cap U_t = \emptyset$ if $s \neq t$ and Fact 2 is proved.

Observe that Fact 2 implies that any linearly ordered space L is Hausdorff and hence T_1 so to establish the Tychonoff property of L, it suffices to prove that L is normal (see Problem 015). We will, in fact, prove that the space L is collectionwise normal. Take any discrete family $\mathcal{F} = \{F_s : s \in S\}$ of closed subsets of L. The set $P_s = \bigcup\{F_t : t \in S \setminus \{s\}\}$ is closed and disjoint from F_s for any $s \in S$. Let $W_s = \bigcup\{(x, y) : x, y \in F_s \text{ and } (x, y) \cap P_s = \emptyset\}$ for each $s \in S$. Then $W_s \cap P_s = \emptyset$ for each $s \in S$ and the family $\mathcal{W} = \{W_s : s \in S\}$ is discrete. Indeed, take any $x \in L$ and any interval $J = (a, b) \ni x$ such that (a, b) meets at most one element, say F_s, of the family $\mathcal{F}$; such an interval exists because $\mathcal{F}$ is discrete. If $t \neq s$ then $(a, b) \cap F_t = \emptyset$; if $(a, b) \cap W_t \neq \emptyset$ then there are $x, y \in F_t$ such that $(x, y) \cap (a, b) \neq \emptyset$ and therefore $y \in (a, b) \cap F_t$ or $x \in (a, b) \cap F_t$ which is a contradiction in both cases. As a consequence, (a, b) can intersect at most the set W_s and hence the family $\mathcal{W}$ is discrete. We leave to the reader to carry out a completely analogous proof if the interval J is $(a, \rightarrow)$ or $(\leftarrow, b)$.

The following step is to show that the set $D = (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{W})$ is discrete. Take any $x \in D$; then $x \in F_s$ for some $s \in S$. Since the family $\mathcal{F}$ is discrete, we can find an interval $J = (a, b) \ni x$ such that $(a, b) \cap P_s = \emptyset$ (we leave to the reader to give a completely analogous proof if the interval J is $(a, \rightarrow)$ or $(\leftarrow, b)$). Thus, only points from F_s can be in (a, b). Suppose that $y, z \in F_s \cap (a, b)$, $y \neq z$ and $x \notin \{y, z\}$. Out of three distinct points x, y, z one lies between the other two; assume, for example that $x < y < z$. Then $(x, z) \cap P_s \subset (a, b) \cap P_s = \emptyset$ and therefore $(x, z) \subset W_s$ so y cannot be in D. This shows that (a, b) is an open neighbourhood of x which intersects at most one element of D distinct from x, i.e., D is discrete.

By Fact 2, we can choose an open $U_x^s \in \tau(x, L)$ for each $x \in D \cap F_s$ in such a way that the family $\{U_x^s : s \in S, x \in D \cap F_s\}$ is disjoint. Our final step is to define a set $V_s = \bigcup \{U_x^s : x \in D \cap F_s\} \cup W_s$ for each $s \in S$. It is immediate that $\{V_s : s \in S\}$ is a disjoint family of open subsets of L and $F_s \subset V_s$ for every $s \in S$. Apply Fact 1 to conclude that L is collectionwise normal and finish our solution.

S.303. *Prove that $t(X) = \psi(X) = \chi(X)$ for any linearly ordered topological space X.*

Solution. Let $\leqslant$ be an arbitrary linear order on X which generates $\tau(X)$. Assume that $\psi(X) = \kappa$. Given $x \in X$ we can find a family $\mathcal{B} = \{(a_\alpha, b_\alpha) : \alpha < \kappa\}$ such that $\{x\} = \bigcap \mathcal{B}$. It can happen that $\mathcal{B} = \{(\leftarrow, b_\alpha) : \alpha < \kappa\}$ or $\mathcal{B} = \{(a_\alpha, \rightarrow) : \alpha < \kappa\}$ but the proof in any of these cases is essentially the same or easier.

Let $\mathcal{C}$ be the family of all finite intersections of the elements from $\mathcal{B}$; we claim that $\mathcal{C}$ is a local base at x. Indeed, take any $U \in \tau(x, X)$; there are $a, b \in X$ such that $x \in (a, b) \subset U$. Since $a \notin \bigcap \mathcal{B}$, there is $\alpha < \kappa$ such that $a \notin (a_\alpha, b_\alpha)$ and hence $a < a_\alpha$. Analogously, $b \notin \bigcap \mathcal{B}$ so there exists $\beta < \kappa$ with $b \notin (a_\beta, b_\beta)$ and therefore $b_\beta < b$. Then $W = (a_\alpha, b_\alpha) \cap (a_\beta, b_\beta) \in \mathcal{C}$ and $x \in W \subset (a_\alpha, b_\beta) \subset (a, b) \subset U$ so $\mathcal{C}$ is a local base at x with $|\mathcal{C}| \leqslant |\mathcal{B}| \leqslant \psi(X)$. The point x being taken arbitrarily, we proved that $\chi(X) \leqslant \psi(X)$ so $\chi(X) = \psi(X)$.

Since $t(X) \leqslant \chi(X)$ (Problem 156), it suffices to show that $\chi(X) = \psi(X) \leqslant t(X)$. Take any $x \in X$; we must consider the following cases:

(a) $x \notin \overline{(\leftarrow, x)} \cup \overline{(x, \rightarrow)}$; then x is an isolated point of the space X so we have $\psi(x, X) = 1 \leqslant t(X)$.
(b) $x \notin L = \overline{(\leftarrow, x)}$ but $x \in \overline{(x, \rightarrow)}$. Take any $A \subset (x, \rightarrow)$ with $|A| \leqslant t(X)$ and $x \in \overline{A}$. Then the family $\mathcal{B} = \{X \backslash L\} \cup \{(\leftarrow, a) : a \in A\}$ consists of open subsets of X and $|\mathcal{B}| \leqslant t(X)$. If $y \in X \backslash \{x\}$ and $y \in L$ then $y \notin X \backslash L$ so $y \notin \bigcap \mathcal{B}$. If $y > x$ then $W = (\leftarrow, y)$ is an open neighbourhood of x so there is a point $a \in A \cap W$. Therefore $y \notin (\leftarrow, a)$, i.e., $y \notin \bigcap \mathcal{B}$. We proved that $\bigcap \mathcal{B} = \{x\}$ and hence $\psi(x, X) \leqslant t(X)$.
(c) $x \notin R = \overline{(x, \rightarrow)}$ but $x \in \overline{(\leftarrow, x)}$. This case is analogous to the case (b).
(d) $x \in \overline{(\leftarrow, x)}$ and $x \in \overline{(x, \rightarrow)}$. Fix $A \subset (\leftarrow, x)$ and $B \subset (x, \rightarrow)$ such that $x \in \overline{A} \cap \overline{B}$ and $|A| + |B| \leqslant t(X)$. The family $\mathcal{B} = \{(a, b) : a \in A \text{ and } b \in B\}$ consists of open subsets of X and $|\mathcal{B}| \leqslant t(X)$. If $y \neq x$ and $y < x$ then $(y, \rightarrow) \in \tau(x, X)$ and hence there is $a \in A$ with $a > y$; this implies $y \notin (a, b)$ for any $b \in B$ and hence $y \notin \bigcap \mathcal{B}$. Now, if $y > x$ then $(\leftarrow, y) \in \tau(x, X)$ and therefore there is $b \in B$ with $b < y$; this shows that $y \notin (a, b)$ for any $a \in A$ so $y \notin \bigcap \mathcal{B}$ and we proved that $\bigcap \mathcal{B} = \{x\}$ so $\psi(X) \leqslant t(X)$ and our solution is complete.

S.304. *Prove that $d(X) = hd(X) = \pi w(X)$ for any linearly ordered topological space X.*

Solution. The following easy fact is formulated for further references.

Fact 1. For any space Z we have $d(Z) \leqslant \pi w(Z)$.

Proof. Take any π-base $\mathcal{B}$ in Z with $|\mathcal{B}| \leqslant \pi w(Z)$. For each $V \in \mathcal{B}$ choose a point $x_V \in V$ and let $D = \{x_V : V \in \mathcal{B}\}$. It is immediate that D is dense in Z and $|D| \leqslant \pi w(X)$; hence $d(Z) \leqslant \pi w(Z)$ and Fact 1 is proved.

Fact 2. For any linearly ordered space L, we have $s(L) = c(L)$.

Proof. Take any discrete subspace D of the space L; Fact 2 of S.302 says that there exists a disjoint family $\mathcal{U} = \{U_d : d \in D\} \subset \tau(L)$ such that $d \in U_d$ for each $d \in D$. It is clear that $|D| \leqslant |\mathcal{U}| \leqslant c(L)$ and hence $s(L) \leqslant c(L)$. Since the inverse inequality holds for any space (Problem 156(ii)), Fact 2 is proved.

Let $\leqslant$ be an arbitrary linear order on X which generates $\tau(X)$. Assume that $d(X) = \kappa$ and take a set $D \subset X$ with $\overline{D} = X$ and $|D| = \kappa$. Denote by E the set of isolated points of X (which can be empty but in any case $E \subset D$). The family $\mathcal{B} = \{(x, y) : x < y$ and $x, y \in D\}$ consists of open subsets of X and $|\mathcal{B}| = \kappa$. Let $\mathcal{B}' = \{U \in \mathcal{B} : U \neq \emptyset\} \cup \{\{x\} : x \in E\}$; then $\mathcal{B}' \subset \tau^*(X)$ and $|\mathcal{B}'| \leqslant \kappa$. We claim that $\mathcal{B}'$ is a π-base in X. To see this, take any $V \in \tau^*(X)$; if V has an isolated point d then $d \in E \subset D$ and hence $\{d\} \subset V$ witnesses that $\mathcal{B}'$ is a π-base.

Now assume that $V \cap E = \emptyset$; then V is an infinite set so there is $x \in V$ which is neither minimal nor maximal element of L. Therefore there exist $a, b \in X$ such that $x \in (a, b) \subset V$. The non-empty open set (a, b) has to be infinite because it has no isolated points. This implies $(a, b) \cap D$ is also infinite and hence we can choose points $p, q, r \in (a, b) \cap D$ such that $p < q < r$; then $q \in (p, r)$ which implies $(p, r) \in \mathcal{B}'$ and $(p, r) \subset (a, b) \subset V$ so $\mathcal{B}'$ is a π-base in X. This proves that $\pi w(X) \leqslant d(X)$. Applying Fact 1, we conclude that $\pi w(X) = d(X)$.

Since it is evident that $d(X) \leqslant hd(X)$, it suffices to show that $d(Y) \leqslant \kappa = d(X)$ for any $Y \subset X$. Pick any $D \subset X$ with $\overline{D} = X$ and $|D| = \kappa$. Given $a, b \in D$, $a < b$, take any point $w(a, b) \in (a, b) \cap Y$ if such a point exists. The set $E = \{w(a, b) : a, b \in D, a < b \text{ and } (a, b) \cap Y \neq \emptyset\}$ has cardinality $\leqslant \kappa$; let us prove that $P = E \cup I$ is dense in Y, where I is the set of all isolated points of Y. If $W \in \tau^*(Y)$ and W has an isolated point y then y is isolated in Y so $y \in I \subset P \cap W$. If W has no isolated points then it is infinite and hence we can fix $x \in W$ such that x is neither minimal nor maximal element of L. Thus there exist $a, b \in X$ such that $x \in (a, b) \cap Y \subset W$. Since $(a, b) \cap Y$ is an open subspace of W, it has no isolated points; therefore it is infinite and hence we can choose points $x_1, x_2, x_3, x_4, x_5 \in (a, b) \cap Y$ such that $x_1 < x_2 < x_3 < x_4 < x_5$. The intervals (x_1, x_3) and (x_3, x_5) being non-empty, there exist $c \in (x_1, x_3) \cap D$ and $d \in (x_3, x_5) \cap D$. Since $x_3 \in (c, d) \cap Y$, we have $w(c, d) \in (c, d) \cap Y \subset (a, b) \cap Y \subset W$ which proves that P is dense in Y. Observe that I is a discrete subspace of X so $|I| \leqslant s(X) = c(X) \leqslant d(X)$ (we applied Fact 2 and Problem 156(i)). Thus $|P| \leqslant |D| + |I| \leqslant d(X)$ which proves that $hd(X) \leqslant d(X)$ and therefore $d(X) = hd(X) = \pi w(X)$ so our solution is complete.

S.305. *Prove that a linearly ordered space X is compact if and only if any non-empty closed subset of X has the smallest and the largest element under the order that generates $\tau(X)$.*

Solution. Let $\leqslant$ be the linear order that generates the topology of X. Suppose that X is compact and $F \neq \emptyset$ is a closed subset of X. The set $F_x = \{y \in F : y \leqslant x\}$ is closed in F for any $x \in F$. It is straightforward that the family $\mathcal{F} = \{F_x : x \in F\}$ is centered; by compactness of F we have $\bigcap F \neq \emptyset$. It is easy to see that there is a unique $y \in \bigcap \mathcal{F}$ and this y is the smallest point of F. The existence of the largest point of F is proved analogously, considering the family $\mathcal{G} = \{G_x : x \in F\}$ where $G_x = \{y \in F : x \leqslant y\}$.

Now suppose that any non-empty closed subset of X has the smallest and the largest element with respect to the order $\leqslant$. In particular, X has the maximal element m. Given any $x \in X$, let $L_x = \{y \in X : y \leqslant x\}$. Take any infinite $A \subset X$ and consider the set $P = \{x \in X : |A \cap L_x| = |A|\}$. The set P is non-empty because $L_m \cap A = X \cap A = A$ and therefore $m \in P$. The set $F = \overline{P}$ is closed and non-empty so it has a smallest element $x \in F$.

(1) If $|L_x \cap A| = |A|$ then, for any $U \in \tau(x, X)$, we can choose $y \in X$ such that $(y, x) \subset U$. It is clear that $y \notin P$ and hence the set $L_y \cap A$ has a smaller cardinality than A. Thus $(y, x) \cap A = (L_x \cap A) \backslash ((L_y \cap A) \cup \{x\})$ has the same cardinality as A and hence $|U \cap A| = |A|$ because $U \cap A \supset (y, x) \cap A$.
(2) If $|L_x \cap A| < |A|$ then, for any $U \in \tau(x, X)$ we can choose $y \in X$ such that $(x, y) \subset U$. The set $(\leftarrow, y)$ is an open neighbourhood of $x \in \overline{P}$ so there is $z \in P \cap (x, y)$. Thus $(x, z) \cap A = (L_z \cap A) \backslash ((L_x \cap A) \cup \{z\})$ has the same cardinality as A and hence $|U \cap A| = |A|$ because $U \cap A \supset (x, z) \cap A$.

We proved that in all possible cases $|U \cap A| = |A|$ for every $U \in \tau(x, X)$, i.e., x is a complete accumulation point of A. The infinite set A was chosen arbitrarily, so we proved that any infinite subset of X has a complete accumulation point. Applying Problem 118(x) we conclude that X is compact.

S.306. *Suppose that a well order $\leqslant$ generates the topology of a space X. Prove that X is compact if and only if $(X, \leqslant)$ has the largest element.*

Solution. If X is compact then X has the largest element by Problem 305. Now assume that m is the largest element of X. Given a non-empty closed $F \subset X$ the set F has the smallest element because X is well ordered. Consider the set $P = \{x \in X : y \leqslant x \text{ for every } y \in F\}$. Since $m \in P$, the set P is non-empty so we can choose a minimal element $z \in P$ of the set P. If $z \notin F$ then $X \backslash F$ is an open neighbourhood of the point z so there is $u < z$ such that $(u, z) \cap F = \emptyset$. However, this implies that $u < z$ and $y \leqslant u$ for any $y \in F$ which contradicts the minimality of z. Thus $z \in F$ is the largest element of F and we proved that any non-empty closed $F \subset X$ has the smallest and the largest elements. Applying Problem 305 again, we conclude that X is compact.

S.307. *Let X be an arbitrary product of separable spaces. Prove that every pseudocompact subspace of $C_p(X)$ is metrizable (and hence compact).*

Solution. The proof is not easy at all so we will first establish some facts.

Fact 1. Let Z be a space with $Z = \bigcup\{Z_n : n \in \omega\}$; suppose that $C_p(Z)$ has a non-metrizable pseudocompact subspace. Then $C_p(Z_n)$ has a non-metrizable pseudocompact subspace for some $n \in \omega$.

Proof. Let P be a non-metrizable pseudocompact subspace of $C_p(Z)$. The identity map on each Z_n defines a continuous onto map $\varphi : Z' = \bigoplus \{Z_n : n \in \omega\} \to Z$. Thus φ^* embeds $C_p(Z)$ into $C_p(Z')$ (see Problems 113 and 163). Therefore P also embeds in $C_p(Z') = \prod\{C_p(Z_n) : n \in \omega\}$ (Problem 114). Let $\pi_n : C_p(Z') \to C_p(Z_n)$ be the natural projection. If the subspace $\pi_n(P) \subset C_p(Z_n)$ is metrizable for each $n \in \omega$ then the product $\prod\{\pi_n(P) : n \in \omega\}$ is also metrizable (207) and hence the space $P \subset \prod\{\pi_n(P) : n \in \omega\}$ is metrizable as well. This contradiction shows that there exists $n \in \omega$ such that the pseudocompact subspace $\pi_n(P)$ of the space $C_p(Z_n)$ is not metrizable so Fact 1 is proved.

Fact 2. If K is a countably compact space then any closed pseudocompact subspace of $C_p(K)$ is compact.

Proof. Let F be a closed pseudocompact subspace of $C_p(K)$. For any $x \in K$ the set $e_x(F) = \{f(x) : f \in F\} \subset \mathbb{R}$ is a continuous image of F (Problem 167); therefore $e_x(F)$ is compact being a pseudocompact metrizable space (Problem 212). It is easy to see that $F \subset Q = \prod\{e_x(F) : x \in K\}$; since Q is compact (Problem 125), it suffices to show that F is closed in $\mathbb{R}^K$.

Suppose not, and fix any $f \in [F] \backslash F$ (the brackets denote the closure in $\mathbb{R}^K$). Since F is closed in $C_p(K)$, the function f must be discontinuous so take any point $a \in K$

and $A \subset K$ such that $a \in \mathrm{cl}_K(A)$ while $f(a) \notin \overline{f(A)}$ (the bar denotes the closure in $\mathbb{R}$). Take $O, G \in \tau(\mathbb{R})$ such that $f(a) \in O, f(A) \subset G$ and $\overline{O} \cap \overline{G} = \emptyset$. We will construct sequences $\{f_n : n \in \omega\} \subset F$, $\{U_n : n \in \omega\} \subset \tau(a, K)$ and $\{a_n : n \in \omega\} \subset A$ with the following properties:

(1) $\mathrm{cl}_K(U_{n+1}) \subset U_n$ and $a_n \in U_n$ for all $n \in \omega$.
(2) $f_n(U_n) \subset O$ for all $n \in \omega$.
(3) $f_{n+1}(a_i) \in G$ for all $n \in \omega$ and $i \leqslant n$.

Since $f \in [F]$, there is $f_0 \in F$ such that $f_0(a) \in O$; the function f_0 being continuous there exists $U_0 \in \tau(a, K)$ such that $f_0(U_0) \subset O$. The point a belongs to the closure of A so there exists $a_0 \in A \cap U_0$. It is evident that (1)–(2) are fulfilled for a_0, f_0 and U_0. The property (3) is fulfilled vacuously.

Assume that we have a_i, f_i and U_i with the properties (1)–(3) for all $i \leqslant n$. Since $A_n = \{a_0, \dots, a_n\} \subset A$, we have $f(A_n) \subset G$; it follows from $f \in [F]$ that there exists $f_{n+1} \in F$ such that $f_{n+1}(a) \in O$ and $f_{n+1}(A_n) \subset G$. The function f_{n+1} being continuous there exists $U_{n+1} \in \tau(a, K)$ such that $\mathrm{cl}_K(U_{n+1}) \subset U_n$ and $f_{n+1}(U_{n+1}) \subset O$. Take any point $a_{n+1} \in U_{n+1} \cap A$ and observe that (1)–(3) are fulfilled for the sequence $\{a_i, f_i, U_i : i \leqslant (n+1)\}$ so our inductive construction can be carried out for all $n \in \omega$.

Once we have the sequences $\{f_n : n \in \omega\} \subset F$, $\{U_n : n \in \omega\} \subset \tau(a, K)$ and $S = \{a_n : n \in \omega\} \subset A$ with (1)–(3) take an accumulation point b of the sequence $\{a_n : n \in \omega\}$ (which exists because K is countably compact). Note that, for any $x \in K \backslash (\bigcap\{U_n : n \in \omega\})$ we have $x \in V = K \backslash \mathrm{cl}_K(U_n)$ for some $n \in \omega$ and hence V is a neighbourhood of x which intersects only finitely many points of the sequence S so x cannot be an accumulation point of S. This shows that $b \in P = \bigcap\{U_n : n \in \omega\} = \bigcap\{\mathrm{cl}_K(U_n) : n \in \omega\}$. If $Y = \{b\} \cup S$ then Y is countable so $\pi_Y(F) \subset C_p(Y)$ is a pseudocompact second countable space, i.e., $\pi_Y(F)$ is compact (see Problems 152, 210 and 138). Therefore there exists an accumulation point $g \in \pi_Y(F)$ of the sequence $\{g_n = \pi_Y(f_n) : n \in \omega\}$. Thus $g(b)$ has to be in the closure of the set $\{g_n(b) : n \in \omega\} = \{f_n(b) : n \in \omega\}$. But $f_n(b) \in f_n(P) \subset f_n(U_n) \subset O$ for all $n \in \omega$ so $\overline{\{f_n(b) : n \in \omega\}} \subset \overline{O}$ and hence $g(b) \in \overline{O}$.

On the other hand, it immediately follows from continuity of the function g that $g(b) \in \overline{\{g(a_n) : n \in \omega\}}$; since $f_k(a_n) \in G$ for all $k > n$, we have $g(a_n) \in \overline{G}$ for each $n \in \omega$. An immediate consequence is that $g(b) \in \overline{G}$, i.e., $g(b) \in \overline{O} \cap \overline{G} = \emptyset$ which is a contradiction. We proved that F is closed in $\mathbb{R}^K$ so F is compact and Fact 2 is proved.

Fact 3. Suppose that M_t is a second countable compact space for each $t \in T$. Given an arbitrary point $a \in M = \prod\{M_t : t \in T\}$, consider the space $\Sigma(a) = \{x \in M :$ the set $\mathrm{supp}(x) = \{t \in T : x(t) \neq a(t)\}$ is at most countable$\}$. Then $\Sigma(a)$ is a dense subspace of M such that, for any countable $A \subset \Sigma(a)$, we have $\mathrm{cl}_M(A) \subset \Sigma(a)$ and the space $\mathrm{cl}_M(A)$ is second countable. In particular, $\mathrm{cl}_M(A) = \mathrm{cl}_{\Sigma(a)}(A)$ is a compact metrizable space for any countable $A \subset \Sigma(a)$.

Proof. For any $W \subset T$ denote by $p_W : M \to M_W = \prod\{M_t : t \in W\}$ the natural projection to the face M_W defined by $p_W(x) = x|W$ for each $x \in M$. Let

$U = \prod_{t\in T} U_t \neq \emptyset$ be a standard open set in M, i.e., $U_t \in \tau(M_t)$ for all $t \in T$ and the set $S = \text{supp}(U) = \{t \in T : U_t \neq M_t\}$ is finite. Take any $x_t \in U_t$ for all $t \in S$ and let $x(t) = x_t$ if $t \in S$ and $x(t) = a(t)$ for all $t \in T\setminus S$. It is immediate that $x \in U \cap \Sigma(a)$ which shows that $\Sigma(a)$ is dense in M.

Given a point $x \in \Sigma(a)$, let $\text{supp}(x) = \{t \in T : x(t) \neq a(t)\}$; then $\text{supp}(x)$ is countable for each $x \in \Sigma(a)$. The set $E = \bigcup\{\text{supp}(x) : x \in A\}$ is also countable because $|A| \leqslant \omega$. The space $Y = \{\pi_{T\setminus E}(a)\} \times M_E \subset \Sigma(a)$ is homeomorphic to M_E so Y is a metrizable compact (and hence closed) subspace of M. It is evident that $A \subset Y$ so $\text{cl}_M(A) \subset Y$ which proves that $\text{cl}_M(A)$ is a metrizable compact subspace of $\Sigma(a)$. The rest is clear so Fact 3 is proved.

Fact 4. We have $w(Z) = nw(Z)$ for any compact space Z.

Proof. The inequality $nw(Z) \leqslant w(Z)$ is true for any space Z (Problem 156). If $nw(Z) = \kappa$ then $iw(Z) \leqslant nw(Z) = \kappa$ (Problem 156(iii)) and hence there is a condensation $f : Z \to Y$ of Z onto a space Y with $w(Y) \leqslant \kappa$. Every condensation of a compact space is a homeomorphism (Problem 123) so Z is homeomorphic to Y and hence $w(Z) = w(Y) \leqslant \kappa = nw(Z)$. Hence $w(Z) = nw(Z)$ and Fact 4 is proved.

Fact 5. Any continuous image of a metrizable compact space is a metrizable compact space.

Proof. If Z is a metrizable compact space then Z has a countable base (see Problems 209 and 212). If Z' is a continuous image of Z then Z' is a compact space with $w(Z') = nw(Z') \leqslant nw(Z) = w(Z) = w$ by Fact 4 and Problem 157(iii). Therefore Z' is metrizable by Problems 209 and 207. Fact 5 is proved.

Fact 6. Suppose that M_t is a second countable compact space for each $t \in T$ and let $M = \prod\{M_t : t \in T\}$. Then

(i) For any $a \in M$, if a compact space Z is a continuous image of the space $\Sigma(a) = \{x \in M : \text{the set } \text{supp}(x) = \{t \in T : x(t) \neq a(t)\} \text{ is at most countable}\}$ then Z is metrizable.
(ii) If a compact space Z of countable tightness is a continuous image of M then Z is metrizable.

Proof. (i) Take any continuous onto map $g : \Sigma(a) \to Z$; we can assume that $Z \subset \mathbb{I}^\kappa$ for some cardinal κ (Problem 209). For each $\alpha < \kappa$, let $q_\alpha : \mathbb{I}^\kappa \to \mathbb{I}$ be the natural projection of $\mathbb{I}^\kappa$ to its αth factor. Since $\Sigma(a)$ is dense in M (Fact 3), we can apply Problem 299 to the map $q_\alpha \circ g : \Sigma(a) \to \mathbb{I}$ to find a countable $S_\alpha \subset T$ and a continuous map $h_\alpha : \pi_{S_\alpha}(\Sigma(a)) \to \mathbb{I}$ such that $h_\alpha \circ (\pi_{S_\alpha}|\Sigma(a)) = q_\alpha \circ g$. It is immediate that $\pi_{S_\alpha}(\Sigma(a)) = M_{S_\alpha} = \prod\{M_t : t \in S_\alpha\}$ so h_α is defined on the space M_{S_α} and hence the map $G_\alpha = h_\alpha \circ \pi_{S_\alpha}$ is defined on the whole space M and $G_\alpha|\Sigma(a) = q_\alpha \circ g$.

For any $x \in M$, let $f(x)(\alpha) = G_\alpha(x)$ for all $\alpha < \kappa$. Then $f(x) \in \mathbb{I}^\kappa$, i.e., we have a map $f : M \to \mathbb{I}^\kappa$. Since $q_\alpha \circ f = G_\alpha$ is a continuous map for all $\alpha < \kappa$, the map f is continuous (Problem 102). If $x \in \Sigma(a)$ then $q_\alpha(f(x)) = G_\alpha(x) = q_\alpha \circ g(x) = q_\alpha(g(x))$

for all $\alpha < \kappa$ which implies $f(x) = g(x)$. Therefore, $f(\Sigma(a)) = g(\Sigma(a)) \subset Z$ and hence $f(M) \subset \overline{f(\Sigma(a))} = Z$ (the closure is taken in $\mathbb{I}^\kappa$ and it coincides with Z because Z is compact). We proved that there exists a continuous map $f : M \to Z$ such that $f|\Sigma(a) = g$; this makes it possible for us to forget about g and only use the map f until the end of the proof of (ii). Now it is easy to prove that Z has the following property:

($*$) For any countable $P \subset Z$ the space $\overline{P}$ is a compact metrizable (and hence a second countable) space.

Indeed, take any countable $A \subset \Sigma(a)$ with $f(A) \supset P$; it is possible to find such A because $f(\Sigma(a)) = Z$. Then $F = \mathrm{cl}_M(A)$ is a compact metrizable subset of $\Sigma(a)$ (Fact 3) and therefore $f(F)$ is a compact metrizable (by Fact 5) subspace of Z with $P \subset f(F)$. Now it is clear that $\overline{P} \subset f(F)$ is also a compact metrizable subset of Z so ($*$) is proved.

Our next step is to establish the property

($**$) If $R \subset Z$ and $|R| \leqslant \omega_1$ then there is a countable $P \subset Z$ such that $R \subset \overline{P}$ and, in particular, the space R is second countable.

To prove ($**$), take $A \subset \Sigma(a)$ such that $|A| \leqslant \omega_1$ and $f(A) \supset R$; then $|E| \leqslant \omega_1$ for the set $E = \bigcup\{\mathrm{supp}(x) : x \in A\}$ because $|A| \leqslant \omega_1$ and $\mathrm{supp}(x)$ is countable for any $x \in A$. The space $Y = \{\pi_{T\setminus E}(a)\} \times M_E$ is homeomorphic to M_E so Y is a product of $\leqslant \omega_1 \leqslant \mathfrak{c}$ of separable spaces; an immediate consequence of Problem 108 is that Y is separable. Pick any dense countable $B \subset Y$ and observe that $A \subset Y$; thus $R \subset f(Y)$ and hence $R \subset \overline{f(B)}$. Since $f(B)$ is a countable subset of Z, we can apply ($*$) to see that $\overline{f(B)} \supset R$ is second countable finishing the proof of ($**$).

Now, assume that Z is not separable and take any $z_0 \in Z$; if $\beta < \omega_1$ and we have constructed a set $C = \{z_\alpha : \alpha < \beta\}$ observe that C is countable and hence $\overline{C} \neq Z$, so we can choose $z_\beta \in Z\setminus\overline{\{z_\alpha : \alpha < \beta\}}$. It is clear that this inductive construction can be carried out for all $\beta < \omega_1$ and, as a result, we will have a set $Y = \{z_\alpha : \alpha < \omega_1\} \subset Z$ such that $z_\beta \notin \overline{\{z_\alpha : \alpha < \beta\}}$ for all $\beta < \omega_1$. Apply ($**$) to conclude that Y is second countable and hence separable; this implies that $Y \subset \overline{\{z_\alpha : \alpha < \beta\}}$ for some $\beta < \omega_1$. However, $z_\beta \notin Y\setminus\overline{\{z_\alpha : \alpha < \beta\}}$ which is a contradiction showing that Z is separable. If P is a countable dense subset of Z then apply ($*$) again to conclude that $Z = \overline{P}$ is second countable and hence metrizable.

(ii) Fix any continuous onto map $f : M \to Z$ and a point $a \in M$; consider the set $Q = f(\Sigma(a))$. It is clear that Q is dense in Z; in particular, for any $z \in Z$ we have $z \in \overline{Q}$ and hence $z \in \overline{R}$ for some countable $R \subset Q$ because $t(Z) \leqslant \omega$. Take any countable $A \subset \Sigma(a)$ with $f(A) \supset R$. Then $F = \mathrm{cl}_M(A)$ is a compact metrizable subset of $\Sigma(a)$ (Fact 3) and therefore $f(F)$ is a compact metrizable (by Fact 5) subspace of Z with $R \subset f(F)$. Thus $z \in \overline{R} \subset f(F) \subset Q$ which shows that $Q = Z$. Thus Z is a continuous image of $\Sigma(a)$ so we can apply (i) to conclude that Z is metrizable. This settles (ii) so Fact 6 is proved.

Now assume that $X = \prod\{X_t : t \in T\}$ where each X_t is separable; fix a countable dense $C_t \subset X_t$. It is straightforward that the space $C = \prod\{C_t : t \in T\}$ is dense in X. Let $C_t = \{c^t_n : n \in \omega\}$ be an enumeration of the set C_t for each $t \in T$. The space $C^t_n = \{c^t_i : i \leqslant n\}$ is compact being finite so $C_n = \prod\{C^t_n : t \in T\}$ is compact.

Our next step is to prove that $D = \bigcup\{C_n : n \in \omega\}$ is dense in C. Take any standard open set $U = \prod_{t\in T} U_t \neq \emptyset$ in the space X and let $B = \mathrm{supp}(U)$. For any $t \in \mathrm{supp}(U)$ the set C_t is dense in X_t so there is $n(t) < \omega$ such that $c^t_{n(t)} \in U_t$. If $m = \max\{n(t) : t \in \mathrm{supp}(U)\}$ then $C^t_m \cap U_t \neq \emptyset$ for all $t \in \mathrm{supp}(U)$. As a consequence, $D \cap (U \cap C) \supset C_m \cap (U \cap C) \neq \emptyset$ which proves that D is dense in C and hence in X. This means that the map $\pi_D : C_p(X) \to \pi_D(C_p(X)) \subset C_p(D)$ is a condensation.

Suppose that P is a non-metrizable pseudocompact subspace of $C_p(X)$. Then $\pi_D(P)$ is a non-metrizable pseudocompact subspace of $C_p(D)$ because any condensation of a pseudocompact space onto a metrizable space is a condensation onto a second countable space (Problem 212) and hence a homeomorphism (Problem 140). Thus the space $C_p(D)$ has a non-metrizable pseudocompact subspace and we have $D = \bigcup\{C_n : n \in \omega\}$; apply Fact 1 to conclude that $C_p(C_n)$ has a non-metrizable pseudocompact subspace Q for some $n \in \omega$. It is an easy exercise to show that any space, which has a dense pseudocompact subspace, is pseudocompact, so $\overline{Q}$ is a closed pseudocompact subspace of $C_p(C_n)$. Since C_n is compact, Fact 2 is applicable to see that $F = \overline{Q}$ is a compact subspace of $C_p(C_n)$.

For each $x \in C_n$ define a map $\varphi(x) : F \to \mathbb{R}$ by $\varphi(x)(f) = f(x)$ for all $f \in F$. Then $\varphi : C_n \to C_p(F)$ is a continuous map (Problem 166) and the compact space $K = \varphi(C_n)$ has countable tightness because $t(C_p(F)) = \omega$ (Problem 149). Since C_n is a product of second countable spaces, Fact 6 is applicable to conclude that K is a metrizable compact space and hence $w(K) = \omega$. It is easy to see that K separates the points of F. For each $y \in F$ define a map $\psi(y) : K \to \mathbb{R}$ by $\psi(y)(f) = f(y)$ for all $f \in K$. Then $\psi : F \to C_p(K)$ is a continuous map (Problem 166) and it immediately follows from the fact that K separates the points of F, that ψ is a condensation. Any condensation of a compact space is a homeomorphism so ψ embeds F into $C_p(K)$ and hence $w(F) = nw(F) \leqslant nw(C_p(K)) = nw(K) \leqslant w(K) = \omega$ (we applied Fact 4 and Problem 172) so F is second countable and hence metrizable. As a consequence, $Q \subset F$ is also metrizable which is a contradiction showing that our solution is complete.

S.308. *Let X be an arbitrary product of separable spaces. Suppose that Y is a dense subspace of X. Is it true that every compact subspace of $C_p(Y)$ is metrizable?*

Solution. No, it is not true. Let $Y = C_p(K)$ where $K = A(\omega_1)$. Then Y is a dense subspace of $X = \mathbb{R}^K$ (Problem 111) and it is clear that X is a product of second countable (and hence separable) spaces. The compact space K is not metrizable because it is not second countable. Thus a non-metrizable compact space K embeds in $C_p(Y) = C_p(C_p(K))$ (Problem 167) which shows that not every compact subspace of $C_p(Y)$ is metrizable.

S.309. *Suppose that $C_p(X)$ has a dense σ-pseudocompact subspace. Does it necessarily have a dense σ-countably compact subspace?*

Solution. No, not necessarily. To construct a relevant example, let us first establish several facts.

Fact 1. Let cZ be a compact extension of a space Z such that, for any function $f \in C(Z, \mathbb{I})$ there is $g \in C(cZ, \mathbb{I})$ such that $g|Z = f$. Then there exists a homeomorphism $\varphi : cZ \to \beta Z$ such that $\varphi(z) = z$ for every $z \in Z$.

Proof. Take any compact space Y and any continuous map $u : Z \to Y$. We can consider that Y is a subspace of $\mathbb{I}^A$ for some A (Problem 209). For each $a \in A$ let $p_a : \mathbb{I}^A \to \mathbb{I}$ be the natural projection onto the ath factor. The map $p_a \circ u : Z \to \mathbb{I}$ is continuous for each $a \in A$ and hence there is a continuous function $g_a : cZ \to \mathbb{I}$ such that $g_a|Z = p_a \circ u$. Letting $g(z)(a) = g_a(z)$ for each $z \in cZ$ and $a \in A$ we define a map $g : cZ \to \mathbb{I}^A$; the map g is continuous because $p_a \circ g = g_a$ is a continuous map for every $a \in A$ (Problem 102). If $z \in Z$ then $g(z)(a) = g_a(z) = p_a \circ u(z) = u(z)(a)$ for all $a \in A$ and therefore $g(z) = u(z)$ which shows that $g|Z = u$. The set $g(Z)$ is dense in $g(cZ)$ and Y is closed in $\mathbb{I}^A$ so $g(cZ) \subset \overline{g(Z)} = \overline{u(Z)} \subset \overline{Y} = Y$ and hence $g : cZ \to Y$ is a continuous extension of u. Now apply Problem 258 to finish the proof of Fact 1.

Fact 2. Let M_t be a second countable compact space for all $t \in T$. Suppose that Z is a dense pseudocompact subspace of $M = \Pi\{M_t : t \in T\}$. Then $\beta Z = M$ in the sense that there exists a homeomorphism $\varphi : \beta Z \to M$ such that $\varphi(z) = z$ for any $z \in Z$.

Proof. Given $S \subset T$, denote by $p_S : M \to M_S = \Pi\{M_t : t \in S\}$ the natural projection to the S-face M_S of the space M. Take any continuous map $f : Z \to \mathbb{I}$. There exists a countable $S \subset T$ and a continuous map $g : p_S(Z) \to \mathbb{I}$ such that $g \circ p_S = f$ (Problem 299). Observe that $p_S(Z)$ is second countable and hence compact; since $p_S(Z)$ is dense in M_S, we have $p_S(Z) = M_S$. This shows that the continuous map $h = g \circ p_S$ is defined on the whole space M and $h|P = f$. It turns out that M is a compact extension of Z such that any continuous $f : Z \to \mathbb{I}$ has a continuous extension $h : M \to \mathbb{I}$ so we can apply Fact 1 to conclude that $M = \beta Z$. Fact 2 is proved.

Fact 3. Let Z be a pseudocompact space. Then $Z \cap H \neq \emptyset$ for any non-empty G_δ-set H of the space βZ.

Proof. Pick any $z \in H$ and $\{U_n : n \in \omega\} \subset \tau(\beta Z)$ such that $H \bigcap \{U_n : n \in \omega\}$. It is routine to construct a sequence $\{V_n : n \in \omega\} \subset \tau(z, \beta Z)$ such that $\overline{V}_{n+1} \subset V_n$ and $V_n \subset U_0 \cap \cdots \cap U_n$ for each $n \in \omega$. If $W_n = V_n \cap Z$ then $W_n \neq \emptyset$ and $\mathrm{cl}_Z(W_{n+1}) \subset W_n$ for each $n \in \omega$. The space Z being pseudocompact, we have $P = \bigcap\{\mathrm{cl}_Z(W_n) : n \in \omega\} = \bigcap\{W_n : n \in \omega\} \neq \emptyset$ (Problem 136) and hence $H \cap Z \supset P \neq \emptyset$ so Fact 3 is proved.

Fact 4. If Z is a pseudocompact space let $\pi : C_p(\beta Z) \to C_p(Z)$ be the restriction map, i.e., $\pi(h) = h|Z$ for each $h \in C_p(\beta Z)$. Then π is a bijection and hence, for any function $f \in C_p(Z)$ there exists a unique $e(f) \in C_p(\beta Z)$ such that $e(f)|Z = f$.

Proof. Since Z is pseudocompact, each $f \in C_p(Z)$ is bounded and hence f maps Z into a compact space $[-n, n]$ for some $n \in \omega$. Apply Problem 257 to conclude that there exists $e(f) \in C(\beta Z)$ with $e(f)|Z = f$. The uniqueness of $e(f)$ follows from the fact that Z is dense in βZ and hence the map π is injective (Problem 152). Fact 4 is proved.

Fact 5. Given a pseudocompact space Z, denote by $e : C_p(Z) \to C_p(\beta Z)$ the extension map from Fact 4. Then, for any $f \in C_p(Z)$ and any countable set $A \subset C_p(Z)$ with $f \in \overline{A}$, we have $e(f) \in [e(A)]$ (the brackets denote the closure in $C_p(\beta Z)$). In particular, if f is an accumulation point of A then $e(f)$ is an accumulation point of $e(A)$.

Proof. Given a function $g \in C_p(\beta Z)$, an arbitrary finite set $K \subset \beta Z$ and $\varepsilon > 0$, let $O(g, K, \varepsilon) = \{h \in C_p(\beta Z) : |g(x) - i(x)| < \varepsilon$ for all $x \in K\}$. It is clear that the family $\mathcal{B}_g = \{O(g, K, \varepsilon) : K$ is a finite subset of βZ and $\varepsilon > 0\}$ is a local base of $C_p(\beta Z)$ at g. Assume that $g = e(f) \notin [e(A)]$ and fix a finite set $K \subset \beta Z$ and $\varepsilon > 0$ such that $O(g, K, \varepsilon) \cap e(A) = \emptyset$. For each point $x \in K$ the set $H_x = (\bigcap\{h^{-1}(h(x)) : h \in e(A)\}) \cap g^{-1}(g(x))$ is a G_δ-subset of βZ with $x \in H_x$. Apply Fact 3 to observe that $H_x \cap Z \neq \emptyset$ and hence we can choose $z_x \in Z \cap H_x$ for each $x \in K$. Then $L = \{z_x : x \in K\} \subset Z$ and $W = \{h \in C_p(Z) : |h(y) - f(y)| < \varepsilon$ for all $y \in L\}$ is an open neighbourhood of the function f in $C_p(Z)$. If $h \in A$ then $h(z_x) = e(h)(x)$ for each $x \in K$. There exists $x \in K$ such that $|e(h)(x) - g(x)| \geqslant \varepsilon$ whence $|h(z_x) - f(z_x)| \geqslant \varepsilon$ so $h \notin W$. As a consequence $W \cap A = \emptyset$ which is a contradiction with $f \in \overline{A}$. Fact 5 is proved.

Fact 6. Given a pseudocompact space Z, denote by $e : C_p(Z) \to C_p(\beta Z)$ the extension map from Fact 4. Then $e(P)$ is a countably compact subset of $C_p(\beta Z)$ for each countably compact $P \subset C_p(Z)$.

Proof. If $e(P)$ is not countably compact then there is a countably infinite closed discrete $B \subset e(P)$. The set $A = \pi(B) \subset P$ is infinite (Fact 4), so it has an accumulation point f in the countably compact space P. By Fact 5, $e(f) \in e(P)$ is an accumulation point of the set $B = e(A)$ which is a contradiction. Fact 6 is proved.

We are now ready for presenting the solution. In Fact 4 of S.286 it was proved that there exists a pseudocompact space X with the following properties:

(1) X is a dense subspace of $\mathbb{I}^{\mathfrak{c}}$.
(2) $C_p(X, \mathbb{I})$ is pseudocompact.
(3) Every countable $B \subset X$ is closed and discrete so X is not second countable.

Since $[-n, n]$ is homeomorphic to $\mathbb{I}$, the space $C_p(X, [-n, n])$ is homeomorphic to $C_p(X, \mathbb{I})$ for each $n \in \mathbb{N}$; as a consequence, $C_p(X, [-n, n])$ is pseudocompact for each $n \in \mathbb{N}$ so the space $C_p(X) = \bigcup\{C_p(X, [-n, n]) : n \in \mathbb{N}\}$ is σ-pseudocompact.

Now take any countably compact subspace P of the space $C_p(X)$. Observe that $\beta X = \mathbb{I}^{\mathfrak{c}}$ by (1) and Fact 2. Let $e : C_p(X) \to C_p(\beta X)$ be the extension map constructed in Fact 4 for $Z = X$. Apply Fact 6 to conclude that $e(P)$ is a countably compact subspace of $C_p(\beta X) = C_p(\mathbb{I}^{\mathfrak{c}})$. The space $e(P)$ has to be metrizable and second countable by Problem 307 and 212. The map $\pi : e(P) \to P$ is continuous so $nw(P) \leqslant nw(e(P)) \leqslant w(P) = \omega$ (see Problem 157(iii)). By Problem 156(i) the space P is separable and hence we proved that every countably compact subspace of $C_p(X)$ is separable. It is evident that any countable union of separable spaces is separable, so if $C_p(X)$ has a dense σ-countably compact subspace then $C_p(X)$ is separable. As a consequence X can be condensed onto a second countable space (Problem 174). This condensation must be a homeomorphism by Problem 140 so X is second countable which is a contradiction. Thus $C_p(X)$ has no dense σ-countably compact subspace and our solution is complete.

S.310. *Suppose that $C_p(X)$ has a dense σ-countably compact subspace. Does it necessarily have a dense σ-compact subspace?*

Solution. No, not necessarily. To construct a relevant example, let us first establish some facts. Say that Z is *a P-space* if every G_δ-subset of Z is open. Call a space Z ω-bounded if $\overline{A}$ is compact for any countable $A \subset Z$.

Fact 1. Every ω-bounded space is countably compact.

Proof. If A is an infinite subset of an ω-bounded space Z then take any countably infinite $B \subset A$. Then $\overline{B}$ is compact so B has an accumulation point b in $\overline{B}$. It is clear that b is also an accumulation point of A in Z so Z is countably compact (Problem 132).

Fact 2. If Z is a P-space then $C_p(Z, \mathbb{I})$ is ω-bounded and hence countably compact.

Proof. Since $\mathbb{I}^Z$ is compact and $C_p(Z, \mathbb{I}) \subset \mathbb{I}^Z$, it suffices to prove that $\overline{A} \subset C_p(Z, \mathbb{I})$ for every countable $A \subset C_p(Z, \mathbb{I})$ (the bar denotes the closure in $\mathbb{I}^Z$). So take any countable $A \subset C_p(Z, \mathbb{I})$ and any $f \in \overline{A}$. Given any $z \in Z$, we will prove that f is continuous at the point z. Note that the set $\{h(z)\}$ is a G_δ-set in the space $\mathbb{I}$ for any $h \in A$ so $W = \bigcap\{h^{-1}(h(z)) : h \in A\}$ is a G_δ-set in Z. Since Z is a P-space, the set W is an open neighbourhood of z; we claim that $f(W) = \{f(z)\}$. To see this, suppose that $w \in W$ and $|f(w) - f(z)| > \varepsilon$ for some $\varepsilon > 0$. Since $f \in \overline{A}$, there is $h \in A$ such that $|h(z) - f(z)| < \frac{\varepsilon}{2}$ and $|h(w) - f(w)| < \frac{\varepsilon}{2}$. However, $h(w) = h(z)$ so we have $|f(w) - f(z)| \leqslant |f(w) - h(w)| + |h(z) - f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ which is a contradiction. Thus for any $\varepsilon > 0$ we have $f(W) = \{f(z)\} \subset (f(z) - \varepsilon, f(z) + \varepsilon)$, i.e., f is continuous at z. Fact 2 is proved.

Fact 3. The set $C^*(Z)$ is dense in $C_p(Z)$ for any space Z.

Proof. It suffices to show that, for any function $f \in C_p(Z)$ and any finite $K \subset Z$ there is $g \in C^*(Z)$ such that $g|K = f|K$. Let $N = \max\{|f(z)| : z \in K\} + 1$; define $c_N, d_N \in C(Z)$ by the equalities $c_N(z) = N$ and $d_N(z) = -N$ for all $z \in Z$. There exists $h \in C_p(X)$ such that $h(x) = f(x)$ for any $x \in K$ (Problem 034). The function $h_1 = \min(h, c_N)$ is continuous (Problem 028) and $h_1(z) \leqslant N$ for all $z \in Z$. Finally, $g = \max(h_1, d_N)$ is also continuous (Problem 028) and $g(z) \in [-N, N]$ for all $z \in Z$, i.e., $g \in C^*(Z)$. Note finally that $g(z) = h(z) = f(z)$ for all $z \in K$ so Fact 3 is proved.

Fact 4. If Z is a P-space then $C_p(Z)$ has a dense σ-countably compact subspace.

Proof. The space $C_p(Z, \mathbb{I})$ is countably compact by Fact 2. Since $[-n, n]$ is homeomorphic to $\mathbb{I}$ for each $n \in \mathbb{N}$, the space $C_n = C_p(Z,[-n, n])$ is countably compact for all $n \in \mathbb{N}$. As a consequence the set $\bigcup\{C_n : n \in \mathbb{N}\} = C^*(Z)$ is σ-countably compact and dense in $C_p(Z)$ by Fact 3 so Fact 4 is proved.

Fact 5. If $C_p(Z)$ has a dense σ-compact subspace then there is a compact subspace of $C_p(Z)$ which separates the points of Z.

Proof. The space $C = C_p(Z, (0, 1))$ is homeomorphic to $C_p(Z)$ (Fact 1 of S.295) so there is a set $D = \bigcup\{K_n : n \in \mathbb{N}\} \subset C$ such that D is dense in C and K_n is compact for all $n \in \mathbb{N}$. Let $L_n = \left\{\frac{1}{n} \cdot f : f \in K_n\right\}$ for each $n \in \mathbb{N}$. It is easy to see that L_n is compact being homeomorphic to the compact space K_n. Denote by u_0 the function which is identically zero on Z; we claim that $K = \bigcup\{L_n : n \in \mathbb{N}\} \cup \{u_0\}$ is a compact subset of $C_p(Z)$ which separates the points of Z.

To see that K is compact, take any open cover $\mathcal{U}$ of the space K. Take an arbitrary $U \in \mathcal{U}$ with $u_0 \in U$; there exists a finite set $P \subset Z$ and $\varepsilon > 0$ such that $W = \{f \in C_p(Z) : |f(z)| < \varepsilon$ for all $z \in P\} \subset U$. There is $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$; if $n \geqslant m$ then $|f(z)| < \frac{1}{n} \leqslant \frac{1}{m} < \varepsilon$ for any $f \in L_n$ and $z \in Z$. Thus, $L_n \subset U$ for all $n \geqslant m$. This shows that the space $K \backslash U$ is compact being a closed subspace of the compact space $L_1 \cup \cdots \cup L_m$. Therefore, there is a finite $\mathcal{U}' \subset \mathcal{U}$ with $\bigcup \mathcal{U}' \supset K \backslash U$. Hence $\mathcal{U}' \cup \{U\}$ is a finite subcover of K which proves that K is compact.

If $z, t \in Z$ and $z \neq t$ then $V = \{f \in C_p(Z, (0, 1)) : f(z) \neq f(t)\}$ is an open subset of $C_p(Z, (0, 1))$. Since D is dense in $C_p(Z, (0, 1))$, there is $n \in \mathbb{N}$ and $f \in K_n$ such that $f(z) \neq f(t)$. Now if $g = \frac{1}{n}$ then $g \in K$ and $g(z) \neq g(t)$; thus K separates the points of Z and Fact 5 is proved.

Fact 6. Let K be a compact space. Then $|\overline{A}| \leqslant \mathfrak{c}c$ for any countable $A \subset C_p(K)$.

Proof. Define a map $\varphi : K \to \mathbb{R}^A$ by the formula $\varphi(z)(f) = f(z)$ for every function $f \in A$. It is evident that φ is a continuous map so $L = \varphi(K)$ is a second countable compact space. Therefore $nw(C_p(L)) = nw(L) = \omega$ which implies $iw(C_p(L)) = \omega$ (Problem 156(iii)). This shows that there is an injective map of $C_p(L)$ into a second countable space Y which in turn embeds in $\mathbb{I}^\omega$ (see Problem 209). As a consequence $|C_p(L)| \leqslant |Y| \leqslant |\mathbb{I}^\omega| = \mathfrak{c}^\omega = \mathfrak{c}$. Let $\varphi^*(f) = f \circ \varphi$ for each function $f \in C_p(L)$. It was proved in Problem 163 that φ^* is an embedding; the set $F = \varphi^*(C_p(L))$ is closed in $C_p(Z)$ because φ is a closed map (see Problems 163(iii) and 122). It is immediate that $A \subset F$ so $\overline{A} \subset F$ and hence $|\overline{A}| \leqslant |F| \leqslant \mathfrak{c}$ so Fact 6 is proved.

Returning to our solution take any set A of cardinality $\mathfrak{c}$ and consider the space X whose underlying set is $\mathbb{I}^A$ and whose topology consists of all G_δ-sets of $\mathbb{I}^A$. It is an easy exercise to see that X is Tychonoff space. Since a countable intersection of G_δ-sets is still a G_δ-set, the space X is a P-space. Therefore the space $C_p(X)$ has a dense σ-countably compact subspace by Fact 4. Consider the space $\Sigma = \{x \in \mathbb{I}^A :$ the set $\{a \in A : x(a) \neq 0\}$ is countable$\}$. Since $|\mathbb{I}| = \mathfrak{c}$, we have $|\mathbb{I}^B| = \mathfrak{c}^\omega = \mathfrak{c}$ for any countable set B. Therefore, for any countable $B \subset A$, the set $G_B = \{x \in \mathbb{I}^A : x(a) = 0$ for all $a \in A \backslash B\}$ has cardinality $\mathfrak{c}$. Since $\Sigma = \bigcup\{G_B : B$ is a countable subset of $A\}$, we have $|\Sigma| = \mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$.

Recall that a set $U = \prod_{a \in A} U_a$ is called *standard in* $\mathbb{I}^A$ if $U_a \in \tau(\mathbb{I})$ for all $a \in A$ and the set $\text{supp}(U) = \{a \in A : U_a \neq \mathbb{I}\}$ is finite. Standard sets form a base in $\mathbb{I}^A$ (Problem 101). The set Σ is dense in X; to see this take any point $x \in X$ and any G_δ-set $H \ni x$. There are standard sets U_n, $n \in \omega$ such that $x \in G = \bigcap\{U_n : n \in \omega\} \subset H$. If $B = \bigcup\{\text{supp}(U_n) : n \in \omega\}$ then B is a countable subset of A and hence the point $y \in \mathbb{I}^A$

defined by $y|B = x|B$ and $y(a) = 0$ for all $a \in A \setminus B$, belongs to Σ. It is immediate that $y \in G \subset H$ so $y \in \Sigma \cap H$ which proves that Σ is dense in X.

Suppose that $C_p(X)$ has a σ-compact dense subspace. Apply Fact 5 to find a compact $K \subset C_p(X)$ which separates the points of X. Define a map $e : X \to C_p(K)$ by the formula $e(x)(f) = f(x)$ for all $f \in K$. The map e is continuous (Problem 166) and it follows immediately from the fact that K separates the points of X that e is a condensation onto $Y = e(X)$. Therefore $|Y| = |X| = 2^{\mathfrak{c}}$; since Σ is dense in X, the set $S = e(\Sigma)$ is dense in Y. Tightness of the space $C_p(K)$ is countable (Problem 149) so $Y \subset \bigcup \{\overline{P} : P$ is a countable subset of $S\}$. However, $|\,\overline{P}\,| \leqslant \mathfrak{c}$ for any countable $P \subset C_p(K)$ by Fact 6. This shows that $|Y| \leqslant \Sigma\{|\,\overline{P}\,| : P$ is a countable subset of $S\} \leqslant \mathfrak{c} \cdot |S^\omega| = \mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$ which is a contradiction. Thus $C_p(X)$ has no dense σ-compact subset and our solution is complete.

S.311. *Prove that any space X is a continuous image of a space Y such that $C_p(Y)$ has a dense σ-compact subspace.*

Solution. Let Y be the discrete space with the underlying set X. Then the identity map $i : Y \to X$ is continuous and onto so X is a continuous image of Y. Observe that $C_n = C_p(Y, [-n, n]) = [-n, n]^Y$ is compact for any $n \in \mathbb{N}$. Hence $C^*(Y) = \bigcup \{C_n : n \in \mathbb{N}\}$ is a dense σ-compact subspace of $C_p(Y)$ (see Fact 3 of S.310).

S.312. *Is it true that any space X is an $\mathbb{R}$-quotient image of a space Y such that $C_p(Y)$ has a dense σ-compact subspace?*

Solution. Yes, it is true. We will prove even more, namely, that any space X is a quotient image of a space Y such that $C_p(Y)$ has a dense σ-compact subspace. Since every quotient map is $\mathbb{R}$-quotient, this will give a solution in a stronger form.

Call a topological property $\mathcal{P}$ *complete* if it satisfies the following conditions:

(1) Any metrizable compact space has $\mathcal{P}$.
(2) If $n \in \mathbb{N}$ and Z_i has $\mathcal{P}$ for all $i = 1, \dots, n$ then $Z_1 \times \cdots \times Z_n$ has $\mathcal{P}$.
(3) If Z has $\mathcal{P}$ then every continuous image of Z has $\mathcal{P}$.

For an arbitrary space A, denote by A^0 the one-point space $\{0\} \subset \mathbb{R}$ and let $\mathcal{C}(A) = \{A^m \times Y : m \in \omega$ and Y is a second countable compact space$\}$.

Given a set $A \subset C_p(Z)$, let $P(A) = \{f_1 \cdot \dots \cdot f_n : n \in \mathbb{N}, f_i \in A$ for all $i \leqslant n\}$ and $R(A) = \{\lambda_0 + \lambda_1 \cdot g_1 + \cdots + \lambda_m \cdot g_m : m \in \mathbb{N}, \lambda_i \in \mathbb{R}$ and $g_i \in P(A)$ for all $i \leqslant m\}$.

Fact 1. For any $A \subset C_p(Z)$ the space $R(A)$ is an algebra in $C_p(Z)$ which contains A and can be represented as a countable union of continuous images of spaces from $\mathcal{C}(A)$.

Proof. It is evident that $R(A)$ is an algebra and $A \subset R(A)$; given $m \in \mathbb{N}$ and $n_0, \dots, n_m \in \omega$, the space $P(n_0, \dots, n_m) = [-n_0, n_0]^{m+1} \times A^{n_1} \times \cdots \times A^{n_m}$ belongs to $\mathcal{C}(A)$. For each $i \in \omega$, define a map $p_i : A^i \to C_p(X)$ as follows: $p_0(0)$ is the function identically zero on Z; $p_1(f) = f$ for all $f \in A = A^1$ and $p_n(f_1, \dots, f_n) = f_1 \cdot \dots \cdot f_n$ for each $(f_1, \dots, f_n) \in A^n$ for all $n > 1$.

Now, we are going to define a map $\varphi = \varphi_{n_0}, \ldots, n_m : P(n_0, \ldots, n_m) \to C_p(Z)$ as follows: for every point $(\lambda, f) = (\lambda, f^1, \ldots, f^m) \in P(n_0, \ldots, n_m)$, where $\lambda = (\lambda_0, \ldots, \lambda_m) \in [-n_0, n_0]^{m+1}$ and $f^i = \left(f_1^i, \ldots, f_{n_i}^u\right) \in A^{n_i}$ for each natural number $i \leqslant m$, let $\varphi(\lambda, f) = \lambda_0 + \lambda_1 \cdot p_{n_1}(f^1) + \cdots + \lambda_m \cdot p_{n_m}(f^m) \in C_p(Z)$. It easily follows from the results in 115 and 116 that the map $\varphi_{n_0, \ldots, n_m}$ is continuous for any $(n_0, \ldots, n_m)$. If we denote by $Q(n_0, \ldots, n_m)$ the image of $P(n_0, \ldots, n_m)$ under the map $\varphi_{n_0, \ldots, n_m}$ then it is clear that $R(A) = \bigcup \{Q(n_0, \ldots, n_m) : m \in \mathbb{N}$ and $n_0, \ldots, n_m \in \omega\}$ so Fact 1 is proved.

Fact 2. Let $\mathcal{P}$ be a complete property. If $A \subset C_p(Z)$ and A has $\mathcal{P}$ then $R(A)$ has σ-$\mathcal{P}$, i.e., $R(A)$ is a countable union of spaces with the property $\mathcal{P}$.

Proof. It is straightforward that all elements of $\mathcal{C}(A)$ have $\mathcal{P}$. Fact 1 says that $R(A)$ is a countable union of continuous images of spaces from $\mathcal{C}(A)$ and every such image also has $\mathcal{P}$ by (3), so Fact 2 is proved.

Fact 3. For any space Z, there is a compact $K \subset C_p(Z)$ which separates the points of Z if and only if $C_p(Z)$ has a σ-compact dense subspace.

Proof. Sufficiency is proved in Fact 5 of S.310. It is evident that compactness is a complete property so, if $K \subset C_p(Z)$ is compact then the algebra $R(K)$ must be σ-compact by Fact 2. Furthermore, if K separates the points of Z then $R(K)$ is an algebra which also separates the points of Z; applying 192 we conclude that $R(K)$ is dense in $C_p(Z)$. Therefore $R(K)$ is a σ-compact dense subspace of $C_p(Z)$ and Fact 3 is proved.

Given a space X and $a \in X$, let X_a be a space with the underlying set X in which every point $x \in X\backslash\{a\}$ is isolated and $\tau(a, X)$ is the local base at the point a. The space X_a is Tychonoff for each $a \in X$ and a is the unique non-isolated point of X_a (see Fact 1 of S.293). Now let $Y = \bigoplus\{X_a \times \{a\} : a \in X\}$; we consider $X_a \times \{a\}$ to be a clopen subspace of Y (see Problem 113). Given $y \in Y$ we have $y = (x, a)$ for some $a \in X$; let $\varphi(y) = x$.

The map $\varphi : Y \to X$ is quotient. It is immediate that φ is continuous and onto. Take any $U \subset X$ such that $\varphi^{-1}(U)$ is open in Y; for any $a \in U$ the set $U \times \{a\} = \varphi^{-1}(U) \cap (X_a \times \{a\})$ has to be a neighbourhood of the point (a, a) in Y and hence U has to be a neighbourhood of a in X_a. Thus U is a neighbourhood of any $a \in U$, i.e., U is open in X. This proves that the map f is quotient.

If $B \subset Y$, denote by χ_B the characteristic function of B defined by $\chi_B(y) = 1$ if $y \in B$ and $\chi_B(y) = 0$ for all $y \in Y\backslash B$. It is an easy exercise that χ_B is a continuous function on Y if and only if B is a clopen subset of Y. Given $y \in Y$, we write χ_y instead of $\chi_{\{y\}}$. The function χ_y is continuous if y is an isolated point of Y. Denote by u_0 the function which is identically zero on Y; let $U(a) = X_a \times \{a\}$ for all $a \in X$ and let $K = \{u_0\} \cup \{\chi_y : y = (x, a)$ and $x \neq a\} \cup \{\chi_{U(a)} : a \in X\}$. It is straightforward that $K \subset C_p(Y)$; let us check that K separates the points of Y.

If $y \neq z$, $y \in U(a)$, $z \notin U(a)$ for some $a \in X$, then $f = \chi_{U(a)} \in K$ and $f(y) = 1 \neq 0 = f(z)$. Now if there is $a \in X$ with $y, z \in U(a)$ then one of the points, say y, is distinct from (a, a) and therefore $g = \chi_y \in K$ and $g(y) = 1 \neq 0 = g(z)$.

Let us establish that K is compact. Given an open cover $\mathcal{U}$ of the set K, take any $U \in \mathcal{U}$ with $u_0 \in U$. There exists a finite $P \subset Y$ and $\varepsilon > 0$ such that $\{h \in C_p(Y) : |h(z)| < \varepsilon$ for all $z \in P\} \subset U$. Let $\gamma_K = \{B \subset Y : \chi_B \in K\}$; it is easy to see that every $y \in Y$ belongs to at most two elements of γ_K. This implies that there is a finite $Q \subset K$ such that $f|P \equiv 0$ for all $f \in K \backslash Q$ and therefore $K \backslash Q \subset U$. The set Q being finite, there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that $Q \subset \bigcup \mathcal{U}'$. Then $\mathcal{U}' \cup \{U\}$ is a finite subcover of the cover $\mathcal{U}$; this proves that K is compact.

Finally, apply Fact 3 to conclude that $C_p(Y)$ has a dense σ-compact subspace and finish our solution.

S.313. *Let X be a metrizable space. Prove that $C_p(X)$ has a dense σ-compact subspace.*

Solution. Take any base $\mathcal{B}$ of the space X such that $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ where each $\mathcal{B}_n$ is a discrete family (see Problem 221). Take an arbitrary metric d on X such that $\tau(d) = \tau(X)$ and $d(x, y) \leqslant 1$ for all $x, y \in X$ (see Problem 206). For each $U \in \mathcal{B}$ let $\varphi_U(x) = \inf\{d(x, y) : y \in X \backslash U\}$. The function φ_U is continuous on X and $\varphi_U^{-1}(0) = X \backslash U$ for all $U \in \mathcal{B}$ (see Fact 1 of S.212). Denote by u_0 the function identically zero on X and let $K_n = \{\varphi_U : U \in \mathcal{B}_n\} \cup \{u_0\}$ for all $n \in \omega$.

Let us prove that K_n is compact for every $n \in \omega$. Take any open cover $\mathcal{U}$ of the set K_n; pick any $W \in \mathcal{U}$ with $u_0 \in W$. There is a finite $P \subset X$ and $\varepsilon > 0$ such that $\{h \in C_p(X) : |h(x)| < \varepsilon$ for all $x \in P\} \subset W$. The family $\mathcal{B}_n$ is discrete and hence only finitely many elements of $\mathcal{B}_n$ intersect the set P; fix a finite $\gamma \subset \mathcal{B}_n$ such that $U \cap P = \emptyset$ for all $U \in \mathcal{B}_n \backslash \gamma$. This implies $\varphi_U(x) = 0$ for all $x \in P$ and $U \in \mathcal{B}_n \backslash \gamma$, i.e., $\varphi_U \in W$ for all $U \in \mathcal{B}_n \backslash \gamma$. Since the family γ is finite, there exists a finite $\mathcal{U}' \subset \mathcal{U}$ such that $\varphi_U \in \bigcup \mathcal{U}'$ for all $U \in \gamma$. As a consequence, $\mathcal{U}' \cup \{W\}$ is a finite subcover of the cover $\mathcal{U}$ so K_n is compact.

If x and y are distinct points of X then there is $U \in \mathcal{B}$ such that $x \in U$ and $y \notin U$. Then $\varphi_U(x) \neq 0 = \varphi_U(y)$ This shows that $S = \bigcup\{K_n : n \in \omega\}$ separates the points of X. Therefore the algebra $R(S)$ is dense in $C_p(X)$ (see S.312 for the definition of $R(S)$ and Problem 192 to be convinced that $R(S)$ is dense in $C_p(X)$). It is immediate that the property of being σ-compact is complete (see the definition of a complete property in S.312) and hence Fact 2 of S.312 can be applied to conclude that $R(S)$ is a countable union of σ-compact spaces, i.e., $R(S)$ is σ-compact. Thus $R(S)$ is a dense σ-compact subspace of $C_p(X)$ and our solution is complete.

S.314. *Show that the space ω_1 is countably compact and $(\omega_1 + 1)$ is compact. Prove that, for every continuous function $f: \omega_1 \to \mathbb{R}$, there exists $\alpha_0 < \omega_1$ such that $f(\alpha) = f(\alpha_0)$ for every $\alpha \geqslant \alpha_0$. Deduce from this fact that $\beta\omega_1 = \omega_1 + 1$.*

Solution. Call a space Z ω-bounded if $\overline{A}$ is compact for any countable $A \subset Z$.

Fact 1. Every ω-bounded space is countably compact.

Proof. If A is an infinite subset of an ω-bounded space Z then take any countably infinite $B \subset A$. Then $\overline{B}$ is compact so B has an accumulation point b in $\overline{B}$. It is clear that b is also an accumulation point of A in Z so Z is countably compact (Problem 132). Fact 1 is proved.

In Fact 2 of S.232 it was proved that the set $[0, \alpha] = \{\beta < \omega_1 : \beta \leqslant \alpha\}$ is a compact subspace of ω_1 for each $\alpha < \omega_1$. Since every countable $A \subset \omega_1$ is contained in $[0, \alpha]$ for some $\alpha < \omega_1$, the space ω_1 is ω-bounded and hence countably compact by Fact 1. The space $\omega_1 + 1$ is compact because it is well ordered and has the largest element ω_1 (Problem 306).

Now take any continuous function $f : \omega_1 \to \mathbb{R}$. Assume that, for any $\alpha < \omega_1$, there is $\beta = \beta(\alpha) > \alpha$ such that $f(\beta) \neq f(\alpha)$; let us also fix $n = n(\alpha) \in \mathbb{N}$ with $|f(\beta) - f(\alpha)| \geqslant \frac{1}{n}$. There exists $m \in \mathbb{N}$ and an uncountable $A \subset \omega_1$ such that $n(\alpha) = m$ for each $\alpha \in A$. Take any $\alpha_0 \in A$ and $\beta_0 = \beta(\alpha_0)$; if we have α_i, β_i for all $i \leqslant n$, find $\alpha_{n+1} \in A$ with $\alpha_{n+1} > \max\{\alpha_i, \beta_i : i \leqslant n\}$ and let $\beta_{n+1} = \beta(\alpha_{n+1})$. This inductive construction gives us sequences $\{\alpha_i : i \in \omega\}$ and $\{\beta_i : i \in \omega\}$ such that $\beta_i = \beta(\alpha_i)$ and $\beta_i < \alpha_{i+1}$ for all $i \in \omega$. Let $\alpha = \min\{\beta < \omega_1 : \alpha_i < \beta$ for all $i \in \omega\}$. It is evident that α is well defined; the function f being continuous at the point α, there is $\beta < \alpha$ such that $|f(\gamma) - f(\alpha)| < \frac{1}{2m}$ for all $\gamma \in (\beta, \alpha)$. There is $n \in \omega$ such that $\beta < \alpha_n < \beta_n$ which implies that

$$|f(\alpha_n) - f(\beta_n)| \leqslant |f(\alpha_n) - f(\alpha)| + |f(\beta_n) - f(\alpha)| < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m},$$

a contradiction with $|f(\alpha_n) - f(\beta_n)| \geqslant \frac{1}{m}$. This contradiction proves that, for some α_0, there will be no $\alpha > \alpha_0$ with $f(\alpha) \neq f(\alpha_0)$, i.e., $f(\alpha) = f(\alpha_0)$ for all $\alpha \geqslant \alpha_0$.

Finally, let $f : \omega_1 \to \mathbb{I}$ be a continuous function; we proved that there is $\alpha_0 < \omega_1$ such that $r_0 = f(\alpha) = f(\alpha_0)$ for all $\alpha \geqslant \alpha_0$. Letting $g(\omega_1) = r_0$ and $g|\omega_1 = f|\omega_1$ we obtain a continuous function $g : (\omega_1 + 1) \to \mathbb{I}$ such that $g|\omega_1 = f|\omega_1$. Since $\omega_1 + 1$ is a compact extension of ω_1, we can apply Fact 1 of S.309 to conclude that $\omega_1 + 1 = \beta\omega_1$ and finish our solution.

S.315. *Prove that $C_p(\omega_1)$ has no dense σ-compact subspace.*

Solution. Assume that $C_p(\omega_1)$ has a dense σ-compact subspace. Apply Fact 3 of S.312 to conclude that there is a compact $K \subset C_p(\omega_1)$ which separates the points of ω_1. Let $\pi : C_p(\omega_1 + 1) \to C_p(\omega_1)$ be the restriction map, i.e., $\pi(f) = f|\omega_1$ for every $f \in C_p(\omega_1 + 1)$. Since ω_1 is countably compact and $\omega_1+1 = \beta\omega_1$ (314), for each $f \in C(\omega_1)$ there exists a unique $e(f) \in C(\omega_1 + 1)$ such that $e(f)|\omega_1 = f$ (Fact 4 of S.309). In particular, π is an onto map. Observe that $e(K)$ is a countably compact subset of $C_p(\omega_1 + 1)$ by Fact 6 of S.309; besides, $e(K) = \pi^{-1}(K)$ is closed in $C_p(\omega_1 + 1)$ because the map π is continuous (Problem 152). Thus $L = e(K)$ is compact by Fact 2 of S.307 and it is immediate that L separates the points of ω_1, i.e., for any distinct $\alpha, \beta < \omega_1$, there is $f \in L$ such that $f(\alpha) \neq (\beta)$. By Fact 3 of S.256, the set L cannot separate the points of $\omega_1 + 1$ so there is $\alpha < \omega_1$ such that $f(\alpha) = f(\omega_1)$ for all $f \in L$. Now, if $\beta < \omega_1$ and $\beta \neq \alpha$ then there is $g \in L$ such that $g(\alpha) \neq g(\beta)$. Since $g(\omega_1) = g(\alpha)$, we have $g(\beta) \neq g(\omega_1)$ and hence L separates all pairs of points except α and ω_1. Taking any $h \in C_p(\omega_1 + 1)$ with $h(\alpha) = 0$ and

$h(\omega_1) = 1$, we obtain a compact set $L \cup \{h\}$ which separates the points of $\omega_1 + 1$; this is again a contradiction with Fact 3 of S.256 so our solution is complete.

S.316. *Prove that $C_p(\omega_1)$ is Lindelöf.*

Solution. For any function $f \in C_p(\omega_1)$ and a finite subset K of ω_1, we will need the set $O(f, K, \varepsilon) = \{g \in C_p(\omega_1) : |f(x) - g(x)| < \varepsilon$ for all $x \in K\}$ for every $\varepsilon > 0$. The family $\mathcal{U} = \{O(f, K, \varepsilon) : f \in C_p(\omega_1), K$ is a finite subset of ω_1 and $\varepsilon > 0\}$ is a base in $C_p(\omega_1)$ and $\mathcal{U}_f = \{O(f, K, \varepsilon) : K$ is a finite subset of ω_1 and $\varepsilon > 0\}$ is a local base at f in the space $C_p(\omega_1)$. Given $U = O(f, K, \varepsilon) \in \mathcal{U}$, let $\mathrm{supp}(U) = \{\alpha < \omega_1 :$ there is $\beta \in K$ such that $\alpha \leqslant \beta\}$. Observe that $\mathrm{supp}(U)$ can be identified with the ordinal $\beta + 1$ where $\beta = \max\{\alpha : \alpha \in K\}$. For technical reasons, we let $\mathrm{supp}(U) = 0$ if $U = \emptyset$. For any $\alpha < \omega_1$, let $r_\alpha(\beta) = \beta$ if $\beta \leqslant \alpha$ and $r_\alpha(\beta) = \alpha$ for all $\beta > \alpha$. It is evident that $r_\alpha : \omega_1 \to (\alpha + 1)$ is a continuous map; the map $r^*_\alpha : C_p(\alpha + 1) \to C_p(\omega_1)$, defined by $r^*_\alpha(f) = f \circ r_\alpha$ for each $f \in C_p(\alpha + 1)$, is also continuous (Problem 163). If $R_\alpha = r^*_\alpha\big(C_p(\alpha + 1)\big)$ then R_α is a closed subset of $C_p(\omega_1)$ homeomorphic to the space $C_p(\alpha + 1)$ for each ordinal $\alpha < \omega_1$ (Problem 163). Observe also that the map $s_\alpha : C_p(\omega_1) \to C_p(\omega_1)$ defined by $s_\alpha(f) = r^*_\alpha(f|(\alpha + 1))$ is continuous and $s_\alpha(C_p(\omega_1)) = R_\alpha$; besides $s_\alpha(f) = f$ for any $f \in R_\alpha$.

Denote by $\mathcal{O}$ the family of all non-trivial rational intervals in $\mathbb{R}$ and let $\mathcal{B}_\alpha$ be a countable base in the space $(\alpha + 1)$. Given $U_1, \ldots, U_n \in \mathcal{B}_\alpha$ and $O_1, \ldots, O_n \in \mathcal{O}$, let $M_\alpha(U_1, \ldots, U_n; O_1, \ldots, O_n) = \big\{f \in R_\alpha : f\big(r^{-1}_\alpha(U_i)\big) \subset O_i$ for each $i \leqslant n\big\}$.

Let us prove that

(1) The family $\mathcal{N}_\alpha = \{M_\alpha(U_1, \ldots, U_n; O_1, \ldots, O_n) : n \in \mathbb{N}, U_i \in \mathcal{B}_\alpha$ and $O_i \in \mathcal{O}$ for all $i \leqslant n\}$ is a countable network in the space R_α.

It is evident that $\mathcal{N}_\alpha$ is countable. Take any $f \in R_\alpha$, any finite $K \subset \omega_1$ and any $\varepsilon > 0$. Then $K = K_0 \cup K_1$ where $K_0 = K \cap \alpha = \{x_1, \ldots, x_n\}$ and $K_1 = K \backslash K_0$. For any $i \leqslant n$ pick a rational interval J_i such that $f(x_i) \in J_i \subset (f(x_i) - \varepsilon, f(x_i) + \varepsilon)$; there exists a rational interval J_{n+1} such that $f(\alpha) \in J_{n+1} \subset (f(\alpha) - \varepsilon, f(\alpha) + \varepsilon)$. Use continuity of f to find a disjoint family $\gamma = \{U_i : i \leqslant n + 1\} \subset \mathcal{B}_\alpha$ such that $x_i \in U_i$ for all $i \leqslant n$, $\alpha \in U_{n+1}$ and $f(U_i) \subset J_i$ for all $i \leqslant n + 1$. Observing that $r^{-1}_\alpha(U_i) = U_i$ for each $i \leqslant n$ and $f(x) = f(\alpha)$ for any $x \in K_1$, we convince ourselves that $f \in M_\alpha(U_1, \ldots, U_n, U_{n+1}; J_1, \ldots, J_n, J_{n+1}) \subset O(f, K, \varepsilon)$ which proves that $\mathcal{N}_\alpha$ is a network in R_α.

We will also need the following property of the map s_α.

(2) If $U \in \mathcal{U}$ and $\mathrm{supp}(U) \subset \alpha$ then $s_\alpha(U) = U \cap R_\alpha$ and $s^{-1}_\alpha(U \cap R_\alpha) = U$.

Let $U = O(f, K, \varepsilon)$; then $K \subset \alpha$ and we have $s_\alpha(g) = g$ for any $g \in R_\alpha$ so it is evident that $U \cap R_\alpha \subset s_\alpha(U)$. Now, if $g \in U$ then $s_\alpha(g)(\beta) = g(\beta)$ for all $\beta \leqslant \alpha$ and $s_\alpha(g)(\beta) = g(\alpha)$ for all $\beta > \alpha$. This shows that $s_\alpha(g)(x) = g(x)$ for any $x \in K$ and therefore $s_\alpha(U) \subset U$, i.e., $s_\alpha(U) = U \cap R_\alpha$. If we take any $h \in C_p(\omega_1)$ with $s_\alpha(h) \in U$ then $|s_\alpha(h)(x) - f(x)| < \varepsilon$ for all $x \in K$; but $s_\alpha(h)(x) = h(x)$ for all $x \in K$ so $|h(x) - f(x)| < \varepsilon$ for all $x \in K$ and hence $h \in U$. Property (2) is proved.

Suppose that, for each function $f \in C_p(\omega_1)$, we are given $U_f \in \mathcal{U}$ such that $f \in U_f$; the open cover $\{U_f : f \in C_p(\omega_1)\}$ is called *pointwise*. Now, if $\mathcal{V}$ is an arbitrary open cover of $C_p(\omega_1)$ then, for each $f \in C_p(\omega_1)$, there is $U_f \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $f \in U_f$

$\subset V$. This shows that every open cover of $C_p(\omega_1)$ has a pointwise refinement, so to prove the Lindelöf property of $C_p(\omega_1)$, it suffices to show that any pointwise open cover of $C_p(\omega_1)$ has a countable subcover.

Fix $U_f \in \mathcal{U} \cap \tau(f, C_p(\omega_1))$ for each function $f \in C_p(\omega_1)$ and consider the family $\mathcal{W} = \{U_f : f \in C_p(\omega_1)\}$. For each element N of the countable family $\mathcal{N}_\omega$ choose $W(N) \in \mathcal{W}$ with $N \subset W(N)$; if such $W(N)$ does not exist then let $W(N) = \emptyset$. Let $\nu_0 = \bigcup\{\text{supp}(W(N) : N \in \mathcal{N}_\omega\}$ and $\alpha_0 = \max\{\omega, \nu_0\} + 1$.

If we have a sequence $\alpha_0 < \cdots < \alpha_m$, choose a set $W(N) \in \mathcal{W}$ such that $N \subset W(N)$ for each $N \in \mathcal{N}_{\alpha_m}$; if such $W(N)$ does not exist then let $W(N) = \emptyset$. Let $\nu_{m+1} = \bigcup\{\text{supp}(W(N) : N \in \mathcal{N}_{\alpha_m}\}$ and $\alpha_{m+1} = \max\{\alpha_m, \nu_{m+1}\} + 1$. The ordinal $\alpha = \sup\{\alpha_m : m \in \omega\}$ and the family $\mathcal{V} = \{W(N) : N \in \mathcal{N}_{\alpha_m}$ for some $m \in \omega\}$ have the following properties:
(3) The family $\mathcal{V}$ is a countable subfamily of $\mathcal{W}$ and $\text{supp}(V) \subset \alpha$ for all $V \in \mathcal{V}$.
(4) $\{s_\alpha(V) : V \in \mathcal{V}\}$ is a cover of R_α.

Since (3) is evident, we will only prove the property (4). Take any function $f \in R_\alpha$; then $U_f = O(f, K, \varepsilon)$ for some finite $K \subset \omega_1$ and $\varepsilon > 0$. Consider the set $K_0 = K \cap \alpha = \{x_1, \ldots, x_n\}$; then $K_0 \subset \alpha_m$ for some $m < \omega$ and there exist rational intervals $J_1, \ldots, J_{n+1}$ such that $f(x_i) \in J_i \subset (f(x_i) - \varepsilon, f(x_i) + \varepsilon)$ for all $i \leqslant n$ and $f(\alpha) \in J_{n+1} \subset (f(\alpha) - \varepsilon, f(\alpha) + \varepsilon)$. Since the function f is continuous, there exist disjoint $U_1, \ldots, U_{n+1} \in \tau(\alpha + 1)$ such that $x_i \in U_i$ for all $i \leqslant n$, $\alpha \in U_{n+1}$ and $f(U_i) \subset J_i$ for all $j \leqslant n + 1$. There exists $\beta < \alpha$ such that $(\beta, \alpha] = \{\nu < \omega_1 : \beta < \nu \leqslant \alpha\} \subset U_{n+1}$. Take any $k \geqslant m$ such that $\beta < \alpha_k$ and find $W \in \mathcal{B}_{\alpha k}$ with $\alpha_k \in W \subset (\beta, \alpha_k]$. It is easy to see that $f(\alpha) \in f\left(r_{\alpha_k}^{-1}(W)\right) \subset J_{n+1}$ and hence $f \in N \in \mathcal{N}_{\alpha_k}$ where $N = M_{\alpha_k}(U_1, \ldots, U_n, W, J_1, \ldots, J_n, J_{n+1}) \subset U_f$ which shows that $f \in N \subset W(N)$ and the property (4) is proved.

Now it is very easy to finish our solution. Observe that $C_p(\omega_1) = s_\alpha^{-1}(R_\alpha) = s_\alpha^{-1}(\bigcup\{V \cap R_\alpha : V \in \mathcal{V}\}) = \bigcup\{s_\alpha^{-1}(V \cap R_\alpha) : V \in \mathcal{V}\} = \bigcup \mathcal{V}$ because we have $s_\alpha^{-1}(V \cap R_\alpha) = V$ for each $V \in \mathcal{V}$ by the property (2). Thus, $\mathcal{V}$ is a countable subcover of $\mathcal{W}$ and our solution is complete.

S.317. *Prove that* $C_p(\omega_1 + 1)$ *does not have a dense Lindelöf subspace.*

Solution. Any dense subspace of $C_p(\omega_1 + 1)$ separates the points of $(\omega_1 + 1)$ and there is no Lindelöf subspace of $C_p(\omega_1 + 1)$ which separates the points of $(\omega_1 + 1)$ by Fact 3 of S.256.

S.318. *Prove that* $C_p(\omega_1 + 1)$ *embeds into* $C_p(\omega_1)$. *Is it possible to embed* $C_p(\omega_1)$ *into* $C_p(\omega_1 + 1)$?

Solution. Define a map $\varphi : \omega_1 \to (\omega_1 + 1)$ as follows: $\varphi(0) = \omega_1$, $\varphi(n) = n - 1$ for all $n \in \omega$ and $\varphi(\alpha) = \alpha$ for all $\alpha \geqslant \omega$. It is immediate that $\varphi : \omega_1 \to (\omega_1 + 1)$ is a continuous onto map and hence φ^* embeds $C_p(\omega_1 + 1)$ into $C_p(\omega_1)$ (Problem 163).

It is not possible to embed $C_p(\omega_1)$ into $C_p(\omega_1 + 1)$ because $t(C_p(\omega_1 + 1)) = \omega$ due to compactness of $(\omega_1 + 1)$ (see Problems 314 and 149) while tightness of $C_p(\omega_1)$ is uncountable because ω_1 is not Lindelöf (see Problems 138, 149, 159(vi) and 314).

S.319. *Prove that $C_p(\omega_1 + 1)$ is a Fréchet–Urysohn space.*

Solution. It turns out that second countable continuous images of $(\omega_1 + 1)$ are countable.

Fact 1. Given a second countable space M, let $\varphi : (\omega_1 + 1) \to M$ be a continuous onto map. Then M is countable.

Proof. If $x = \varphi(\omega_1)$ then there is a sequence $\{U_n : n \in \omega\} \subset \tau(M)$ such that $\{x\} = \bigcap\{U_n : n \in \omega\}$. By continuity of φ at the point ω_1, there is $\alpha_n < \omega_1$ such that $f((\alpha_n, \omega_1]) \subset U_n$ for each $n \in \omega$. For $\alpha = \sup\{\alpha_n : n \in \omega\} + 1$ we have $\varphi((\alpha, \omega_1]) \subset \bigcap\{U_n : n \in \omega\} = \{x\}$ i.e., $\varphi(\beta) = \varphi(\alpha)$ for all $\beta \geqslant \alpha$. This shows that $M = \varphi([0, \alpha])$ so M is countable. Fact 1 is proved.

Now take any $A \subset C_p(\omega_1)$ and any $f \in \overline{A}$. Tightness of $C_p(\omega_1 + 1)$ is countable by Problem 149, so there is a countable set $B \subset A$ with $f \in \overline{B}$. Let $\varphi(x)(g) = g(x)$ for all $g \in B$ and $x \in (\omega_1 + 1)$. Then $\varphi : (\omega_1 + 1) \to C_p(B)$ is a continuous map (Problem 166). The space $M = \varphi(C_p(\omega_1 + 1))$ is second countable because $C_p(B)$ is second countable (Problem 169). Applying Fact 1, we conclude that M is countable and hence $\varphi^*(C_p(M))$ is a closed second countable subspace of $C_p(\omega_1 + 1)$ (Problem 163(iii)). It is straightforward that $B \subset \varphi^*(C_p(M))$ so $\overline{B} \subset \varphi^*(C_p(M))$ which shows that $\overline{B}$ is second countable. The space $\{f\} \cup B$ is also second countable and hence Fréchet–Urysohn; this makes it possible to find a sequence $\{f_n : n \in \omega\} \subset B$ with $f_n \to f$. Since $B \subset A$, we have $\{f_n : n \in \omega\} \subset A$ so, for every $A \subset C_p(\omega_1)$ and every $f \in \overline{A}$, we found a sequence $\{f_n : n \in \omega\} \subset A$ with $f_n \to f$. This proves that $C_p(\omega_1 + 1)$ is a Fréchet–Urysohn space so our solution is complete.

S.320. *Prove that $C_p(\omega_1 + 1)$ is not normal.*

Solution. We will need the following statement.

Fact 1. There exists a closed discrete uncountable subspace in the space $C_p(\omega_1 + 1)$, i.e., $\text{ext}(C_p(\omega_1 + 1)) = \omega_1$.

Proof. Let $f_\alpha(\beta) = 1$ for all ordinals $\beta \leqslant \alpha$ and $f_\alpha(\beta) = 0$ for all ordinals $\beta > \alpha$. It is clear that $f_\alpha \in C_p(\omega_1 + 1)$ for all $\alpha < \omega_1$; we will prove that $D = \{f_\alpha : \alpha < \omega_1\}$ is a closed discrete subspace of $C_p(\omega_1 + 1)$.

The set D is closed. Indeed, take any $f \in C_p(\omega_1 + 1)\backslash D$. If there is some point $x \in (\omega_1 + 1)$ with $f(x) \notin \{0, 1\}$ then $O_f = \{g \in C_p(\omega_1 + 1) : g(x) \neq 1 \text{ and } g(x) \neq 0\}$ is an open set in $C_p(\omega_1 + 1)$ such that $f \in O_f \subset C_p(\omega_1 + 1)\backslash D$. Thus, if $f \in \overline{D}$ then $f(\omega_1 + 1) \subset \{0,1\}$. Now if $\alpha < \beta$, $f(\alpha) = 0$ and $f(\beta) = 1$ then $U_f = \{g \in C_p(\omega_1 + 1) : g(\alpha) \in \left(-\frac{1}{2}, \frac{1}{2}\right), g(\beta) \in \left(\frac{1}{2}, \frac{3}{2}\right)\} \in \tau\big(f, C_p(\omega_1 + 1)\big)$ and $f \in U_f \subset C_p(\omega_1 + 1)\backslash D$. This shows that if $f(\beta) = 1$ for some $\beta \leqslant \omega_1$ then $f(\alpha) = 1$ for all $\alpha \leqslant \beta$; hence if $f(\omega_1) = 1$ then $f \equiv 1$. In this case consider the set $W_f = \{g \in C_p(\omega_1 + 1) : g(\omega_1) > 0\} \in \tau(f, C_p(\omega_1 + 1))$ and note that $f \in W_f \in C_p(\omega_1 + 1)\backslash D$. Finally, if $\alpha = \sup\{\beta : f(\beta) = 1\}$ then $f(\alpha) = 1$ by continuity of f and hence $f = f_\alpha$ which is a contradiction showing that D is closed in $C_p(\omega_1 + 1)$.

To see that set D is discrete, take an arbitrary $f_\alpha \in D$ and consider the set $W_\alpha = \{g \in C_p(\omega_1 + 1) : g(\alpha) > 0$ and $g(\alpha + 1) < 1\}$. It is immediate that we have $W_\alpha \in \tau(C_p(\omega_1 + 1))$ and $W_\alpha \cap D = \{f_\alpha\}$ so D is discrete and Fact 1 is proved.

Returning to our solution, suppose for contradiction that $C_p(\omega_1 + 1)$ is normal. Fix and uncountable closed and discrete $D \subset C_p(\omega_1 + 1)$ which exists by Fact 1. Apply Problem 295 to conclude that the space $C_p(\omega_1 + 1)$ is collectionwise normal; therefore there exists a disjoint family $\mathcal{U} = \{U_d : d \in D\} \subset \tau(C_p(\omega_1 + 1))$ such that $d \in U_d$ for all $d \in D$. This clearly contradicts the fact that $c(C_p(\omega_1 + 1)) = \omega$ (Problem 111) and shows that $C_p(\omega_1 + 1)$ is not normal so our solution is complete.

S.321. *Let X be an arbitrary space. Supposing that all compact subspaces of $C_p(X)$ are first countable, prove that they are all metrizable.*

Solution. If we are given a function $f \in C_p(X)$, $\varepsilon > 0$ and a finite set $K \subset X$, let $O(f, K, \varepsilon) = \{g \in C_p(X) : |f(x) - g(x)| < \varepsilon$ for all $x \in K\}$. Then the family $\mathcal{U}_f = \{O(f, K, \varepsilon) : K$ is a finite subset of X and $\varepsilon > 0\}$ is a local base at f in the space $C_p(X)$.

Take any compact set $P \subset C_p(X)$. The space $P \times P$ is also compact; if a mapping $\varphi : P \times P \to C_p(X)$ is defined by the formula $\varphi(f, g) = f - g$ for all $f, g \in P$, then φ is a continuous map (see Problems 115 and 116) so the space $L = \varphi(P \times P)$ is compact and hence first countable. Denote by w the function which is identically zero on X; it is evident that $w \in L$. All compact subspaces of $C_p(X)$ are first countable so $\chi(w, L) \leqslant \omega$; therefore there exists a sequence $\{U_n = O(w, K_n, \varepsilon_n) : n \in \omega\} \subset \mathcal{U}_w$ such that $(\bigcap\{U_n : n \in \omega\}) \cap L = \{w\}$. The set $A = \bigcup\{K_n : n \in \omega\}$ is countable and hence the restriction map $\pi_A : C_p(X) \to C_p(A)$ maps $C_p(X)$ into a second countable space $C_p(A)$ (see Problems 152 and 169). If $f, g \in P$ and $f \neq g$ then there is $n \in \omega$ such that $f - g \notin U_n$ which implies that $f|K_n \neq g|K_n$ and therefore $\pi_A(f) \neq \pi_A(g)$. This shows that $\pi_A : P \to \pi_A(P)$ is a condensation; since every condensation of a compact space is a homeomorphism (Problem 123), the space P is second countable so our solution is complete.

S.322. *Does there exist a space X such that all countably compact subspaces of $C_p(X)$ are first countable but not all of them are metrizable?*

Solution. No, such a space does not exist, i.e., if all countably compact subsets of $C_p(X)$ are first countable then they are all metrizable. Given a function $f \in C_p(X)$, a number $\varepsilon > 0$ and a finite set $K \subset X$, we let $O(f, K, \varepsilon) = \{g \in C_p(X) : |f(x) - g(x)| < \varepsilon$ for all $x \in K\}$. Then the family $\mathcal{U}_f = \{O(f, K, \varepsilon) : K$ is a finite subset of the space X and $\varepsilon > 0\}$ is a local base at f in the space $C_p(X)$. A square of a countably compact space can fail to be countably compact but the following fact shows that countable compactness is often preserved by finite products.

Fact 1. Let Z and T be Fréchet–Urysohn countably compact spaces. Then $Z \times T$ is countably compact.

Proof. Let $p_Z : Z \times T \to Z$ be the natural projection. Suppose that $Z \times T$ is not countably compact and take any countably infinite closed discrete $D \subset Z \times T$. Since T is countably compact and $T_z = \{z\} \times T$ is homeomorphic to T for all $z \in Z$, the set

$D \cap T_z$ has to be finite for each $z \in Z$. Analogously, if $Z_t = Z \times \{t\}$ then $D \cap Z_t$ is finite for all $t \in T$. This shows that there is an infinite $E \subset D$ such that the sets $E \cap T_z$ and $E \cap Z_t$ have each one at most one point for all $z \in Z$ and $t \in T$. In particular, the set $p_Z(E)$ is infinite and hence it has an accumulation point $a \in Z$. Since the space Z is Fréchet–Urysohn, we can choose a sequence $\{(z_n, t_n) : n \in \omega\} \subset E$ and a point $a \in Z$ such that $z_n \to a$ and $z_n \neq a$ for all $n \in \omega$. The set $\{t_n : n \in \omega\} \subset T$ is infinite so there is $b \in T$ and an increasing sequence $\{n_k : k \in \omega\} \subset \omega$ such that $t_{n_k} \to b$ and $t_{n_k} \neq b$ for all $k \in \omega$. If we let $a_k = z_{n_k}$ and $b_k = t_{n_k}$ for all $k \in \omega$ then it is straightforward that $S = \{(a_k, b_k) : k \in \omega\} \subset D$ and the sequence S converges to the point $e = (a, b) \notin S$, i.e., e is an accumulation point of D which is a contradiction. Fact 1 is proved.

Take any countably compact $P \subset C_p(X)$. The space $P \times P$ is also countably compact because P is first countable (Fact 1). If $\varphi : P \times P \to C_p(X)$ is defined by $\varphi(f, g) = f - g$ for all $f, g \in P$, then φ is a continuous map (see Problems 115 and 116) so the space $L = \varphi(P \times P)$ is countably compact and hence first countable. Denote by w the function which is identically zero on X; it is evident that $w \in L$. All countably compact subspaces of $C_p(X)$ are first countable so $\chi(w, L) \leqslant \omega$; therefore there exists a sequence $\{U_n = O(w, K_n, \varepsilon_n) : n \in \omega\} \subset \mathcal{U}_w$ such that $(\bigcap\{U_n : n \in \omega\}) \cap L = \{w\}$. The set $A = \bigcup\{K_n : n \in \omega\}$ is countable and hence the restriction map $\pi_A : C_p(X) \to C_p(A)$ maps $C_p(X)$ into a second countable space $C_p(A)$ (see Problems 152 and 169). If $f, g \in P$ and $f \neq g$ then there is $n \in \omega$ such that $f - g \notin U_n$ which implies that $f|K_n \neq g|K_n$ and therefore $\pi_A(f) \neq \pi_A(g)$. This shows that $\pi_A : P \to \pi_A(P)$ is a condensation; since every condensation of a countably compact space onto a second countable space is a homeomorphism (Problem 140), the space P is second countable so our solution is complete.

S.323. *Suppose that all countably compact subspaces of $C_p(X)$ are metrizable. Is the same true for all pseudocompact subspaces of $C_p(X)$?*

Solution. No, this is not true. In Fact 4 of S.286 it was proved that there exists a pseudocompact space X with the following properties:

(1) X is a dense subspace of $\mathbb{I}^{\mathfrak{c}}$ and hence X is infinite.
(2) $C_p(X, \mathbb{I})$ is pseudocompact.
(3) Every countable $B \subset X$ is closed and discrete so X is not second countable.

The set $C_p(X, \mathbb{I})$ is a pseudocompact non-metrizable subspace of $C_p(X)$ because otherwise the space $C_p(X, \mathbb{I})$ would be compact implying $C_p(X, \mathbb{I}) = \mathbb{I}^X$ and discreteness of X which is a contradiction because only finite discrete spaces can be pseudocompact.

Now take any countably compact subspace P of the space $C_p(X)$. Observe that $\beta X = \mathbb{I}^{\mathfrak{c}}$ by (1) and Fact 2 of S.309. Let $e : C_p(X) \to C_p(\beta X)$ be the extension map constructed in Fact 4 of S.309 for $Z = X$. Apply Fact 6 of S.309 to conclude that $e(P)$ is a countably compact subspace of $C_p(\beta X) = C_p(\mathbb{I}^{\mathfrak{c}})$. The space $e(P)$ has to be metrizable and second countable by Problems 307 and 212. The map $\pi : e(P) \to P$ is continuous so $nw(P) \leqslant nw(e(P)) \leqslant w(P) = \omega$ (see Problem 157(iii)).

By Problem 156(iii), the space P condenses onto a second countable space and hence P is second countable itself (Problem 140). This shows that every countably compact subspace of $C_p(X)$ is second countable and hence metrizable so our solution is complete.

S.324. *Is it true that any compact space X can be embedded into $C_p(Y)$ for some pseudocompact space Y?*

Solution. No, it is not true. To see this, take the space $X = \{0,1\}^{\omega_1}$. Suppose that $X \subset C_p(Y)$ for some pseudocompact Y. Given any $y \in Y$, let $\varphi(y)(f) = f(y)$ for any $f \in X$ (remember that $X \subset C_p(Y)$ and hence each element of X is a function defined on Y). The map $\varphi : Y \to Z = \varphi(Y) \subset C_p(X)$ is continuous (Problem 166) and Z is a pseudocompact subspace of $C_p(X)$ which separates the points of X. Since X is a product of second countable spaces, Z has to be second countable (Problem 307 and S.307). However, Fact 5 of S.256 (where the letter D is used to denote the space $X = \{0,1\}^{\omega_1}$) says that no Lindelöf subspace of $C_p(X)$ separates the points of X. This contradiction shows that X can not be embedded into $C_p(Y)$.

S.325. *Is it true that any compact space X can be embedded into $C_p(Y)$ for some space Y with $c(Y) = \omega$?*

Solution. Yes, this is true. To see it, observe that any space X embeds into $C_p(Y)$ where $Y = C_p(X)$ (see Problem 167). Since $c(Y) = c(C_p(X)) = \omega$ for any space X (Problem 111), we conclude that any (not necessarily compact) space X can be embedded into $C_p(Y)$ for some space Y which has the Souslin property.

S.326. *Is it true that any compact space X can be embedded into $C_p(Y)$ for some space Y with* $\text{ext}(Y) = \omega$?

Solution. No, it is not true for the space $X = \omega_1 + 1$. Given $\alpha \in X$ and $H \in \tau(\mathbb{R})$, let $O(\alpha, H) = \{f \in C_p(X) : f(\alpha) \in H\}$. Given any set R, denote by $\text{Fin}(R)$ the family of all finite subsets of R. We first prove a simple fact which will be used in this solution and needed for further references.

Fact 1. Let Z be any space; suppose that $\mathcal{F}$ is a family of compact subsets of Z and $G = \cap\, \mathcal{F}$. Then, for any $U \in \tau(G, Z)$ there is a finite $\mathcal{F}' \subset \mathcal{F}$ such that $\cap\, \mathcal{F}' \subset U$.

Proof. The set $H = Z \setminus U$ is closed so $F \cap H$ is compact for any $F \in \mathcal{F}$. For the family $\mathcal{G} = \{F \cap H : F \in \mathcal{F}\}$, we have $\cap \mathcal{G} = (\cap \mathcal{F}) \cap H = G \cap H = \emptyset$ so $\mathcal{G}$ cannot be centered (Problem 118). Thus there is a finite $\mathcal{F}' \subset \mathcal{F}$ such that $\cap \{F \cap H : F \in \mathcal{F}'\} = \emptyset$ which implies $(\cap\, \mathcal{F}') \cap H = \emptyset$ and hence $\cap\, \mathcal{F}' \subset U$ so Fact 1 is proved.

Suppose that $\text{ext}(Y) = \omega$ and X is embedded in $C_p(Y)$. For each $y \in Y$ let $\varphi(y)(f) = f(y)$ for every $f \in X$ (remember that the elements of X are continuous functions on Y). The mapping $\varphi : Y \to L = \varphi(Y) \subset C_p(X)$ is continuous (Problem 166) and the set L separates the points of $\omega_1 + 1 = X$. Since extent cannot be increased in a continuous image (Problem 157(v)), we have $\text{ext}(L) = \omega$.

Since the map $f \longmapsto (-f)$ is a homeomorphism of $C_p(\omega_1 + 1)$ onto itself, the sets $-L = \{-f : f \in L\}$ and $L \cup (-L)$ also have countable extent (it is an easy exercise that a union of two (or even countably many) spaces of countable extent is again a space of countable extent). This shows that we can assume that $(-f) \in L$ for any $f \in L$.

For each $\alpha < \omega_1$ fix rational numbers s_α, t_α and a function $f_\alpha \in L$ such that $f_\alpha(\alpha) < s_\alpha < t_\alpha < f_\alpha(\omega_1)$ or $f_\alpha(\alpha) > s_\alpha > t_\alpha > f_\alpha(\omega_1)$. However, if we have the second inequality then, for the function $(-f_\alpha) \in L$, we have the first one. Therefore we can assume that $f_\alpha(\alpha) < s_\alpha < t_\alpha < f_\alpha(\omega_1)$ for all $\alpha < \omega_1$. Since each f_α is continuous, there exists $\beta_\alpha < \alpha$ such that $f_\alpha(\gamma) < s_\alpha$ for each $\gamma \in (\beta_\alpha, \alpha]$.

The map $r : \omega_1 \to \omega_1$ defined by $r(\alpha) = \beta_\alpha$ satisfies the hypothesis of Fact 2 of S.256, so there is $\beta < \omega_1$ and an uncountable $R \subset \omega_1$ such that $\beta_\alpha = \beta$ for all $\alpha \in R$. Passing to a smaller uncountable subset of R if necessary, we can assume that there are $s, t \in \mathbb{Q}$ such that $s_\alpha = s$ and $t_\alpha = t$ for all $\alpha \in R$; let $E = \{f_\alpha : \alpha \in R\}$ and choose any $s' \in (s, t)$.

Given $f \in L$, if $f(\omega_1) < t$ let $O_f = \{g \in C_p(\omega_1 + 1) : g(\omega_1) < t\}$. Then O_f is an open neighbourhood of f in L such that $O_f \cap E = \emptyset$ because $g(\omega_1) > t$ for any $g \in E$. If $f(\omega_1) \geqslant t$ observe that, by continuity of f, there is $\gamma > \beta$ such that $f(\gamma) > s' > s$. The set $O_f = \{g \in C_p(\omega_1 + 1) : g(\gamma) > s'\}$ is an open neighbourhood of f in the space L. If $\alpha > \gamma$ then $\gamma \in (\beta, \alpha] = (\beta_\alpha, \alpha]$ which implies, by the choice of β_α, that $f_\alpha(\gamma) < s < s'$ whence $f_\alpha \notin O_f$. As a consequence, $O_f \cap E \subset \{f_\alpha : \alpha \leqslant \gamma\}$ and therefore $O_f \cap E$ is a countable set.

The family $\mathcal{U} = \{O_f : f \in L\}$ is an open cover of the space L such that every $U \in \mathcal{U}$ intersects only countably many elements of E. We proved, in fact, that for any function $f \in C_p(\omega_1 + 1)$ with $f(\omega_1) \geqslant t$ there is $v_f < \omega_1$ such that $f(v_f) \in H = (s', +\infty)$ and the open set $O_f = \mathrm{O}(v_f, H)$ intersects only countably many elements of the set E; let $N = \{v_f : f \in L\}$.

Take $h_0 \in E$ arbitrarily and let $N_0 = \emptyset$; suppose that, for some $\alpha < \omega_1$, we have the set $F_\alpha = \{h_\beta : \beta < \alpha\} \subset E$ and the family $\{N_\beta : \beta < \alpha\}$ of countable subsets of N. For each finite $P \subset F$ define a function $e_P : (\omega_1 + 1) \to \mathbb{R}^P = C_p(P)$ by $e_P(\gamma)(f) = f(\gamma)$ for each $f \in P$; the map e_P is continuous (Problem 166) and $w(\mathbb{R}^P) = \omega$ which implies that $e_P(\omega_1 + 1)$ is countable (Fact 1 of S.319). This makes it possible to find a countable set $N_\alpha(P) \subset N$ such that $e_P(N_\alpha(P)) = e_P(N)$. Since $N_\alpha(P)$ is countable for each finite $P \subset F_\alpha$, the set

$$N_\alpha = \bigcup\{N_\alpha(P) : P \in \mathrm{Fin}(F_\alpha)\} \cup \left(\bigcup\{N_\beta : \beta<\alpha\}\right) \subset N$$

is also countable. As a consequence, the set $W_\alpha = \bigcup\{O(v, H) : v \in N_\alpha\}$ can intersect only countably many elements of E; choose $h_\alpha \in E \setminus W_\alpha$. This inductive construction gives us a set $F = \{h_\alpha : \alpha < \omega_1\} \subset L$ and a family $\{N_\alpha : \alpha < \omega_1\}$ of subsets of N with the following properties:

(*) $e_P(N_\alpha) = e_P(N)$ for any $\alpha < \omega_1$ and any finite $P \subset F_\alpha = \{h_\beta : \beta < \alpha\}$.
(**) $h_\alpha \in F \setminus (\bigcup\{O(v, H) : v \in N_\alpha\})$ for any $\alpha < \omega_1$.

The set F is closed and discrete in L; to see this, take any accumulation point $f \in L$ for the set F. Then $f \in \overline{E}$ and hence $f(\omega_1) \geqslant t$; tightness of $C_p(\omega_1 + 1)$ is countable so f has to be an accumulation point of some countable $F' \subset F$. It is clear that f is also an accumulation point of the set $F'' = F' \cap O_f$. Let $\beta = \min\{\alpha < \omega_1 : f$ is an accumulation point of the set $F''_\alpha = F'' \cap \alpha\}$. Then f is an accumulation point for the set $F^* = F'' \cap \beta$ while not so for the set $F^* \cap \beta'$ for any $\beta' < \beta$.

The set $T = \cap\{g^{-1}(g(v_f)) : g \in F^*\}$ is non-empty because $v_f \in T$; suppose first that $T \setminus f^{-1}(H) \neq \emptyset$. Take any $\gamma \in T \setminus f^{-1}(H)$ and observe that $g(\gamma) = g(v_f)$ for any $g \in F^*$ while $v_f \in f^{-1}(H)$ γ and hence $f(\gamma) \neq f(v_f)$ which contradicts the fact that f is in the closure of the set F^*.

Thus, $T \subset W = f^{-1}(H)$; by Fact 1 there is a finite set $P \subset F^*$ such that $S = \cap\{g^{-1}(g(v_f)) : g \in P\} \subset f^{-1}(H)$. Take any $\beta' < \beta$ with $P \subset \{h_\alpha : \alpha < \beta'\}$; the property $(*)$ for $\alpha = \beta'$ implies that there is $\gamma \in N_{\beta'}$ such that $e_P(\gamma) = e_P(v_f)$. Since $S = e_P^{-1}(v_f) \subset f^{-1}(H)$, we have $\gamma \in f^{-1}(H)$, i.e., $f(\gamma) \in H$ which is equivalent to $f \in O(\gamma, H)$. However, $h_\alpha \notin O(\gamma, H)$ for any $\alpha > \beta'$ by the property $(**)$. This shows that $O(\gamma, H) \cap O(v_f, H) \cap F^* \subset \{h_\alpha : \alpha < \beta'\}$, i.e., f has to be an accumulation point for the set $F^* \cap \beta'$ which is a contradiction. We finally proved that F is a closed discrete uncountable subset of L; this final contradiction with $\text{ext}(L) = \omega$ shows that L cannot separate the points of $\omega_1 + 1$ and hence $\omega_1 + 1$ cannot be embedded in a $C_p(Y)$ for a space Y with $\text{ext}(Y) = \omega$. Our solution is complete.

S.327. *Prove that, for any compact space X, we have $\psi(F, X) = \chi(F, X)$ for any closed $F \subset X$. In particular, $\chi(X) = \psi(X)$.*

Solution. If $\psi(F, X) \leqslant \kappa$, take any family $\mathcal{B} \subset \tau(X)$ with $|\mathcal{B}| \leqslant \kappa$ and $\cap\,\mathcal{B} = F$. By normality of X, for each $U \in \mathcal{B}$ there is $V_U \in \tau(F, X)$ such that $\overline{V}_U \subset U$. If $\mathcal{U}$ is the family of all finite intersections of the elements of the family $\{V_U : U \in \mathcal{B}\}$ then $|\mathcal{U}| \leqslant \kappa$ so it suffices to show that $\mathcal{U}$ is an outer base of F in X.

Take any $W \in \tau(F, X)$. Then $\cap\{\overline{V}_U : U \in \mathcal{B}\} \subset \cap\,\mathcal{B} = F$ and therefore $\cap\{\overline{V}_U : U \in \mathcal{B}\} = F \subset W$. Applying Fact 1 of S.326 we can find $U_1, \cdots, U_n \in \mathcal{B}$ such that $\overline{V}_{U_1} \cap \cdots \cap \overline{V}_{U_n} \subset W$. Now it is clear that $V = V_{U_1} \cap \cdots \cap V_{U_n} \in \mathcal{U}$ and $F \subset V \subset W$ so our solution is complete.

S.328. *Let X be a space. Call a set $F = \{x_\alpha : \alpha < \kappa\} \subset X$ a free sequence of length κ if $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha \geqslant \beta\}} = \emptyset$ for every $\beta < \kappa$. Prove that, for any compact space X, tightness of X is equal to the supremum of the lengths of free sequences in X.*

Solution. Assume that $t(X) = \kappa$ and $S = \{x_\alpha : \alpha < \kappa^+\}$ is a free sequence in X. If $F_\beta = \overline{\{x_\alpha : \beta \leqslant \alpha\}}$ then the family $\{F_\beta : \beta < \kappa^+\}$ consists of decreasing closed subsets of X. Since X is compact, there is $y \in \bigcap\{F_\beta : \beta < \kappa^+\}$; then $y \in \overline{S}$ while $y \notin \overline{A}$ for any $A \subset S$ with $|A| \leqslant \kappa$. This contradiction with $t(X) \leqslant \kappa$ shows that X has no free sequences of length κ^+ so the supremum of lengths of free sequences in X does not exceed $\kappa = t(X)$.

A set $A \subset X$ is called *κ-closed* if $\overline{B} \subset A$ for any $B \subset A$ with $|B| \leqslant \kappa$.

Fact 1. For any space Z we have $t(Z) \leqslant \kappa$ if and only if any κ-closed subset of Z is closed.

Proof. If $t(Z) \leqslant \kappa$ and $x \in \overline{A} \setminus A$ then there is $B \subset A$ with $|B| \leqslant \kappa$ and $x \in \overline{B}$. As $\overline{B} \not\subset A$ the set A is not κ-closed so we proved necessity.

Now assume that every κ-closed set is closed and take an arbitrary $A \subset X$. If $D = \bigcup\{\overline{B} : B \subset A \text{ and } |B| \leqslant \kappa\}$ then $A \subset D \subset \overline{A}$; given a set $C \subset D$ with $|C| \leqslant \kappa$, for every $c \in C$ there is $B_c \subset A$ with $|B_c| \leqslant \kappa$ and $c \in \overline{B}_c$. It is evident that $B = \bigcup\{B_c : c \in C\}$ has cardinality $\leqslant \kappa$ and $\overline{C} \subset \overline{B} \subset D$ whence D is κ-closed. Since any κ-closed subset of Z is closed, we have $D = \overline{A}$ so $\overline{A} = \bigcup\{\overline{B} : B \subset A \text{ and } |B| \leqslant \kappa\}$ which shows that $t(Z) \leqslant \kappa$ and hence Fact 1 is proved.

Let Z be any space; given any $A \subset Z$ let $[A]_\kappa = \bigcup\{\overline{B} : B \subset A \text{ and } |B| \leqslant \kappa\}$ and $[A]^\kappa = \{x \in X : \text{if } H \text{ is a } G_\kappa\text{-subset of } Z \text{ and } x \in H \text{ then } H \cap A \neq \emptyset\}$.

Fact 2. Let Z be any space. Given an infinite cardinal κ and any G_κ-subset H of the space Z, for any $x \in H$ there is a closed G_κ-set G such that $x \in G \subset H$.

Proof. Fix a family $\mathcal{C} \subset \tau(Z)$ such that $|\,\mathcal{C}\,| \leqslant \kappa$ and $H = \bigcap \mathcal{C}$. By regularity of Z, for each $U \in \mathcal{C}$ there is a sequence $\{V(U,n) : n \in \omega\} \subset \tau(x, Z)$ such that $V(U,0) = U$ and $\overline{V(U,n+1)} \subset V(U,n)$ for each $n \in \omega$. The family $\mathcal{B} = \{V(U,n) : U \in \mathcal{C}, n \in \omega\}$ has cardinality $\leqslant \kappa$ and

$$G = \bigcap \mathcal{B} = \bigcap\Big\{\bigcap\{V(U,n) : n \in \omega\} : U \in \mathcal{C}\Big\} = \bigcap\Big\{\bigcap\Big\{\overline{V(U,n+1)} : n \in \omega\Big\} : U \in \mathcal{C}\Big\}$$

is a closed set; it is clear that $x \in G \subset H$ so Fact 2 is proved.

Fact 3. Let Z be any compact space. Then $[[A]_\kappa]^\kappa = \overline{A}$ for any $A \subset Z$ and any infinite cardinal κ.

Proof. Since it is evident that $[[A]_\kappa]^\kappa \subset \overline{A}$, let us prove that $\overline{A} \subset D = [[A]_\kappa]^\kappa$. If $x \in \overline{A} \backslash D$ then there is a family $\mathcal{C} \subset \tau(Z)$ with $|\,\mathcal{C}\,| \leqslant \kappa$ and $x \in H = \bigcap \mathcal{C} \subset Z \backslash [A]_\kappa$. By Fact 2 there is a closed G_κ-set F such that $x \in F \subset H$. By Problem 327 there is an outer base $\mathcal{D}$ for the set F of cardinality $\leqslant \kappa$. Since $x \in F$, for every $W \in \mathcal{D}$ we can choose $y_W \in W \cap A$ because $x \in \overline{A}$ and W a neighbourhood of x. The set $B = \{x_W : W \in \mathcal{D}\} \subset A$ has cardinality $\leqslant \kappa$ so $F \cap \overline{B} \subset (\bigcap \mathcal{C}) \cap [A]_\kappa = \emptyset$. Since $\mathcal{D}$ is an outer base of F in Z, there is $W \in \mathcal{D}$ with $W \cap \overline{B} = \emptyset$ which is a contradiction with the fact that $x_W \in W \cap B$. Fact 3 is proved.

Now assume that κ is the supremum of the lengths of all free sequences of X. If $t(X) > \kappa$ then there is a κ-closed non-closed set $A \subset X$ by Fact 1. Then $[A]_\kappa = A$ and hence $\overline{A} = [A]^\kappa$ by Fact 3. Fix any $x \in \overline{A} \backslash A$; then $x \in [A]^\kappa$ so

(*) $H \cap A \neq \emptyset$ for any G_κ-set $H \ni x$.

Take $a_0 \in A$ arbitrarily and let $H_0 = X$. Suppose that $\alpha < \kappa^+$ and we have constructed points $\{a_\beta : \beta < \alpha\} \subset A$ and closed G_κ-sets $\{H_\beta : \beta < \alpha\}$ with the following properties:

(1) $\{x, a_\beta\} \subset H_\beta$ for all $\beta < \alpha$.
(2) $H_\beta \subset H_{\beta'}$ if $\beta' < \beta < \alpha$.
(3) $\overline{\{a_\gamma : \gamma < \beta\}} \cap H_\beta = \emptyset$ for all $\beta < \alpha$.

Since $x \notin P = \overline{\{a_\gamma : \gamma < \alpha\}}$, there exists a closed G_κ-set $H \ni x$ such that $H \cap P = \emptyset$ (we used Fact 2 applied to the set $X \setminus P$ and the point $x \in X \setminus P$). If we let $H_\alpha = H \cap (\cap \{H_\beta : \beta < \alpha\})$ and take any $a_\alpha \in H_\alpha \cap A$ (this choice is possible because of (∗)), then the same conditions are fulfilled for all $\beta \leqslant \alpha$ and hence the inductive construction can go on providing a set $S = \{a_\alpha : \alpha < \kappa^+\} \subset A$. We claim that S is a free sequence. Indeed, if $\beta < \kappa^+$ then $\overline{\{a_\gamma : \gamma < \beta\}} \cap H_\beta = \emptyset$ while $\{a_\gamma : \gamma \geqslant \beta\} \subset H_\beta$ by (2) and (3). The set H_β being closed, we have $\overline{\{a_\gamma : \gamma < \beta\}} \cap \overline{\{a_\gamma : \gamma \geqslant \beta\}} \subset \overline{\{a_\gamma : \gamma < \beta\}} \cap H_\beta = \emptyset$. As a consequence, S is a free sequence of length $> \kappa$; this contradiction shows that $t(X) \leqslant \kappa$ so our solution is complete.

S.329. *Prove that $|X| \leqslant 2^{\chi(X)}$ for any compact space X. In particular, the cardinality of a first countable compact space does not exceed* $\mathfrak{c}$*.*

Solution. Let $\chi(X) = \kappa$; fix a local base $\mathcal{B}_x$ of cardinality $\leqslant \kappa$ at each point $x \in X$. If B is a set then $P_\kappa(B)$ is the family of all subsets of B of cardinality $\leqslant \kappa$.

Fact 1. If $A \subset X$ and $|A| \leqslant 2^\kappa$ then $|\overline{A}| \leqslant 2^\kappa$.

Proof. Since $|P_\kappa(A)| = |A|^\kappa \leqslant (2^\kappa)^\kappa = 2^\kappa$, we have $|P_\kappa(P_\kappa(A))| \leqslant (2^\kappa)^\kappa = 2^\kappa$ so it suffices to construct an injection $\varphi : \overline{A} \to P_\kappa(P_\kappa(A))$. Given $x \in \overline{A}$, for each $U \in \mathcal{B}_x$ choose a point $x_U \in U \cap A$ and let $C_x = \{x_U : U \in \mathcal{B}_x\}$. It is clear that $C_x \subset A$, $x \in \overline{C_x}$ and $|C_x| \leqslant \kappa$. Therefore $\mathcal{D}_x = \{C_x \cap U : U \in \mathcal{B}_x\} \in P_\kappa(P_\kappa(A))$; let $\varphi(x) = \mathcal{D}_x$ for any $x \in \overline{A}$. Note that $\bigcap\{\overline{U} : U \in \mathcal{B}_x\} = \{x\}$; since $x \in \overline{C_x \cap U}$ for any $U \in \mathcal{B}_x$, we have $x \in \bigcap\{\overline{D} : D \in \mathcal{D}_x\} \subset \bigcap\{\overline{U} : U \in \mathcal{B}_x\} = \{x\}$ and therefore $\bigcap\{\overline{D} : D \in \mathcal{D}_x\} = \{x\}$. Thus, given distinct $x, y \in \overline{A}$, the families $\mathcal{D}_x$ and $\mathcal{D}_y$ cannot coincide because the intersections of the closures of their elements do not coincide. This proves that φ is an injection so $|\overline{A}| \leqslant 2^\kappa$ and Fact 1 is proved.

Take any $x_0 \in X$ and let $H_0 = \{x_0\}$. Suppose that $\alpha < \kappa^+$ and we have sets $\{H_\beta : \beta < \alpha\}$ with the following properties:

(1) H_β is a closed subset of X with $|H_\beta| \leqslant 2^\kappa$.
(2) $H_\beta \subset H_\gamma$ if $\beta < \gamma < \alpha$.
(3) If $\beta < \alpha$ and $\mathcal{C}_\beta = \bigcup\{\mathcal{B}_x : x \in \bigcup\{H_\gamma : \gamma < \beta\}\}$ then, for any finite $\mathcal{U} \subset \mathcal{C}_\beta$ with $\bigcup\mathcal{U} \neq X$, we have $H_\beta \setminus (\bigcup \mathcal{U}) \neq \emptyset$.

Let $\mathcal{C}_\alpha = \bigcup\{\mathcal{B}_x : x \in \bigcup\{H_\gamma : \gamma < \alpha\}\}$; then $|\mathcal{C}_\alpha| \leqslant 2^\kappa$. If $\mathcal{U}$ is a finite subfamily of $\mathcal{C}_\alpha$ with $\bigcup \mathcal{U} \neq X$ then choose a point $x(\mathcal{U}) \in X \setminus (\bigcup\mathcal{U})$ and let $A_\alpha = \{x(\mathcal{U}) : \mathcal{U}$ is a finite subfamily of $\mathcal{C}_\alpha$ such that $\bigcup\mathcal{U} \neq X\}$. Then $|A_\alpha| \leqslant 2^\kappa$ and hence $H_\alpha = \overline{\bigcup\{H_\beta : \beta < \alpha\} \cup A_\alpha}$ also has cardinality $\leqslant 2^\kappa$ by Fact 1. It is clear that (1)–(3) now hold for all $\beta \leqslant \alpha$ and hence we can construct a family $\{H_\beta : \beta < \kappa^+\}$ with the properties (1)–(3).

If $H = \bigcup\{H_\beta : \beta < \kappa^+\}$ then $|H| \leqslant \kappa^+ \cdot 2^\kappa = 2^\kappa$. We prove next that the set H is closed in X. Observe first that $t(X) \leqslant \chi(X) \leqslant \kappa$ (Problem 156(iv)); thus, for any $x \in \overline{H}$, there is $A \subset H$ with $|A| \leqslant \kappa$ and $x \in \overline{A}$. The set A has to be contained in some H_β so $x \in \overline{A} \subset H_\beta \subset H$ because H_β is a closed subset of X.

The last step is to show that $H = X$. To obtain contradiction, suppose not. Pick any $p \in X\backslash H$ and, for any $x \in H$, choose $U_x \in \mathcal{B}_x$ such that $p \notin U_x$. Since H is a compact set, there are $x_1,\ldots, x_n \in H$ such that $H \subset U = U_{x_1} \cup \cdots \cup U_{x_n}$. Take an arbitrary $\beta < \kappa^+$ with $\{x_1,\ldots, x_n\} \subset H_\beta$; the property (3) implies that $\mathcal{U} = \{U_{x_i} : i \leqslant n\}$ is a finite subfamily of $\mathcal{C}_{\beta+1}$ such that $p \notin \bigcup\mathcal{U}$ and hence $\bigcup\mathcal{U} \neq X$. Thus $H_{\beta+1}\backslash U \neq \emptyset$ which is a contradiction with $H_{\beta+1} \subset H \subset U$. As a consequence, $|X| = |H| \leqslant 2^\kappa$ so our solution is complete.

S.330. *Given an infinite cardinal κ, suppose that X is a compact space such that $\chi(x, X) \geqslant \kappa$ for any $x \in X$. Prove that $|X| \geqslant 2^\kappa$.*

Solution. As is usual in set theory, we identify each ordinal α with the set of all preceding ordinals, i.e., $\alpha = \{\beta : \beta < \alpha\}$. In particular, $n = \{0, \ldots, n - 1\}$ for any $n \in \mathbb{N}$. We have already used this identification many times before in our solutions as well as in problem formulations. However, this time it is worth to mention it explicitly because each ordinal will be often used as a set and as a point in the same line of text. If we bear in mind these two possibilities, it will be always clear from the context, how an ordinal is used.

Given an ordinal $\alpha > 0$, let $C_\alpha = \{0,1\}^\alpha$. We consider first the case $\kappa = \omega$. Then no point of X is isolated so X is infinite. For each $k \in \mathbb{N}$ and each function $f \in C_k$ we will construct a non-empty open set U_f so that

(1) For any $m, k \in \mathbb{N}$ with $m < k$, we have $\mathrm{cl}_X(U_{f|m}) \subset U_f$ for all $f \in C_k$;
(2) For any $k \in \mathbb{N}$ and any $f, g \in C_k$ with $f \neq g$, we have $\mathrm{cl}_X(U_f) \cap \mathrm{cl}_X(U_g) = \emptyset$.

To start with, take distinct $x, y \in X$ and choose $U \in \tau(x, X)$, $V \in \tau(y, X)$ such that $\overline{U} \cap \overline{V} = \emptyset$ (the bar denotes the closure in X). We have $C_1 = \{f_0, f_1\}$ where $f_i(0) = i$ for $i \leqslant 1$; let $U_{f\,0} = U$ and $U_{f1} = V$. It is clear that (1) and (2) are satisfied for $k = 1$. Suppose that, for each $k \leqslant n$, we defined U_f for all $f \in C_k$ so that (1) and (2) hold. Any function $f \in C_{n+1}$ is an extension of the function $f|n$ and there are exactly two such extensions. This shows that $C_{n+1} = \left\{f_0^g, f_1^g : g \in C_n\right\}$ where $f_i^g|n = g$ and $f_i^g(n) = i$ for $i = 0, 1$.

Now, take an arbitrary function $g \in C_n$; observe that the set U_g has no isolated points and hence we can take distinct $x, y \in U_g$. It is easy to find non-empty sets $U \in \tau(x, X)$, $V \in \tau(y, X)$ such that $\overline{U} \cup \overline{V} \subset U_g$ and $\overline{U} \cap \overline{V} = \emptyset$. Let $U_{f_0^g} = U$ and $U_{f_1^g} = V$; since the function $g \in C_n$ was taken arbitrarily, we indicated how to construct sets $U_{f_0^g}$ and $U_{f_1^g}$ for all $g \in C_n$. Therefore we obtained the desired family $\{U_f : f \in C_{n+1}\}$. The property (1) is guaranteed by our construction for $m = n$ and $k = n + 1$. Therefore (1) holds for $k = n + 1$ and all $m \leqslant n$ by the inductive hypothesis. The property (2) has only to be checked for $k = n + 1$. Observe that, if $f|n = g|n$ then $\overline{U}_f \cap \overline{U}_g$ by our construction. If $f\,|n \neq g|n$ then $\overline{U}_f \cap \overline{U}_g \subset \mathrm{cl}_X(U_{f|n}) \cap \mathrm{cl}_X\,(U_{g|n}) = \emptyset$ by the induction hypothesis.

Once we have the family $\{U_f : f \in C_n, n \in \mathbb{N}\}$ with the properties (1) and (2), let $P_f = \bigcap\{U_{f|n} : n \in \mathbb{N}\}$ for each $f \in C_\omega$. The property (1) and compactness of X imply $P_f = \bigcap\{\mathrm{cl}_X(U_{f|n}) : n \in \mathbb{N}\} \neq \emptyset$. Observe also that $f \neq g$ implies $f\,|n \neq g|n$

for some $n \in \mathbb{N}$; since $P_f \subset U_{f|n}$, $P_g \subset U_{g|n}$ and $\text{cl}_X(U_{f|n}) \cap \text{cl}_X(U_{g|n}) = \emptyset$, we have $P_f \cap P_g = \emptyset$ for distinct $f, g \in C_\omega$. This shows that, choosing any $\varphi(f) \in P_f$, we obtain an injection $\varphi : C_\omega \to X$; thus $|X| \geqslant |C_\omega| = \mathfrak{c}c$ so the case of $\kappa = \omega$ is settled.

Now assume that $\kappa > \omega$. For any ordinal $\alpha < \kappa$, we will construct a family $\{K_f : f \in C_\alpha\}$ with the following properties:

(3) $K_f \neq \emptyset$ is a closed set which is the intersection of $\leqslant (|\alpha| + \omega)$-many open sets;
(4) If $f \in C_\alpha$ and $\beta < \alpha$ then $K_f \subset K_{f|\beta}$.
(5) If $f, g \in C_\alpha$ and $f \neq g$ then $K_f \cap K_g = \emptyset$.

To start with, take distinct $x, y \in X$ and choose $U \in \tau(x, X)$, $V \in \tau(y, X)$ such that $U \cap V = \emptyset$. Apply Fact 2 of S.328 to find closed G_δ-sets K, L such that $x \in K \subset U$ and $y \in L \subset V$. We have $C_1 = \{f_0, f_1\}$ where $f_i(0) = i$ for $i \leqslant 1$; let $K_{f_0} = K$ and $K_{f_1} = L$. It is clear that the properties (3)–(5) are satisfied for $\alpha = 1$.

Take $\alpha < \kappa$, $\alpha > 0$ and assume that, for each $\beta < \alpha$, the sets K_f are constructed for all $f \in C_\beta$ so that the properties (3)–(5) are satisfied. If α is a limit ordinal, let $K_f = \bigcap\{K_{f|\beta} : \beta < \alpha\}$ for any $f \in C_\alpha$. It is evident that the properties (3) and (4) hold for all $\beta \leqslant \alpha$. Now, if $f, g \in C_\alpha$ and $f \neq g$ then $f|\beta \neq g|\beta$ for some $\beta < \alpha$. As a consequence $K_f \cap K_g \subset K_{f|\beta} \cap K_{g|\beta} = \emptyset$ so the property (5) is also satisfied.

Now consider the case when $\alpha = \gamma + 1$. Any function $f \in C_\alpha$ is an extension of the function $f|\gamma \in C_\gamma$ and there are exactly two such extensions. This shows that $C_\alpha = \left\{f_0^g, f_1^g : g \in C_\gamma\right\}$, where $f_i^g|n = g$ and $f_i^g(n) = i$ for $i = 0, 1$.

Take an arbitrary function $g \in C_\gamma$; observe that the set K_g is the intersection of $|\alpha| + \omega < \kappa$ of open sets. If K_g consists of only one point x then $\chi(x, X) = \psi(x, X) < \kappa$ (Problem 327) which is a contradiction. This shows that we can take distinct points $x, y \in K_g$ and sets $U \in \tau(x, X)$, $V \in \tau(y, X)$ such that $U \cap V = \emptyset$. We have $x \in U \cap K_g$ and $y \in V \cap K_g$; since $U \cap K_g$ and $V \cap K_g$ are intersections of at most $|\alpha| + \omega$ open sets, we can apply Fact 2 of S.328 to find closed K, L such that $x \in K \subset U \cap K_g$, $y \in L \subset V \cap K_g$ and both K and L are intersections of at most $|\alpha| + \omega$ open sets. Let $K_{f_0^g} = K$ and $K_{f_1^g} = L$; since the function $g \in C_\gamma$ was taken arbitrarily, we indicated how to construct sets $U_{f_0^g}$ and $U_{f_1^g}$ for all $g \in C_\gamma$. Therefore, we obtained the desired family $\{U_f : f \in C_{\gamma+1} = C_\alpha\}$. It is easy to see that (3)–(5) are satisfied for all $\beta \leqslant \alpha$ so the inductive step is concluded. Therefore we can construct the families $\{K_f : f \in C_\alpha\}$ with properties (3)–(5) for all $\alpha < \kappa$.

Given any $f \in C_\kappa$, let $P_f = \bigcap\{K_{f|\alpha} : \alpha < \kappa\}$. The property (4) implies $P_f \neq \emptyset$ by compactness of X. Observe also that $f \neq g$ implies $f|\alpha \neq g|\alpha$ for some $\alpha < \kappa$; since $P_f \subset K_{f|\alpha}$, $P_g \subset K_{g|\alpha}$ and $K_{f|\alpha} \cap K_{g|\alpha} = \emptyset$, we have $P_f \cap P_g = \emptyset$ for distinct $f, g \in C_\kappa$. This shows that, choosing any $\varphi(f) \in P_f$, we obtain an injection $\varphi : C_\kappa \to X$ whence $|X| \geqslant |C_\kappa| = 2^\kappa$ which completes our solution.

S.331. *(Shapirovsky's theorem on π-character) Prove that* $\pi\chi(X) \leqslant t(X)$ *for any compact space X.*

Solution. There will be no loss of generality to assume that $t(X) = \kappa$ is an infinite cardinal. Suppose that there exists a point $p \in X$ such that $\pi\chi(p, X) \geqslant \kappa^+$. Let $\mathcal{C}$ be the family of all closed non-empty G_δ-subsets of X. Each $G \in \mathcal{C}$ has a countable outer base $\mathcal{B}_G$ by Problem 327. This shows that $G \subset U \in \tau(X)$ implies that there is $V \in \mathcal{B}_G$ with $V \subset U$.

Suppose that $\mathcal{C}' \subset \mathcal{C}$ and $|\mathcal{C}'| \leqslant \kappa$. If every $U \in \tau(p, X)$ contains some $G \in \mathcal{C}'$ then it also contains some $V \in \mathcal{B}_G$ by the previous remark. This shows that $\bigcup\{\mathcal{B}_G : G \in \mathcal{C}'\}$ is a local π-base at p of cardinality $\leqslant \kappa$ which is a contradiction. This proves that we have the following property:

(∗) For any $\mathcal{C}' \subset \mathcal{C}$ with $|\mathcal{C}'| \leqslant \kappa$ there is $W \in \tau(p, X)$ such that $G \backslash W \neq \emptyset$ for all $G \in \mathcal{C}'$.

We will construct collections $\{A_\gamma : \gamma < \kappa^+\} \subset \mathcal{C}$ and $\{B_\gamma : \gamma < \kappa^+\} \subset \mathcal{C}$ with the following properties:

(1) $p \in A_\gamma$ and $A_\gamma \cap B_\gamma = \emptyset$ for all $\gamma < \kappa^+$;
(2) For any $\gamma < \kappa^+$, if H is a non-empty finite intersection of elements of the family $\{A_\beta : \beta < \gamma\} \cup \{B_\beta : \beta < \gamma\}$ then $H \cap B_\gamma \neq \emptyset$.

The construction is by transfinite induction. Take any $x \neq p$ and choose disjoint $U, V \in \tau(X)$ such that $p \in U$ and $x \in V$; by Fact 2 of S.328 there exist $A_0, B_0 \in \mathcal{C}$ such that $p \in A_0 \subset U$ and $x \in B_0 \subset V$. It is clear that (1) and (2) are satisfied for $\gamma = 0$ and the sets A_0, B_0.

Now fix an arbitrary $\alpha < \kappa^+$ and assume that we have constructed families $\{A_\beta : \beta < \alpha\} \subset \mathcal{C}$ and $\{B_\beta : \beta < \alpha\} \subset \mathcal{C}$ for which (1) and (2) are satisfied. Let $\mathcal{H}$ be the collection of all non-empty finite intersections of the elements of the family $\{A_\beta : \beta < \alpha\} \cup \{B_\beta : \beta < \alpha\}$; then $\mathcal{H} \subset \mathcal{C}$ and $|\mathcal{H}| \leqslant \kappa$, so we can use (∗) to find $U \in \tau(p, X)$ such that $H \backslash U \neq \emptyset$ for every $H \in \mathcal{H}$. Apply again Fact 2 of S.328 to find $A_\alpha \in \mathcal{C}$ such that $p \in A_\alpha \subset U$. If $P = X \backslash U$ then $X \backslash A_\alpha \in \tau(P, X)$; using normality of the space X, we can construct $V_n \in \tau(P, X)$ such that $V_0 = X \backslash A_\alpha$ and $\overline{V}_{n+1} \subset V_n$ for all $n \in \omega$. The set $B_\alpha = \bigcap\{V_n : v \in \omega\} = \bigcap\{\overline{V}_{n+1} : n \in \omega\}$ belongs to $\mathcal{C}$ and $B_\alpha \subset V_0 = X \backslash A_\alpha$, i.e., $A_\alpha \cap B_\alpha = \emptyset$; therefore (1) is fulfilled for $\gamma = \alpha$. Since $H \backslash U \subset B_\alpha$ for all $H \in \mathcal{H}$, the property (2) holds for $\gamma = \alpha$ as well. As a consequence, we can construct families $\{A_\gamma : \gamma < \kappa^+\} \subset \mathcal{C}$ and $\{B_\gamma : \gamma < \kappa^+\} \subset \mathcal{C}$ with the properties (1) and (2).

Given any $\gamma < \kappa^+$, the family $\mathcal{F}_\gamma = \{A_\beta : \beta \leqslant \gamma\} \cup \{B_\beta : \beta > \gamma\}$ is centered. To see this, take finite families $\mathcal{U} \subset \{A_\beta : \beta \leqslant \gamma\}$ and $\mathcal{V} \subset \{B_\beta : \beta > \gamma\}$. We will prove that $(\bigcap\mathcal{U}) \cap (\bigcap\mathcal{V}) \neq \emptyset$ using induction on the number of elements of $\mathcal{V}$. If $\mathcal{V} = \emptyset$ then $(\bigcap\mathcal{U}) \cap (\bigcap\mathcal{V}) = \bigcap\mathcal{U} \neq \emptyset$ because $p \in U$ for every $U \in \mathcal{U}$. Assume that we proved that $(\bigcap\mathcal{U}) \cap (\bigcap\mathcal{V}) \neq \emptyset$ for all families $\mathcal{V} \subset \{B_\beta : \beta > \gamma\}$ with $|\mathcal{V}| < n$. Now if $\mathcal{V} = \{B_{\beta_1}, \ldots, B_{\beta_n}\}, \mathcal{U} = \{A_{\alpha_1}, \ldots, A_{\alpha_k}\}$ and $\beta_n > \beta_i$ for all $i \leqslant n$ then $\beta_n > \gamma \geqslant \alpha_i$ for all $i \leqslant k$. Since $H = B_{\beta_1} \cap \cdots \cap B_{\beta_{n-1}} \cap (\bigcap \mathcal{U}) \neq \emptyset$ by the induction hypothesis, the set H belongs to the family $\mathcal{H}$ of all finite intersections of the family $\{A_\beta : \beta < \beta_n\} \cup \{B_\beta : \beta < \beta_n\}$, so $(\bigcap \mathcal{U}) \cap (\bigcap \mathcal{V}) = B_{\beta_n} \cap H \neq \emptyset$ by (2). This proves that the family F_γ is centered for each $\gamma < \kappa^+$. By compactness of X, we can choose $x_\gamma \in \bigcap \mathcal{F}_\gamma$ for all $\gamma < \kappa^+$.

Observe that the set $\{x_\gamma : \gamma < \kappa^+\}$ is a free sequence of length κ^+ because $\overline{\{x_\gamma : \gamma < \alpha\}} \cap \overline{\{x_\gamma : \gamma \geqslant \alpha\}} \subset A_\alpha \cap B_\alpha = \emptyset$ (the last inclusion is due to the fact that

$\{x_\gamma : \gamma < \alpha\} \subset B_\alpha$ and $\{x_\gamma : \gamma \geqslant \alpha\} \subset A_\alpha$ for each $\alpha < \kappa^+$). This contradiction with $t(X) \leqslant \kappa$ (Problem 328) shows that the inequality $\pi\chi(p, X) > \kappa$ is impossible. Therefore $\pi\chi(X) \leqslant \kappa = t(X)$ and our solution is complete.

S.332. *(Shapirovsky's theorem on π-bases) Suppose that X is a compact space with t (X) $\leqslant \kappa$. Prove that X has a π-base of order $\leqslant \kappa$.*

Solution. Given a space Z, a set $P \subset Z$ and a family $\mathcal{U} \subset \tau^*(Z)$, say that $\mathcal{U}$ is a *π-base for P in Z* if $\mathcal{U}$ is a π-base at z for any $z \in P$. If we have a family $\mathcal{D}$ of subsets of Z and $z \in Z$, let $\text{ord}(z, \mathcal{D}) = |\{D \in \mathcal{D} : z \in D\}|$; then $\text{ord}\,(\mathcal{D}) = \sup\{\text{ord}(z, D) : z \in Z\}$. It is clear that the statement $\text{ord}\,(\mathcal{D}) \leqslant \kappa$ says precisely that the order of $\mathcal{D}$ is $\leqslant \kappa$.

Fact 1. Suppose that Z is a space, $P \subset Z$ and U is a π-base for P in Z. Then

(a) $\mathcal{U}$ is also a π-base for $\overline{P}$ in Z.
(b) If the set $F \subset Z$ is closed and $F \cap P = \emptyset$ then $\mathcal{U}' = \{W \in \mathcal{U} : \overline{W} \cap F = \emptyset\}$ is also a π-base for P in Z.

Proof. (a) Take any $z \in \overline{P}$ and $U \in \tau(z, Z)$. There is $y \in P \cap U$; since $U \in \tau(y, Z)$ and $\mathcal{U}$ is a π-base at y, there exists $W \in \mathcal{U}$ with $W \subset U$ so (a) is proved.

(b) Take any $z \in P$ and $U \in \tau(z, Z)$. Since the family $\mathcal{U}$ is a π-base at z and $U\backslash F \in \tau(z, Z)$, there exists $W \in \mathcal{U}$ with $\overline{W} \subset U\backslash F$. This shows that $W \in \mathcal{U}'$ and hence $\mathcal{U}'$ is a π-base at z so Fact 1 is proved.

Fact 2. Let Z be any compact space with $t(Z) \leqslant \kappa$. Then, for any $A \subset Z$, there exists a π-base for A in Z of order $\leqslant \kappa$.

Proof. Our proof will be by transfinite induction on the cardinality of the set A. More precisely, we will show that the following stronger statement is true:

$(*)$ For any $Q \subset Z$ with $|Q| = \delta$, there exists a π-base $\mathcal{E}\,(Q)$ for the set Q in Z such that $|\,\mathcal{E}\,(Q)| \leqslant \delta \cdot \kappa$ and $\text{ord}\,(\mathcal{E}\,(Q)) \leqslant \kappa$.

Since $t(Z) \leqslant \kappa$, we have $\pi\chi(Z) \leqslant \kappa$ (Problem 331) so we can fix a π-base $\mathcal{B}_z$ at any $z \in Z$ with $|\,\mathcal{B}_z| \leqslant \kappa$. For any set $Q \subset Z$, the family $\mathcal{B} = \bigcup\{\mathcal{B}_z : z \in Q\}$ is a π-base for Q in Z; if, additionally, $|Q| \leqslant \kappa$, then $|\,\mathcal{B}\,| \leqslant \kappa$ and hence $\text{ord}\,(\mathcal{B}) \leqslant \kappa$. This shows that $(*)$ is true for every $Q \subset Z$ with $|Q| \leqslant \kappa$.

Now take a cardinal $\lambda > \kappa$ and assume that $(*)$ has been established for all $Q \subset Z$ with $|Q| < \lambda$. Take any $P \subset Z$ with $|P| = \lambda$ and let $S = \{\alpha < \lambda : \alpha = \beta + 1$ for some $\beta < \lambda\}$; then $|S| = \lambda$ so we can take an enumeration $\{p_\alpha : \alpha \in S\}$ of the set P.

We are going to construct families $\{F_\alpha : \alpha < \lambda\}$ and $\{\mathcal{C}_\alpha : \alpha < \lambda\}$ with the following properties:

(1) F_α is a closed subset of Z for each $\alpha < \lambda$ and $p_\alpha \in F_\alpha$ for all $\alpha \in S$.
(2) $\mathcal{C}_\alpha \subset \tau^*(Z)$ and $|{}_\alpha| \leqslant \kappa \cdot |\alpha|$ for each $\alpha < \lambda$.
(3) $\mathcal{D}_\alpha = \bigcup\{\mathcal{C}_\beta : \beta \leqslant \alpha\}$ is a π-base for F_α in Z and $\text{ord}\,(\mathcal{D}_\alpha) \leqslant \kappa$ for each $\alpha < \lambda$.
(4) If $\alpha < \beta < \lambda$ then $F_\alpha \subset F_\beta$ and, for any $U \in \mathcal{C}_\beta$, we have $\overline{U} \cap F_\alpha = \emptyset$.
(5) If $\alpha < \lambda, \mathcal{U} \subset \mathcal{D}_\alpha$ is finite and $V = (\bigcap \mathcal{U})\backslash F_\alpha \neq \emptyset$ then $F_{\alpha+1} \cap V \neq \emptyset$.

Take any $z_0 \in Z$ and let $F_0 = \{z_0\}$, $\mathcal{C}_0 = \mathcal{B}_{z_0}$. It is clear that (1)–(5) are satisfied (most of them vacuously) for $\alpha = 0$. Assume that $\mu = \nu + 1$ and we have families $\{F_\alpha : \alpha < \mu\}$ and $\{\mathcal{C}_\alpha : \alpha < \mu\}$ such that the properties (1)–(4) hold for all $\alpha, \beta < \mu$ and (5) holds for all $\alpha < \nu$. Let $\mathcal{D}_\nu = \bigcup\{\mathcal{C}_\alpha : \alpha \leqslant \nu\}$; if $\mathcal{U}$ is a finite subfamily of $\mathcal{D}_\nu$ such that $V = (\bigcap \mathcal{U}) \setminus F_\nu \neq \emptyset$, fix any point $x(\mathcal{U}) \in V$. If $p_\mu \in F_\nu$ then $R_\mu = \{x(\mathcal{U}) : \mathcal{U}$ is a finite subfamily of $\mathcal{D}_\nu$ such that $(\bigcap \mathcal{U}) \setminus F_\nu \neq \emptyset\}$. If $p_\mu \notin F_\nu$ then $R_\mu = \{p_\mu\} \cup \{x(\mathcal{U}) : \mathcal{U}$ is a finite subfamily of $\mathcal{D}_\nu$ such that $(\bigcap \mathcal{U}) \setminus F_\nu \neq \emptyset\}$. In both cases let $F_\mu = F_\nu \cup \overline{R}_\mu$.

Observe that $|\mathcal{D}_\nu| \leqslant |\nu| \cdot \kappa < \lambda$ and therefore $|R_\mu| \leqslant |\mu| \cdot \kappa < \lambda$; this implies that there exists a family $\mathcal{C}_\mu \subset \tau^*(Z)$ such that $\mathcal{C}_\mu$ is a π-base for R_μ in Z of order $\leqslant \kappa$ and $|\mathcal{C}_\mu| \leqslant |\mu| \cdot \kappa$. Since $R_\mu \cap F_\nu = \emptyset$, we can assume, without loss of generality, that $\overline{W} \cap F_\nu = \emptyset$ for every $W \in \mathcal{C}_\mu$ (Fact 1(b)). The family $\mathcal{C}_\mu$ is a π-base also for the set $\overline{R}_\mu$ (Fact 1(a)) and therefore (1)–(4) are satisfied for all α, $\beta \leqslant \mu$ and (5) holds for all $\alpha \leqslant \nu$.

Suppose now that μ is a limit ordinal. We let $F_\mu = \overline{\bigcup\{F_\alpha : \alpha < \mu\}}$ and $\mathcal{C}_\mu = \emptyset$. It is evident that the properties (1)–(2) and (4)–(5) are satisfied for all $\alpha, \beta < \mu$; in the property (3) only the statement ord $(\mathcal{D}_\mu) \leqslant \kappa$ needs proof. Assume first that the cofinality of μ is $\leqslant \kappa$. Then there exist families $\{\mathcal{E}_\gamma : \gamma \leqslant \kappa\}$ such that ord $(\mathcal{E}_\gamma) \leqslant \kappa$ for each $\gamma < \kappa$ and $\mathcal{D}_\mu = \bigcup\{\mathcal{E}_\gamma : \gamma \leqslant \kappa\}$. Since any union of $\leqslant \kappa$ families of order $\leqslant \kappa$ has order $\leqslant \kappa$, we have ord $(\mathcal{D}_\mu) \leqslant \kappa$ in this case.

If the cofinality of μ is strictly greater than κ then $F_\mu = \bigcup\{F_\alpha : \alpha < \mu\}$, i.e., the set $\bigcup\{F_\alpha : \alpha < \mu\}$ is closed in Z. To see this, take any $z \in \overline{\bigcup\{F_\alpha : \alpha < \mu\}}$. Since $t(Z) \leqslant \kappa$, there is $C \subset \bigcup\{F_\alpha : \alpha < \mu\}$ such that $|C| \leqslant \kappa$ and $z \in \overline{C}$. Choose $\alpha(c) < \mu$ with $c \in F_{\alpha(c)}$ for any $c \in C$. Since the cofinality of μ is $> \kappa$, the set $\{\alpha(c) : c \in C\}$ cannot be cofinal in μ so there is $\beta < \mu$ such that $\alpha(c) < \beta$ for all $c \in C$. As a consequence, $C \subset F_\beta$ and therefore $\overline{C} \subset F_\beta \subset \bigcup\{F_\alpha : \alpha < \mu\}$ so $z \in \bigcup\{F_\alpha : \alpha < \mu\}$ and hence the set $\bigcup\{F_\alpha : \alpha < \mu\}$ is closed. Assume that ord $(\mathcal{D}_\mu) > \kappa$ and fix $z \in Z$ and a family $\mathcal{W} = \{W_\gamma : \gamma < \kappa^+\} \subset \mathcal{D}_\mu$ of distinct elements of $\mathcal{D}_\mu$ such that $z \in \bigcap \mathcal{W}$. If $z \in F_\mu$ then $z \in F_\alpha$ for some $\alpha < \mu$ and hence $z \notin U$ for any $U \in \bigcup\{\mathcal{C}_\beta : \beta > \alpha\}$ by (4). Thus, $\mathcal{W} \subset \mathcal{D}_\alpha$ while ord $(\mathcal{D}_\alpha) \leqslant \kappa$ which is a contradiction.

Thus, $z \in Z \setminus F_\mu$; observe that the family $\mathcal{F} = \{W \cap F_\mu : W \in \mathcal{W}\}$ is centered. Indeed, if $\mathcal{U} \subset \mathcal{W}$ is a finite family then $\mathcal{U} \subset \mathcal{D}_\alpha$ for some ordinal $\alpha < \mu$. We have $z \in V = (\bigcap \mathcal{U}) \setminus F_\alpha$ and hence $F_{\alpha+1} \cap (\bigcap \mathcal{U}) \neq \emptyset$ by the property (5). As a consequence, $\bigcap\{W \cap F_\mu : W \in \bigcap \mathcal{U}\} \supset F_{\alpha+1} \cap (\bigcap \mathcal{U}) \neq \emptyset$ and we proved that $\mathcal{F}$ is a centered family. Since F_μ is compact, there is $x \in F_\mu$ such that $x \in \bigcap\{\overline{F} : F \in \mathcal{F}\}$. In particular, $x \in \overline{W}$ for all $W \in \mathcal{W}$. As $x \in F_\alpha$ for some $\alpha < \mu$, it is impossible that $W \in \mathcal{C}_\beta$ if $\beta > \alpha$ and $W \in \mathcal{W}$ by the property (4). Thus, $\mathcal{W} \subset \mathcal{D}_\alpha$ while ord $(\mathcal{D}_\alpha) \leqslant \kappa$ which gives us a contradiction again.

Therefore conditions (1)–(5) are satisfied for all $\alpha, \beta < \mu$ also in case when μ is a limit ordinal so our inductive construction can go on. Once we have the families $\{F_\alpha : \alpha < \lambda\}$ and $\{\mathcal{C}_\alpha : \alpha < \lambda\}$, observe that $\mathcal{D}_\lambda = \bigcup\{\mathcal{C}_\beta : \beta \leqslant \lambda\}$ is a π-base for $F_\lambda = \bigcup\{F_\alpha : \alpha < \lambda\}$ and hence for a smaller set P by the property (1). The order of $\mathcal{D}_\lambda$ is $\leqslant \kappa$: to see this, observe that the proof of ord $(\mathcal{D}_\mu) \leqslant \kappa$ we gave for limit ordinals μ, is also valid for $\mu = \lambda$. Finally, the property (1) implies

$|\mathcal{D}_\lambda| \leqslant \lambda \cdot \kappa$ so letting $\mathcal{E}(P) = \mathcal{D}_\lambda$ we bring to an end the proof of (∗). This shows that Fact 2 is proved because (∗) is a stronger statement.

To finish our solution apply Fact 2 to the compact space $Z = X$ and to the set $A = X \subset Z$. This gives a π-base $\mathcal{B}$ for X in X of order $\leqslant \kappa$. Of course, this implies that $\mathcal{B}$ is a π-base in X of order $\leqslant \kappa$ so our solution is complete.

S.333. *Suppose that X has a dense σ-compact subspace. Prove that so does* $C_p(C_p(X))$.

Solution. Call a topological property $\mathcal{P}$ *complete* if it satisfies the following conditions:

(1) Any metrizable compact space has $\mathcal{P}$.
(2) If $n \in \mathbb{N}$ and Z_i has $\mathcal{P}$ for all $i = 1, \cdots, n$ then $Z_1 \times \cdots \times Z_n$ has $\mathcal{P}$.
(3) If Z has $\mathcal{P}$ then every continuous image of Z has $\mathcal{P}$.

It is clear that σ-compactness is a complete property. We consider X to be canonically embedded in $C_p(C_p(X))$, i.e., for every $x \in X$, we have $x \in C_p(C_p(X))$ and $x(f) = f(x)$ for any $f \in C_p(X)$ (see Problem 167).

Let Y be a dense σ-compact subspace of X. Given distinct $f, g \in C_p(X)$, the set $W = \{x \in X : f(x) \neq g(x)\}$ is open in X and non-empty. Therefore, we can find $y \in Y \cap W$; then $y(f) = f(y) \neq g(y) = y(g)$ which shows that the set Y separates the points of $C_p(X)$. Facts 1 and 2 of S.312 applied to the space $Z = C_p(X)$, show that there exists an algebra $R(Y) \subset C_p(Z) = C_p(C_p(X))$ such that $Y \subset R(Y)$. Fact 2 of S.312 applied to the property $\mathcal{P}$ = "σ-compactness" shows that $R(Y)$ is σ–$\mathcal{P}$, i.e., $R(Y)$ is also σ-compact. Since $Y \subset R(Y)$, the algebra $R(Y)$ also separates the points of $C_p(X)$ which shows that $R(Y)$ is dense in $C_p(C_p(X))$ by Problem 192. Thus $R(Y)$ is a dense σ-compact subspace of $C_p(C_p(X))$.

S.334. *Is it true that if* $C_p(C_p(X))$ *has a dense σ-compact subspace, then so has X?*

Solution. No, it is not true; we will prove this for the space $X = \omega_1$. Suppose that K is a compact subspace of X. If K is cofinal in ω_1 then $\{[0, \alpha) : \alpha < \omega_1\}$ is an open cover of K which has no finite subcover, a contradiction. Thus, there is $\alpha < \omega_1$ such that $K \subset \alpha$. Now, if K_n is a compact subspace of ω_1 for each $n \in \omega$, fix $\alpha_n < \omega_1$ with $K_n \subset \alpha_n$; there exists $\alpha < \omega_1$ such that $\alpha_n < \alpha$ for all $n \in \omega$. As a consequence, $\bigcup\{K_n : n \in \omega\} \subset \alpha$ which proves that no σ-compact subspace of ω_1 can be dense in ω_1.

To prove that $C_p(C_p(X))$ has a dense σ-compact subspace we will need the following fact.

Fact 1. If $f : \omega_1 \to \mathbb{R}$ is an arbitrary continuous function then, for any $\varepsilon > 0$, the set $A(f, \varepsilon) = \{\alpha < \omega_1 : |f(\alpha) - f(\alpha + 1)| \geqslant \varepsilon\}$ is finite.

Proof. If $A(f, \varepsilon)$ is infinite then it has an accumulation point β (Problem 314); there exists a sequence $\{\alpha_n : n \in \omega\} \subset A(f, \varepsilon)$ such that $\alpha_n + 1 < \alpha_{n+1}$ for each n and $\alpha_n \to \beta$. By continuity of f at the point β, there exists $\gamma < \beta$ such that $|f(\alpha) - f(\beta)| < \frac{\varepsilon}{3}$ for all $\alpha \in (\gamma, \beta)$. Since $\alpha_n \to \beta$, we can choose $n \in \omega$ with $\alpha_n > \gamma$. Since $\alpha_n \in A(f, \varepsilon)$, we have $\varepsilon \leqslant |f(\alpha_n) - f(\alpha_n + 1)| \leqslant |f(\alpha_n) - f(\beta)| + |f(\beta) - f(\alpha_n + 1)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$ which is a contradiction. Fact 1 is proved.

Returning to our solution, recall that it suffices to prove that there is a compact $K \subset C_p(C_p(X))$ which separates the points of $C_p(X)$ (Fact 3 of S.312). Let $z(f) = f(0)$ and $z_\alpha(f) = f(\alpha) - f(\alpha + 1)$ for any $f \in C_p(X)$ and any $\alpha < \omega_1$. Denote by u the function which is identically zero on $C_p(X)$. We claim that the set $K = \{u\} \cup \{z\} \cup \{z_\alpha : \alpha < \omega_1\} \subset C_p(C_p(X))$ is compact.

Given a function $\varphi \in C_p(C_p(X))$, a number $\varepsilon > 0$ and a finite set $P \subset C_p(X)$, we let $O(\varphi, P, \varepsilon) = \{\delta \in C_p(C_p(X)) : |\delta(f) - \varphi(f)| < \varepsilon$ for all $f \in P\}$. Then the family $\mathcal{U}_\varphi = \{O(\varphi, P, \varepsilon) : P$ is a finite subset of $C_p(X)$ and $\varepsilon > 0\}$ is a local base at φ in the space $C_p(C_p(X))$. If $\mathcal{U} \subset \tau(C_p(C_p(X)))$ is an open cover of the space K then there is $W \in \mathcal{U}$ and a finite set $P \subset C_p(X)$ such that $O(u, P, \varepsilon) \subset W$ for some $\varepsilon > 0$. Apply Fact 1 to conclude that, for each $f \in P$ the set $A_f = \{\alpha < \omega_1 : |z_\alpha(f)| \geqslant \varepsilon\}$ is finite. This shows that the set $A = \bigcup\{A_f : f \in P\}$ is also finite and, for any $\alpha \in X \backslash A$, we have $|z_\alpha(f)| < \varepsilon$ for all $f \in P$, i.e., $z_\alpha \in O(u, P, \varepsilon)$ $\subset W$. As a consequence, $K \backslash W$ is finite so it can be covered by a finite subfamily $\mathcal{U}'$ of the family $\mathcal{U}$. Thus $\{W\} \cup \mathcal{U}'$ is a finite subcover of $\mathcal{U}$ which shows that K is compact.

To prove that K separates the points of $C_p(X)$, take any $f, g \in C_p(X)$ such that $\varphi(f) = \varphi(g)$ for any $\varphi \in K$. In particular, $z(f) = f(0) = z(g) = g(0)$. If $f \neq g$ then the set $B = \{\alpha < \omega_1 : f(\alpha) \neq g(\alpha)\}$ is non-empty; let β be the minimal element of B. If $\beta = \alpha + 1$ for some $\alpha < \omega_1$ then $f(\alpha) = g(\alpha)$ and, since $z_\alpha(f) = f(\alpha) - f(\alpha + 1) = z_\alpha(g) = g(\alpha) - g(\alpha + 1)$, we also have $f(\beta) = f(\alpha + 1) = g(\alpha + 1) = g(\beta)$ which is a contradiction.

Now, since $f(0) = g(0)$, we have $\beta > 0$ so β must be a limit ordinal. However B is an open set by continuity of f and g, so there is $\alpha < \beta$ such that $(\alpha, \beta) \subset B$ which is a contradiction with minimality of β. Therefore $f = g$ and hence K is a compact subset of $C_p(C_p(X))$ which separates the points of X. Applying Fact 3 of S.312 we can conclude that $C_p(C_p(X))$ has a dense σ-compact subspace, so our solution is complete.

S.335. *Suppose that every compact subspace of X is metrizable. Is the same true for $C_p(C_p(X))$?*

Solution. No, it is not true; the space $X = \omega_1$ is the relevant example. Suppose that K is a compact subspace of X. If K is cofinal in ω_1 then $\{[0,\alpha) : \alpha < \omega_1\}$ is an open cover of K which has no finite subcover, a contradiction. Thus there is $\alpha < \omega_1$ such that $K \subset \alpha$. This shows that every compact $K \subset X$ is countable; any countable compact space has countable network weight and hence countable weight (Fact 4 of S.307), so K is metrizable.

However, not every compact subspace of $C_p(C_p(X))$ is metrizable. To see this recall that we proved in S.334 that there is a compact subspace L of $C_p(C_p(X))$ which separates the points of $C_p(X)$. Suppose that the space L is metrizable; then L is second countable (Problem 212). Given $f \in C_p(X)$, let $\varphi(f)(z) = z(f)$ for all points $z \in L$. Then $\varphi : C_p(X) \to C_p(L)$ is a continuous map (Problem 166) and, since L

separates the points of $C_p(X)$, the map φ is a condensation. There is a condensation of $C_p(L)$ onto a second countable space because L is separable (Problem 173). The composition of the two mentioned condensations is also a condensation of $C_p(X)$ onto a second countable space. Therefore $iw(C_p(X)) = \omega$ which implies $d(X) = \omega$ (Problem 173).

In Fact 2 of S.232 it was proved that the set $[0,\alpha] = \{\beta < \omega_1 : \beta \leqslant \alpha\}$ is a compact subspace of ω_1 for each $\alpha < \omega_1$. Since every countable $A \subset \omega_1$ is contained in $[0, \alpha]$ for some $\alpha < \omega_1$, the closure of any countable subset of ω_1 is compact. This shows that $X = \omega_1$ is not separable because it is not compact (Problem 306). This contradiction shows that the compact space $L \subset C_p(C_p(X))$ cannot be metrizable so our solution is complete.

S.336. *Give an example of a compact space X such that $C_p(C_p(X))$ is not Lindelöf.*

Solution. We will prove that, for the compact space $X = \omega_1 + 1$, the space $C_p(C_p(X))$ is not Lindelöf. For any ordinal $\alpha < \omega_1$, let $s_\alpha(\beta) = \beta$ if $\beta \leqslant \alpha$ and $s_\alpha(\beta) = \alpha$ if $\beta > \alpha$. It is clear that $s_\alpha : (\omega_1 + 1) \to [0, \alpha]$ is a continuous map. The map $s_\alpha^* : C_p([0, \alpha]) \to C_p(\omega_1 + 1)$ defined by $s_\alpha^*(f) = f \circ s_\alpha$ for all $f \in C_p([0,\alpha])$, is an embedding (Problem 163) and therefore the space $R_\alpha = s_\alpha^*\big(C_p([0, \alpha])\big)$ is second countable being homeomorphic to $C_p([0, \alpha])$ (Problem 169). Let $\pi_\alpha : C_p(\omega_1 + 1) \to C_p([0, \alpha])$ be the restriction map, i.e., $\pi_\alpha(f) = f|[0, \alpha]$ for each function $f \in C_p(\omega_1 + 1)$. Observe that $r_\alpha = s_\alpha^* \circ \pi_\alpha$ maps $C_p(\omega_1 + 1)$ onto R_α and $r_\alpha(f) = f$ for any $f \in R_\alpha$. It is easy to see that $R_\alpha = \{f \in C_p(\omega_1 + 1) : f(\beta) = f(\alpha)$ for all $\beta \geqslant \alpha\}$. Recall that, for every $f \in C_p(\omega_1 + 1)$, there is $\alpha < \omega_1$ such that $f(\beta) = f(\alpha)$ for all $\beta \geqslant \alpha$ (Problem 314); this shows that $\bigcup\{R_\alpha : \alpha < \omega_1\} = C_p(\omega_1 + 1)$.

Fact 1. Every closed $F \subset R_\alpha$ is C-embedded in $C_p(\omega_1 + 1)$.

Proof. Take any continuous function $\varphi : F \to \mathbb{R}$; since the space R_α is second countable, it is metrizable and hence normal (see Problems 209, 214 and 124). Take a continuous function $\Phi : R_\alpha \to \mathbb{R}$ such that $\Phi|F = \varphi$. Then we have $\tilde{\varphi} = \Phi \circ r_\alpha \in C_p\big(C_p(\omega_1 + 1)\big)$ and $\tilde{\varphi}(f) = \Phi(r_\alpha(f)) = \Phi(f) = \varphi(f)$ for any $f \in F$ (we used the equality $r_\alpha(f) = f$ because it is true for all $f \in R_\alpha$ and $F \subset R_\alpha$). Thus $\tilde{\varphi}$ is a continuous extension of φ over the space $C_p(\omega_1 + 1)$ so Fact 1 is proved.

Given a space Z, call a family $\mathcal{F} \subset \exp(Z)$ *countably centered* if $\bigcap \mathcal{F}' \neq \emptyset$ for any countable $\mathcal{F}' \subset \mathcal{F}$.

Fact 2. A space Z is Lindelöf if and only if any countably centered family of closed subsets of Z has a non-empty intersection.

Proof. Suppose that Z is Lindelöf and $\mathcal{F}$ is a countably centered family of closed subsets of Z with $\bigcap \mathcal{F} = \emptyset$. Then $\mathcal{U} = \{Z \backslash F : F \in \mathcal{F}\}$ is an open cover of the space Z. If $\mathcal{U}' \subset \mathcal{U}$ is a countable subcover of $\mathcal{U}$ then $\mathcal{F}' = \{Z \backslash U : U \in \mathcal{U}'\}$ is a countable subfamily of $\mathcal{F}$ with $\cap\ \mathcal{F}' = \emptyset$ which is a contradiction.

Now assume that any countably centered family of closed subsets of Z has a non-empty intersection and take any open cover $\mathcal{U}$ of the space Z. The family $\mathcal{F} = \{Z \backslash U : U \in \mathcal{U}\}$ consists of closed subsets of Z and $\bigcap \mathcal{F} = \emptyset$. This shows that the family $\mathcal{F}$ cannot be countably centered so there is a countable $\mathcal{F}' \subset \mathcal{F}$ such that $\bigcap \mathcal{F}' = \emptyset$. It is immediate that the family $\mathcal{U}' = \{Z \setminus F : F \in \mathcal{F}'\}$ is a countable subcover of the cover $\mathcal{U}$ so the space Z is Lindelöf and Fact 2 is proved.

Returning to our solution, assume that the space $C_p(C_p(\omega_1 + 1))$ is Lindelöf. Take any closed discrete set $D \subset C_p(\omega_1 + 1)$ of cardinality ω_1 (see Fact 1 of S.320). If $D_\alpha = D \cap R_\alpha$ for each $\alpha < \omega_1$ then there exists $\alpha_0 < \omega_1$ with $D_{\alpha_0} \neq \emptyset$; besides, $D_\alpha \neq D$ for any $\alpha < \omega_1$ for otherwise the uncountable closed discrete set D is contained in a second countable space R_α which is impossible.

Given an arbitrary $\delta : D \to \mathbb{R}$, let $\Phi_\alpha = \{\varphi \in C_p(C_p(\omega_1 + 1)) : \varphi|D_\alpha = \delta|D_\alpha\}$ for all $\alpha \geqslant \alpha_0$. Note first that $\Phi_\beta \subset \Phi_\alpha$ if $\beta > \alpha$. Another observation is that $\Phi_\alpha \neq \emptyset$ for any $\alpha \in [\alpha_0, \omega_1)$. Indeed, the set D_α is closed and discrete in R_α so $\delta|D_\alpha$ is a continuous map. Applying Fact 1, we can see that there is $\varphi \in C_p(C_p(\omega_1 + 1))$ such that $\varphi|D_\alpha = \delta|D_\alpha$ and hence we have $\varphi \in \Phi_\alpha$. It is evident that the family $\mathcal{F} = \{\Phi_\alpha : \alpha_0 \leqslant \alpha < \omega_1\}$ consists of closed subsets of $C_p(C_p(\omega_1 + 1))$. Assume that $\mathcal{F}'$ is a countable subfamily of $\mathcal{F}$. Then there exists an ordinal $\alpha < \omega_1$ such that $\mathcal{F}' \subset \{\Phi_\beta : \beta < \alpha\}$. As a consequence, $\bigcap \mathcal{F}' \supset \Phi_\alpha \neq \emptyset$ which proves that the family $\mathcal{F}$ is countably centered. Since we assume that $C_p(C_p(\omega_1 + 1))$ is Lindelöf, there is a function $\varphi \in \bigcap \mathcal{F}$ by Fact 2; it is straightforward that $\varphi|D = \delta$. As a result, the set D is C-embedded in $C_p(\omega_1 + 1)$.

Let $\pi_D : C_p(C_p(\omega_1 + 1)) \to C_p(D)$ be the restriction map, i.e., $\pi_D(\varphi) = \varphi|D$ for any $\varphi \in C_p(C_p(\omega_1 + 1))$. Then π_D is a continuous map (Problem 152); it follows from the fact that D is C-embedded in $C_p(\omega_1 + 1)$ that $\pi_D(C_p(C_p(\omega_1 + 1))) = C_p(D) = \mathbb{R}^D$. Since the space $\mathbb{R}^D$ is homeomorphic to $\mathbb{R}^{\omega_1}$, we conclude that $\mathbb{R}^{\omega_1}$ is Lindelöf being a continuous image of the space $C_p(C_p(\omega_1 + 1))$. However, $\mathbb{R}^{\omega_1}$ is not even normal (Fact 2 of S.215) which gives us the final contradiction. Therefore $C_p(C_p(\omega_1 + 1))$ is not Lindelöf and our solution is complete.

S.337. *Given a space X prove that, for any $n \in \mathbb{N}$, the space X^n is homeomorphic to a closed subspace C_n of the space $L_p(X)$ (see Problem 078). Therefore, every X^n embeds into $C_p(C_p(X))$ as a closed subspace.*

Solution. We will identify any space Z with the respective closed subset of the space $C_p(C_p(Z))$; this identification treats any $z \in Z$ as a map on $C_p(Z)$ defined by $z(f) = f(z)$ for any $f \in C_p(Z)$ (see Problem 167). Since any $z, z' \in Z$ are functions on $C_p(Z)$, usual arithmetical operations can be applied to them (see Problem 027) to obtain all possible functions $\lambda \cdot z + \lambda' \cdot z'$ where $\lambda, \lambda' \in \mathbb{R}$. Another important point is the equality in Z and in $C_p(C_p(Z))$; as usual, two functions $\varphi, \varphi' \in C_p(C_p(Z))$ are called equal if $\varphi(f) = \varphi'(f)$ for any $f \in C_p(Z)$. Let us denote this by $\varphi \equiv \varphi'$. It is clear that if $y, z \in Z$ and $y = z$ then $y \equiv z$. Now, if $y \equiv z$ and $y \neq z$ then, by the Tychonoff property of Z, there is $f \in C_p(Z)$ such that $f(z) = 1$ and $f(y) = 0$; this implies $y(f) = 0 \neq 1 = z(f)$ which is a contradiction. As a consequence, for any $y, z \in Z$ we have $y = z$ if and only if $y \equiv z$. If $z, y \in Z$ and $z \neq y$, the equality $z + y \equiv y + z$ shows that, in general, it is not true that $y, z, u, v \in Z$ and $y + z \equiv u + v$ implies $y = u$ and $z = v$. However, we have the following fact.

Fact 1. Let $y_1, z_1, \cdots, y_n, z_n$ be any (not necessarily distinct) points of a space Z. If $y_1 + 2 \cdot y_2 + \cdots + 2^{n-1} \cdot y_n \equiv z_1 + 2 \cdot z_2 + \cdots + 2^{n-1} \cdot z_n$ then $y_i = z_i$ for all $i \leqslant n$.

Proof. The proof is by induction on n. In the first paragraph of our solution, we showed that the statement of our Fact is true for $n = 1$. Assume that $k \geqslant 1$ and our fact is true for all $n \leqslant k$; take a set $Y = \{y_i, z_i : 1 \leqslant i \leqslant (k+1)\} \subset Z$ such that $\varphi \equiv \delta$ where $\varphi = y_1 + 2 \cdot y_2 + \cdots + 2^k \cdot y_{k+1}$ and $\delta = z_1 + 2 \cdot z_2 + \cdots + 2^k \cdot z_{k+1}$. Assume first that $y_{k+1} \neq z_{k+1}$. Let $z_{j1}, \ldots, z_{jm}$ be a listing of all points from the set $\{z_1, \ldots, z_{k+1}\}$ coinciding with y_{k+1}; by our assumption we have $j_l \neq k+1$ for all $l \leqslant m$. Take any $f \in C_p(Z, [0, 1])$ such that $f(y_{k+1}) = 1$ and $f(z) = 0$ for all $z \in Y$ such that $z \neq y_{k+1}$. Then

$$\delta(f) = \sum\nolimits_{i=1}^{m} 2^{j_i - 1} \cdot f\left(z_{j_i}\right) = \sum\nolimits_{i=1}^{m} 2^{j_i - 1} \leqslant 2^0 + \cdots + 2^{k-1} = 2^k - 1.$$

On the other hand, $\varphi(f) \geqslant 2^k \cdot f(y_{k+1}) = 2^k$ which is a contradiction with $\varphi(f) = \delta(f) \leqslant 2^k - 1$. Therefore, we have $y_{k+1} = z_{k+1}$; it is clear that this implies $y_1 + 2 \cdot y_2 + \cdots + 2^{k-1} \cdot y_k \equiv z_1 + 2 \cdot z_2 + \cdots + 2^{k-1} \cdot z_k$ so we can apply the induction hypothesis to conclude that $y_i = z_i$ for all $i \leqslant k$. Therefore $y_i = z_i$ for all $i \leqslant k+1$ and Fact 1 is proved.

Fact 2. Given spaces Y, Z and T assume that $p : Y \to Z$ and $q : Z \to T$ are condensations such that $r = q \circ p$ is a homeomorphism. Then both p and q are homeomorphisms.

Proof. The maps $p^{-1} = r^{-1} \circ q$ and $q^{-1} = p \circ r^{-1}$ are continuous because r is a homeomorphism and hence the map r^{-1} is continuous. Fact 2 is proved.

Returning to our solution, fix any $n \in \mathbb{N}$ for any $y = (y_1, \ldots, y_n) \in (\beta X)^n$, let $\Phi_n(y) = y_1 + 2^1 \cdot y_2 + \cdots + 2^{n-1} \cdot y_n \in C_p(C_p(\beta X))$. It follows from the results of Problems 115 and 116 that the map $\Phi_n : (\beta X)^n \to C_p(C_p(\beta X))$ is continuous. It is also injective by Fact 1, so if $B_n = \Phi_n((\beta X)^n)$ then $\Phi_n : (\beta X)^n \to B_n$ is a homeomorphism (Problem 123); being compact the set B_n is closed in $C_p(C_p(\beta X))$.

If $f \in C_p^*(X)$, then $f : X \to [-n, n]$ for some $n \in \mathbb{N}$ by Problem 257 there exists a map $f' \in C_p(\beta X, [-n, n]) \subset C_p(\beta X)$ such that $f'|X = f$. This shows that the restriction map $r : C_p(\beta X) \to C_p^*(X)$ is a condensation (see Problem 152). The dual map $r^* : C_p(C_p^*(X)) \to C_p\left(C_p(\beta X)\right)$ is an embedding (see Problem 163). Observe that $C_p^*(X)$ is a dense subspace of the space $C_p(X)$ (Fact 3 of S.310), so the restriction mapping $\pi : C_p\left(C_p(X)\right) \to C_p\left(C_p^*(X)\right)$ is a condensation onto the space $\pi(C_p(C_p(X)))$ (Problem 152). We will also need the map $\varphi_n : X^n \to C_p(C_p(X))$ defined by $\varphi_n(x) = x_1 + 2^1 \cdot x_2 + \cdots + 2^{n-1} \cdot x_n \in L_p(X) \subset C_p(C_p(X))$ for any point $x = (x_1, \ldots, x_n) \in X^n$. Recalling that $X^n \subset (\beta X)^n$ we claim that

(∗) $r^*(\pi(\varphi_n(x))) = \Phi_n(x)$ for any $x \in X^n$.

Indeed, $\pi(\varphi_n(x))$ is defined by $\pi(\varphi_n(x))(g) = \sum_{i=1}^{n} 2^{i-1} \cdot g(x_i)$ for any function $g \in C_p^*(X)$; hence

$$r^*(\pi(\varphi_n(x)))(f) = \pi(\varphi_n(x))(r(f)) = \sum_{i=1}^{n} 2^{i-1} \cdot r(f)(x_i) = \sum_{i=1}^{n} 2^{i-1} \cdot f(x_i)$$

for any $f \in C_p(\beta X)$. Since $\Phi_n(x)(f) = \sum_{i=1}^{n} 2^{i-1} \cdot f(x_i) = r^*(\pi(\varphi_n(x)))(f)$ for all $f \in C_p(\beta X)$, we have $r{*}(\pi(\varphi_n(x))) = \Phi_n(x)$ for any $x \in X^n$ so $(*)$ is proved.

Let $C_n = \varphi_n(X^n)$ and $D_n = \pi(C_n)$. Since $C_n \subset L_p(X) \subset C_p(C_p(X))$, it suffices to show that $\varphi_n : X^n \to C_n$ is a homeomorphism and C_n is closed in $C_p(C_p(X))$. If $\Phi'_n = \Phi_n|X^n$ then the map $\Phi'_n : X^n \to B'_n = \Phi'_n(X^n)$ is a homeomorphism. It follows from $(*)$ that $r^*(D_n) = B'_n$ and hence $(r^*)^{-1} \circ \Phi'_n : X^n \to D_n$ is a homeomorphism. This shows that we can apply Fact 2 to the maps $\varphi_n : X^n \to C_n$ and $\pi|C_n : C_n \to D_n$ whose composition is the homeomorphism $(r^*)^{-1} \circ \Phi'_n$, to conclude that φ_n is a homeomorphism. Finally note that B_n is closed in $C_p(C_p(\beta X))$ and $C_n = (r^* \circ \pi)^{-1}(B_n)$ so C_n is closed in $C_p(C_p(X))$ and our solution is complete.

S.338. *Say that a space X is $K_{\sigma\delta}$ if there exists a space Y such that $X \subset Y$ and $X = \bigcap\{Y_n : n \in \omega\}$ where each Y_n is a σ-compact subset of Y. Prove that a space X is $K_{\sigma\delta}$ if and only if X embeds as a closed subspace into a countable product of σ-compact spaces. Deduce from this fact that*

(*i*) *Any closed subset of a $K_{\sigma\delta}$-space is a $K_{\sigma\delta}$-space.*
(*ii*) *Any countable product of $K_{\sigma\delta}$-spaces is a $K_{\sigma\delta}$-space.*
(*iii*) *If X is a $K_{\sigma\delta}$-space then X^ω is Lindelöf.*

Solution. Take any space $Y \supset X$ and any family $\{Y_n : n \in \omega\}$ of σ-compact subspaces of Y such that $X = \bigcap\{Y_n : n \in \omega\}$. Apply Fact 7 of S.271 to conclude that X can be embedded as a closed subspace in the space $P = \Pi\{Y_n : n \in \omega\}$. This proves necessity.

Now assume that X is a closed subspace of a space $Z = \Pi\{Z_n : n \in \omega\}$ where Z_n is σ-compact for any $n \in \omega$. Let $B = \Pi\{\beta Z_n : n \in \omega\}$; denote by π_n the natural projection of the space B onto the factor βZ_n. Observe that the space $T_n = \pi_n^{-1}(Z_n) = Z_n \times \prod\{\beta Z_k : k \in \omega\backslash\{n\}\}$ is σ-compact for all $n \in \omega$ and $Z = \bigcap\{T_n : n \in \omega\}$. The set $Y = \mathrm{cl}_B(X) \supset X$ is compact and therefore $Y_n = T_n \cap Y$ is σ-compact; observe also that $Z \cap Y = X$ because X is closed in Z. Now $\bigcap\{Y_n : n \in \omega\} = \bigcap\{T_n : n \in \omega\} \cap Y = Z \cap Y = X$ and hence X is $K_{\sigma\delta}$.

The statement of (i) follows immediately from the fact that if X is a closed subspace of a countable product of σ-compact spaces and F is closed in X then F is a closed subspace of the same product.

To prove (ii) suppose that X is a closed subspace of a countable product P of σ-compact spaces. Then X^ω is a closed subspace of the space P^ω which is also a countable product of σ-compact spaces.

To establish (iii) observe that any countable product of σ-compact spaces is Lindelöf by Fact 6 of S.271. Since any closed subspace of a Lindelöf space is Lindelöf, any $K_{\sigma\delta}$-space X is Lindelöf. Therefore X^ω is Lindelöf because it is also a $K_{\sigma\delta}$-space by (ii).

S.339. *Give an example of a $K_{\sigma\delta}$-space which is not σ-compact.*

Solution. Let $\mathbb{P} \subset \mathbb{R}$ be the space of irrational numbers with the topology induced from $\mathbb{R}$. Observe that $\mathbb{P} = \bigcap\{\mathbb{R}\backslash\{q\} : q \in \mathbb{Q}\}$ and each space $\mathbb{R}\backslash\{q\}$ is σ-compact; thus $\mathbb{P}$ is a $K_{\sigma\delta}$-space. If $\mathbb{P}$ is σ-compact then the space $\mathbb{Q} = \mathbb{R}\backslash \mathbb{P}$ is a G_δ-subset of $\mathbb{R}$ which it is not (276). Therefore $\mathbb{P}$ a $K_{\sigma\delta}$-space that is not σ-compact.

S.340. *Prove that $C_p(X)$ is a $K_{\sigma\delta}$-space for any metrizable compact space X.*

Solution. Since X is compact and metrizable, the space $C_p(X)$ is separable (Problem 213); fix a dense countable $D \subset C_p(X)$. Call a topological property $\mathcal{P}$ *complete* if it satisfies the following conditions:

(1) Any metrizable compact space has $\mathcal{P}$.
(2) If $n \in \mathbb{N}$ and Z_i has $\mathcal{P}$ for all $i = 1,\cdots, n$ then $Z_1 \times \cdots \times Z_n$ has $\mathcal{P}$.
(3) If Z has $\mathcal{P}$ then every continuous image of Z has $\mathcal{P}$.

It is clear that σ-compactness is a complete property. It was proved in Fact 2 of S.312 that if $\mathcal{P}$ is a complete property and $A \subset C_p(X)$ has $\mathcal{P}$ then there exists an algebra $R(A) \supset A$ such that $R(A)$ is σ-$\mathcal{P}$, i.e., $R(A)$ can be represented as a countable union of spaces with the property $\mathcal{P}$. When $\mathcal{P}$ =“σ-compactness” then any space with the property σ-$\mathcal{P}$ is also σ-compact. Applying these remarks to the (countable and hence σ-compact) set $D \subset C_p(X)$, we conclude that there exists an algebra $R(D) \subset C_p(X)$ which is dense in $C_p(X)$ and σ-compact. The density of $R = R(D)$ implies that R separates the points of X. As a consequence, for every $f \in C_p(X)$ there exists a sequence $\{f_n : n \in \omega\} \subset R$ such that $f_n \rightrightarrows f$ (Problem 191). Let $I_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ and $S_n = R \times I_n^X$; it is evident that S_n is σ-compact. If $s = (f, h) \in S_n$ let $\varphi_n(s) = f + h \in \mathbb{R}^X$. Since $\mathbb{R}^X = C_p(D(|X|))$ and $R \times I_n^X \subset \mathbb{R}^X \times \mathbb{R}^X$, we can apply Problem 115 to conclude that φ_n is a continuous map for any $n \in \mathbb{N}$. Consequently, $T_n = \varphi_n(S_n)$ is a σ-compact subset of $\mathbb{R}^X$.

Given any $g \in C_p(X)$ and $n \in \mathbb{N}$, we can find $f \in R$ such that $|g(x) - f(x)| < \frac{1}{n}$ for all $x \in X$. Thus, $h = g - f \in I_n^X$ and $g = f + h = \varphi_n(f, h) \in \varphi_n\left(R \times I_n^X\right) = T_n$. Therefore $C_p(X) \subset T_n$ for all $n \in \mathbb{N}$, i.e., $C_p(X) \subset \bigcap\{T_n : n \in \mathbb{N}\}$.

On the other hand, if $g \in \bigcap\{T_n : n \in \mathbb{N}\}$ then, for every $n \in \mathbb{N}$, we have $g = f_n + h_n$ where $f_n \in R$ and $h_n \in I_n^X$. Therefore $|g(x) - f_n(x)| = |h_n(x)| \leqslant \frac{1}{n}$ for any $x \in X$ which shows that $f_n \rightrightarrows g$. Since the uniform limit of continuous functions $\{f_n : n \in \mathbb{N}\} \subset R \subset C_p(X)$ has to be a continuous function (Problem 029), the function g is continuous. Since we took g arbitrarily, we have $\bigcap\{T_n : n \in \mathbb{N}\} \subset C_p(X)$ and hence $C_p(X) = \bigcap\{T_n : n \in \mathbb{N}\}$. Recalling that each T_n is σ-compact, we conclude that the space $C_p(X)$ is $K_{\sigma\delta}$ so our solution is complete.

S.341. *Prove that $C_p(X)$ is a $K_{\sigma\delta}$-space for every countable metrizable X.*

Solution. To abbreviate complicated formulas, we will only indicate the summation variable omitting the symbol “$\in\mathbb{N}$” in the summation indices; for example, the formula $\bigcap\{\bigcap\{S_{km} : m \in \mathbb{N}\} : k \in \mathbb{N}\}$ will be written as $\bigcap_k \bigcup_m S_{km}$. Fix some metric d on the space $X = \{x_l : l \in \mathbb{N}\}$ with $\tau(d) = \tau(X)$. Given any numbers $k, l, n \in \mathbb{N}$, let $M_{kln} = \{f \in \mathbb{I}^X : |f(x) - f(y)| \leqslant 1/k$ whenever $d(x, x_l) < 1/n$ and $d(y, x_l) < 1/n\}$. We claim that

$(*)\ C_p(X, \mathbb{I}) = \bigcap_k \bigcap_l \bigcup_n M_{kln}$.

To prove it, take any $f \in C_p(X, \mathbb{I})$ and any $k, l \in \mathbb{N}$. Since f is continuous at the point x_l, there is $n \in \mathbb{N}$ such that $d(z, x_l) < 1/n$ implies $|f(z) - f(x_l)| < 1/(2k)$. Now if $d(x, x_l) < 1/n$ and $d(y, x_l) < 1/n$ then $|f(x) - f(y)| \leqslant |f(x) - f(x_l)| + |f(x_l) - f(y)| < 1/(2k) + 1/(2k) = 1/k$ which shows that $f \in M_{kln}$. The function f has been taken arbitrarily, so we proved that, for any $k, l \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $C_p(X, \mathbb{I}) \subset M_{kln}$; this means, of course, that $C_p(X, \mathbb{I}) \subset \bigcap_k \bigcap_l \bigcup_n M_{kln}$. To establish the inverse inclusion, suppose that $f \in \bigcap_k \bigcap_l \bigcup_n M_{kln}$ and take any $\varepsilon > 0$. Choose any $l \in \mathbb{N}$ and any $k \in \mathbb{N}$ with $1/k < \varepsilon$; there exists $n \in \mathbb{N}$ such that $|f(x) - f(y)| \leqslant 1/k < \varepsilon$ if $d(x, x_l) < 1/n$ and $d(y, x_l) < 1/n$. In particular, $|f(x) - f(x_l)| \leqslant 1/k < \varepsilon$ for any $x \in X$ with $d(x, x_l) < 1/n$. This shows that f is continuous at the point x_l for any $l \in \mathbb{N}$, i.e., $f \in C_p(X, \mathbb{I})$ and $(*)$ is established.

The next observation is that the set M_{kln} is compact for all $k, l, n \in \mathbb{N}$. Indeed, if $f \in \mathbb{I}^X \setminus M_{kln}$ then there are $x, y \in X$ such that $d(x, x_l) < 1/n$, $d(y, x_l) < 1/n$ while $|f(x) - f(y)| > 1/k$. Choose $\varepsilon > 0$ such that $|f(x) - f(y)| > 1/k + 3\varepsilon$; the set $W = \{h \in \mathbb{I}^X : |h(x) - f(x)| < \varepsilon$ and $|h(y) - f(y)| < \varepsilon\}$ is open in $\mathbb{I}^X$ and $f \in W$. If $h \in W$ then $|h(x) - h(y)| \geqslant |f(x) - f(y)| - |h(x) - f(x)| - |f(y) - h(y)| > 1/k + 3\varepsilon - \varepsilon - \varepsilon > 1/k$ which shows that $h \notin M_{kln}$. This proves that $\mathbb{I}^X \setminus M_{kln}$ is open in $\mathbb{I}^X$ so M_{kln} is compact being closed in $\mathbb{I}^X$. As a consequence, $C_p(X, \mathbb{I})$ is a countable intersection of σ-compact subspaces of $\mathbb{I}^X$. Observe also that $W_m = \{f \in \mathbb{I}^X : f(x_m) \in (-1, 1)\}$ is an open subset of the metrizable compact space $\mathbb{I}^X$ for any $m \in \mathbb{N}$. Thus, W_m is σ-compact for all $m \in \mathbb{N}$. Observe finally that

$C_p(X, (-1,1)) = (\bigcap_m W_m) \cap C_p(X, \mathbb{I}) = (\bigcap_m W_m) \cap (\bigcap_l \bigcap_k \bigcup_n M_{kln})$ which shows that $C_p(X, (-1, 1))$ is a countable intersection of σ-compact subspaces of $\mathbb{I}^X$. Since $C_p(X)$ is homeomorphic to $C_p(X, (-1, 1))$ (Fact 1 of S.295), the space $C_p(X)$ is $K_{\sigma\delta}$ so our solution is complete.

S.342. *Let M be a separable metrizable space. Prove that there is a countable space Y such that M is homeomorphic to a closed subspace of $C_p(Y)$.*

Solution. Let $\mathcal{B}$ be a countable base of M; call a pair $p = (U, V) \in \mathcal{B} \times \mathcal{B}$ *special* if $\overline{U} \subset V$. By normality of the space M, for each special pair $p = (U, V)$ there exists a function $f_p \in C(M, [0, 1])$ such that $f_p | \overline{U} \equiv 1$ and $f_p|(M \setminus V) \equiv 0$. Let $F = \{f_p : p \in \mathcal{B} \times \mathcal{B}$ is a special pair$\}$. It is clear that the set F is countable; let $Y = \{r_1 f_1 + \cdots + r_n f_n : n \in \mathbb{N}, f_1, \cdots, f_n \in F$ and $r_1, \cdots, r_n \in \mathbb{Q}\}$. The set Y is also countable; we claim that M embeds in $C_p(Y)$ as a closed subspace.

The map $e : M \to C_p(Y)$ defined by $e(z)(f) = f(z)$ for any $z \in M$ and $f \in Y$, is continuous (166); given $z \in M$ and a closed $G \subset M$ with $z \notin G$ there is $V \in \mathcal{B}$ such that $z \in V \subset M \setminus G$. By regularity of M there is $U \in \mathcal{B}$ such that $z \in U \subset \overline{U} \subset V$. This means that the pair $p = (U, V)$ is special so $f = f_p \in F \subset Y$ and $f(z) = 1$ while $f(G) = \{0\}$. As a consequence $f(z)$ is not in the closure (in $\mathbb{R}$) of the set $f(G)$; this proves that the map $e : M \to T = e(M)$ is a homeomorphism (the respective criterion was also formulated in Problem 166). Let us finally show that T is closed in $C_p(Y)$.

Take any function $\varphi \in C_p(Y) \setminus T$; it is easy to see that $u \in Y$, where u is the function which is identically zero on M. Observe that $t(u) = 0$ for every $t \in T$.

If $\varphi(u) = s \neq 0$ then the set $W = \{\delta \in C_p(Y) : |\delta(u) - \varphi(u)| < |s|\}$ is open in $C_p(Y)$, contains φ and $W \cap T = \emptyset$, i.e., φ is not in the closure of T.

Given a finite $K \subset M$ and $\varepsilon > 0$, let $O(K, \varepsilon) = \{g \in C_p(M) : |g(z)| < \varepsilon$ for all $z \in K\}$. It is clear that the family $\{O(K, \varepsilon) : K$ is a finite subset of M and $\varepsilon > 0\}$ is a local base of $C_p(M)$ at u.

Now suppose that $\varphi(u) = 0$. By continuity of φ there exist $V \in \tau(u, Y)$ with $\varphi(V) \subset (-1/3, 1/3)$. Recalling that $Y \subset C_p(M)$ we can find a finite set $K = \{z_1, \cdots, z_n\} \subset M$ and $\varepsilon > 0$ such that $O(K, \varepsilon) \cap Y \subset V$. Let $t_i = e(z_i)$ for all $i \leqslant n$; then $L = e(K) = \{t_1, \cdots, t_n\}$ is a finite subset of T with $\varphi \notin L$. Find disjoint $W_i \in \tau(t_i, T)$, $i \leqslant n$ such that $\varphi \notin \overline{W}_1 \cup \cdots \cup \overline{W}_n$ (the closure is taken in $C_p(Y)$). Since the map e is a homeomorphism, the sets $e^{-1}(W_i)$ are open, disjoint and $z_i \in e^{-1}(W_i)$ for all $i \leqslant n$. Using regularity of M again, we can find $U_i, V_i \in \mathcal{B}$, $i \leqslant n$ such that $z_i \in U_i \subset \overline{U}_i \subset V_i \subset e^{-1}(W_i)$ for all $i \leqslant n$. The pair $p_i = (U_i, V_i)$ is special and hence $f_i = f_{p_i} \in F$ for all numbers $i \leqslant n$. The function $f = 1 - (f_1 + \cdots + f_n)$ belongs to the set Y and $f|L \equiv 0$ while $f(M \setminus \bigcup_{i \leqslant n} V_i) = \{0\}$. Since $f \in O(K, \varepsilon) \cap Y$, we have $\varphi(f) \in (-1/3, 1/3)$ while $t(f) = 1$ for any $t \in P = T \setminus \bigcup_{i \leqslant n} e(V_i)$. As a consequence, φ is not in the closure of P in the space $C_p(Y)$. Since $T \setminus P \subset \bigcup_{i \leqslant n} W_i$, the map φ is not in the closure of T and hence T is closed in $C_p(Y)$. Our solution is complete.

S.343. *Prove that there exists a countable space X for which $C_p(X)$ is not a $K_{\sigma\delta}$-space.*

Solution. Call a space Z *uniformly uncountable* if every non-empty open subset of Z is uncountable.

Fact 1. If M is an uncountable second countable space then there is a closed uniformly uncountable $P \subset M$.

Proof. Let $\mathcal{C} = \{U \in \tau^*(M) : U$ is countable$\}$. Since M is hereditarily Lindelöf, there is a countable $\mathcal{C}' \subset \mathcal{C}$ such that $O = \bigcup \mathcal{C}' = \bigcup \mathcal{C}$. Since every element of $\mathcal{C}'$ is countable, the set O is also countable. The set $P = M \setminus O$ is as promised; indeed, if $W \in \tau^*(P)$ is a non-empty countable set then $W \cup O$ is open in M and countable; this countable open set is strictly larger than O which is a contradiction. Thus P is uniformly uncountable and Fact 1 is proved.

Fact 2. If M be a second countable uncountable $K_{\sigma\delta}$-space then there is a subspace $C \subset M$ which is homeomorphic to the Cantor set $\mathbb{K}$ (see Problem 128).

Proof. Fix a metric d on M with $\tau(d) = \tau(M)$; let $Y \supset M$ be any space such that $M = \bigcap\{Y_n : n \in \mathbb{N}\}$ where each Y_n is a σ-compact subspace of Y. A set $A \subset M$ will be called *n-precompact* if $\text{cl}_Y(A)$ is a compact subset of Y_n. Given a set $A \subset M$, the symbol $\overline{A}$ denotes the closure of A in M. For each $k \in \mathbb{N}$, denote by C_k the set of all functions from $k = \{0, \ldots, k-1\}$ to $\{0, 1\}$. We will construct by induction a family $\{P_f : f \in C_k, k \in \mathbb{N}\}$ of non-empty closed subsets of M with the following properties:

(1) The family $\{P_f : f \in C_k\}$ is disjoint for any $k \in \mathbb{N}$.
(2) If $m, k \in \mathbb{N}$, $m < k$ and $f \in C_k$ then $P_f \subset P_{f|m}$.

(3) For any $k \in \mathbb{N}$, the space P_f is uniformly uncountable, k-precompact and diam $(P_f) \leqslant \frac{1}{k}$ for any $f \in C_k$.

Since the set $Y_1 \supset M$ is a countable union of compact spaces, there is a compact $K_1 \subset Y_1$ such that the set $K_1 \cap M$ is uncountable; it is clear that $K_1 \cap M$ is closed in M. Take a uniformly uncountable closed $M_1 \subset K_1 \cap M$ (this is possible by Fact 1). Pick distinct $x_0, x_1 \in M_1$ and choose $U \in \tau(x_0, M)$, $V \in \tau(x_1, M)$ such that $\overline{U} \cap \overline{V} = \emptyset$. There is $\varepsilon \in (0, 1/2)$ such that $B(x_0, \varepsilon) \subset U$ and $B(x_1, \varepsilon) \subset V$. We have $C_1 = \{f_0, f_1\}$, where $f_i(0) = i$ for $i \leqslant 1$; let $P_{f_0} = \mathrm{cl}_M(B(x_0, \varepsilon) \cap M_1)$ and $P_{f_1} = \mathrm{cl}_M(B(x_1, \varepsilon) \cap M_1)$. It is clear that the properties (1)–(2) are satisfied for the family $\{P_f : f \in C_1\}$.

As to the property (3), observe that the closure of any open set in a uniformly uncountable space is uniformly uncountable so P_{f_i}, is uniformly uncountable for every $i \in \{0,1\}$. Since $P_{f_i} \subset K_1 \subset Y_1$, every P_{f_i} is 1-precompact. To see that diam $(P_{f_i}) \leqslant 1$, observe that, for any $r > 0$ and any $z \in M$ we have

($*$) $\mathrm{diam}(B(z,r)) \leqslant 2r$.

Indeed, if $a, b \in B(z,r)$ then $d(a, b) \leqslant d(a, z) + d(z, b) < 2r$ which implies $\mathrm{diam}(B(z,r)) = \sup\{d(a,b) : a, b \in B(z, r)\} \leqslant 2r$. In our particular case, we have $\mathrm{diam}\left(P_{f_i}\right) \leqslant \mathrm{diam}\overline{(B(x_i, \varepsilon))} = \mathrm{diam}(B(x_i, \varepsilon)) \leqslant 2\varepsilon < 1$ (see ($*$) and Fact 1 of S.236) so (3) is also satisfied.

Suppose that, for each $k \leqslant n$, we defined P_f for all $f \in C_k$ so that the properties (1)–(3) hold. Any function $f \in C_{n+1}$ is an extension of the function $f|n$ and there are exactly two such extensions. This shows that $C_{n+1} = \left\{f_0^g, f_1^g : g \in C_n\right\}$ where $f_i^g|n = g$ and $f_i^g(n) = i$ for $i = 0, 1$.

Now, take an arbitrary function $g \in C_n$; observe that the uncountable set P_g is contained in Y_{n+1} which is σ-compact. Therefore, there is a compact $K_{n+1} \subset Y_{n+1}$ such that $K_{n+1} \cap P_g$ is closed in P_g and uncountable. Apply Fact 1 to find a closed uniformly uncountable $M_{n+1} \subset K_{n+1} \cap P_g$. The space M_{n+1} has no isolated points and hence we can take distinct $x_0, x_1 \in M_{n+1}$. Fix any set $U \in \tau(x_0, M)$, $V \in \tau(x_1, M)$ such that $\overline{U} \cap \overline{V} = \emptyset$. We can find a number $\varepsilon \in (0, 1/(2n))$ such that $B(x_0, \varepsilon) \subset U$ and $B(x_1, \varepsilon) \subset V$. Let $P_{f_0^g} = \mathrm{cl}_M(B(x_0, \varepsilon) \cap M_{n+1})$ and $P_{f_1^g} = \mathrm{cl}_M(B(x_1, \varepsilon) \bigcap M_{n+1})$. Since the function $g \in C_n$ was taken arbitrarily, we indicated how to construct sets $P_{f_0^g}$ and $P_{f_1^g}$ for all $g \in C_n$. This gives the desired family $\{P_f : f \in C_{n+1}\}$. The property (1) has only to be checked for $k = n + 1$. Observe that, if $f \neq g$ and $f|n = g|n$ then $P_f \cap P_g = \emptyset$ by our construction. If $f|n \neq g|n$, then we have $P_f \cap P_g \subset P_{f|n} \cap P_{g|n} = \emptyset$ by the induction hypothesis. The property (2) is guaranteed by our construction for $m = n$ and $k = n + 1$. Therefore (2) holds for $k = n + 1$ and all $m \leqslant n$ by the induction hypothesis.

As to the property (3), observe that the closure of any open set in a uniformly uncountable space is uniformly uncountable so $P_{f_i^g}$ is uniformly uncountable for every $g \in C_n$ and every $i \in \{0,1\}$. Since $P_{f_i^g} \subset K_{n+1} \subset Y_{n+1}$, every set $P_{f_i^g}$ has to be $(n + 1)$-precompact. Applying ($*$) and Fact 1 of S.236 to the sets $P_{f_i}^g = \overline{B(x_i, \varepsilon) \cap M_{n+1}}$ we conclude that

$$\operatorname{diam}\left(P^g_{f_i}\right) = \operatorname{diam}(B(x_i, \varepsilon) \cap M_{n+1}) \leqslant \operatorname{diam}\left(\overline{B(x_i, \varepsilon)}\right) = \operatorname{diam}(B(x_i, \varepsilon)) \leqslant 2\varepsilon < 1/n$$

so (3) is also satisfied.

Once we have at our disposal the family $\{P_f : f \in C_n,\ n \in \mathbb{N}\}$ with the properties (1)–(3), let $Q_f = \bigcap\{\mathrm{cl}_Y(P_{f|n}) : n \in \mathbb{N}\}$ for each $f \in \mathbb{K}$; here we consider that $\mathbb{K} = \{0, 1\}^\omega$ (see Problem 128). Observe that the family $\{\mathrm{cl}_Y(P_{f|n}) : n \in \mathbb{N}\}$ consists of non-empty decreasing closed compact sets by (2) and (3) so $Q_f \neq \emptyset$. Besides, (3) implies that there is a compact set $K_n \subset Y_n$ such that $\mathrm{cl}_Y(P_{f|n}) \subset K_n \subset Y_n$ and consequently $Q_f \subset \bigcap\{Y_n : n \in \mathbb{N}\} = M$. Since $Q_f \subset \mathrm{cl}_Y(P_{f|n})$ for every $n \in \mathbb{N}$, we have $Q_f \subset \mathrm{cl}_Y(P_{f|n}) \cap M = \mathrm{cl}_M(P_{f|n}) = P_{f|n}$. Thus, $Q_f \subset \bigcap\{P_{f|n} : n \in \mathbb{N}\}$. Since $\mathrm{diam}(P_{f|n}) \to 0$, there is $x_f \in M$ such that we have $Q_f = \{x_f\}$ for all $f \in \mathbb{K}$. Letting $\varphi(f) = x_f$ we obtain a map $\varphi : \mathbb{K} \to M$.

The map φ is injective because if $f \neq g$ then $f|n \neq g|n$ for some $n \in \mathbb{N}$; as a consequence, $\varphi(f) \in P_{f|n}$ and $\varphi(g) \in P_{g|n}$. Since $P_{f|n} \cap P_{g|n} = \emptyset$ by (1), we have $\varphi(f) \neq \varphi(g)$.

The map φ is continuous; to see this, take any $f \in \mathbb{K}$ and any $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $1/n < \varepsilon$. The set $W = \{g \in \mathbb{K} : g|n = f|n\}$ is open in $\mathbb{K}$ and $f \in W$. For any $g \in W$ we have $\varphi(g) \in P_{g|n} = P_{f|n}$; since $\mathrm{diam}(P_{f|n}) \leqslant 1/n$, we have $d(\varphi(g), \varphi(f)) \leqslant 1/n < \varepsilon$ and hence $\varphi(W) \subset B(\varphi(f), \varepsilon)$ which proves continuity of φ at the point f. Thus $\varphi : \mathbb{K} \to C = \varphi(\mathbb{K})$ is a condensation and hence homeomorphism. This shows that $\mathbb{K}$ embeds in M so Fact 2 is proved.

Fact 3. There is a subspace $Y \subset \mathbb{K}$ which is not $K_{\sigma\delta}$.

Proof. It follows from Fact 5 of S.151 that there exist disjoint sets $A, B \subset \mathbb{K}$ such that $A \cap P \neq \emptyset\ B \cap P$ for any uncountable compact $P \subset \mathbb{K}$. Let us prove that $Y = A$ is not $K_{\sigma\delta}$. If, on the contrary, Y is a $K_{\sigma\delta}$-space then there is $P \subset A$ which is homeomorphic to $\mathbb{K}$ (Fact 2). Thus, P is an uncountable compact subset of $A \subset \mathbb{K}\backslash B$, which is a contradiction with the fact that B meets every uncountable compact subset of $\mathbb{K}$. Fact 3 is proved.

Now, it easy to finish our solution. Take any second countable space Y such that Y is not $K_{\sigma\delta}$ (see Fact 3). Apply Problem 342 to find a countable space X such that Y embeds in $C_p(X)$ as a closed subspace. If $C_p(X)$ is a $K_{\sigma\delta}$-space then so is Y by Problem 338, which is a contradiction. Thus X is a countable space such that $C_p(X)$ is not a $K_{\sigma\delta}$-space so our solution is complete.

S.344. *Call a subset $A \subset C_p(X)$ strongly (or uniformly) dense if, for every $f \in C_p(X)$, there is a sequence $\{f_n : n \in \omega\} \subset A$ such that $f_n \rightrightarrows f$. In other words, a subset is strongly dense in $C_p(X)$ if it is dense in the uniform convergence topology on $C(X)$. Prove that,*

(i) *If $A \subset C_p(X)$ is strongly dense in $C_p(X)$ then it is dense in $C_p(X)$.*
(ii) *For any compact X, the space $C_p(X)$ has a strongly dense σ-compact subspace if and only if it has a dense σ-compact subspace.*

Solution. (i) Assume that A is strongly dense in $C_p(X)$ and take an arbitrary function $f \in C_p(X)$. There exists a sequence $\{f_n\} \subset A$ with $f_n \rightrightarrows f$. It is immediate that the sequence $\{f_n(x)\}$ converges to $f(x)$ for any $x \in X$ so the sequence $\{f_n\}$ converges to f in the topology of $C_p(X)$ (143). Thus, $f \in \overline{\{f_n\}} \subset \overline{A}$ which shows that $\overline{A} = C_p(X)$, i.e., A is dense in $C_p(X)$.

(ii) It follows from (i) that we only have to prove sufficiency. Suppose that B is a σ-compact dense subset of $C_p(X)$ for a compact space X.

Call a topological property $\mathcal{P}$ *complete* if it satisfies the following conditions:

(1) Any metrizable compact space has $\mathcal{P}$.
(2) If $n \in \mathbb{N}$ and Z_i has $\mathcal{P}$ for all $i = 1,\ldots, n$ then $Z_1 \times \cdots \times Z_n$ has $\mathcal{P}$.
(3) If Z has $\mathcal{P}$ then every continuous image of Z has $\mathcal{P}$.

It is evident that σ-compactness is a complete property; apply Fact 2 of S.312 to conclude that there exists an algebra $A = R(B) \subset C_p(X)$ such that A is a countable union of σ-compact spaces. This means, of course, that the set A is a dense σ-compact algebra in $C_p(X)$. Since A separates the points of X, we can apply Problem 191 to conclude that A is strongly dense in $C_p(X)$.

S.345. *Give an example of a space X such that $C_p(X)$ has a dense σ-compact subspace while there is no strongly dense σ-compact subspace in $C_p(X)$.*

Solution. Any countable space is σ-compact, so any separable $C_p(X)$ has a dense σ-compact subspace. Let X be the set $\mathbb{N}$ with the discrete topology. Then $C_p(X)$ is separable being second countable. Assume that A is a strongly dense subspace of $C_p(X)$ with $A = \bigcup\{K_n : n \in \mathbb{N}\}$, where each K_n is compact. For any $x \in X$, the map $e_x : C_p(X) \to \mathbb{R}$ defined by $e_x(f) = f(x)$ for any $f \in C_p(X)$, is continuous (Problem 166). Therefore, for each $n \in \mathbb{N}$, the set $P_n = e_n(K_n)$ is compact being a continuous image of K_n. Any compact subset of $\mathbb{R}$ is bounded, so we can choose $r_n \in \mathbb{R}$ such that $z + 1 < r_n$ for any $z \in P_n$. The function $f : X \to \mathbb{R}$ defined by $f(n) = r_n$, is continuous because X is discrete. Since A is strongly dense in $C_p(X)$, there is $g \in A$ with $|g(x) - f(x)| < 1/2$ for all $x \in X$. If $g \in K_n$ then $g(n) = e_n(g) \in P_n$; thus $|g(n) - f(n)| = r_n - g(n) > 1$ which is a contradiction. Therefore $C_p(X)$ is a separable space that has no strongly dense σ-compact subspace.

S.346. *Prove that $C_p(A(\kappa))$ has a strongly dense σ-compact subspace for any cardinal κ.*

Solution. By Problem 344 it suffices to show that $C_p(A(\kappa))$ has a dense σ-compact subspace. By Fact 3 of S.312 this is equivalent to having a compact $K \subset C_p(A(\kappa))$ which separates the points of $A(\kappa)$. Denote by u the function which is identically zero on $A(\kappa)$; for each $\alpha \in \kappa$, let $f_\alpha(\alpha) = 1$ and $f_\alpha(z) = 0$ for all $z \in A(\kappa)\backslash\{\alpha\}$. Then $f_\alpha \in C_p(A(\kappa))$ for each $\alpha \in \kappa$; consider the set $K = \{f_\alpha : \alpha < \kappa\} \cup \{u\}$.

The set K separates the points of $A(\kappa)$. To see this, take any distinct points $x, y \in A(\kappa)$. One of them, say x, is distinct from the point $a \in A(\kappa)$; if $x = \alpha \in \kappa$ then $f_\alpha(x) = 1 \neq 0 = f_\alpha(y)$.

To see that the subspace K is compact, take any open cover $\mathcal{U}$ of the set K. Since the point $u \in K$ belongs to some $U \in \mathcal{U}$, there is a finite $P \subset A(\kappa)$ and $\varepsilon > 0$ such that $W = \{f \in C_p(A(\kappa)) : |f(x)| < \varepsilon$ for all $x \in P\} \subset U$. This shows that $f_\alpha \in U$ for all $\alpha \in \kappa \backslash P$ and therefore the set $K' = K \backslash U$ is finite. Choosing a finite $\mathcal{U}' \subset \mathcal{U}$ to cover the set $K \backslash U$, we get a finite subcover $\mathcal{U}' \cup \{U\}$ of the cover $\mathcal{U}$. As a consequence K is a compact subspace of $C_p(A(\kappa))$ that separates the points of $A(\kappa)$ so our solution is complete.

S.347. *Suppose that there exists a strongly dense subset $A \subset C_p(X)$ such that $t(A) \leqslant \kappa$. Prove that $t(C_p(X)) \leqslant \kappa$.*

Solution. Recall that $\mathcal{U} \subset \tau(X)$ is called *an open ω-cover* of X if, for any finite $K \subset X$, there is $U \in \mathcal{U}$ with $K \subset U$. It suffices to prove that for any open ω-cover $\mathcal{U}$ of the space X, there is $\mathcal{U}' \subset \mathcal{U}$ such that $\mathcal{U}'$ is an ω-cover of X and $|\mathcal{U}'| \leqslant \kappa$ (see Problem 148 and 149).

Given $f \in A$, let $S(f) = \{x \in X : f(x) \geqslant 1/3\}$; consider the set $P = \{f \in A : S(f) \subset U$ for some $U \in \mathcal{U}\}$. Observe that the function $u_1 \equiv 1$ is in the closure of P. Indeed, if K is a finite subset of X and $\varepsilon > 0$, then there is $U \in \mathcal{U}$ such that $K \subset U$; take any $h \in C_p(X)$ with $h|K \equiv 1$ and $h|(X \backslash U) \equiv 0$. Since A is strongly dense in $C_p(X)$, there is $f \in A$ such that $|f(x) - h(x)| < \min\{1/3, \varepsilon\}$ for all points $x \in X$. Then $f(x) < 1/3$ for all points $x \in X \backslash U$ and $|f(x) - 1| < \varepsilon$ for all $x \in K$, which shows that $S(f) \subset U$ and $f(x)$ is ε-close to $u_1(x)$ for all $x \in K$. Therefore, $u_1 \in \overline{P}$; since $t(A) \leqslant \kappa$, there is $B \subset A$ with $|B| \leqslant \kappa$ such that $u_1 \in \overline{B}$. For each $f \in B$ fix $U_f \in \mathcal{U}$ such that $S(f) \subset U_f$; then $\mathcal{U}' = \{U_f : f \in B\} \subset \mathcal{U}$ has cardinality $\leqslant \kappa$. Given a finite $K \subset X$ there is $f \in B$ such that $|f(x) - 1| < 1/3$ for all $x \in K$; thus $f(x) > 2/3$ for all $x \in K$ and hence $K \subset U_f$. This shows that $\mathcal{U}'$ is an ω-cover of X so our solution is complete.

S.348. *Suppose that there exists a strongly dense Fréchet-Urysohn subspace A in the space $C_p(X)$. Prove that $C_p(X)$ is a Fréchet-Urysohn space.*

Solution. Recall that $\mathcal{U} \subset \tau(X)$ is called *an open ω-cover* of X if, for any finite $K \subset X$, there is $U \in \mathcal{U}$ with $K \subset U$. We say that $\mathcal{U} \to X$ if, for every $x \in X$, the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

Take any open ω-cover $\mathcal{U}$ of the space X. Given function $f \in A$, let $S(f) = \{x \in X : f(x) \geqslant 1/3\}$; consider the set $P = \{f \in A : S(f) \subset U$ for some $U \in \mathcal{U}\}$. Observe that the function $u_1 \equiv 1$ is in the closure of P. Indeed, if K is a finite subset of X and $\varepsilon > 0$ then there is $U \in \mathcal{U}$ such that $K \subset U$; take any $h \in C_p(X)$ with $h|K \equiv 1$ and $h|(X \backslash U) \equiv 0$. Since A is strongly dense in $C_p(X)$, there is $f \in A$ such that $|f(x) - h(x)| < \min\{1/3, \varepsilon\}$ for all $x \in X$. Then $f(x) < 1/3$ for all $x \in X \backslash U$ and $|f(x) - 1| < \varepsilon$ for all $x \in K$, which shows that $S(f) \subset U$ and $f(x)$ is ε-close to $u_1(x)$ for all $x \in K$. Therefore, $u_1 \in \overline{P}$; since A is Fréchet–Urysohn, there is a sequence $T = \{f_n : n \in \omega\} \subset A$ such that $f_n \to u_1$. For each $f \in T$ fix $U_f \in \mathcal{U}$ such that $S(f) \subset U_f$; then the family $\mathcal{U}' = \{U_f : f \in T\} \subset \mathcal{U}$ is countable. Given any $x \in X$, there is $m \in \omega$ such that $f_n(x) > 2/3$ for all $n \geqslant m$. This shows that $x \in U_{f_n}$ for all $n \geqslant m$ and hence $\mathcal{U}' \to X$.

This proves that, for any open ω-cover $\mathcal{U}$ of the space X, there exists a countable $\mathcal{U}' \subset \mathcal{U}$ such that $\mathcal{U}' \to X$. Applying Problem 144(ii) we conclude that $C_p(X)$ is a Fréchet–Urysohn space.

S.349. *Suppose that there exists a strongly dense subspace $A \subset C_p(X)$ with $\psi(A) \leqslant \omega$. Is it true that $C_p(X)$ has countable pseudocharacter?*

Solution. No, it is not true. We will prove this for the space $A(\mathfrak{c})$. Fix any set T with $|T| = \mathfrak{c}$; take any point $a \notin T$ and introduce in the set $X = T \cup \{a\}$ a topology like in $A(\mathfrak{c})$, i.e., all points of T are declared isolated in X and $U \in \tau(a, X)$ if and only if $a \in U$ and $T \setminus U$ is finite. It is evident that the space X is homeomorphic to $A(\mathfrak{c})$.

Denote by Fin(T) the family of all finite subsets of T. Observe that the set $R = \bigcup\{\mathbb{R}^P : P \in \text{Fin}(T)\}$ has cardinality $\mathfrak{c}$ and fix an enumeration $\{h_\alpha : \alpha < \mathfrak{c}\}$ of the set R. For any $\alpha < c$, let S_α be the finite set P for which $h_\alpha \in \mathbb{R}^P$. It is easy to find a disjoint family $\mathcal{T} = \{T_\alpha : \alpha < \mathfrak{c}\}$ such that $T = \bigcup \mathcal{T}$ and each T_α is countably infinite, i.e., $T_\alpha = \{t_n^\alpha : n \in \mathbb{N}\}$ where $t_n^\alpha \neq t_m^\alpha$ if $n \neq m$.

For each $\alpha < \mathfrak{c}$ and $m \in \mathbb{N}$ define a point $w_\alpha^m \in \mathbb{R}^X$ as follows: $w_\alpha^m(a) = 0$ and $w_\alpha^m(t) = h_\alpha(t)$ for all $t \in S_\alpha$; if $t \in \{t_n^\alpha : n \leqslant m\} \cup (T \setminus (S_\alpha \cup T_\alpha))$ then $w_\alpha^m(t) = 0$; if $t = t_n^\alpha \in T_\alpha \setminus S_\alpha$ and $n > m$ then $w_\alpha^m(t) = 1/n$. Let $B_m = \{w_\alpha^m : \alpha < \mathfrak{c}\}$ and $B = \bigcup\{B_m : m \in \mathbb{N}\}$. If $Q_\alpha^m = S_\alpha \cup \{t_i^\alpha : i > m\}$ then it is immediate that $\{w_\alpha^m(t) : t \in Q_\alpha^m\}$ is a sequence which converges to zero and $w_\alpha^m(t) = 0$ for all $t \in T \setminus Q_\alpha^m$. This shows that $w_\alpha^m \in C_p(X)$ for each $m \in \mathbb{N}$ and $\alpha < \mathfrak{c}$. Given any $r \in \mathbb{R}$, let u_r be the function on X with $u_r(x) = r$ for each $x \in X$. Consider the set $A = \{w + u_r : w \in B \text{ and } r \in \mathbb{R}\}$.

The set A is strongly dense in $C_p(X)$. To prove this, take any $f \in C_p(X)$; if $f(a) = r$, then there exists a set $D = \{s_n : n \in \mathbb{N}\} \subset T$ such that $f(s_n) \to r$ and $f(t) = r$ for all points $t \in T \setminus D$. For each $m \in \mathbb{N}$ the function $f_m = (f - u_r)|\{s_i : i \leqslant m\}$ belongs to the set R; take $\alpha_m < \mathfrak{c}$ with $S_{\alpha_m} = \{s_i : i \leqslant m\}$ and $h_{\alpha_m} = f_m$. Then $g_m = w_{\alpha_m}^m + u_r \in A$ for all $m \in \mathbb{N}$ and $g_m \rightrightarrows f$. Indeed, fix $\varepsilon > 0$; there exists $k \in \mathbb{N}$ such that $1/k < \varepsilon/2$ and $|f(s_i) - r| < \varepsilon/2$ for all $i \geqslant k$. Now, if $m \geqslant k$ then we have $g_m(s_i) = f(s_i)$ for all $i \leqslant m$. As a consequence $|g_m(t) - f(t)| = 0 < \varepsilon$ for all $t \in S_{\alpha_m}$. For any $t \in T \setminus S_{\alpha_m} = T \setminus \{s_i : i \leqslant m\}$, we have $|g_m(t) - r| = |w_{\alpha_m}^m(t)| \leqslant 1/m \leqslant 1/k < \varepsilon/2$. Thus, if $t \in T \setminus D$ then $|g_m(t) - f(t)| = |g_m(t) - r| < \varepsilon/2 < \varepsilon$. If $t = s_i$ for some $i > m$ then $|g_m(t) - f(t)| \leqslant |g_m(t) - r| + |r - f(t)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus, we proved that, for any $m \geqslant k$, we have $|f(x) - g_m(x)| < \varepsilon$ for all $x \in X$; hence $g_m \rightrightarrows f$.

To see that $\psi(A) \leqslant \omega$, take any $f = w_\alpha^m + u_r \in A$. The set $B_f = T_\alpha \cup S_\alpha \cup \{a\}$ is countable; suppose that $g = w_\beta^k + u_s \in A$ and $g|B_f = f|B_f$. Then $g(a) = r$ and hence $s = r$. If $\alpha \neq \beta$ then $T_\alpha \cap T_\beta = \emptyset$; take any $t = t_n^\alpha \in T_\alpha \setminus (S_\alpha \cup S_\beta)$ with $n > m$. Then $t \in B_f$ and $f(t) = \frac{1}{n} \neq 0 = g(t)$ which is a contradiction. This shows that $\alpha = \beta$.

It is an easy exercise that the set $E = \{h \in C_p(X) : h|B_f = f|B_f\}$ is a G_δ-set in $C_p(X)$. We proved that $f \in E \cap A$ and $E \cap A \subset \{w_\alpha^m + u_r : m \in \mathbb{N}\}$. It turns out that for any $f \in A$ there is a G_δ set $E' = E \cap A$ in A such that $f \in E'$ and E' is countable. This implies that $\{f\}$ is a G_δ-set in A, i.e., $\psi(f, A) \leqslant \omega$. Since the function $f \in A$ was chosen arbitrarily, we proved that $\psi(A) \leqslant \omega$. Therefore, A is a strongly dense subspace of $C_p(X)$ with $\psi(A) \leqslant \omega$. However, $\psi(C_p(X)) > \omega$ because X is not separable (Problem 173) so our solution is complete.

S.350. *Suppose that there exists a strongly dense σ-pseudocompact subspace A in the space $C_p(X)$. Prove that X is pseudocompact.*

Solution. We will need the following simple facts for further references.

Fact 1. A space Z is not pseudocompact if and only if there exists a countably infinite closed discrete $D \subset Z$ such that D is C-embedded in Z.

Proof. If a set $D = \{d_n : n \in \omega\} \subset Z$ is faithfully indexed, closed, discrete and C-embedded in Z, let $f(d_n) = n$ for all $n \in \omega$. The function $f : D \to \mathbb{R}$ is continuous as is any function on a discrete space. Since D is C-embedded, there is $g \in C(Z)$ such that $g|D = f$. The function g being unbounded, the space Z is not pseudocompact so sufficiency is proved.

Now, if Z is not pseudocompact then, by Problem 136, there exists a discrete family $\mathcal{U} = \{U_n : n \in \omega\} \subset \tau^*(Z)$. Take a point $x_n \in U_n$ for each $n \in \omega$ and fix a function $f_n \in C(Z, [0, 1])$ such that $f_n(x_n) = 1$ and $f_n(Z \setminus U_n) = \{0\}$. It is clear that $D = \{x_n : n \in \omega\}$ is a discrete and closed subspace of Z. If $f : D \to \mathbb{R}$ then the function $g = \Sigma_{n\in\omega} f(x_n) \cdot f_n$ is continuous on Z. Indeed, if $y \in Z$ then there is a set $U \in \tau(y, Z)$ which meets at most one element of $\mathcal{U}$, say, U_k. Then $g|U = (f(x_k) \cdot f_k)|U$ is a continuous function. This implies continuity of g at the point y and hence g is continuous. This shows that D is C-embedded in Z so Fact 1 is proved.

Fact 2. Let Z be any space. If Y is C-embedded in Z and D is strongly dense in $C_p(Z)$ then $\pi_Y(D)$ is strongly dense in $C_p(Y)$. Here $\pi_Y : C_p(Z) \to C_p(Y)$ is the restriction map defined by $\pi_Y(f) = f|Y$ for all $f \in C_p(Z)$.

Proof. Take any $f \in C(Y)$ and any $\varepsilon > 0$; there exists $g \in C(Z)$ with $g|Y = f$. Since D is strongly dense in $C_p(Z)$, there is $h \in D$ such that $|h(x) - g(x)| < \varepsilon$ for all $x \in Z$. If $h_1 = \pi_Y(h)$ then $|h_1(y) - f(y)| = |h(y) - g(y)| < \varepsilon$ for all $y \in Y$ which proves that $\pi_Y(D)$ is strongly dense in $C_p(Y)$. Fact 2 is proved.

Suppose that $A = \bigcup\{K_n : n \in \mathbb{N}\}$ is strongly dense in $C_p(X)$ and each K_n is pseudocompact. If X is not pseudocompact then, by Fact 1, there is a closed discrete $D = \{d_n : n \in \omega\} \subset X$ which is C-embedded in X. Apply Fact 2 to conclude that $E = \pi_D(A)$ is strongly dense in $C_p(D)$. Observe that $E = \bigcup\{L_n : n \in \omega\}$ where $L_n = \pi_D(K_n)$ is compact being a pseudocompact second countable space for each $n \in \omega$.

For any $x \in D$, the map $e_x : C_p(D) \to \mathbb{R}$ defined by $e_x(f) = f(x)$ for any $f \in C_p(D)$, is continuous (Problem 166). Therefore, for each $n \in \mathbb{N}$, the set $P_n = e_{d_n}(L_n)$ is compact being a continuous image of L_n. Any compact subset of $\mathbb{R}$ is bounded, so we can choose $r_n \in \mathbb{R}$ such that $z + 1 < r_n$ for any $z \in P_n$. The function $f : D \to \mathbb{R}$ defined by $f(d_n) = r_n$, is continuous because D is discrete. Since E is strongly dense in $C_p(D)$, there is $g \in E$ with $|g(x) - f(x)| < 1/2$ for all $x \in D$. If $g \in L_n$ then $g(d_n) = e_{d_n}(g) \in P_n$; thus $1/2 > |g(d_n) - f(d_n)| = r_n - g(d_n) > 1$ which is a contradiction. Therefore X is pseudocompact and our solution is complete.

S.351. *Suppose that there exists a strongly dense σ-countably compact set $A \subset C_p(X)$. Prove that X is compact.*

Solution. The author's politics is to never refer to anything which will be proved in the future solutions. Well, they are not future for the reader, but they are so for the author who is writing the solutions one by one in the order the problems come. There are some results in the future problems (I will give no reference partly because I haven't written their solutions yet) which will easily imply the statement of this problem. When I formulated it I hoped to be able to find a direct solution which would not involve the heavy artillery. However, I failed to do this and the solution I present is really difficult because it has to develop some methods which give rise to a variety of famous and non-trivial results of C_p-theory. I still have hope that some reader will find a simpler solution before reading mine. If I am alive by then, send it to me to include it in the future versions of this book.

If T is a space then a continuous map $r : T \to T$ is called *a retraction* if $r \circ r = r$. A subspace $R \subset T$ is called *a retract of* T if there is a retraction $r : T \to T$ such that $r(T) = R$. Given a continuous map $\varphi : T \to T'$, the dual map $\varphi^* : C_p(T') \to C_p(T)$ is defined by $\varphi^*(f) = f \circ \varphi$ for any $f \in C_p(T')$. If φ is onto then φ^* is an embedding (see Problem 163). If $A \subset T$ and $B \subset C_p(T)$ say that *B separates the points of A* if, for any distinct $a_1, a_2 \in A$, there is $f \in B$ such that $f(a_1) \neq f(a_2)$. A map $\varphi : Z \to Y$ is called *a condensation of Z into Y* if φ is an injective (but not necessarily surjective) continuous map. If there exists a condensation of Z into Y we say that Z condenses into Y.

Given a function $f \in C_p(T)$, a number $\varepsilon > 0$ and a finite set $K \subset T$, we let $O_T(f, K, \varepsilon) = \{g \in C_p(T) : |f(x) - g(x)| < \varepsilon \text{ for all } x \in K\}$. Then the family $\mathcal{U}_f = \{O_T(f, K, \varepsilon) : K \text{ is a finite subset of } T \text{ and } \varepsilon > 0\}$ is a local base at f in the space $C_p(T)$. If I is an infinite set then $\Sigma_*(I) = \{x \in \mathbb{R}^I : \text{for any } \varepsilon > 0, \text{ the set } \{i \in I : |x(i)| \geqslant \varepsilon\} \text{ is finite}\}$. The space $\Sigma_*(I)$ is a linear subspace of $\mathbb{R}^I$, i.e., $ax + by \in \Sigma_*(I)$ for any $x, y \in \Sigma_*(I)$ and $a, b \in \mathbb{R}$. Thus it is also a linear space. Given any $x \in \Sigma_*(I)$, let $||x|| = \sup\{|x(i)| : i \in I\}$. It is clear that $||x||$ is finite for any $x \in \Sigma_*(I)$. If Z is a compact space and $f \in C_p(Z)$ then f is a bounded function so the number $||f|| = \sup\{|f(z)| : z \in Z\}$ is also finite. We use the same symbol $||\cdot||$ of norm for the points of $\Sigma_*(I)$ and for elements of function spaces because it is convenient for our notation and never leads to confusion. If M and L are linear spaces then a map $\varphi : M \to L$ is called *linear* if $\varphi(ax + by) = a\varphi(x) + b\varphi(y)$ for any $x, y \in M$ and $a, b \in \mathbb{R}$. Since any $C_p(Z)$ is a linear space (see Problem 027), this gives a definition of a linear map $\varphi : C_p(Z) \to C_p(Y)$ or a linear map $\varphi : C_p(Z) \to \Sigma_*(I)$.

Suppose that Z is a compact space and Y is a compact subspace of $C_p(Z)$ which separates the points of Z. Given $A \subset Z$ and $B \subset Y$, let $\pi_A : C_p(Z) \to C_p(A)$ be the restriction map defined by $\pi_A(f) = f|A$ for every $f \in C_p(Z)$. Now, $e_B : Z \to C_p(B)$ is the evaluation map defined by $e_B(z)(f) = f(z)$ for any $f \in B$ and $z \in Z$. Both maps are continuous (see Problems 152 and 166 for related information). Call the sets A and B *conjugate* if $\pi_A(B) = \pi_A(Y)$ and $e_B(A) = e_B(Z)$. The sets A and B will be called *preconjugate* if $\pi_A(B)$ is dense in $\pi_A(Y)$ and $e_B(A)$ is dense in $e_B(Z)$. The notion of (pre)conjugacy, obviously, depends on spaces Z and Y so we might say that A and B are (Z, Y)-(pre)conjugate if Z and Y are not clear from the context.

Fact 0. Let Y and Z be any spaces. Suppose that D is dense in Y and $f, g : Y \to Z$ are continuous maps such that $f|D = g|D$. Then $f = g$.

Proof. Suppose not and take any $y \in Y$ with $z = f(y) \neq t = g(y)$. Choose disjoint $U, V \in \tau(Z)$ such that $z \in U$ and $t \in V$. If $U' = f^{-1}(U)$ and $V' = g^{-1}(V)$ then $U', V' \in \tau(Y)$ and $y \in U' \cap V'$, i.e., $U' \cap V' \neq \emptyset$. Since D is dense in Y, there is $d \in D \cap U' \cap V'$. Then $f(d) \in U$ and $g(d) \in V$; recalling that we have $f|D = g|D$, we conclude that $g(d) = f(d) \in V \cap U = \emptyset$ which is a contradiction. Fact 0 is proved.

Fact 1. For any space T, a continuous map $r : T \to T$ is a retraction if and only if there exists a closed set $R \subset T$ such that $r(T) = R$ and $r(z) = z$ for every $z \in R$. In particular, we can define a retraction r as a continuous map from T to R with $r(z) = z$ for all $z \in R$.

Proof. If $R \subset T$ is a closed set as in the hypothesis then, for any point $t \in T$, we have $(r \circ r)(t) = r(r(t)) = r(t)$ because $z = r(t) \in R$. Therefore $(r \circ r)(t) = r(t)$ for any $t \in T$ which means that $r \circ r = r$, i.e., r is a retraction.

Let $r : T \to T$ be a retraction. Take any $z \in R = r(T)$; then $z = r(t)$ for some $t \in T$ and hence $r(z) = r(r(t)) = (r \circ r)(t) = r(t) = z$. Thus, $r(z) = z$ for any $z \in R$; in particular, $r(z) \neq z$ implies $z \notin R$.

To see that R is closed, take any $t \in T \backslash R$; then $r(t) \neq t$ and therefore there exist disjoint $U, V \in \tau(T)$ such that $t \in U$ and $r(t) \in V$. By continuity of r there exists a set $W \in \tau(t, T)$ such that $W \subset U$ and $r(W) \subset V$. The sets W and $V \supset r(W)$ are disjoint so $r(t') \neq t'$ for any $t' \in W$. By the last remark of the second paragraph, we have $W \cap R = \emptyset$ so $T \backslash R$ is open in T. Fact 1 is proved.

Fact 2. Let T be any space; assume that $A \subset T$ and $B \subset C_p(T)$. Then

(a) If B separates the points of A and B' is dense in B then B' also separates the points of A;
(b) The map $e_B|A$ is injective if and only if B separates the points of A;
(c) The map $\pi_A|B$ is injective if and only if $e_B(A)$ separates the points of B;
(d) If $e_B(A)$ is dense in $e_B(T)$ then $e_B(A)$ separates the points of B. As a consequence if $e_B(A)$ is dense in $e_B(T)$ then the map $\pi_A|B$ is an injection.

Proof. (a) Take any distinct $a_1, a_2 \in A$; since B separates the points of A, there is $f \in B$ such that $f(a_1) \neq f(a_2)$. Then $\varepsilon = |f(a_1) - f(a_2)| > 0$ and, for any $g \in B' \cap O_T(f, \{a_1, a_2\}, \varepsilon/3)$, we have

$$|g(a_1) - g(a_2)| \geqslant |f(a_1) - f(a_2)| - |f(a_1) - g(a_1)| - |f(a_2) - g(a_2)| \geqslant \varepsilon - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} > 0,$$

and therefore $g(a_1) \neq g(a_2)$, i.e., B' also separates the points of A.

(b) The map e_B is injective if and only if, for any distinct $a_1, a_2 \in A$, we have $e_B(a_1) \neq e_B(a_2)$ which is equivalent to the existence of some $f \in B$ with $e_B(a_1)(f) \neq e_B(a_2)(f)$, i.e., $f(a_1) \neq f(a_2)$; hence e_B is injective if and only if B separates the points of A.

(c) The map $\pi_A|B$ is injective if and only if, for any distinct $f, g \in B$, we have $f|A \neq g|A$ which is equivalent to existence of a point $a \in A$ with $f(a) \neq g(a)$, i.e., $e_B(a)(f) \neq e_B(a)(g)$. It is evident that the last statement is equivalent to the fact that $e_B(A)$ separates the points of B.

(d) It is evident that $e_B(T)$ separates the points of B so we can apply (a) to conclude that $e_B(A)$ also separates the points of B. Fact 2 is proved.

Fact 3. Suppose that Z is compact and Y is a compact subspace of $C_p(Z)$ which separates the points of Z. If compact sets $A \subset Z$ and $B \subset Y$ are conjugate then

(a) The maps $\pi_A|B : B \to \pi_A(B)$ and $e_B|A : A \to e_B(A)$ are homeomorphisms.
(b) If $p_A = (e_B|A)^{-1} \circ e_B$ and $q_B = (\pi_A|B)^{-1} \circ (\pi_A|Y)$ then the maps $p_A : Z \to A$ and $q_B : Y \to B$ are retractions.
(c) $f(p_A(z)) = f(z)$ for any $z \in Z$ and $f \in B$.
(d) $q_B = ((p_A)^* \circ \pi_A)|Y$.

Proof. Take any distinct points $a_1, a_2 \in A$; since Y separates the points of Z, there is $f \in Y$ such that $f(a_1) \neq f(a_2)$. There is a function $g \in B$ with $g|A = f|A$ which shows that $g(a_1) \neq g(a_2)$. As a consequence, B separates the points of A so the map $e_B|A$ is injective (Fact 2). The space A being compact, the mapping $e_B|A : A \to e_B(A)$ is a homeomorphism and therefore $(e_B|A)^{-1} : e_B(A) = e_B(Z) \to A$ is also a continuous map. As a consequence, the composition $(e_B|A)^{-1} \circ e_B$ makes sense and the map $p_A = (e_B|A)^{-1} \circ e_B : Z \to A$ is continuous. It is clear that A is a closed subset of Z such that $p_A : Z \to A$ and $p_A(a) = a$ for any $a \in A$. Thus, $p_A : Z \to A$ is a retraction (Fact 1).

Since the set $e_B(Z)$ separates the points of B and $e_B(Z) = e_B(A)$, the set $e_B(A)$ also separates the points of B and therefore $\pi_A|B$ is an injection by Fact 2. The space B being compact, the map $\pi_A|B : B \to \pi_A(B)$ is a homeomorphism; therefore the map $(\pi_A|B)^{-1} : \pi_A(B) = \pi_A(Y) \to B$ is also continuous. As a consequence, the composition $(\pi_A|B)^{-1} \circ (\pi_A|Y)$ makes sense and the mapping $q_B = (\pi_A|B)^{-1} \circ (\pi_A|Y) : Y \to B$ is continuous. It is immediate that $q_B(f) = f$ for any $f \in B$; the set B being compact, it is closed in Y so q_B is indeed a retraction (Fact 1). This completes the proof of (a) and (b).

To prove (c), observe that, for each point $z \in Z$, the definition of the retraction p_A implies $e_B(p_A(z)) = e_B((e_B|A)^{-1}(e_B(z))) = e_B(z)$; thus $e_B(p_A(z)) = e_B(z)$ which is equivalent to $f(p_A(z)) = e_B(p_A(z))(f) = e_B(z)(f) = f(z)$ for all $f \in B$ and this is precisely what (c) says.

Take any $g \in B$; then $g(p_A(z)) = g(z)$ for any $z \in Z$ by (c). This is equivalent to $(g|A) \circ p_A = g$, i.e., $(p_A)^*(\pi_A(g)) = g$. Now, if $f \in Y$ then $g = q_B(f) \in B$. We have $\pi_A(g) = \pi_A(f)$ by definition of the map q_B; therefore $(p_A)^*(\pi_A(f)) = (p_A)^*(\pi_A(g)) = g = q_B(f)$ which shows that $q_B = ((p_A)^* \circ \pi_A)|Y$ and this is what was promised in (d). Fact 3 is proved.

Fact 4. Suppose that Z is compact and Y is a compact subspace of $C_p(Z)$ which separates the points of Z. If $A' \subset Z$ and $B' \subset Y$ are preconjugate then $A = \mathrm{cl}_Z(A')$ and $B = \mathrm{cl}_Y(B')$ are conjugate.

Proof. Since $A \supset A'$ is compact and $e_{B'}(A')$ is dense in $e_{B'}(Z)$, we have the equality $e_{B'}(A) = e_{B'}(Z)$. For any $z \in Z$, there is $a \in A$ such that $e_{B'}(a) = e_{B'}(z)$; the maps $e_B(z)$ and $e_B(a)$ are continuous on B and coincide on a dense set B' of the space B. By Fact 0, they coincide on the whole B, i.e., $e_B(a) = e_B(z)$ which shows that $e_B(A) = e_B(Z)$.

Analogously, since $B \supset B'$ is compact and $\pi_{A'}(B')$ is dense in $\pi_{A'}(Y)$, we have $\pi_{A'}(B) = \pi_{A'}(Y)$. For any $f \in Y$, there is $g \in B$ such that $\pi_{A'}(f) = \pi_{A'}(g)$; the maps $\pi_A(f)$ and $\pi_A(g)$ are continuous on A and coincide on a dense set A' of the space A. By Fact 0, they coincide on the whole A, i.e., $\pi_A(f) = \pi_A(g)$ which shows that $\pi_A(B) = \pi_A(Y)$. Fact 4 is proved.

Fact 5. Suppose that Z is compact and Y is a compact subspace of $C_p(Z)$ which separates the points of Z. Given a limit ordinal β, assume that we have defined sets $A_\alpha \subset Z, B_\alpha \subset Y$ for all $\alpha < \beta$ in such a way that

(a) $A_\alpha \subset A_{\alpha'}$ and $B_\alpha \subset B_{\alpha'}$ for all $\alpha < \alpha' < \beta$.
(b) $\pi_{A_\alpha}(B_{\alpha+1})$ is dense in $\pi_{A_\alpha}(Y)$ and $e_{B_\alpha}(A_{\alpha+1})$ is dense in $e_{B_\alpha}(Z)$ for all $\alpha < \beta$.

Then $A = \mathrm{cl}_Z(\bigcup\{A_\alpha : \alpha < \beta\})$ and $B = \mathrm{cl}_Y(\bigcup\{B_\alpha : \alpha < \beta\})$ are conjugate.
Proof. It suffices to show that $A' = \bigcup\{A_\alpha : \alpha < \beta\}$ and $B' = \bigcup\{B_\alpha : \alpha < \beta\}$ are preconjugate (Fact 4). Take any $f \in Y$ and any $W \in \tau(\pi_{A'}(f), C_p(A'))$. There exists a finite $K \subset A'$ and $\varepsilon > 0$ such that $O_{A'}(\pi_{A'}(f), K, \varepsilon) \subset W$; take $\alpha < \beta$ such that $K \subset A_\alpha$. Since the set $\pi_{A_\alpha}(B_{\alpha+1})$ is dense in $\pi_{A_\alpha}(Y)$, we can find $g \in B_{\alpha+1} \subset B'$ such that $\pi_{A_\alpha}(g) \in O_{A\alpha}(\pi_{A_\alpha}(f), K, \varepsilon)$. This implies that $|g(z) - f(z)| < \varepsilon$ for each $z \in K$ and therefore $\pi_{A'}(g) \in O_{A'}(\pi_{A'}(f), K, \varepsilon)$; thus $\pi_{A'}(g) \in \pi_{A'}(B') \cap W$ which shows that $\pi_{A'}(f)$ is in the closure of the set $\pi_{A'}(B')$. Since the function $f \in Y$ was taken arbitrarily, we conclude that $\pi_{A'}(B')$ is dense in $\pi_{A'}(Y)$.

Take any $z \in Z$ and any $V \in \tau(e_{B'}(z), C_p(B'))$. There exists a finite $L \subset B'$ and $\varepsilon > 0$ such that $O_{B'}(e_{B'}(z), L, \varepsilon) \subset V$; take $\alpha < \beta$ such that $L \subset B_\alpha$. Since the set $e_{B_\alpha}(A_{\alpha+1})$ is dense in the space $e_{B_\alpha}(Z)$, we can find $t \in A_{\alpha+1} \subset A'$ such that $e_{B_\alpha}(t) \in O_{B_\alpha}(e_{B_\alpha}(z), L, \varepsilon)$. This implies $|e_{B'}(z)(f) - e_{B'}(t)(f)| = |f(z) - f(t)| = |e_{B_\alpha}(z)(f) - e_{B_\alpha}(t)(f)| < \varepsilon$ for each $f \in L$ and therefore we have $e_{B'}(t) \in O_{B'}(e_{B'}(z), L, \varepsilon)$; thus $e_{B'}(t) \in e_{B'}(A') \cap V$ which shows that $e_{B'}(z)$ is in the closure of the set $e_{B'}(A')$. Since the point $z \in Z$ was taken arbitrarily, we conclude that $e_{B'}(A')$ is dense in $e_{B'}(Z)$. Fact 5 is proved.

Fact 6. Suppose that Z is compact and Y is a compact subspace of $C_p(Z)$ which separates the points of Z. Given a limit ordinal β, assume that we have defined conjugate sets $A_\alpha \subset Z$ and $B_\alpha \subset Y$ for all $\alpha < \beta$ so that $A_\alpha \subset A_{\alpha'}$ and $B_\alpha \subset B_{\alpha'}$ for all $\alpha < \alpha' < \beta$. Then, the sets $A = \mathrm{cl}_Z(\bigcup\{A_\alpha : \alpha < \beta\})$ and $B = \mathrm{cl}_Y(\bigcup\{B_\alpha : \alpha < \beta\})$ are conjugate.

Proof. For each $\alpha < \beta$, we have $\pi_{A_\alpha}(B_{\alpha+1}) \supset \pi_{A_\alpha}(B_\alpha) = \pi_{A_\alpha}(Y)$. Analogously $e_{B_\alpha}(A_{\alpha+1}) \supset e_{B_\alpha}(A_\alpha) = e_{B_\alpha}(Z)$ and therefore the families $\{A_\alpha : \alpha < \beta\}$ and $\{B_\alpha : \alpha < \beta\}$ satisfy the hypothesis of Fact 5. As a consequence, the sets A and B are conjugate so Fact 6 is proved.

Fact 7. Suppose that Z is compact and Y is a compact subspace of $C_p(Z)$ which separates the points of Z. Given an infinite cardinal κ, assume that we have $P \subset Z$

and $Q \subset Y$ such that $|P| \leqslant \kappa$ and $|Q| \leqslant \kappa$. Then there exist sets $A' \subset Z$ and $B' \subset Y$ such that $P \subset A'$, $Q \subset B'$, $|A'| \leqslant \kappa$, $|B'| \leqslant \kappa$ and the sets $A = \text{cl}_Z(A')$ and $B = \text{cl}_Y(B')$ are conjugate.

Proof. Let $P_0 = P$ and $Q_0 = Q$. Assume that we have sets $P_0, \ldots, P_n$ and $Q_0, \ldots, Q_n$ with the following properties:

(1) $p_i \subset Z$, $Q_i \subset Y$, $|P_i| \leqslant \kappa$ and $|Q_i| \leqslant \kappa$ for all $i \leqslant n$.
(2) $p_i \subset p_{i+1}$ and $Q_i \subset Q_{i+1}$ for all $i < n$.
(3) $\pi_{P_i}(Q_{i+1})$ is dense in $\pi_{P_i}(Y)$ and $e_{Q_i}(P_{i+1})$ is dense in $e_{Q_i}(Z)$ for all $i < n$.

Since weight of the space $\pi_{P_n}(Y)$ does not exceed κ, there exists $Q_{n+1} \subset Y$ such that $Q_n \subset Q_{n+1}$, $|Q_{n+1}| \leqslant \kappa$ and $\pi_{P_n}(Q_{n+1})$ is dense in $\pi_{P_n}(Y)$. Analogously, since weight of the space $e_{Q_n}(Z)$ does not exceed κ, there exists $P_{n+1} \subset Z$ such that $P_n \subset P_{n+1}$, $|P_{n+1}| \leqslant \kappa$ and $e_{Q_n}(P_{n+1})$ is dense in $e_{Q_n}(Z)$. This shows that the inductive construction can go on to give us sequences of sets $\{P_i : i \in \omega\}$ and $\{Q_i : i \in \omega\}$ which satisfy (1)–(3). Applying Fact 5 we can conclude that the sets $A' = \bigcup\{p_i : i \in \omega\}$ and $B' = \bigcup\{Q_i : i \in \omega\}$ are as promised. Fact 7 is proved.

Fact 8. Suppose that Z is compact and Y is a compact subspace of $C_p(Z)$ which separates the points of Z. Suppose that we have sequences of sets $\{A_n : n \in \omega\}$ and $\{B_n : n \in \omega\}$ with the following properties:

(a) $A_n \subset Z$, $B_n \subset Y$, $A_n \subset A_{n+1}$ and $B_n \subset B_{n+1}$ for all $n \in \omega$.
(b) The sets A_n and B_n are conjugate for all $n \in \omega$.

Denote by p_n the retraction $p_{A_n} : Z \to A_n$ defined in Fact 3. For the set $A = \text{cl}_Z(\bigcup\{A_n : n \in \omega\})$ assume that $\{z_n : n \in \omega\} \subset A$ and z_n converges to a point z for some $z \in Z$ (it is clear that the point z has to belong to A). Then the sequence $\{p_n(z_n) : n \in \omega\}$ also converges to z.

Proof. Let $B' = \bigcup\{B_n : n \in \omega\}$; the sets A and $B = \text{cl}_Y(B')$ are conjugate by Fact 6 and therefore the mapping $e_B|A : A \to e_B(A)$ is a homeomorphism (see Fact 3). Since B' is dense in B, the restriction map $\pi : C_p(B) \to C_p(B')$ is injective (Problem 152). Therefore $\pi|e_B(A)$ is also a homeomorphism of $e_B(A)$ onto $\pi(e_B(A))$. It is straightforward that $\pi \circ e_B$ coincides with $e_{B'}$ so A is mapped homeomorphically onto $e_{B'}(A)$. This shows that the sequence $\{p_n(z_n) : n \in \omega\}$ converges to z if and only if the sequence $S = \{e_{B'}(p_n(z_n)) : n \in \omega\} \subset C_p(B')$ converges to $e_{B'}(z)$.

Fix any $g \in B'$; there is $\kappa \in \omega$ such that $g \in B_\kappa$. For any $n \geqslant \kappa$ we have $g \in B_k \subset B_n$; since the sets A_n and B_n are conjugate and $g \in B_n$, we have $g(p_n(z_n)) = g(p_{A_n}(z_n)) = g(z_n)$ by property (c) of Fact 3. The function g being continuous, we have $g(z_n) \to g(z)$; since the sequences $\{g(p_n(z_n)) : n \geqslant \kappa\}$ and $\{g(z_n) : n \geqslant \kappa\}$ coincide, we conclude that we have $g(p_n(z_n)) \to g(z)$. The function g was taken arbitrarily, so we established that $g(p_n(z_n)) \to g(z)$ for all $g \in B'$.

It remains to observe that $e_{B'}(z)(g) = g(z)$ and $e_{B'}(p_n(z_n))(g) = g(p_n(z_n))$ for each $n \in \omega$ to conclude that the sequence $S(g) = \{e_{B'}(p_n(z_n))(g) : n \in \omega\}$ converges to $e_{B'}(z)(g)$ for each $g \in B'$. Therefore $S = \{e_{B'}(p_n(z_n)) : n \in \omega\}$ converges to $e_{B'}(z)$

in $C_p(B')$ (Problem 143) and hence $\{p_n(z_n) : n \in \omega\}$ converges to z. Fact 8 is proved.

Fact 9. Let T be a pseudocompact space. Then $nw(\overline{P}) \leqslant \omega$ for any countable $P \subset C_p(T)$.

Proof. Define a map $\varphi : T \to \mathbb{R}^P$ by the formula $\varphi(z)(f) = f(z)$ for all $z \in T$ and $f \in P$. It is evident that φ is a continuous map so $L = \varphi(T)$ is a second countable compact space. Therefore, $nw(C_p(L)) = nw(L) = \omega$. Let $\varphi^*(f) = f \circ \varphi$ for each $f \in C_p(L)$. It was proved in Problem 163 that $\varphi^* : C_p(L) \to C_p(T)$ is an embedding; the set $F = \varphi^*(C_p(L))$ is closed in $C_p(T)$ because φ is an $\mathbb{R}$-quotient map (see Problem 163(iii) and Fact 3 of S.154). It is immediate that $P \subset F$ so $\overline{P} \subset F$ and hence $nw(\overline{P}) \leqslant nw(F) = nw(C_p(L)) \leqslant \omega$ so Fact 9 is proved.

Fact 10. Let T be a compact space. Then any compact subspace Y of the space $C_p(T)$ is a Fréchet–Urysohn space.

Proof. We have $t(Y) \leqslant t(C_p(T)) = \omega$ (Problem 149). Given any $P \subset Y$ and any $y \in \overline{P}$, find a countable $D \subset P$ such that $y \in \overline{D}$. The space $\overline{D}$ has a countable network by Fact 9; being compact, $\overline{D}$ is second countable (Fact 4 of S.307). Thus $\overline{D}$ is a Fréchet– Urysohn space and therefore there is a sequence $S = \{d_n : n \in \omega\} \subset D$ with $d_n \to y$. It is clear that $S \subset P$ and $S \to y$ whence Y is Fréchet–Urysohn so Fact 10 is proved.

Fact 11. Suppose that Z is a compact space of density $\kappa \geqslant \omega$ and Y is a compact subspace of $C_p(Z)$ which separates the points of Z. Then it is possible to find families $\{A_\alpha : \alpha < \kappa\}$ and $\{B_\alpha : \alpha < \kappa\}$ such that

(a) A_α is a compact subset of Z and B_α is a compact subset of Y for each $\alpha < \kappa$.
(b) A_α and B_α are conjugate for all $\alpha < \kappa$.
(c) $A_\alpha \subset A_\beta$ and $B_\alpha \subset B_\beta$ for all $\alpha < \beta < \kappa$.
(d) $d(A_\alpha) \leqslant \omega \cdot |\alpha|$ and $d(B_\alpha) \leqslant \omega \cdot |\alpha|$ for all $\alpha < \kappa$.
(e) $\mathrm{cl}_Z(\bigcup\{A_\alpha : \alpha < \kappa\}) = Z$ and $A_\beta = \mathrm{cl}_Z(\bigcup\{A_\alpha : \alpha < \beta\})$ for any limit $\beta < \kappa$.

Proof. Fix a dense set $D = \{z_\alpha : \alpha < \kappa\}$ in the space Z. We will construct families $\{A_\alpha : \alpha < \kappa\}$ and $\{B_\alpha : \alpha < \kappa\}$ by transfinite induction so that
(f) $z_\alpha \in A_\beta$ for all $\alpha < \beta < \kappa$

and the properties (a)–(e) are satisfied.

Let $P = \{z_0\}$ and $Q = \{f_0\}$ for some $f_0 \in Y$. Apply Fact 7 to find compact conjugate sets $A_0 \subset Z$, $B_0 \subset Y$ such that $P \subset A_0$, $Q \subset B_0$, $d(A_0) \leqslant \omega$ and $d(B_0) \leqslant \omega$. It is clear that $z_0 \in A_0$ and the conditions (a)–(e) are satisfied (most of them vacuously) for the sets A_0 and B_0.

Assume that $\beta < \kappa$ and we have constructed families $\{A_\alpha : \alpha < \beta\}$ and $\{B_\alpha : \alpha < \beta\}$ with the properties (a)–(e) where applicable. If β is a limit ordinal, let $A_\beta = \mathrm{cl}_Z(\{A_\alpha : \alpha < \beta\})$ and $B_\beta = \mathrm{cl}_Y(\{B_\alpha : \alpha < \beta\})$. The sets A_β and B_β are conjugate by Fact 6. Applying the property (d) we can find $C_\alpha \subset A_\alpha$ and $E_\alpha \subset B_\alpha$ such that $\mathrm{cl}_Z(C_\alpha) = A_\alpha$, $\mathrm{cl}_Y(E_\alpha) = B_\alpha$ and $|C_\alpha| \leqslant \omega \cdot |\alpha|$, $|E_\alpha| \leqslant \omega \cdot |\alpha|$ for all ordinals $\alpha < \beta$. If $C = \bigcup\{C_\alpha : \alpha < \beta\}$ and $E = \bigcup\{E_\alpha : \alpha < \beta\}$ then $\mathrm{cl}_Z(C) = A_\beta$, $\mathrm{cl}_Y(E) = B_\beta$ and

$|C| \leqslant \omega \cdot |\beta|$ as well as $|E| \leqslant \omega \cdot |\beta|$. Thus, the condition (d) is satisfied for A_β and B_β; it is evident that (f) also holds.

Now, if $\beta = \alpha + 1$ then we can find sets $C_\alpha \subset A_\alpha$ and $E_\alpha \subset B_\alpha$ such that $\mathrm{cl}_Z(C_\alpha) = A_\alpha$, $\mathrm{cl}_Y(E_\alpha) = B_\alpha$ and $|C_\alpha| \leqslant \omega \cdot |\alpha|$, $|E_\alpha| \leqslant \omega \cdot |\alpha|$. By Fact 7 applied to the sets $P = C_\alpha \cup \{z_\alpha\}$ and $Q = E_\alpha$, there exist $C_\beta \supset P$, $E_\beta \supset Q$ such that $|C_\beta| \leqslant \omega \cdot |\beta|$, $|E_\beta| \leqslant \omega \cdot |\beta|$ and the sets $A_\beta = \mathrm{cl}_Z(C_\beta)$, $B_\beta = \mathrm{cl}_Y(E_\beta)$ are conjugate. It is obvious that the conditions (a)–(f) are satisfied where applicable also for the families $\{A_\alpha : \alpha \leqslant \beta\}$ and $\{B_\alpha : \alpha \leqslant \beta\}$. Thus our inductive construction can be carried out giving us the families $\{A_\alpha : \alpha < \beta\}$ and $\{B_\alpha : \alpha < \beta\}$ with the properties (a)–(f). The first part of (e) is also fulfilled because $D \subset \bigcup\{A_\alpha : \alpha < \kappa\}$ by property (f). Fact 11 is proved.

Call a compact space Z *Eberlein compact* if there is a compact $Y \subset C_p(Z)$ which separates the points of Z. A family $\{p_\alpha : \alpha < \kappa\}$ of retractions on a compact space Z with $d(Z) = \kappa$ is called *resolving* if it satisfies the following conditions:

(r1) For any $\alpha < \kappa$, the space $p_\alpha(Z)$ is either separable or the density of $p_\alpha(Z)$ is strictly less than κ.
(r2) $p_\alpha(Z) \subset p_\beta(Z)$ if $\alpha < \beta$.
(r3) If $\beta < \kappa$ is a limit ordinal then $p_\beta(Z) = \mathrm{cl}_Z(\bigcup\{p_\alpha(Z) : \alpha < \beta\})$.
(r4) If $\{z_n : n \in \omega\} \subset Z$, $z_n \to z \in Z$ and we are given an increasing sequence $\{\alpha_n : n \in \omega\} \subset \kappa$ with $\{z_n : n \in \omega\} \subset \bigcup\{p_{\alpha_n}(Z) : n \in \omega\}$ then the sequence $\{p_{\alpha_n}(z_n) : n \in \omega\}$ also converges to z.
(r5) $Z = \mathrm{cl}_Z(\bigcup\{p_\alpha(Z) : \alpha < \kappa\})$.

A compact space Z is called *resolvable* if every closed $F \subset Z$ has a resolving family of retractions.

Fact 12. (a) A compact space Z is Eberlein compact if and only if it embeds into $C_p(K)$ for some compact space K.

(b) Any closed subspace of an Eberlein compact space is Eberlein compact.

(c) Any Eberlein compact space is Fréchet–Urysohn and resolvable.

Proof. (a) If Z is Eberlein compact then there is a compact $K \subset C_p(Z)$ which separates the points of Z. The map $e_K : Z \to C_p(K)$ is an injection by Fact 2(b); therefore it embeds Z into $C_p(K)$. To prove sufficiency, observe that, if $Z \subset C_p(K)$ is compact then the space $Y = e_Z(K)$ is a compact subspace of $C_p(Z)$ which separates the points of Z.

(b) This is an easy consequence of (a) because if Z is a subspace of $C_p(K)$ for some compact K then any closed subset of Z embeds in the same $C_p(K)$.

(c) If Z is Eberlein compact then $Z \subset C_p(K)$ for some compact K by (a); apply Fact 10 to conclude that Z is Fréchet–Urysohn. Now, if $d(Z) = \kappa$ and $Y \subset C_p(Z)$ is compact and separates the points of Z then there exist families $\{A_\alpha : \alpha < \kappa\}$ and $\{B_\alpha : \alpha < \kappa\}$ with the properties (a)–(e) of Fact 11. If we let $p_\alpha = p_{A_\alpha}$, where $p_{A_\alpha} : Z \to A_\alpha$ is the retraction determined by the conjugate pair (A_α, B_α), then the family $\{p_\alpha : \alpha < \kappa\}$ is as promised. Indeed, (r1) follows from $p_\alpha(Z) = A_\alpha$ and Fact 11(d). The property (r2) is a consequence of Fact 11(c); (r3) and (r5) follow from Fact 11(e). Finally (r4) follows from Fact 8. We showed that every Eberlein compact Z

has a resolving family of retractions. Since every closed subspace of Z is also an Eberlein compact by (b), the space Z is resolvable so Fact 12 is proved.

Fact 13. Let Z be a compact Fréchet–Urysohn space. Suppose that $\{p_\alpha : \alpha < \kappa\}$ is a family of retractions on Z with the properties (r2)–(r5). Let $A_\alpha = p_\alpha(Z)$ for each ordinal $\alpha < \kappa$; denote by h_{-1} the restriction map π_{A_0}. For any ordinal $\alpha < \kappa$, let $h_\alpha(f) = \pi_{A_\alpha+1} \circ (f - (p_\alpha)^*(\pi_{A_\alpha}(f)))$. In other words, $h_\alpha : C_p(Z) \to C_p(A_{\alpha+1})$ and for every $z \in A_{\alpha+1}$, we have $h_\alpha(f)(z) = f(z) - f(p_\alpha(z))$. Then we have

(a) If $f, g \in C(Z)$ and $f \neq g$ then there is $\alpha \in [-1,\kappa)$ such that $h_\alpha(f) \neq h_\alpha(g)$.
(b) For any $f \in C(Z)$ and for any $\varepsilon > 0$ the set $K(f, \varepsilon) = \{\alpha < \kappa :$ there exists a point $z \in A_{\alpha+1}$ with $|h_\alpha(f)(z)| \geqslant \varepsilon\}$ is finite.

Proof. (a) We have $Z = \overline{\bigcup\{A_\alpha : \alpha<\kappa\}}$ by (r5); apply Fact 0 to conclude that $f|(\bigcup\{A_\alpha : \alpha < \kappa\}) \neq g|(\bigcup\{A_\alpha : \alpha < \kappa\})$. Therefore, there exists $\alpha < \kappa$ for which $f|A_\alpha \neq g|A_\alpha$; let β be the minimal of such α. Applying the property (r3) in the same way we applied (r5), we can see that β cannot be a limit ordinal. If $\beta = 0$ then $h_{-1}(f) = f|A_0 \neq g|A_0 = h_{-1}(g)$ so $\alpha = -1$ is suitable for checking (a).

If $\beta > 0$ then $\beta = \alpha + 1$ for some $\alpha < \kappa$; take any $z \in A_\beta$ with $f(z) \neq g(z)$. By the choice of β we have $f|A_\alpha = g|A_\alpha$ whence $f(p_\alpha(z)) = g(p_\alpha(z))$. As a consequence, $f(z) - f(p_\alpha(z)) \neq g(z) - g(p_\alpha(z))$, i.e., $h_\alpha(f) \neq h_\alpha(g)$ so (a) is proved.

(b) If the set $K(f, \varepsilon)$ is infinite then there exist sequences $\{z_n : n \in \omega\} \subset Z$ and $\{\alpha_n : n \in \omega\} \subset \kappa$ such that $z_n \in A_{\alpha_n+1}$, $\alpha_n \neq \alpha_m$ if $m \neq n$ and $|h_{\alpha n}(f)(Z_n)| \geqslant \varepsilon$ for all $n \in \omega$. Now, κ is a well-ordered set and Z is Fréchet–Urysohn; this implies that we can assume that $\alpha_n < \alpha_{n+1}$ for all $n \in \omega$ and the sequence $\{Z_n : n \in \omega\}$ converges to some point $z \in Z$ for if it is not so, we can restrict our consideration to the respective subsequences. We have $z_n \in A_{\alpha_n+1} \subset A_{\alpha_n+1}$ because $\alpha_n + 1 \leqslant \alpha_{n+1}$. This shows that $\{Z_n : n \in \omega\} \subset \bigcup\{A_{\alpha_n} : n \in \omega\}$; applying (r4) we can see that the sequence $\{p_{\alpha_n}(z_n) : n \in \omega\}$ also converges to z. The function f being continuous, we have $f(p_{\alpha_n}(z_n)) \to f(z)$ and $f(z_n) \to f(z)$. Therefore there exists $m \in \omega$ such that $|f(z_m) - f(p_{\alpha_m}(z_m))| < \varepsilon$, i.e., $|h_{\alpha_m}(f)(x_m)| < \varepsilon$ which is a contradiction with the choice of the point z_m. Fact 13 is proved.

Fact 14. If Z is a separable compact space then $C_p(Z)$ condenses linearly onto a subspace of $\Sigma_*(\mathbb{N})$ without increasing the norm. In other words, there exists a linear injective continuous map $\varphi : C_p(Z) \to \Sigma_*(\mathbb{N})$ such that $||\varphi(f)|| \leqslant ||f||$ for all $f \in C_p(Z)$.

Proof. Take any dense set $D = \{z_n : n \in \mathbb{N}\}$ in the space Z. For any $f \in C_p(Z)$ let $\varphi(f)(n) = \frac{1}{n}f(z_n)$ for all $n \in \mathbb{N}$. It is straightforward that $\varphi(f) \in \mathbb{R}^{\mathbb{N}}$ for each function $f \in C_p(Z)$ and the map $\varphi : C_p(Z) \to \mathbb{R}^{\mathbb{N}}$ is linear and continuous. Take any $f \in C_p(Z)$; the space Z being compact, the function f is bounded, so there is $L > 0$ such that $|f(z)| \leqslant L$ for all $z \in Z$. If $\varepsilon > 0$ then $\frac{1}{m} < \frac{\varepsilon}{L}$ for some $m \in \mathbb{N}$. Thus $|\varphi(f)(n)| = \frac{1}{n}|f(z_n)| \leqslant \frac{1}{n} \cdot L \leqslant \frac{1}{m} \cdot L < \varepsilon$ for all $n \geqslant m$ which shows that $\varphi(f) \in \Sigma_*(\mathbb{N})$. Therefore φ is a linear map from $C_p(Z)$ to $\Sigma_*(\mathbb{N})$. If $f \neq g$ then $f|D \neq g|D$ (Fact 0) so $f(Z_n) \neq g(Z_n)$ for some $n \in \mathbb{N}$. Hence $\varphi(f)(n) \neq \varphi(g)(n)$ so $\varphi(f) \neq \varphi(g)$; this proves that φ is a condensation. Observe finally that $|f(Z_n)| \leqslant ||f||$ and $|\varphi(f)(n)| = \frac{1}{n}|f(z_n)| \leqslant \|f\|$ for all $n \in \mathbb{N}$. This shows that $||\varphi(f)|| \leqslant ||f||$ so Fact 14 is proved.

Fact 15. Let Z be a resolvable compact Fréchet–Urysohn space. Then the space $C_p(Z)$ can be linearly condensed without increasing norm into $\Sigma_*(I)$ for some I of cardinality $d(Z)$. In other words, there exists a linear continuous injective map $\varphi : C_p(Z) \to \Sigma_*(I)$ such that $||\varphi(f)|| \leqslant ||f||$ for each $f \in C_p(Z)$.

Proof. The proof will be by induction on the cardinal number $\kappa = d(Z)$. Our statement is true for $\kappa = \omega$ by Fact 14. Assume that $d(Z) = \kappa$ for some $\kappa > \omega$ and the statement of our Fact is true for any resolvable compact Fréchet–Urysohn space of density $< \kappa$.

There exists a resolving family $\{p_\alpha : \alpha < \kappa\}$ of retractions on the space Z. Let $A_\alpha = p_\alpha(Z)$ for each $\alpha < \kappa$; denote the restriction map $\pi_{A_0} : C_p(Z) \to C_p(A_0)$ by g_{-1}. For any ordinal $\alpha < \kappa$, let $g_\alpha(f) = \frac{1}{2} \cdot \left(\pi_{A_{\alpha+1}} \circ (f - (p_\alpha) * (\pi_{A_\alpha}(f)))\right)$. In other words, $g_\alpha : C_p(Z) \to C_p(A_{\alpha+1})$ and for every point $z \in A_{\alpha+1}$, we have $g_\alpha(f)(z) = \frac{1}{2}(f(z) - f(p_\alpha(z)))$. It is easy to see that g_α is a linear continuous map with $||g_\alpha(f)|| \leqslant ||f||$ for any $f \in C_p(Z)$ and any $\alpha \in [-1, \kappa)$. Here the norm of $g_\alpha(f)$ is taken in the space $C_p(A_{\alpha+1})$. The condition (rl) implies that the density of the compact space $A_{\alpha+1}$ is strictly less than κ; since the space $A_{\alpha+1}$ is resolvable, by the induction hypothesis, there exists a linear injective continuous map $\varphi_\alpha : C_p(A_{\alpha+1}) \to \Sigma_*(I_{\alpha+1})$ with $||\varphi_\alpha(f)|| \leqslant ||f||$ for any function $f \in C_p(A_{\alpha+1})$; here $I_{\alpha+1}$ is a set with $|I_{\alpha+1}| = d(A_{\alpha+1}) < \kappa$. Without loss of generality we can assume that $I_{\alpha+1} \cap I_{\beta+1} = \emptyset$ if $\alpha \neq \beta$. Observe also that the cardinality of the set $I = \bigcup\{I_{\alpha+1} : -1 \leqslant \alpha < \kappa\}$ is equal to κ.

For any $f \in C_p(Z)$ let us define $\varphi(f) \in \Sigma_*(I)$ as follows. If $i \in I$ then there is a unique $\alpha \in [-1, \kappa)$ such that $i \in I_{\alpha+1}$; let $\varphi(f)(i) = \varphi_\alpha(g_\alpha(f))(i)$. It is immediate that $\varphi(f) \in \Pi = \prod\{\Sigma_*(I_{\alpha+1}) : \alpha \in [-1,\kappa)\} \subset \mathbb{R}^I$ and, for every $\alpha \in [-1, \kappa)$, we have $w_\alpha \circ \varphi = \varphi_\alpha \circ g_\alpha$, where $w_\alpha : \Pi \to \Sigma_*(I_{\alpha+1})$ is the respective natural projection. The map $\varphi_\alpha \circ g_\alpha$ being continuous for each $\alpha \in [-1,\kappa)$, this shows that φ is a continuous map (Problem 101). The linearity of φ follows easily from the fact that $\varphi_\alpha \circ g_\alpha$ is linear for each $\alpha \in [-1,\kappa)$. Observe, also that $||h|| = \sup\{||w_\alpha(h)|| : \alpha \in [-1,\kappa)\}$ for each $h \in \Pi$. Since $||w_\alpha(\varphi(f))|| = ||\varphi_\alpha(g_\alpha(f))|| \leqslant ||g_\alpha(f)|| \leqslant ||f||$, we have $||\varphi(f)|| \leqslant ||f||$ for each $f \in C_p(Z)$.

Now, if we take any $f, h \in C_p(Z)$ with $f \neq h$ then there is $\alpha \in [-1,\kappa)$ such that $g_\alpha(f) \neq g_\alpha(h)$ (see Fact 13(a) applied to $g_\alpha = \frac{1}{2} \cdot h_\alpha$). Since the map φ_α is a condensation, we have $\varphi_\alpha(g_\alpha(f)) \neq \varphi_\alpha(g_\alpha(h))$ whence $\varphi(f) \neq \varphi(h)$, i.e., the map φ is injective. The last thing we must prove is that $\varphi(f) \in \Sigma_*(I)$ for each $f \in C_p(Z)$. To obtain a contradiction suppose not. Then there is $\varepsilon > 0$ and an infinite $J \subset I$ such that $|\varphi(f)(i)| \geqslant \varepsilon$ for each $i \in J$. If $J \cap I_{\alpha+1}$ is infinite for some $\alpha \in [-1, \kappa)$ then we have the desired contradiction because $w_\alpha(\varphi(f)) \in \Sigma_*(I_{\alpha+1})$, which means that the number of i's with $i \in I_{\alpha+1}$ with $|\varphi(f)(i)| \geqslant \varepsilon$, is finite.

Now, if $J \cap I_{\alpha+1}$ is finite for each $\alpha \in [-1, \kappa)$ then there are infinitely many ordinals $\alpha \in [-1,\kappa)$ such that, for some $i \in I_{\alpha+1}$, we have $|\varphi(f)(i)| = |\varphi_\alpha(g_\alpha(f))(i)| \geqslant \varepsilon$. By the definition of the norm in $\Sigma_*(I_{\alpha+1})$ we have $||\varphi_\alpha(g_\alpha(f))|| \geqslant \varepsilon$ and hence

$||g_\alpha(f)|| \geqslant \varepsilon$ because the map φ_α does not increase the norm. As a consequence, for infinitely many α's there exists a point $z \in A_{\alpha+1}$ with $|g_\alpha(f)(z)| \geqslant \varepsilon$. This contradicts Fact 13(b) applied to the functions $h_\alpha = 2 \cdot g_\alpha$ and the number 2ε. The obtained contradiction shows that $\varphi\,(C_p(Z)) \subset \Sigma_*(I)$ so Fact 15 is proved.

Fact 16. (a) If Z is an Eberlein compact space then $C_p(Z)$ condenses into $\Sigma_*(I)$ for some set I.
(b) A compact space Z is Eberlein if and only if Z embeds in $\Sigma_*(I)$ for some set I.

Proof. (a) Any Eberlein compact space Z is Fréchet–Urysohn and resolvable (Fact 12(c)) so we can apply Fact 15 to conclude that $C_p(Z)$ condenses into $\Sigma_*(I)$ for some set I.

(b) Suppose that Z is a compact subspace of $\Sigma_*(I)$ for some I. For any $i \in I$, denote by π_i the natural projection of $\mathbb{R}^I$ to the ith factor. Let $u \in C_p(Z)$ be the function equal to zero at all points of Z and $f_i = \pi_i|Z$ for all $i \in I$. Observe that the set $K = \{u\} \cup \{f_i : i \in I\}$ separates the points of Z. Indeed, if $y, z \in Z$ and $y \neq z$ then, recalling that $Z \subset \mathbb{R}^I$, we conclude that there is $i \in I$ such that $\pi_i(y) \neq \pi_i(z)$, i.e., $f_i(y) \neq f_i(z)$.

To see that K is compact take any open (in $C_p(Z)$) cover $\mathcal{U}$ of the set K. There is $U \in \mathcal{U}$ such that $u \in U$; choose $\varepsilon > 0$ and a finite $P = \{z_1, \ldots, z_n\} \subset Z$ such that $O_Z(u, P, \varepsilon) \subset U$. Since $z_k \in \Sigma_*(I)$ for all $k \leqslant n$, there is a finite $J \subset I$ such that $|\pi_i(z_k)| < \varepsilon$ for all $i \in I \backslash J$ and $k \leqslant n$. Therefore $|f_i(z_k)| < \varepsilon$ for all $i \in I \backslash J$ and $k \leqslant n$ which implies $f_i \in O_Z(u, P, \varepsilon) \subset U$ for all $i \in I \backslash J$. Thus the set $K \backslash U$ is finite and can be covered by some finite subfamily $\mathcal{U}' \subset \mathcal{U}$. Since the family $\{U\} \cup \mathcal{U}'$ is a finite subcover of $\mathcal{U}$, we proved that K is a compact subset of $C_p(Z)$ which separates the points of Z. This shows that Z is an Eberlein compact so we proved sufficiency.

Now, if Z is an Eberlein compact, fix a compact $Y \subset C_p(Z)$ which separates the points of Z. The map $e_Y : Z \to C_p(Y)$ is injective by Fact 2 so $e_Y : Z \to e_Y(Z)$ is a homeomorphism, i.e., Z embeds into $C_p(Y)$. Since Y is a compact subspace of $C_p(Z)$, it is also Eberlein compact by Fact 12. Now apply (a) to find a condensation φ of $C_p(Y)$ into $\Sigma_*(I)$ for some I. It is clear that $\varphi \circ e_Y$ is an embedding of Z into $\Sigma_*(I)$ so Fact 16 is proved.

Fact 17. Let Z be an Eberlein compact space. Then, for any $x \in Z$, there is a family $\mathcal{V} = \{V_n : n \in \mathbb{N}\} \subset \tau^*(Z)$ which converges to x in the sense that, for any $U \in \tau(x, Z)$, there is $m \in \mathbb{N}$ such that $V_n \subset U$ for all $n \geqslant m$.

Proof. We can consider that $Z \subset \Sigma_*(I)$ for some set I (Fact 16). Let u be the element of $\Sigma_*(I)$ with $u(i) = 0$ for all indices $i \in I$. Observe that, $a, b \in \Sigma_*(I)$ implies $a + b \in \Sigma_*(I)$ and $a - b \in \Sigma_*(I)$. If $a \in \Sigma_*(I)$, define a map $T_a : \mathbb{R}^I \to \mathbb{R}^I$ by the formula $T_a(c) = c - a$ for all $c \in \mathbb{R}^I$. Then $T_a : \mathbb{R}^I \to \mathbb{R}^I$ is a homeomorphism (Problem 079) such that $T_a(\Sigma_*(I)) \subset \Sigma_*(I)$. Thus, considering the space $T_x(Z)$ instead of Z, we do not loose generality and reduce our task to the case when $x = u \in Z$. So let us show that there is a family $\mathcal{U} = \{U_n : n \in \omega\} \subset \tau^*(Z)$ which converges to u.

The following property is crucial for our proof.

$(*)$ For any $W \in \tau(u, Z)$ and any $\varepsilon > 0$, there exists $V \in \tau^*(Z)$ and a finite $J \subset I$ such that $V \subset W$ and $|z(i)| \leqslant \varepsilon$ for any $z \in V$ and $i \in I \backslash J$.

Given a finite $K \subset I$, let $O(K) = \{z \in Z : |z(i)| > \varepsilon$ for every $i \in K\}$; it is evident that $O(K)$ is an open subset of Z for every finite $K \subset I$. Assume that $(*)$ does not hold and take $W \in \tau(u, Z)$ which witnesses this. Let $W_0 = W$ and $J_0 = \emptyset$. Suppose that we have $W_0, \ldots, W_n \in \tau^*(Z)$ and finite sets $J_0, \ldots, J_n \subset I$ with the following properties:

(1) $W = W_0 \supset \cdots \supset W_n$ and $J_0 \subset \cdots \subset J_n$.
(2) $W_{k+1} = W_k \cap O(J_{k+1})$ and $J_{k+1} \backslash J_k \neq \emptyset$ for every $k \leqslant n - 1$.

If, for every $i \in I \backslash J_n$ and every $w \in W_n$, we have $|w(i)| \leqslant \varepsilon$ then we can take $V = W_n$ and $J = J_n$ to show that $(*)$ holds for W, a contradiction. Therefore there exists $i \in I \setminus J_n$ and $y \in W_n$ such that $|y(i)| > \varepsilon$. Let $J_{n+1} = J_n \cup \{i\}$; the set $W_{n+1} = O(J_{n+1}) \cap W_n$ is non-empty because $y \in W_{n+1}$ so our inductive construction can go on giving us sequences $\{W_n : n \in \omega\}$ and $\{J_n : n \in \omega\}$ with properties (1) and (2) for every $n \in \omega$. The set $P = \bigcup\{J_n : n \in \omega\}$ is infinite by (2); for each $n \in \mathbb{N}$, the set $F_n = \{z \in Z : |z(i)| \geqslant \varepsilon$ for all $i \in J_n\}$ is non-empty because $\emptyset \neq W_n \subset O(J_n) \subset F_n$ by (2). Besides, F_n is closed and $F_{n+1} \subset F_n$ for each $n \in \omega$. As a consequence, $F = \bigcap\{F_n : n \in \omega\} \neq \emptyset$ because Z is compact. But if $y \in F$ then $|y(i)| \geqslant \varepsilon$ for every $i \in P$ which contradicts the fact that $y \in \Sigma_*(I)$ and P is infinite. The property $(*)$ is proved.

Given a non-empty finite set $J \subset I$ and any $n \in \mathbb{N}$, we will need the set $W(J, n) = \{z \in Z : |z(i)| < \frac{1}{n}$ for each $i \in J\}$; if $J = \emptyset$ then we let $W(J, n) = Z$. It is clear that $W(J, n)$ is an open subset of Z which contains u for all $n \in \mathbb{N}$.

Applying $(*)$ to the set $W = Z$, we can find a set $V_1 \in \tau^*(Z)$ and a finite $B_1 \subset I$ such that $|z(i)| \leqslant 1$ for any $i \in I \backslash B_1$ and $z \in V_1$.

Suppose that $n \in \mathbb{N}$ and we have constructed sets $V_1, \ldots, V_n \in \tau^*(Z)$ and $B_1, \ldots, B_n \subset I$ with the following properties:

(4) $B_1 \subset \cdots \subset B_n$.
(5) $V_{k+1} \subset W(B_k, k + 1)$ for each $k = 1, \ldots, n - 1$.
(6) For any $k \leqslant n$, we have $|z(i)| \leqslant 1/k$ for any $i \in I \backslash B_k$ and $z \in V_k$.

Apply $(*)$ to the set $W = W(B_n, n + 1)$ to find $V_{n+1} \in \tau^*(Z)$ and a finite set $P \subset I$ such that $V_{n+1} \subset W$ and $|z(i)| \leqslant \frac{1}{n+1}$ for all $z \in V_{n+1}$ and $i \in I \backslash P$. Letting $B_{n+1} = B_n \cup P$, we accomplish our inductive construction getting sequences $\{V_n : n \in \mathbb{N}\}$ and $\{B_n : n \in \mathbb{N}\}$ with properties (4)–(6) for all $n \in \mathbb{N}$. The sequence $\{V_n : n \in \mathbb{N}\}$ converges to u. To see this, take any $U \in \tau(u, Z)$; there exists a finite $J \subset I$ and $k \in \mathbb{N}$ such that $W(J, k) \subset U$. The set J is finite, so there is $m \in \mathbb{N}$ such that $m > k$ and $J \cap B_m = J \cap (\bigcup\{B_n : n \in \mathbb{N}\})$.

For any $n \geqslant m + 1$ and any $z \in V_n$, if $i \in J \cap B_m$ then $i \in B_{n-1}$ and hence $|z(i)| < \frac{1}{n} < \frac{1}{m} < \frac{1}{k}$ by (5). If $i \in J \backslash B_m$ then $i \notin B_n$ which implies $|z(i)| \leqslant \frac{1}{n} < \frac{1}{m} < \frac{1}{k}$ by (6). As a consequence, $|z(i)| < \frac{1}{k}$ for all $i \in J$, i.e., $z \in W(J, k)$. The point $z \in V_n$ was chosen arbitrarily, so $V_n \subset W(J, k) \subset U$ for all $n \geqslant m + 1$ and therefore the sequence $\{V_n : n \in \mathbb{N}\}$ converges to u. Fact 17 is proved.

Fact 18. If a space Z has a dense pseudocompact subspace then Z is also pseudocompact.

Proof. Let Y be a dense pseudocompact subspace of Z. If $f : Z \to \mathbb{R}$ is a continuous function then $f(Z) = f(\text{cl}_Z(Y)) \subset \overline{f(Y)}$ (the bar denotes the closure in $\mathbb{R}$). However, $f(Y)$ is compact being a pseudocompact second countable space (Problem 212) so $f(Y)$ is closed and bounded in $\mathbb{R}$ whence $f(Z) \subset \overline{f(Y)} = f(Y)$. Thus $f(Z)$ is bounded and hence Z is also pseudocompact. Fact 18 is proved.

Fact 19. If Z is an Eberlein compact space then any pseudocompact subspace of Z is compact and hence closed in Z.

Proof. Assume that $P \subset Z$ is a pseudocompact non-closed subspace of Z. Fix any point $z \in \overline{P} \backslash P$; the space $Y = \overline{P}$ is Eberlein compact (Fact 12) and $Y \backslash \{z\}$ is pseudocompact because it has a dense pseudocompact subspace P (Fact 18). Apply Fact 17 to find a sequence $\{V_n : n \in \mathbb{N}\} \subset \tau^*(Y)$ which converges to z. Since $P \subset Y \backslash \{z\}$ is dense in Y, the sequence $\mathcal{V} = \{V_n \backslash \{z\} : n \in \mathbb{N}\}$ consists of non-empty open subsets of $Y \backslash \{z\}$. We claim that the family $\mathcal{V}$ is locally finite in $Y \backslash \{z\}$. Indeed, if $y \in Y \backslash \{z\}$ then take $U, V \in \tau(Y)$ such that $z \in U, y \in V$ and $U \cap V = \emptyset$. Since $\mathcal{V}$ converges to z, there exists $m \in \mathbb{N}$ for which $V_n \subset U$ for all $n \geqslant m$. This shows that V is a neighbourhood of y which intersects at most m elements of $\mathcal{V}$. Thus $\mathcal{V}$ is an infinite locally finite family of non-empty open subsets of $Y \backslash \{z\}$ which is a contradiction with pseudocompactness of $Y \backslash \{z\}$. Fact 19 is proved.

We are, at last, ready to present our solution. Let X be a space with a strongly dense σ-countably compact set $A \subset C_p(X)$. The space X is pseudocompact by Problem 350. Let $A = \bigcup \{A_n : n \in \omega\}$ where each A_n is countably compact. For each $f \in C_p(X)$ there is a unique $e(f) \in C_p(\beta X)$ such that $e(f)|X = f$ (Fact 4 of S.309). The set $B_n = e(A_n)$ is also a countably compact subspace of $C_p(\beta X)$ for each $n \in \omega$ (Fact 6 of S.309). If C_n is the closure of B_n in $C_p(\beta X)$ then C_n is compact for each $n \in \omega$ (see Fact 18 of this solution and Fact 2 of S.307).

The set $B = \bigcup \{B_n : n \in \omega\}$ is dense in $C_p(\beta X)$. Indeed, if $f \in C_p(\beta X)$ then $g = f|X$ is a uniform limit of some sequence $S \subset A$. In particular, g is in the closure of S in the space $C_p(X)$. Now apply Fact 5 of S.309 to conclude that f is in the closure of $e(S)$ in the space $C_p(\beta X)$. Since $e(S) \subset B$, we have $f \in \overline{B}$; the function f was chosen arbitrarily so $\overline{B} = C_p(\beta X)$. If $C = \bigcup \{C_n : n \in \omega\}$ then $\overline{C} \supset \overline{B} = C_p(\beta X)$ which shows that a σ-compact set C is dense in $C_p(\beta X)$. As a consequence, there is a compact subset of $C_p(\beta X)$ which separates the points of βX (Fact 5 of S.310). Therefore βX is an Eberlein compact; finally apply Fact 19 to conclude that X is closed in βX and hence $X = \beta X$ is compact. Our solution is complete.

S.352. *Suppose that there exists a strongly dense countable $A \subset C_p(X)$. Prove that X is compact and metrizable.*

Solution. Every countable space is, evidently, σ-compact so A is a strongly dense σ-compact subspace of $C_p(X)$; hence X is compact by Problem 351. Since $C_p(X)$ is separable, the space X is metrizable by Problem 213.

S.353. *Give an example of a non-compact space X for which there exists a strongly dense σ-pseudocompact $A \subset C_p(X)$.*

Solution. There exists an infinite pseudocompact space X such that $C_p(X, \mathbb{I})$ is pseudocompact and every countable subset of X is closed and discrete in X (see Facts 1 and 4 of S.286). This shows that X is not even countably compact because in a countably compact space every closed and discrete subspace is finite.

Since $[-n, n]$ is homeomorphic to $\mathbb{I}$, the space $C_p(X, [-n, n])$ is homeomorphic to $C_p(X, \mathbb{I})$ for each $n \in \mathbb{N}$; as a consequence, $C_p(X, [-n, n])$ is pseudocompact for each $n \in \mathbb{N}$ so the space $C_p(X) = \bigcup \{C_p(X, [-n, n]) : n \in \mathbb{N}\}$ is σ-pseudocompact and, of course, strongly dense in itself.

S.354. *Prove that $L(\kappa)^\omega$ is a Lindelöf space for any κ.*

Solution. A family of sets $\mathcal{V}$ is *inscribed* in a family of sets $\mathcal{U}$ if, for any $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$. The family $\mathcal{V}$ is *a refinement* of $\mathcal{U}$ if it is inscribed in $\mathcal{U}$ and $\bigcup \mathcal{V} = \bigcup \mathcal{U}$. Call a space X *completely screenable* if, for any $\mathcal{U} \subset \tau(X)$, there exits a σ-disjoint family $\mathcal{V} \subset \tau(X)$ such that $\mathcal{V}$ is inscribed in $\mathcal{U}$ and $\bigcup \mathcal{V} = \bigcup \mathcal{U}$. In other words a space X is completely screenable if any family of open subsets of X has a σ-disjoint open refinement.

Fact 1. The space $L(\kappa)^\omega$ is completely screenable for any uncountable cardinal κ.

Proof. Let us first prove by induction that $L(\kappa)^n$ is completely screenable for any $n \in \mathbb{N}$. If $n = 1$ and $\mathcal{U} \subset \tau(L(\kappa))$ then we have two cases:

(1) $a \notin \bigcup \mathcal{U}$. Then the family $\mathcal{V} = \{\{\alpha\} : \alpha \in U$ for some $U \in \mathcal{U}\} \subset \tau(L(\kappa))$ is disjoint, inscribed in $\mathcal{U}$ and $\bigcup \mathcal{V} = \bigcup \mathcal{U}$.
(2) $a \in \bigcup \mathcal{U}$. Pick any $W \in \mathcal{U}$ with $a \in W$; then the family $\mathcal{V} = \{W\} \cup \{\{\alpha\} : \alpha \in U$ for some $U \in \mathcal{U}\} \subset \tau(L(\kappa))$ is σ-disjoint (in fact, it is a union of two disjoint families), inscribed in $\mathcal{U}$ and $\bigcup \mathcal{V} = \bigcup \mathcal{U}$. Thus, we proved that $L(\kappa)$ is completely screenable.

Assume that $L(\kappa)^n$ is completely screenable for some $n \in \mathbb{N}$ and take any family $\mathcal{U} \subset \tau(L(\kappa)^{n+1})$. For any $x = (x_1, \ldots, x_n, x_{n+1}) \in L(\kappa)^{n+1}$, the point $\pi(x) = (x_1, \ldots, x_n, a)$ belongs to $L(\kappa)^{n+1}$; it is evident that $\pi : L(\kappa)^{n+1} \to L(\kappa)^{n+1}$ is a continuous map and $F = \pi(L(\kappa)) = L(\kappa)^n \times \{a\}$ is a closed set of $L(\kappa)^{n+1}$ homeomorphic to $L(\kappa)^n$.

Observe that $L(\kappa)^{n+1} \backslash F = \bigcup \{V_\alpha : \alpha < \kappa\}$, where $V_\alpha = L(\kappa)^n \times \{\alpha\}$ for each $\alpha \in \kappa$. It is clear that each V_α is open in $L(\kappa)^{n+1}$ and the family $\{V_\alpha : \alpha < \kappa\}$ is disjoint. Of course, each V_α is also homeomorphic to $L(\kappa)^n$. By the induction hypothesis there exists a σ-disjoint family $\mathcal{V}_\alpha \subset \tau(V_\alpha)$ such that $\mathcal{V}_\alpha$ is inscribed in $\mathcal{U}_\alpha = \{U \cap V_\alpha : U \in \mathcal{U}\}$ and $\bigcup \mathcal{U}_\alpha = \bigcup \mathcal{V}_\alpha$ for each $\alpha < \kappa$. Let $\mathcal{V}_\alpha = \bigcup \{\mathcal{V}_\alpha^m : m \in \omega\}$ where $\mathcal{V}_\alpha^m$ is disjoint for each $m \in \omega$. Then the family $\mathcal{V}^m = \bigcup \{\mathcal{V}_\alpha^m : \alpha < \kappa\}$ is disjoint, inscribed in $\mathcal{U}$ and consists of open subsets of $L(\kappa)^{n+1}$. It is easy to see that $\mathcal{V}' = \bigcup \{\mathcal{V}^m : m \in \omega\}$ is a σ-disjoint family of open subsets of $L(\kappa)^{n+1}$ such that $(\bigcup \mathcal{U}) \cap (L(\kappa)^{n+1} \backslash F) = \bigcup \mathcal{V}'$.

The space F is also homeomorphic to $L(\kappa)^n$ so there is a σ-disjoint family $\mathcal{W}$ of open subsets of F such that $\mathcal{W}$ is inscribed in $\mathcal{U}_F = \{U \cap F : U \in \mathcal{U}\}$ and $\bigcup \mathcal{W} = \bigcup \mathcal{U}_f$. Since $\mathcal{W}$ is also inscribed in $\mathcal{U}$, for each $W \in \mathcal{W}$ we can find $O(W) \in \mathcal{U}$ with $W \subset O(W)$. Observe that $\pi(z) = z$ for any $z \in F$ which shows that $W \subset \pi^{-1}(W)$ for each $W \in \mathcal{W}$. Besides, the family $\{\pi^{-1}(W) : W \in \mathcal{W}\}$ is σ-disjoint and consists of open subsets of $L(\kappa)^{n+1}$. To make it inscribed in $\mathcal{U}$, let $\mathcal{V}'' = \{\pi^{-1}(W) \cap O(W) : W \in \mathcal{W}\}$. Then the family $\mathcal{V}''$ is σ-disjoint, inscribed in $\mathcal{U}$, consists of open subsets of $L(\kappa)^{n+1}$ and $\bigcup \mathcal{V}'' \supset (\bigcup \mathcal{U}) \cap F$. Now, the family $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$ is σ-disjoint, inscribed in $\mathcal{U}$, consists of open subsets of $L(\kappa)^{n+1}$ and $\bigcup \mathcal{V} = \bigcup \mathcal{U}$. Therefore $L(\kappa)^n$ is completely screenable for any $n \in \mathbb{N}$.

Now take any family $\mathcal{U} \subset \tau(L(\kappa)^\omega)$; let $p_n : L(\kappa)^\omega \to L(\kappa)^n$ be the respective natural projection of $L(\kappa)^\omega$ onto the product of its first n factors. Since it suffices to prove our property for any open refinement of $\mathcal{U}$, we can assume that every $U \in \mathcal{U}$ is a standard set, i.e., there is $n \in \omega$ and $V \in \tau(L(\kappa)^n)$ such that $U = p_n^{-1}(V)$; let $i(U) = n$. Then $\mathcal{U} = \bigcup \{\mathcal{U}_n : n \in \omega\}$ where $\mathcal{U}_n = \{U \in \mathcal{U} : i(U) = n\}$. For any $n \in \omega$, the family $\mathcal{U}'_n = \{p_n(U) : U \in \mathcal{U}_n\}$ consists of open subsets of $L(\kappa)^n$; since we proved that $L(\kappa)^n$ is completely screenable, we can find an open (in $L(\kappa)^n$) σ-disjoint refinement $\mathcal{W}_n$ of U'_n. Then the family $\mathcal{V}_n = \left\{p_n^{-1}(W) : W \in \mathcal{W}_n\right\}$ is σ-disjoint, inscribed in $\mathcal{U}$, consists of open subsets of $L(\kappa)^\omega$ and $\bigcup \mathcal{V}_n = \bigcup \mathcal{U}_n$. As a consequence, the family $\mathcal{V} = \bigcup \{\mathcal{V}_n : n \in \omega\}$ is an open σ-disjoint refinement of $\mathcal{U}$ so Fact 1 is proved.

Claim. $\text{ext}(L(\kappa)^\omega) = \omega$, i.e., every closed discrete subset of $L(\kappa)^\omega$ is at most countable.

Proof. Suppose that, on the contrary, there is a set $D = \{x_\alpha : \alpha < \omega_1\} \subset L(\kappa)^\omega$ which is closed and discrete. Denote by q_n the natural projection of $L(\kappa)^\omega$ onto its nth factor. If $q_0(D)$ is countable then there exist $c_0 \in L(\kappa)$ such that the set $D_0 = q_0^{-1}(c_0) \cap D$ is uncountable. If $q_0(D)$ is uncountable then let $c_0 = a$, choose $x_d \in q_0^{-1}(d) \cap D$ for every $d \in q_0(D)$ and let $D_0 = \{x_d : d \in q_0(D)\}$. Observe that in both cases $\left|D_0 \backslash q_0^{-1}(U)\right| \leqslant \omega$ for any open neighbourhood U of the point c_0 in the space $L(\kappa)$.

Assume that we have chosen points $c_0, \ldots, c_n \in L(\kappa)$ and uncountable subsets $D_0 \supset \cdots \supset D_n$ of the set D such that, for every $k \leqslant n$, the set D_k is *concentrated around* c_k, i.e., $\left|D_k \backslash q_k^{-1}(U)\right| \leqslant \omega$ for every $U \in \tau(c_k, L(\kappa))$.

We have to consider two cases:

(1) The set $q_{n+1}(D_n) \subset L(\kappa)$ is countable. Then there is $c_{n+1} \in L(\kappa)$ such that the set $D_{n+1} = D_n \cap q_{n+1}^{-1}(x)$ is uncountable.

(2) The set $q_{n+1}(D_n) \subset L(\kappa)$ is uncountable. Then choose $x_d \in q_{n+1}^{-1}(d) \cap D_n$ for every $d \in q_{n+1}(D_n)$, let $c_{n+1} = a$ and $D_{n+1} = \{x_d : d \in q_{n+1}(D_n)\}$.

It is clear that in both cases the set $D_{n+1} \subset D_n$ is also concentrated around c_{n+1} so our inductive construction can be continued to give us a point $c \in L(\kappa)^\omega$ such that $c(n) = c_n$ for all $n \in \omega$. The point c is an accumulation point for D: to see it, take any $W \in \tau(c, L(\kappa)^\omega)$. There is $n \in \omega$ and $U_0, \ldots, U_n \in \tau(L(\kappa))$ such that $c_k \in U_k$ for each $k \leqslant n$ and $U = \bigcap \left\{q_k^{-1}(U_k) : k \leqslant n\right\} \subset W$. The set D_n is uncountable and

$E_j = D_n \backslash q_j^{-1}(U_j) \subset D_j \backslash q_j^{-1}(U_j)$ is countable for any $j \leqslant n$. As a consequence, the set $D_n \backslash U = \bigcup\{E_j : j \leqslant n\}$ is countable whence $W \cap D \supset U \cap D \supset U \cap D_n$ is uncountable. Since $W \in \tau(c, L(\kappa)^\omega)$ was chosen arbitrarily, we proved that c is an accumulation point of D which is contradiction with the fact that D is closed and discrete so our Claim is proved.

A family γ of subsets of X is called *point-countable* if every point of X belongs to at most countably many elements of γ. Call a space X *metalindelöf* if every open cover $\mathcal{U}$ of the space of X has a point-countable open refinement. Observe that every completely screenable as well as any Lindelöf space is metalindelöf.

Fact 2. Every metalindelöf space of countable extent is Lindelöf.

Proof. Suppose that X is a metalindelöf space, $\text{ext}(X) \leqslant \omega$ and $\mathcal{U}$ is an open cover of X which has no countable subcover and hence no countable open refinement. Fix a point-countable open refinement $\mathcal{V}$ for the cover $\mathcal{U}$. Take any $x_0 \in X$ and let $\mathcal{V}_0 = \bigcup\{V \in \mathcal{V} : x_0 \in V\}$. Then $\mathcal{V}_0$ is countable because $\mathcal{V}$ is point-countable. Assume that $\beta < \omega_1$ and we have $\{x_\alpha : \alpha < \beta\} \subset X$ and $\{\mathcal{V}_\alpha : \alpha < \beta\}$ with the following properties:

(1) $\mathcal{V}_\alpha = \{V \in \mathcal{V} : x_\alpha \in V\}$ for each $\alpha < \beta$.
(2) $x_\gamma \notin \bigcup\{\bigcup \mathcal{V}_\alpha : \alpha < \gamma\}$ for each $\gamma < \beta$.

The family $\mathcal{V}$ being point-countable, each $\mathcal{V}_\alpha$ is countable and hence the family $\mathcal{V}'_\beta = \bigcup\{\mathcal{V}_\alpha : \alpha < \beta\}$ is also countable. Thus $\bigcup \mathcal{V}'_\beta \neq X$ and hence we can find a point $x_\beta \in X \backslash \left(\bigcup \mathcal{V}'_\beta\right)$. Letting $\mathcal{V}_\beta = \{V \in \mathcal{V} : x_\beta \in V\}$, we finish our inductive construction which gives us a set $D = \{x_\alpha : \alpha < \omega_1\} \subset X$ and families $\{\mathcal{V}_\alpha : \alpha < \omega_1\}$ with the properties (1)–(2). The set D is closed and discrete; to show this, let $x \in X$. Then $x \in V$ for some $V \in \mathcal{V}$; if $x_\alpha \in V$ then $V \in \mathcal{V}_\alpha$ and hence $x_\beta \notin V$ for any $\beta \neq \alpha$ by (2). Therefore every $x \in X$ has a neighbourhood that intersects at most one element of D. Hence D is an uncountable closed and discrete subset of X, a contradiction with $\text{ext}(X) = \omega$. Thus X is Lindelóf and Fact 2 is proved.

Observe finally that $L(\kappa)^\omega$ is metalindelöf because it is completely screenable by Fact 1. Applying Fact 2 and claim we conclude that $L(\kappa)^\omega$ is Lindelöf so our solution is complete.

S.355. *Prove that every G_δ-subset of X is open if and only if for any countable $A \subset C_p(X)$ we have $\overline{A} \subset C_p(X)$ (the closure is taken in $\mathbb{R}^X$).*

Solution. Call a space X *a P-space* if every G_δ-subset of X is open. Suppose that X is a P-space and $A \subset C_p(X)$ is countable. Take any $f \in [A]$ (the brackets denote the closure in $\mathbb{R}^X$). Given any $x \in X$, we will prove that f is continuous at the point x. Note that the set $\{h(x)\}$ is a G_δ-set in the space $\mathbb{R}$ for any $h \in A$ so $W = \bigcap\{h^{-1}(h(x)) : h \in A\}$ is a G_δ-set in X. Since X is a P-space, the set W is an open neighbourhood of x; we claim that $f(W) = \{f(z)\}$. To see this, suppose that $y \in W$ and $|f(y) - f(x)| > \varepsilon$ for some $\varepsilon > 0$. Since $f \in [A]$, there is $h \in A$ such that $|h(x) - f(x)| < \frac{\varepsilon}{2}$ and $|h(y) - f(y)| < \frac{\varepsilon}{2}$. However, $h(y) = h(x)$ so we have

$|f(y) - f(x)| \leqslant |f(y) - h(y)| + |h(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ which is a contradiction. As a consequence, for any $\varepsilon > 0$ we have $f(W) = \{f(x)\} \subset (f(x) - \varepsilon, f(x) + \varepsilon)$, i.e., f is continuous at x and we proved necessity.

Now, suppose that $[A] \subset C_p(X)$ for any countable $A \subset C_p(X)$ and take any G_δ-set $H \subset X$; fix a family $\mathcal{O} = \{O_n : n \in \omega\} \subset \tau(X)$ with $H = \bigcap \mathcal{O}$ and any $z \in H$. It is easy to construct a family $\{U_n : n \in \omega\} \subset \tau(z, X)$ such that $\overline{U}_n \subset O_n$ and $\overline{U}_{n+1} \subset U_n$ for each $n \in \omega$. Let $f_0 : X \to [0, 1]$ be any continuous function with $f_0(z) = 1$ and $f_0(X \setminus U_0) \subset \{0\}$. If we have a continuous function $f_n : X \to [0, 1]$ with $f_n(x) = 1$ and $f_n(X \setminus U_n) \subset \{0\}$ take any $g_n \in C(X, [0, 1])$ with $g_n(x) = 1$ and $g_n(X \setminus U_{n+1}) \subset \{0\}$ and let $f_{n+1} = f_n \cdot g_n$. This construction provides a sequence $\{f_n : n \in \omega\} \subset C(X, [0, 1])$ such that $f_{n+1}(x) \leqslant f_n(x)$ for any $x \in X$ and $n \in \omega$; besides, $f_n(x) = 1$ and $f_n(X \setminus U_n) \subset \{0\}$ for each $n \in \omega$. The sequence $\{f_n(x)\}$ is bounded and monotonous for each $x \in X$ so it has to converge to some $f(x) \in [0, 1]$. If Y is the set X with the discrete topology, then $\mathbb{R}^X = C_p(Y)$ and the sequence $A = \{f_n : n \in \omega\} \subset C_p(X)$ converges to f in the space $C_p(Y) = \mathbb{R}^X$ (Problem 143). Therefore, $f \in [A]$ and hence f is continuous by our assumption on $C_p(X)$. Thus the set $H_z = f^{-1}((0, 1])$ is open in X; it is immediate that $H_z \subset \bigcap\{U_n : n \in \omega\} \subset \bigcap\{O_n : n \in \omega\} = H$ which implies that $H = \bigcup \{H_z : z \in H\}$ is an open set. This settles sufficiency and makes our solution complete.

S.356. *Prove that $C_p(L(\kappa), \mathbb{I})$ is countably compact.*

Solution. Say that X is *a P-space* if every G_δ-subset of X is open. It is easy to see that $L(\kappa)$ is a P-space for each cardinal κ. It was proved in Fact 2 of S.310 that $C_p(X,\mathbb{I})$ is countably compact for any P-space X. Thus $C_p(L(\kappa), \mathbb{I})$ is countably compact.

S.357. *Prove that $C_p(L(\kappa))$ has a dense σ-compact subspace.*

Solution. Observe that the spaces $L(\kappa)$ and $A(\kappa)$ have the same underlying set $X = \{a\} \cup \kappa$; it is immediate that the identity map $\varphi : X \to X$ is a condensation of $L(\kappa)$ onto $A(\kappa)$. Define a map $\varphi^* : C_p(A(\kappa)) \to C_p(L(\kappa))$ by the formula $\varphi^*(f) = f \circ \varphi$ for any $f \in C_p(A(\kappa))$. Then φ^* is an embedding and $\varphi^*(C_p(A(\kappa)))$ is dense in $C_p(L(\kappa))$ (Problem 163). Since $C_p(A(\kappa))$ has a dense σ-compact subspace P (Problem 346), the set $\varphi^*(P)$ will be also σ-compact and dense in $C_p(L(\kappa))$.

S.358. *Given an uncountable cardinal κ, let $\Sigma(\kappa) = \{x \in \mathbb{R}^\kappa :$ the set $x^{-1}(\mathbb{R}\setminus\{0\})$ is countable$\}$. Prove that, if a compact space X is a continuous image of $\Sigma(\kappa)$ then X is metrizable.*

Solution. Fix a continuous onto map $\varphi : \Sigma = \Sigma(\kappa) \to X$. We will need the set $Y = \{x \in X : \chi(x, X) \leqslant \omega\}$. Define a point $u \in \Sigma$ by $u(\alpha) = 0$ for all $\alpha \in \kappa$. If $A \subset \kappa$ then $q_A : \Sigma \to \Sigma_A = \mathbb{R}^A$ is the natural projection defined by $q_A(x) = x|A$ for any $x \in \Sigma$. Given a space Z, call a set $P \subset Z$ *a zero-set* if there exists $f \in C(Z)$ such that $P = f^{-1}(0)$.

Fact 1. (Vedenissov's lemma) If Z is a normal space (in particular, if Z is compact) then a closed $P \subset Z$ is a zero-set in Z if and only if P is a G_δ-set in Z.

Proof. If P is a zero-set, take any $f \in C(Z)$ with $P = f^{-1}(0)$ and observe that $U_n = f^{-1}\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right)$ is an open subset of Z for every $n \in \mathbb{N}$ and $P = \bigcap\{U_n : n \in \mathbb{N}\}$. This settles necessity.

Now, assume that P is a G_δ-set in the space Z; it is not difficult to find a sequence $\mathcal{U} = \{U_n : n \in \mathbb{N}\} \subset \tau(P, Z)$ such that $P = \bigcap \mathcal{U}$ and $\overline{U}_{n+1} \subset U_n$ for every $n \in \mathbb{N}$ (normality of Z must be used here). Applying again normality of Z we can find $f_n \in C(Z, [0, 1])$ such that $f_n|P \equiv 0$ and $f_n|(Z\backslash U_n) \equiv 1$ for each $n \in \mathbb{N}$. If $g_n = \sum_{i=1}^n 2^{-i} \cdot f_i$ then the sequence $\{g_n : n \in \mathbb{N}\}$ converges uniformly to a function $g \in C(Z)$ (see Problem 030). It is clear that $g(z) \geqslant 0$ for every $z \in Z$. If $z \in Z\backslash P$, then there is $n \in \mathbb{N}$ such that $z \notin U_n$ and hence $f_n(z) = 1$. Consequently, $g(z) \geqslant g_n(z) \geqslant 2^{-n} > 0$. On the other hand, $g_n(z) = 0$ for any $z \in P$ which implies $g(z) = 0$. This shows that $P = g^{-1}(0)$ so sufficiency is established and hence Fact 1 is proved.

Fact 2. Suppose that κ is an infinite cardinal and P is a G_κ-set of a space Z. If Q is a G_κ-set in the space P then Q is also a G_κ-set in Z.

Proof. Take a family $\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \subset \tau(P, Z)$ such that $P = \bigcap \mathcal{U}$ and a family $\mathcal{V} = \{V_\alpha : \alpha < \kappa\} \subset \tau(Q, P)$ such that $Q = \bigcap \mathcal{V}$. There exists $V'_\alpha \in \tau(Z)$ such that $V'_\alpha \cap P = V_\alpha$ for all $\alpha < \kappa$. Then, $W_\alpha = U_\alpha \cap V'_\alpha$ is open in Z for each $\alpha < \kappa$ and $\bigcap\{W_\alpha : \alpha < \kappa\} = Q$ so Fact 2 is proved.

Fact 3. If L is a Lindelöf subspace of a space Z then, for every $z \in Z \setminus L$, there exists a closed G_δ-set P of the space Z such that $z \in P \subset Z \setminus L$.

Proof. For any $y \in L$ fix $U_y \in \tau(y, Z)$ such that $z \notin \overline{U}_y$. The family $\{U_y : y \in L\}$ is an open cover of the Lindelöf space L. Therefore there is a countable $M \subset L$ such that $L \subset \bigcup\{U_y : y \in M\}$. It is clear that $H = \bigcap\{Z \setminus \overline{U}_y : y \in M\}$ is a G_δ-set in Z and $z \in H \subset Z \setminus L$. Finally, apply Fact 2 of S.328 to see that there exists a G_δ-set P of the space Z which is closed in Z and $z \in P \subset H$. Then $z \in P \subset Z\backslash L$ and Fact 3 is proved.

Claim. If F is a non-empty G_δ-subset of X then $F \cap Y \neq \emptyset$.

Proof. If not then there is a non-empty G_δ-set $F \subset X \setminus Y$. Apply Fact 2 of S.328 to see that there exists a non-empty G_δ-set $F' \subset F$ which is closed in X. This shows that, without loss of generality, we can assume that F is closed in X and hence compact. No point $x \in F$ can be a G_δ-set in F because otherwise $\{x\}$ is a G_δ-set in X (Fact 2) and therefore $\chi(x, X) \leqslant \omega$ (Problem 327) which contradicts $x \in X\backslash Y$. Take any $x_0 \in F \setminus\{\varphi(u)\}$; the set $F \setminus\{\varphi(u)\}$ is a non-empty G_δ-set in F (it is even open in F) so it is a G_δ-set in X (Fact 2) and hence there is a compact G_δ-set F_0 such that $x_0 \in F_0 \subset F\backslash\{\varphi(u)\}$ (Fact 2 of S.328). Apply Fact 1 to find $f_0 \in C(X)$ such that $F_0 = f_0^{-1}(0)$. The function $f_0 \circ \varphi : \Sigma \to \mathbb{R}$ is continuous on Σ which is dense in $\mathbb{R}^\kappa$; this makes it possible to apply Problem 299 to conclude that there is a non-empty countable $A_0 \subset \kappa$ and a continuous map $h_0 : \Sigma_{A_0} \to \mathbb{R}$ such that $h_0 \circ q_{A_0} = f_0 \circ \varphi$. It is evident that $q_{A_0}(u) \notin h_0^{-1}(0)$; since $\varphi^{-1}(F_0) = (f_0 \circ \varphi)^{-1}(0) = q_{A_0}^{-1}\left(h_0^{-1}(0)\right) \not\ni u$, we have $\varphi^{-1}(F_0) \subset q_{A_0}^{-1}(H_0)$ where $H_0 = \Sigma_{A_0}\backslash\{q_{A_0}(u)\}$.

Suppose that $\beta < \omega_1$ and we have families $\{A_\alpha : \alpha < \beta\}$ and $\{f_\alpha : \alpha < \beta\}$ with the following properties:

(1) A_α is a non-empty countable subset of κ for all $\alpha < \beta$.
(2) $A_\alpha \cap A_\gamma = \emptyset$ if $\alpha, \gamma < \beta$ and $\alpha \neq \gamma$.
(3) F_α is a non-empty G_δ-subset of F and hence of X.
(4) $F_\alpha \subset F_\gamma$ if $\gamma < \alpha < \beta$.
(5) If $H_\alpha = \Sigma_{A_\alpha} \backslash \{q_{A_\alpha}(u)\}$ for all $\alpha < \beta$ then $\varphi^{-1}(F_\gamma) \subset \bigcap\left\{q_{A_\alpha}^{-1}(H_\alpha) : \alpha \leqslant \gamma\right\}$ for every $\gamma < \beta$.

If $F'_\beta = \bigcap\{F_\alpha : \alpha<\beta\}$ and $B_\beta = \bigcup\{A_\alpha : \alpha < \beta\}$ then the space $G_\beta = \{x \in \Sigma : q_{A_\alpha}(x) \in H_\alpha$ for all $\alpha < \beta$ and $x(v) = 0$ for all $v \in \kappa \backslash B_\beta\} \subset \Sigma$ is homeomorphic to $\Pi\{H_\alpha : \alpha < \beta\}$ and therefore $nw(G_\beta) = \omega$. The space $L = \varphi(G_\beta) \cap F'_\beta$ is Lindelöf because it has a countable network; apply Fact 3 to find a non-empty closed $F_\beta \subset F'_\beta$ such that F_β a G_δ-subset of X and $F_\beta \cap L = \emptyset$. There exists $f_\beta \in C(X)$ such that $F_\beta = f_\beta^{-1}(0)$ (Fact 1). Apply Problem 299 to find a countable $A'_\beta \subset \kappa$ and $h_\beta \in C(\Sigma_{A'_\beta})$ such that $h_\beta \circ q_{A'_\beta} = f_\beta \circ \varphi$. To see that $A_\beta = A'_\beta \backslash B_\beta \neq \emptyset$ note that (5) implies that $q_{A_\alpha}(\varphi^{-1}(F_\beta)) \subset H_\alpha$ for all $\alpha < \beta$; as a consequence $q_{B_\beta}(\varphi^{-1}(F_\beta)) \subset q_{B_\beta}(G_\beta)$. Take any $x \in G_\beta$ such that $q_{B_\beta}(x) = q_{B_\beta}(y)$ for some $y \in \varphi^{-1}(F_\beta)$. If $A'_\beta \subset B_\beta$ then $q_{A'_\beta}(x) = q_{A'_\beta}(y)$ so $f_\beta(\varphi(x)) = h_\beta(q_{A'_\beta}(x)) = h_\beta(q_{A'_\beta}(y)) = f_\beta(\varphi(y)) = 0$ which is a contradiction with $f_\beta(\varphi(z)) \neq 0$ for all $z \in G_\beta$. Therefore $A_\beta \neq \emptyset$ and it is straightforward that (1)–(4) are satisfied for the sets $\{A_\alpha : \alpha \leqslant \beta\}$ and $\{F_\alpha : \alpha \leqslant \beta\}$. To see that (5) also holds, suppose that $q_{A_\beta}(\varphi^{-1}(F_\beta)) \ni q_{A_\beta}(u)$; take any $x \in \varphi^{-1}(F_\beta)$ with $q_{A_\beta}(x) = q_{A_\beta}(u)$. Since $q_{B_\beta}(\varphi^{-1}(F_\beta)) \subset q_{B_\beta}(G_\beta))$, we can find $y \in G_\beta$ such that $q_{B_\beta}(y) = q_{B_\beta}(x)$. It is immediate that $q_{A'_\beta}(y) = q_{A'_\beta}(x)$ and hence $0 = f_\beta(\varphi(x)) = f_\beta(\varphi(y))$ which is a contradiction. This proves that $q_{A_\beta}(\varphi^{-1}(F_\beta)) \subset H_\beta = \sum_{A_\beta}\{q_{A_\beta}(u)\}$ so (5) also holds.

This inductive procedure can be carried out for all countable ordinals providing families $\{A_\alpha : \alpha < \omega_1\}$ and $\{F_\alpha : \alpha < \omega_1\}$ with the properties (1)–(5). Since X is compact, there is $y \in \cap\{F_\alpha : \alpha < \omega_1\}$. If $x \in \varphi^{-1}(y)$ then $q_{A_\alpha}(x) \neq q_{A_\alpha}(u)$ for each $\alpha < \omega_1$ by (5). This shows that $x(v_\alpha) \neq 0$ for some $v_\alpha \in A_\alpha$. Since the family $\{A_\alpha : \alpha < \omega_1\}$ is disjoint, the set $x^{-1}(\mathbb{R}\backslash\{0\})$ contains the uncountable set $\{v_\alpha : \alpha < \omega_1\}$ which contradicts the fact that $x \in \Sigma$. Thus our claim is proved.

Now it is very easy to finish our solution. Apply Problem 299 to conclude that $nw(Y) \leqslant \omega$ and hence Y is Lindelöf. If $x \in X\backslash Y$ then there is a non-empty G_δ-set $F \subset X\backslash Y$ by Fact 3. However, $Y \cap F \neq \emptyset$ by our claim which is a contradiction. Thus $X = Y$ and hence $w(X) = nw(X) = \omega$ (Fact 4 of S.307) so our solution is complete.

S.359. *Prove that a dyadic compact space of countable tightness is metrizable.*

Solution. If X is a dyadic compact space then, by definition, X is a continuous image of $\{0, 1\}^\kappa$ for some κ. The space $\{0, 1\}^\kappa$ is a product of metrizable compact spaces so if $t(X) \leqslant \omega$ then Fact 6 of S.307 is applicable to conclude that X is metrizable.

S.360. *Suppose that X is a dyadic space and the set $\{x \in X : \chi(x, X) \leqslant \omega\}$ is dense in X. Prove that X is metrizable.*

Solution. Let $Y = \{x \in X : \chi(x, X) \leqslant \omega\}$. By definition of a dyadic space, there is a continuous onto map $\varphi : K = \{0, 1\}^\kappa \to X$ for some cardinal κ. Given $A \subset \kappa$, let $\pi_A : K \to K_A = \{0, 1\}^A$ be the natural projection defined by $\pi_A(y) = y|A$ for all $y \in K$. The space K is a product of second countable spaces so we can apply Problem 299 to conclude that there is a countable $A \subset \kappa$, a closed $L \subset K$, a closed $M \subset K_A$, and a continuous map $h : M \to X$ such that $\pi_A(L) = M$, $\varphi(y) = h(\pi_A(y))$ for every $y \in L$ and $\varphi^{-1}(Y) \subset L$. It follows from $\varphi^{-1}(Y) \subset L$ that $Y \subset \varphi(L) = h(M)$; since M is a second countable compact space, the space $h(M)$ is also a second countable compact space which contains a dense subset of X. Therefore, $h(M) = X$ so X is second countable and hence metrizable.

S.361. *Show that any hereditarily normal dyadic compact space is metrizable.*

Solution. Given a cardinal κ and a set $A \subset \kappa$ let $\pi_A : \{0, 1\}^\kappa \to \{0, 1\}^A$ be the natural projection onto the face determined by A. We write π_α instead of $\pi_{\{\alpha\}}$ and B_α instead of $\kappa\backslash\{\alpha\}$ for all $\alpha \in \kappa$. Given any $x \in \{0, 1\}^\kappa$ and any $\alpha < \kappa$, let $r_\alpha(x)(\alpha) = 1 - x(\alpha)$ and $r_\alpha(x)(\beta) = x(\beta)$ for any $\beta \in B_\alpha$. In other words, $r_\alpha(x)$ coincides with x at all coordinates except α.

Fact 1. If X is a dyadic space and $x \in X$ is not an isolated point then there is a sequence $\{x_n : n \in \omega\} \subset X\backslash\{x\}$ which converges to x.

Proof. Fix a cardinal κ such that there is a continuous onto map $\varphi : \{0, 1\}^\kappa \to X$. The set $F = \varphi^{-1}(x)$ is compact; let $A = \{\alpha < \kappa :$ there is a point $y_\alpha \in \{0, 1\}^\kappa\backslash F$ such that $r_\alpha(y_\alpha) \in F\}$. We claim that
(∗) $F = \pi_A(F) \times \{0,1\}^{\kappa\backslash A}$.

Of course, it suffices to show that $\pi_A(F) \times \{0, 1\}^{\kappa\backslash A} \subset \overline{F} = F$. Take any $y \in \pi_A(F) \times \{0, 1\}^{\kappa\backslash A}$; to prove that $y \in \overline{F}$ we can restrict ourselves to showing that, for any finite $S \subset \kappa$, there is $z \in F$ such that $\pi_S(z) = \pi_S(y)$.

There exists a point $z_0 \in F$ with $\pi_A(y) = \pi_A(z_0)$; if $S \subset A$ then $\pi_S(z_0) = \pi_S(y)$ and the proof of the property (∗) is over. If not, let $S\backslash A = \{\alpha_1, \ldots, \alpha_n\}$. It follows from the definition of the set A, that, for any ordinal $\beta \in \kappa\backslash A$ and any $w \in F$, we have $r_\beta(w) \in F$. If $z_0(\alpha_1) = y(\alpha_1)$ then we let $z_1 = z_0$; if not then $z_1 = r_{\alpha_1}(z_0)$. In both cases we change z_0 at most at α_1 and we obtain $z_1 \in F$ such that $z_1|(A \cup \{\alpha_1\}) = y|(A \cup \{\alpha_1\})$. Assuming that we have a point $z_k \in F$ with $z_k|(A \cup \{\alpha_1, \ldots, \alpha_k\}) = y|(A \cup \{\alpha_1, \ldots, \alpha_k\})$, let $z_{k+1} = z_k$ if $z_k(\alpha_{k+1}) = y(\alpha_{k+1})$; if not, then $z_{k+1} = r_{\alpha_{k+1}}(z_k)$. It is straightforward that in both cases we obtain $z_{k+1} \in F$ such that

$$z_{k+1}|(A \cup \{\alpha_1, \cdots, \alpha_k, \alpha_{k+1}\}) = y|(A \cup \{\alpha_1, \cdots, \alpha_k, \alpha_{k+1}\}),$$

so the inductive construction can go on until $k = n$. It is evident that, for $z = z_n$, we have $z|S = y|S$. Thus $y \in \overline{F} = F$; since $y \in \pi_A(F) \times \{0, 1\}^{\kappa\backslash A}$ has been chosen arbitrarily, we finished the proof of (∗).

The following step is to prove that the set $D = \{y_\alpha : \alpha \in A\}$ is infinite (see the definition of A where the points y_α are introduced for each $\alpha \in A$). Observe first that

A is infinite because otherwise $\{0, 1\}^A$ is finite and hence discrete; therefore $\pi_A(F)$ is open in $\{0, 1\}^A$ and hence F is open in $\{0, 1\}^\kappa$ by (*). The map φ is closed and hence quotient so the set $\{x\}$ has to be open in X which is a contradiction because x was assumed to be a non-isolated point of X. So the only possibility for the set D to be finite is when there is some $y \in \{0, 1\}^\kappa$ such that $y = y_\alpha$ for each $\alpha \in B$ where $B \subset A$ is an infinite set. Take any finite $C \subset \kappa$. Since B is infinite, there is $\alpha \in B$ such that $\alpha \notin C$. It is evident that $r_\alpha(y)|C = y|C$ which proves that $y \in \overline{\{r_\alpha(y) : \alpha \in B\}}$. Since $r_\alpha(y) = r_\alpha(y_\alpha) \in F$ for all $\alpha \in B$, we obtain $y \in \overline{F} = F$ which is a contradiction.

This proves that D is an infinite set so we can choose a faithfully indexed set $A' = \{\alpha_n : n \in \omega\} \subset A$ such that $y_{\alpha_n} \neq y_{\alpha_m}$ if $m \neq n$. Let $z_n = r_{\alpha_n}(y_{\alpha_n})$ for each $n \in \omega$. Then $E = \{y_{\alpha_n} : n \in \omega\} \subset \{0, 1\}^\kappa \setminus F$, $\{z_n : n \in \omega\} \subset F$ and we claim that (**) for any $V \in \tau(F, \{0, 1\}^\kappa)$ the set $E \setminus V$ is finite.

Indeed, assume that $B \subset \omega$ is infinite and $\{y_{\alpha_n} : n \in B\} \subset \{0, 1\}^\kappa \setminus V$. The set $\{Z_n : n \in B\}$ can be finite but then the same point will be repeated infinitely many times in the sequence $\{z_n : n \in B\}$. This shows that, infinite or not, the sequence $\{z_n : n \in B\}$ has an accumulation point z in the sense that, for every set $W \in \tau(z, \{0, 1\}^\kappa)$ there are infinitely many $n \in B$ with $z_n \in W$. Evidently, $z \in F \subset V$; given any finite $C \subset \kappa$, the set $\{n \in B : z_n|C = z|C\}$ is infinite and hence there is $n \in B$ such that $\alpha_n \notin C$ and $z_n|C = z|C$. Then, $z_n|C = y_{\alpha_n}|C = z|C$ which shows that z is also an accumulation point for the set $E \setminus V$. However, V is a neighbourhood of z which does not meet $E \setminus V$; this contradiction proves (**).

Let $x_n = \varphi(y_{\alpha_n})$ for all $n \in \omega$; since $E \subset D \subset \{0, 1\}^\kappa \setminus \varphi^{-1}(x)$, the sequence $\varphi(E) = \{x_n : n \in \omega\}$ is contained in $X \setminus \{x\}$. To show that $x_n \to x$, take any set $U \in \tau(x, X)$. Then $F \subset V = \varphi^{-1}(U)$ so the set $E \setminus \varphi^{-1}(U)$ is finite by (**). As a consequence, $x_n \in U$ for all but finitely many $n \in \omega$. Hence $x_n \to x$ and Fact 1 is proved.

Returning to our solution consider the set $\Sigma = \{x \in \{0, 1\}^\kappa : |x^{-1}(1)| \leqslant \omega\}$. Assume that X is a hereditarily normal dyadic space and $\varphi : \{0, 1\}^\kappa \to X$ is a continuous onto map. If $\varphi(\Sigma) = X$ then X is metrizable by Fact 6 of S.307.

If $f(\Sigma) \neq X$ then choose any $x \in X \setminus f(\Sigma)$. Since Σ is dense in $\{0, 1\}^\kappa$ (Fact 3 of S.307), the set $f(\Sigma)$ is dense in X and therefore x is not an isolated point of X. Apply Fact 1 to find a sequence $\{x_n : n \in \omega\} \subset X \setminus \{x\}$ such that $x_n \to x$. The space $X \setminus \{x\}$ contains a dense countably compact subspace $f(\Sigma)$ (see Fact 3 of S.307, Fact 1 of S.310 and S.133) so it is pseudocompact by Fact 18 of S.351. Since X is hereditarily normal, the space $X \setminus \{x\}$ is normal and hence countably compact by Problem 137. However, it is easy to see that $\{x_n : n \in \omega\}$ is an infinite closed and discrete subset of $X \setminus \{x\}$; this contradiction shows that $f(\Sigma) = X$ and hence X is metrizable so our solution is complete.

S.362. *Let X be a dyadic compact space. Prove that, if $C_p(X)$ is Lindelöf then X is metrizable.*

Solution. Since $C_p(X)$ is Lindelöf, we have $t(X) = \omega$ by Problem 189. Now apply Problem 359 to conclude that X is metrizable.

S.363. *Let X be a dyadic compact space and suppose that $C_p(X)$ has a dense σ-pseudocompact subspace. Prove that X is metrizable.*

Solution. Take any family $\{P_n : n \in \omega\}$ of pseudocompact subspaces of $C_p(X)$ such that $\bigcup\{P_n : n \in \omega\}$ is dense in $C_p(X)$. The set $Q_n = \overline{P}_n$ is also pseudocompact by Fact 18 of S.351. Since Q_n is closed in $C_p(X)$, we can apply Fact 2 of S.307 to conclude that Q_n is compact. Of course, the σ-compact set $Q = \bigcup\{Q_n : n \in \omega\}$ is also dense in $C_p(X)$ and hence there is a compact $K \subset C_p(X)$ which separates the points of X (Fact 5 of S.310). Consider the map $e_K : X \to C_p(K)$ defined by $e_K(x)(f) = f(x)$ for any $x \in X$ and $f \in K$. Then e_K is continuous (Problem 166) and injective (Fact 2 of S.351); thus e_K embeds X in $C_p(K)$. Consequently, $t(X) \leqslant t(C_p(K)) \leqslant \omega$ (see Problem 149) so we can apply Problem 359 to conclude that X is metrizable.

S.364. *Prove that the Alexandroff double $AD(X)$ of a compact space X is a compact space which is metrizable if and only if X is countable.*

Solution. We omit an easy proof that the definition of $AD(X)$ indeed gives us a topology τ on the set $AD(X) = u_0(X) \cup u_1(X)$. Given any $y = (x, i) \in AD(X)$, let $\pi(y) = x$; then $\pi : AD(X) \to X$ is a map with $\pi^{-1}(x) = \{u_0(x), u_1(x)\}$ for every $x \in X$. To check that τ is Hausdorff, take distinct $x, y \in AD(X)$. If $\pi(x) \neq \pi(y)$ then there are $U' \in \tau(\pi(x), X)$, $V' \in \tau(\pi(y), X)$ with $U \cap V = \emptyset$. It is immediate that $U = \pi^{-1}(U')$ and $V = \pi^{-1}(V')$ are disjoint open (in $AD(X)$) neighbourhoods of x and y, respectively. If $\pi(x) = \pi(y)$ then, without loss of generality, there is $z \in X$ such that $x = u_0(z)$ and $y = u_1(z)$. Then $U = AD(X)\backslash\{u_1(z)\}$ and $V = \{u_1(z)\}$ are disjoint open (in $AD(X)$) neighbourhoods of the points x and y, respectively.

We prove next that, for any $U \in \tau(u_0(X), AD(X))$, the set $F = u_1(X)\backslash U$ is finite. Suppose that $Z = \{z_n : n \in \omega\} \subset F$ and $z_i \neq z_j$ if $i \neq j$. The infinite set $\{\pi(z_n) : n \in \omega\} \subset X$ has an accumulation point $x \in X$ because the space X is compact. It is straightforward that $z = (x, 0)$ is an accumulation point for the set Z in $AD(X)$ which is a contradiction with the fact that $U \in \tau(z, AD(X))$ and $U \cap Z = \emptyset$. The map $u_0 : X \to u_0(X)$ is clearly injective and continuous; thus u_0 is a homeomorphism and hence $u_0(X)$ is a compact subspace of $AD(X)$ homeomorphic to X.

If $\mathcal{U}$ is an open cover of the space $AD(X)$ then there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that $u_0(X) \subset \bigcup\mathcal{U}'$. The observation of the previous paragraph shows that the set $AD(X)\backslash(\bigcup \mathcal{U}')$ is finite and hence can be covered with a finite $\mathcal{U}'' \subset \mathcal{U}$. Then $\mathcal{V} = \mathcal{U}' \cup \mathcal{U}''$ is a finite subcover of $\mathcal{U}$ which shows that $AD(X)$ is compact and hence Tychonoff by Problem 124.

If the space X is countable then $AD(X)$ is also a countable compact and hence metrizable space. Now, if $AD(X)$ is metrizable then $w(AD(X)) \leqslant \omega$ and hence $c(AD(X)) \leqslant w(AD(X)) \leqslant \omega$. The family $\{\{z\}: z \in u_1(X)\}$ is disjoint, consists of non-empty open sets of $AD(X)$ and has cardinality $|X|$; thus $|X| \leqslant c(AD(X)) \leqslant \omega$ and our solution is complete.

S.365. *Let X be a metrizable compact space. Denote by $AD(X)$ the Alexandroff double of the space X. Prove that $C_p(AD(X))$ has a dense σ-compact subspace.*

Solution. Given any $y = (x,i) \in AD(X)$, let $\pi(y) = x$; then $\pi : AD(X) \to X$ is a continuous map with $\pi^{-1}(x) = \{u_0(x),u_1(x)\}$ for every $x \in X$. Fix a countable base $\{U_n : n \in \omega\}$ in the space X. It is evident that the set $X \setminus U_n$ is a G_δ-set in X for any $n \in \omega$. Thus, there is $f_n \in C(X)$ such that $X \setminus U_n = f_n^{-1}(0)$ for all $n \in \omega$.

For any $x \in X$, let $h_x : AD(X) \to \{0, 1\}$ be the characteristic function of the set $\{u_1(x)\}$, i.e., $h_x(u_1(x)) = 1$ and $h_x(z) = 0$ for all $z \in AD(X) \setminus \{u_1(x)\}$. Since the point $u_1(x)$ is isolated in $AD(X)$, the map h_x is continuous for any $x \in X$. If $h(z) = 0$ for all $z \in AD(X)$, then $K = \{h\} \cup \{h_x : x \in X\}$ is a compact subspace of $C_p(AD(X))$. Indeed, take any $\mathcal{U} \subset \tau(C_p(AD(X)))$ with $K \subset \bigcup \mathcal{U}$. Then $h \in U$ for some $U \in \mathcal{U}$; there exists a finite $P \subset AD(X)$ and $\varepsilon > 0$ such that $W = \{f \in C_p(AD(X)) : |f(y)| < \varepsilon$ for each $y \in P\} \subset U$. If $x \in X \setminus \pi(P)$ then $h_x(y) = 0$ for any $y \in P$ and hence $h_x \in W \subset U$. Consequently, the set $K \setminus U$ is finite and can be covered by a finite $\mathcal{U}' \subset \mathcal{U}$. It is clear that the family $\{U\} \cup \mathcal{U}'$ is a finite subcover of $\mathcal{U}$ so we proved that K is compact.

We next show that the set $A = \{f_n \circ \pi : n \in \omega\} \cup K$ separates the points of $AD(X)$. Let y, z be distinct points of $AD(X)$. If $\pi(y) \neq \pi(z)$ then there is $n \in \omega$ such that $\pi(y) \in U_n$ and $\pi(z) \notin U_n$ (because $\pi(y)$ and $\pi(z)$ are elements of X and $\{U_n : n \in \omega\}$ is a base of X). Then $f_n(\pi(y)) \neq 0 = f_n(\pi(z))$, i.e., $(f_n \circ \pi)(y) \neq (f_n \circ \pi)(z)$. If $\pi(y) = \pi(z) = x$ then we can assume, without loss of generality, that $y = u_0(x)$ and $z = u_1(x)$. Then $h_x(y) = 0 \neq 1 = h_x(z)$ and we established that A separates the points of $AD(X)$. It is evident that A is σ-compact.

Call a topological property $\mathcal{P}$ *complete* if it satisfies the following conditions:

(1) Any metrizable compact space has $\mathcal{P}$.
(2) If $n \in \mathbb{N}$ and Z_i has $\mathcal{P}$ for all $i = 1, \ldots, n$ then $Z_1 \times \cdots \times Z_n$ has $\mathcal{P}$.
(3) If Z has $\mathcal{P}$ then every continuous image of Z has $\mathcal{P}$.

It is clear that σ-compactness is a complete property. It was proved in Fact 2 of S.312 that if $\mathcal{P}$ is a complete property and $A \subset C_p(Z)$ has $\mathcal{P}$ then there exists an algebra $R(A) \supset A$ such that $R(A) \subset C_p(Z)$ is σ-$\mathcal{P}$, i.e., $R(A)$ can be represented as a countable union of spaces with the property $\mathcal{P}$. When $\mathcal{P}=$ "σ-compactness" then any space with the property σ–$\mathcal{P}$ is also σ-compact. Applying these remarks to the set $A \subset C_p(AD(X))$, we conclude that there is an algebra $R \subset C_p(AD(X))$ such that $A \subset R$ and R is σ-compact. It follows from $A \subset R$ that R separates the points of $AD(X)$. Applying Problem 192 we can see that R is dense in $C_p(AD(X))$. Thus R is the promised dense σ-compact subset of $C_p(AD(X))$ so our solution is complete.

S.366. *Let X and Y be any spaces. Given a perfect map $f : X \to Y$, prove that there is a closed $F \subset X$ such that $f(F) = Y$ and $f|F$ is an irreducible map. As a consequence, the same is true for any continuous surjective map between compact spaces.*

Solution. Suppose that there exists no set $F \subset X$ such that $f(F) = Y$ and the map $f|F$ is irreducible. The family $\mathcal{F} = \{F \subset X : F$ is closed in X and $f(F) = Y\}$ is non-empty because $X \in \mathcal{F}$; let $F_0 = X$. Suppose that $\alpha < \kappa = |X|^+$ and we have a family $\{F_\beta : \beta < \alpha\} \subset \mathcal{F}$ such that $F_{\beta'} \subset F_\beta$ and $F_\beta \setminus F_{\beta'} \neq \emptyset$ if $\beta < \beta'$. Let $G_\alpha = \bigcap \{F_\beta : \beta < \alpha\}$;

then G_α is closed in X. It is less evident but true that $G_\alpha \in \mathcal{F}$; to prove it observe that, for any $y \in Y$, the family $\mathcal{G}_Y = \{F_\beta \cap f^{-1}(y) : \beta < \alpha\}$ consists of decreasing non-empty compact subsets of X. Therefore $G_\alpha \cap f^{-1}(y) = \cap \mathcal{G}_Y \neq \emptyset$ which means that $f(G_\alpha) \notin y$. Since the point $y \in Y$ has been taken arbitrarily, we proved that $f(G_\alpha) = Y$, i.e., $G_\alpha \in \mathcal{F}$. We assumed that the map $f|G_\alpha$ cannot be irreducible so there is $F_\alpha \subset G_\alpha$ with $F_\alpha \in \mathcal{F}$ and $G_\alpha \backslash F_\alpha \neq \emptyset$. As a consequence, we can continue our inductive construction to obtain a family $\{F_\alpha : \alpha < \kappa\}$ such that $F_{\beta'} \subset F_\beta$ and $F_\beta \backslash F_{\beta'} \neq \emptyset$ if $\beta < \beta' < \kappa$. This provides a disjoint family $\{F_\alpha \backslash F_{\alpha+1} : \alpha < \kappa\}$ of non-empty subsets of X of cardinality $\kappa > |X|$ which is a contradiction.

S.367. *Given a cardinal κ, let $\Sigma(\kappa) = \{x \in \mathbb{R}^\kappa :$ the set $x^{-1}(\mathbb{R}\backslash\{0\})$ is countable$\}$. Prove that any compact space of countable tightness admits an irreducible continuous map onto a subspace of $\Sigma(\kappa)$ for some κ.*

Solution. Let X be a compact space such that $t(X) \leqslant \omega$. Then the space X has a faithfully indexed point-countable π-base $\mathcal{B} = \{U_\alpha : \alpha < \kappa\}$, where "point-countable" means that each point $x \in X$ belongs to at most countably many elements of $\mathcal{B}$ (Problem 332). For each $\alpha < \kappa$ fix a function $f_\alpha \in C(X)$ such that $f_\alpha(X \backslash U_\alpha) \subset \{0\}$ and $f_\alpha(x_\alpha) = 1$ for some $x_\alpha \in U_\alpha$. Let $f(x)(\alpha) = f_\alpha(x)$ for any $x \in X$ and $\alpha < \kappa$. This defines a map $f : X \to \mathbb{R}^\kappa$; if $q_\alpha : \mathbb{R}^\kappa \to \mathbb{R}$ is the natural projection onto the α-th factor then $q_\alpha \circ f = f_\alpha$ is a continuous map for each $\alpha < \kappa$. Therefore the map f is continuous (Problem 102).

Next observe that $Y = f(X) \subset \Sigma(\kappa)$. Indeed, if $x \in X$ then there is a countable $A \subset \kappa$ such that $x \notin U_\alpha$ for any $\alpha \in \kappa \backslash A$ because $\mathcal{B}$ is a point-countable family. Thus $f_\alpha(x) = 0$ for all ordinals $\alpha \in \kappa \backslash A$ and therefore $(f(x))^{-1}(\mathbb{R}\backslash\{0\}) \subset A$ is countable, i.e., $f(x) \in \Sigma(\kappa)$. To finish our solution, it suffices to show that $f : X \to Y$ is an irreducible map. Take any closed $F \subset X$ with $X \backslash F \neq \emptyset$. Since $\mathcal{B}$ is a π-base in X, there is $\alpha < \kappa$ with $U_\alpha \subset X \backslash F$. This implies $f_\alpha(x) = 0$ for all $x \in F$ and therefore $f(x)(\alpha) = 0$ for every $x \in F$. However, $f(x_\alpha)(\alpha) = 1$ which shows that there is no $x \in F$ with $f(x) = f(x_\alpha)$, i.e., $f(F) \neq Y$. Thus $f : X \to Y \subset \Sigma(\kappa)$ is an irreducible map.

S.368. *Prove that $w(\beta\omega) = \mathbf{c}$ and $|\beta\omega| = 2^{\mathbf{c}}$.*

Solution. Given a space X and $U \in \tau(X)$, call the set U *regular open* if $U = \text{Int}(\overline{U})$. Regular open sets form a base in X; indeed, if $x \in U \in \tau(X)$ then, by regularity of X there is $V \in \tau(x, X)$ such that $\overline{V} \subset U$. Then $W = \text{Int}(\overline{V})$ is a regular open set and $x \in W \subset U$.

Fact 1. Suppose that X is any space and Y is a dense subspace of X. If U and V are regular open subsets of X and $U \cap Y = V \cap Y$ then $U = V$.

Proof. We have $\overline{U} = \overline{U \cap Y} = \overline{V \cap Y} = \overline{V}$. Therefore $U = \text{Int}(\overline{U}) = \text{Int}(\overline{V}) = V$ and Fact 1 is proved.

Fact 2. For any space X, we have $w(X) \leqslant 2^{d(X)}$.

Proof. Fix a dense set $D \subset X$ with $|D| = d(X)$. Let $\mathcal{B}$ be the family of all regular open subsets of the space X; we know that the family $\mathcal{B}$ is a base of X. Given any $U \in \mathcal{B}$, let $\varphi(U) = U \cap D$. Then the map $\varphi : \mathcal{B} \to \exp(D)$ is an injection by Fact 1 and therefore $w(X) \leqslant |\mathcal{B}| \leqslant |\exp(D)| = 2^{d(X)}$ so Fact 2 is proved.

Fact 3. We have $w(\mathbb{I}^\kappa) = \kappa$ and $|\mathbb{I}^\kappa| = 2^\kappa$ for any infinite cardinal κ.

Proof. That $w(\mathbb{I}^\kappa) \leqslant \kappa$ is an immediate consequence of the results of Problem 209. To see that $w(\mathbb{I}^\kappa) \geqslant \kappa$ observe that $w(D(\kappa)) = \kappa$ and $D(\kappa)$ embeds in $\mathbb{I}^\kappa$ by Problem 209. Thus $w(\mathbb{I}^\kappa) \geqslant w(D(\kappa)) = \kappa$ (see Problem 159) and hence $w(\mathbb{I}^\kappa) = \kappa$.

Since $\{0, 1\}^\kappa$ is a subset of $\mathbb{I}^\kappa$, we have $|\mathbb{I}^\kappa| \geqslant |\{0, 1\}^\kappa| = 2^\kappa$. On the other hand, $\chi(\mathbb{I}^\kappa) \leqslant w(\mathbb{I}^\kappa) \leqslant \kappa$ implies $|\mathbb{I}^\kappa| \leqslant 2^\kappa$ by Problem 329 so $|\mathbb{I}^\kappa| = 2^\kappa$ and Fact 3 is proved.

Fact 4. If X is compact and Y is a continuous image of X then $w(Y) \leqslant w(X)$.

Proof. We have $nw(Y) \leqslant nw(X)$ by Problem 157 and hence $w(Y) = nw(Y) \leqslant nw(X) = w(X)$ (see Fact 4 of S.307) so Fact 4 is proved.

Returning to our solution, note that, since the countable set ω is dense in $\beta\omega$, we have $w(\beta\omega) \leqslant 2^{d(\beta\omega)} \leqslant 2^\omega = \mathfrak{c}$ by Fact 2. Besides, $\chi(\beta\omega) \leqslant w(\beta\omega) \leqslant \mathfrak{c}$ and hence $|\beta\omega| \leqslant 2^{\chi(\beta\omega)} \leqslant 2^\mathfrak{c}$ (see Problem 329).

The space $\mathbb{I}^\mathfrak{c}$ is separable (Problem 108); fix a countable dense $D \subset \mathbb{I}^\mathfrak{c}$. There is a surjective map $f : \omega \to D$ which is continuous because ω is discrete. There exists a continuous map $g : \beta\omega \to \mathbb{I}^\mathfrak{c}$ such that $g|D = f$ (see Problem 257). Since D is dense in $\mathbb{I}^\mathfrak{c}$, we have $g(\beta\omega) = \mathbb{I}^\mathfrak{c}$ which proves that $|\beta\omega| \geqslant |\mathbb{I}^\mathfrak{c}| = 2^\mathfrak{c}$ (Fact 3). Since $\mathbb{I}^\mathfrak{c}$ is a continuous image of $\beta\omega$, we can apply Fact 4 to conclude that $w(\beta\omega) \geqslant w(\mathbb{I}^\mathfrak{c}) = \mathfrak{c}$ (Fact 3) so our solution is complete.

S.369. *Prove that $\beta\omega\backslash\{x\}$ is countably compact for any $x \in \beta\omega$.*

Solution. Since ω is discrete and dense in $\beta\omega$, all points of ω have to be isolated in $\beta\omega$. Therefore, the case of $x \in \omega$ is trivial because the set $\beta\omega\backslash\{x\}$ is closed in $\beta\omega$ and hence compact.

Fact 1. Suppose that X is a space and a subspace $E = \{x_n : n \in \omega\} \subset X$ is discrete and faithfully indexed, i.e., $m \neq n$ implies $x_m \neq x_n$. Then there exists a disjoint family $\{U_n : n \in \omega\} \subset \tau(X)$ such that $x_n \in U_n$ for all $n \in \omega$.

Proof. By discreteness of E there exists $V_i \in \tau(x_i, X)$ such that $V_i \cap E = \{x_i\}$ for all $i \in \omega$. Use regularity of X to find $U_0 \in \tau(x_0, X)$ with $\overline{U}_0 \subset V_0$. Assume that we have a family $\{U_0, \ldots, U_n\} \subset \tau(X)$ such that

(1) $x_i \in U_i \subset \overline{U}_i \subset V_i$ for all $i \leqslant n$.
(2) $\overline{U}_i \cap \overline{U}_j = \emptyset$ for all $i, j \leqslant n$ with $i \neq j$.

Since $x_{n+1} \notin V_i$ for each $i \leqslant n$, we have $x_{n+1} \notin \overline{U}_i$ for all $i \leqslant n$. Therefore, $F = \overline{U}_0 \cup \cdots \cup \overline{U}_n$ is a closed set which does not contain x_{n+1}. By regularity of X there exists $U_{n+1} \in \tau(x, X)$ such that $\overline{U}_{n+1} \subset V_{n+1}\backslash F$. It is clear that the family $\{U_0, \ldots, U_{n+1}\}$ satisfies the conditions (1) and (2) so this inductive construction can be continued giving us a disjoint family $\{U_n : n \in \omega\} \subset \tau(X)$ with $x_n \in U_n$ for all $n \in \omega$. Fact 1 is proved.

Fact 2. If A and B are disjoint subsets of ω then $\overline{A} \cap \overline{B} = \emptyset$ (the bar denotes the closure in $\beta\omega$.

Proof. The function $f : \omega \to \{0, 1\}$ defined by $f(n) = 0$ if $n \in A$ and $f(n) = 1$ for all $n \in \omega \backslash A$, is continuous because ω is a discrete space. There exists a continuous map $g : \beta\omega \to \{0, 1\}$ such that $g|\omega = f$ (Problem 257). The sets $g^{-1}(0)$ and $g^{-1}(1)$ are closed in $\beta\omega$ so $\overline{A} \cap \overline{B} = \overline{f^{-1}(0)} \cap \overline{f^{-1}(1)} \subset g^{-1}(0) \cap g^{-1}(1) = \emptyset$ so Fact 2 is proved.

Returning to our solution assume that $x \in \beta\omega \backslash \omega$. If $\beta\omega \backslash \{x\}$ is not countably compact then there exists a sequence $\{x_n : n \in \omega\} \subset \beta\omega \backslash \{x\}$ such that $x_i \neq x_j$ for $i \neq j$ and the set $D = \{x_n : n \in \omega\}$ is closed and discrete. By Fact 1 there exists a disjoint family $\{U_n : n \in \omega\} \subset \tau(\beta\omega)$ such that $x_n \in U_n$ for all $n \in \omega$.

Given any $U \in \tau(x, \beta\omega)$, if $D \backslash U$ is infinite then the set $D \backslash U$ has an accumulation point in the compact space $\beta\omega \backslash U$ which is impossible because $D \backslash U$ is closed and discrete in $\beta\omega \backslash U$, a contradiction. Thus, $D \backslash U$ is finite for any $U \in \tau(x, \beta\omega)$, i.e., the sequence D converges to x. The set ω is dense in $\beta\omega$ and therefore $W_n = U_n \cap \omega \neq \emptyset$ for every $n \in \omega$. The sets $A = \bigcup\{W_{2n} : n \in \omega\}$ and $B = \bigcup\{W_{2n+1} : n \in \omega\}$ are contained in ω, disjoint and dense in the sets $A' = \bigcup\{U_{2n} : n \in \omega\}$ and $B' = \bigcup\{U_{2n+1} : n \in \omega\}$, respectively. Since both sequences $\{x_{2n} : n \in \omega\}$ and $\{x_{2n+1} : n \in \omega\}$ converge to x, we have $x \in \overline{B'} \cap \overline{A'} = \overline{A} \cap \overline{B}$ which contradicts Fact 2 and shows that our solution is complete.

S.370. *Prove that every non-empty G_δ-subset of $\omega^* = \beta\omega \backslash \omega$ has a non-empty interior.*

Solution. Let $\omega^* = \beta\omega \backslash \omega$; given a set $A \subset \omega$, let $[A] = \overline{A} \cap \omega^*$ (the bar denotes the closure in $\beta\omega$).

Fact 1. (a) The set ω is open in $\beta\omega$ and hence ω^* is compact.
(b) The set $\overline{A}$ is clopen in $\beta\omega$ for any $A \subset \omega$.
(c) The set $[A]$ is clopen in ω^* for any $A \subset \omega$.
(d) If $A, B \subset \omega$ then $[A] \subset [B]$ if and only if $A \backslash B$ is finite.
(e) If $A, B \subset \omega$ then $[A] = [B]$ if and only if $A \Delta B = (A \backslash B) \cup (B \backslash A)$ is finite.
In particular, $[A] = \emptyset$ if and only if $A = A \Delta \emptyset$ is finite.

Proof. (a) Since ω is a discrete subspace of $\beta\omega$, for every $n \in \omega$ there is $U_n \in \tau(\beta\omega)$ such that $U_n \cap \omega = \{n\}$. Now, ω is dense in $\beta\omega$, so the set $U_n \cap \omega$ must be dense in U_n for each $n \in \omega$. Thus, the closed set $\{n\}$ is dense in U_n and hence $U_n = \{n\}$, i.e., $\{n\}$ is open in $\beta\omega$ for each $n \in \omega$. An immediate consequence is that ω is a union of open subsets of $\beta\omega$; thus ω is open and ω^* is closed in $\beta\omega$. Being closed in the compact space $\beta\omega$, the space ω^* is compact so (a) is proved.

(b) It is evident that $\overline{A}$ is closed in $\beta\omega$; since $\beta\omega = \overline{A} \cup \overline{\omega\backslash A}$ and $\overline{A} \cap \overline{\omega\backslash A} = \emptyset$ (Fact 2 of S.369), we convince ourselves that the complement of $\overline{A}$ in $\beta\omega$ is also closed, i.e., $\overline{A}$ is open in $\beta\omega$.

(c) The intersection of a clopen set of $\beta\omega$ with ω^* must be clopen in ω^* so $[A] = \overline{A} \cap \omega^*$ is clopen in ω^* by (b).

(d) If $C \subset \omega$ is finite then $\overline{C} = C$ and hence $[C] = \emptyset$. Since $\overline{A \cup B} = \overline{A} \cup \overline{B}$, we have $[A] = [(A \backslash B) \cup (A \cap B)] = [A \backslash B] \cup [A \cap B] = [A \cap B] \subset [B]$ if $A \backslash B$ is finite. On the other hand, if $A \backslash B$ is infinite then it is not compact so the compact set $\overline{A \backslash B}$ cannot coincide with $A \backslash B$. Thus, $[A \backslash B] \neq \emptyset$ and $[A \backslash B] \cap [B] \subset \overline{A \backslash B} \cap \overline{B} = \emptyset$ by Fact 2 of S.369. Since $[A \backslash B] \subset [A]$, this shows that $[A]$ is not contained in $[B]$ if $A \backslash B$ is infinite.

(e) The equality $[A] = [B]$ holds if and only if $[A] \subset [B]$ and $[B] \subset [A]$ which, by (d), holds if and only if both sets $A \backslash B$ and $B \backslash A$ are finite, which, of course, is equivalent to $A \Delta B$ being finite. Fact 1 is proved.

Fact 2. The family $\mathcal{O} = \{[A] : A \subset \omega\}$ is a base in ω^*.

Proof. We already saw that $\mathcal{O} \subset \tau(\omega^*)$ (Fact 1); take any $x \in \omega^*$ and any set $U \in \tau(x, \omega^*)$. There is $U' \in \tau(\beta\omega)$ such that $U = U' \cap \omega^*$; use regularity of $\beta\omega$ to find a set $V \in \tau(x, \beta\omega)$ with $\overline{V} \subset U'$. Since $A = V \cap \omega$ is dense in V, we have $x \in \overline{A} = \overline{V} \subset U'$ and therefore $x \in [A] \subset U' \cap \omega^* = U$ so Fact 2 is proved.

Returning to our solution, take any non-empty $H \subset \omega^*$ which is a G_δ-set in ω^* and fix a family $\{U_n : n \in \omega\} \subset \tau(\omega^*)$ such that $H = \bigcap\{U_n : n \in \omega\}$. Pick any $x \in H$ and apply Fact 2 to choose a sequence $\{A_n : n \in \omega\} \subset \exp(\omega)$ such that $x \in [A_n] \subset U_n$ and $[A_{n+1}] \subset [A_n]$ for all $n \in \omega$. We have $[A_n] \subset [A_i]$ for all $i \leqslant n$ and therefore $A_n \backslash A_i$ is finite for all $i \leqslant n$ by Fact 1. Therefore the set $B_n = (A_0 \cap \cdots \cap A_n) \backslash \{0, \ldots, n\}$ is obtained from A_n by cutting off a finite set $(\bigcup\{A_n \backslash A_i : i < n\}) \cup \{0, \ldots, n\}$ for every $n \in \omega$. We have a sequence $\{B_n : n \in \omega\}$ such that $[B_n] = [A_n]$, $B_{n+1} \subset B_n$ for every $n \in \omega$ and $\bigcap\{B_n : n \in \omega\} = \emptyset$. Take a point $x_n \in B_n$ for all $n \in \omega$ and let $B = \{x_n : n \in \omega\}$. Since $x_n > n$ for each $n \in \omega$, the set B is infinite and hence $[B] \neq \emptyset$. The set $B \backslash B_n \subset \{x_0, \ldots, x_n\}$ is finite for all $n \in \omega$ and therefore $[B] \subset [B_n]$ for all $n \in \omega$ by Fact 1. Hence the non-empty open set $[B]$ is contained in $\bigcap\{[B_n] : n \in \omega\} = \bigcap\{[A_n] : n \in \omega\} \subset \bigcap\{U_n : n \in \omega\} = H$ so our solution is complete.

S.371. *Prove that* $c(\beta\omega \backslash \omega) = \mathfrak{c}$.

Solution. Let $\omega^* = \beta\omega \backslash \omega$; then $c(\omega^*) \leqslant w(\omega^*) \leqslant w(\beta\omega) = \mathfrak{c}$ (see Problem 368). Given any $A \subset \omega$, let $[A] = \overline{A} \cap \omega^*$ (the bar denotes the closure in $\beta\omega$); if $A, B \subset \omega$ then $A \Delta B = (A \backslash B) \cup (B \backslash A)$.

Fact 1. Given $A, B \subset \omega$, we have $[A] \cap [B] = \emptyset$ if and only if $A \cap B$ is finite.

Proof. If $A \cap B$ is infinite then $[A \cap B] \neq \emptyset$ (Fact 1 of S.370) and $[A \cap B] \subset [A] \cap [B]$ which proves necessity. The sets $A' = A \backslash B$ and $B' = B \backslash A$ are disjoint so if $A \cap B$ is finite, we have $A \Delta A' = B \Delta B' = A \cap B$ which implies $[A] = [A']$ and $[B] = [B']$ by Fact 1 of S.370. Since A' and B' are disjoint, we have $\overline{A'} \cap \overline{B'} = \emptyset$ (Fact 2 of S.369). As a consequence $[A] \cap [B] = [A'] \cap [B'] \subset \overline{A'} \cap \overline{B'} = \emptyset$ so Fact 1 is proved.

Returning to our solution, apply Problem 141(iii) to find an almost disjoint family $\mathcal{C} \subset \exp(\omega)$ with $|\mathcal{C}| = \mathfrak{c}$. "Almost disjoint" means that all elements of $\mathcal{C}$ are infinite and the set $A \cap B$ is finite for any distinct $A, B \in \mathcal{C}$. The family

$\mathcal{U} = \{[C] : C \in \mathcal{C}\}$ is disjoint by Fact 1, consists of non-empty open subsets of ω^* (Fact 1 of S.370) and $|\mathcal{U}| = |\mathcal{C}| = \mathfrak{c}$ so $c(\omega^*) \geqslant \mathfrak{c}$ and our solution is complete.

S.372. *Prove that $\beta\omega$ admits an irreducible map onto a subspace of $\Sigma(\kappa)$ for some κ.*

Solution. Consider the space $S = \{0\} \cup \left\{\frac{1}{n+1} : n \in \omega\right\} \subset \mathbb{R}$; let $f(n) = \frac{1}{n+1}$ for all $n \in \omega$. The map $f : \omega \to S$ is continuous because ω is discrete. Since S is a compact space, there exists a continuous map $g : \beta\omega \to S$ such that $g|\omega = f$ (Problem 257). Observe first that $g(\beta\omega \setminus \omega) = \{0\}$. Indeed, $\beta\omega\setminus\omega \subset \overline{\{m : n \leqslant m\}}$ for all $n \in \omega$ (this closure is taken in $\beta\omega$). Therefore, for any $x \in \beta\omega \setminus \omega$, we have

$$g(x) \in \bigcap\left\{\overline{\{g(m) : m \geqslant n\}} : n \in \omega\right\} = \bigcap\left\{\left\{\frac{1}{m+1} : m \geqslant n\right\} \cup \{0\} : n \in \omega\right\} = \{0\}$$

and hence $g(x) = 0$ (the last closure is taken in the space S).

As a consequence, $g^{-1}(g(n)) = \{n\}$ for any $n \in \omega$. Now, it is easy to see that the map g is irreducible; indeed, if F is a closed subset of $\beta\omega$ with $g(F) = S$ then $F \cap g^{-1}\left(\frac{1}{n+1}\right) \neq \emptyset$ and hence $n \in F$ for each $n \in \omega$. The set F being closed, we have $\beta\omega = \overline{\omega} \subset \overline{F} = F$ and therefore $F = \beta\omega$, i.e., the map g is irreducible. The space S is second countable so it embeds in $\mathbb{R}^\omega$ which in turn embeds in $\Sigma(\kappa)$ for any infinite κ. Indeed, if $\kappa = \omega$ then $\Sigma(\kappa) = \mathbb{R}^\omega$; if $\kappa > \omega$ then the space $\{x \in \Sigma(\kappa) : x(\alpha) = 0 \text{ for all } \alpha \geqslant \omega\}$ lies in $\Sigma(\kappa)$ and is homeomorphic to $\mathbb{R}^\omega$. This proves that $\beta\omega$ maps irreducibly onto a subspace of $\Sigma(\kappa)$ for any infinite κ so our solution is complete.

S.373. *Prove that $\beta\omega \setminus \omega$ does not admit an irreducible map onto a subspace of $\Sigma(\kappa)$ for any κ.*

Solution. Let $\mathcal{B} = \{(p, q) : p, q \in \mathbb{Q}, p < q \text{ and } pq > 0\}$; in other words, $\mathcal{B}$ is the family of all non-empty rational intervals of $\mathbb{R}$, which do not contain zero. It is evident that any $U \in \tau^*(\mathbb{R})$ contains an element of $\mathcal{B}$, i.e., the family $\mathcal{B}$ is a π-base in $\mathbb{R}$. In fact, $\mathcal{B}$ has even stronger property: for any $a \in \mathbb{R}\setminus\{0\}$ and any $U \in \tau(a, \mathbb{R})$, there exists $B \in \mathcal{B}$ such that $a \in B \subset U$, i.e., $\mathcal{B}$ is a base in $\mathbb{R}\setminus\{0\}$.

Fix any infinite cardinal κ; given $\alpha_1, \ldots, \alpha_n \in \kappa$, and $O_1, \ldots, O_n \in \tau(\mathbb{R})$, let $[\alpha_1, \ldots, \alpha_n; O_1, \ldots, O_n] = \{x \in \Sigma(\kappa) : x(\alpha_i) \in O_i \text{ for each } i \leqslant n\}$. It is clear that $\mathcal{U} = \{[\alpha_1, \ldots, \alpha_n; O_1, \ldots, O_n] : n \in \mathbb{N}, \alpha_i < \kappa \text{ and } O_i \in \tau(\mathbb{R}) \text{ for all } i \leqslant n\}$ is a base in $\Sigma(\kappa)$. If $U = [\alpha_1, \ldots, \alpha_n; O_1, \ldots, O_n] \in \mathcal{U}$ and $B_i \subset O_i$, $B_i \in \mathcal{B}$ for all $i \leqslant n$ then $V = [\alpha_1, \ldots, \alpha_n; B_1, \ldots, B_n] \subset U$, i.e., the family $\mathcal{V} = \{[\alpha_1, \ldots, \alpha_n; B_1, \ldots, B_n] : n \in \mathbb{N}, \alpha_i < \kappa \text{ and } B_i \in \mathcal{B} \text{ for all } i \leqslant n\}$ is a π-base of $\Sigma(\kappa)$. A family $\mathcal{C}$ of subsets of a set X is called *point-countable* if every $x \in X$ belongs to at most ω-many elements of $\mathcal{C}$.

Fact 1. Any subspace X of the space $\Sigma(\kappa)$ has a point-countable π-base.

Proof. Denote by u the element of $\Sigma(\kappa)$ for which $u(\alpha) = 0$ for all $\alpha < \kappa$. It suffices to prove our Fact for the space $X\setminus\{u\}$. Indeed, the case when $u \notin X$ is clear; if $u \in X$

and u is an isolated point of X then for any point-countable π-base $\mathcal{C}$ in the space $X\backslash\{u\}$, the family $\{\{u\}\} \cup \mathcal{C}$ is a point-countable π-base in the space X. If u is not isolated in X then any π-base for $X\backslash\{u\}$ is also a π-base for X so again it suffices to find a point-countable π-base for the space $X\backslash\{u\}$. To simplify the notation we will assume, without loss of generality, that $u \notin X$.

The family $\mathcal{V}$ is point-countable. To see this, consider the set $S = x^{-1}(\mathbb{R}\backslash\{0\})$. If $V = [\alpha_1, \dots, \alpha_n; B_1, \dots, B_n] \in \mathcal{V}$ and $\alpha_i \notin S$ for some $i \leqslant n$ then $x(\alpha_i) = 0 \notin B_i$ which implies $x \notin V$. Thus, the family $\{V \in \mathcal{V} : x \in V\}$ is contained in the family $\mathcal{V}' = \{[\alpha_1, \dots, \alpha_n; B_1, \dots, B_n] : n \in \mathbb{N}, \alpha_i \in S$ and $B_i \in \mathcal{B}$ for all $i \leqslant n\}$ which is countable because S and $\mathcal{B}$ are countable.

We claim that the family $\mathcal{V}|X = \{V \cap X : V \in \mathcal{V}$ and $V \cap X \neq \emptyset\}$ is a π-base in X. Since $\mathcal{V}$ is point-countable, this gives us a point-countable π-base in X. Take any $U \in \tau^*(X)$ and $x \in U$. Then $x(\alpha) \neq 0$ for some $\alpha < \kappa$; there exist ordinals $\alpha_1, \dots, \alpha_n \in \kappa$ and sets $O_1, \dots, O_n \in \tau^*(\mathbb{R})$ such that $\alpha_1 = \alpha$ and $x \in V = [\alpha_1, \dots, \alpha_n; O_1, \dots, O_n] \cap X \subset U$. If $x(\alpha_i) \neq 0$ then there is $B_i \in \mathcal{B}$ such that $x(\alpha_i) \in B_i \subset O_i$; if $x(\alpha_i) = 0$ then $0 \in O_i$. This shows that we do not lose generality if we assume that there is a number $k \in \{1, \dots, n\}$ such that $O_i = B_i \in \mathcal{B}$ for all $i \leqslant k$ and $x(\alpha_i) = 0 \in O_i$ for $i = k+1, \dots, n$.

Call a set $K' = \{\alpha_{i_1}, \dots, \alpha_{i_m}\} \subset K = \{\alpha_{k+1}, \dots, \alpha_n\}$ *marked* if there exists a point $y \in X$ such that $y(\alpha_i) \in B_i$ for all $i \leqslant k$ and $y(\alpha_{ij}) \in O_{ij} \backslash\{0\}$ for every $j \leqslant m$. Since the set K is finite, there exists a maximal marked set $M = \{\alpha_{i_1}, \dots, \alpha_{i_m}\} \subset K$ (which is possibly empty). This means that there is $y \in X$ with $y(\alpha_i) \in B_i$ for all $i \leqslant k$ and $y(\alpha_{i_j}) \in O_{i_j} \backslash\{0\}$ for all $j \leqslant m$ while for any $z \in X$ such that $z(\alpha_i) \in B_i$ for all $i \leqslant k$ and $z(\alpha_{i_j}) \in O_{i_j} \backslash\{0\}$ for all $j \leqslant m$, we have $z(\beta) = 0$ for every $\beta \in K\backslash M$.

Changing the enumeration of K if necessary, we can restrict ourselves to the case when $M = \{\alpha_{k+1}, \dots, \alpha_m\}$ for some $m \leqslant n$. Since $0 \neq y(\alpha_i) \in O_i$ for all $i \leqslant m$, we can choose $B_i \in \mathcal{B}$ such that $y(\alpha_i) \in B_i \subset O_i$ for all $i \in \{k+1, \dots, m\}$. Then $W = [\alpha_1, \dots, \alpha_m; B_1, \dots, B_m] \cap X \in \mathcal{V}|X$ because $y \in W$ and hence $W \neq \emptyset$; besides, for any $z \in W$ we have $z(\alpha_i) = 0 \in O_i$ for all $i \in \{m+1, \dots, n\}$. As a consequence, any $z \in W$ belongs to V, i.e., $W \subset V \subset U$. This shows that, for any $U \in \tau^*(X)$ we have $W \in \mathcal{V}|X$ with $W \subset U$; therefore $\mathcal{V}|X$ is a π-base in X so Fact 1 is proved.

Fact 2. If $f : X \to Y$ is a closed and irreducible map then, for any π-base $\mathcal{B}$ of the space Y, the family $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{B}\}$ is a π-base in X.

Proof. Take any $W \in \tau^*(X)$; the set $F = X\backslash W$ is closed and $F \neq X$. Since the map f is irreducible, we have $f(F) \neq Y$, i.e., the set $U' = Y\backslash f(F)$ is an open non-empty subset of Y. Take any $U \in \mathcal{B}$ with $U \subset U'$; it is straightforward that $V = f^{-1}(U) \in \mathcal{V}$ and $V \subset W$. Hence $\mathcal{V}$ is a π-base in X and Fact 2 is proved.

Fact 3. The space $\beta\omega\backslash\omega$ does not have a point-countable π-base.

Proof. Observe first that no point $x \in \beta\omega\backslash\omega$ can be isolated in $\beta\omega\backslash\omega$. Indeed, if $\{x\} \in \tau(\beta\omega\backslash\omega)$ then there is an infinite $A \subset \omega$ such that $[A] = \overline{A} \cap (\beta\omega\backslash\omega) = \{x\}$ (Fact 2 of S.370). Take $B, C \subset A$ such that B and C are infinite and $B \cap C = \emptyset$. Then $[B] \cap [C] = \emptyset$ (Fact 1 of S.371) and $[B] \neq \emptyset \neq [C]$ (Fact 1 of S.370). Since the non-empty disjoint sets $[B]$ and $[C]$ are contained in $[A]$, it is impossible that $[A] = \{x\}$, which is a contradiction.

Since the space $\beta\omega\setminus\omega$ has no isolated points, for any $U \in \tau^*(\beta\omega\setminus\omega)$ there exists an infinite $A \subset \omega$ and $U' \in \mathcal{B}$ such that $[A] \cup U' \subset U$ and $[A] \neq U \neq U'$.

Let $\mathcal{B}$ be any π-base in $\omega^* = \beta\omega\setminus\omega$. Take $U_0 \in \mathcal{B}$ arbitrarily. By Fact 2 of S.370 and our previous observation there exists an infinite set $A \subset \omega$ such that $\emptyset \neq [A] = \overline{A} \cap \omega^* \subset U_0$ (the bar denotes the closure in $\beta\omega$ and the inclusion is strict, i.e., $[A] \neq U_0$). The set $V_0 = [A] \subset U_0$ is a non-empty clopen subset of ω^* by Fact 1 of S.370. Assume that α is a countable ordinal and we have families $\{U_\beta : \beta < \alpha\} \subset \mathcal{B}$ and $\{V_\beta : \beta < \alpha\}$ with the following properties:

(1) V_β is a non-empty clopen subset of ω^* for all $\beta < \alpha$.
(2) $U_{\beta'} \subset V_\beta \subset U_\beta$ (all inclusions are strict) for all $\beta < \beta' < \alpha$.

Since V_β is compact for each $\beta < \alpha$, the set $V'_\alpha = \bigcap\{V_\beta : \beta < \alpha\}$ is a non-empty G_δ-subset of ω^*. Therefore $\mathrm{Int}(V'_\alpha) \neq \emptyset$ by Problem 370 and hence there exists $U_\alpha \in \mathcal{B}$ with $U_\alpha \subset V'_\alpha$ (the inclusion is strict). Apply Fact 2 of S.370 again to find $A \subset \omega$ such that $\emptyset \neq [A] \subset U_\alpha$ (this inclusion is also strict). Letting $V_\alpha = [A]$, we conclude our inductive construction obtaining families $\{U_\beta : \beta < \omega_1\} \subset \mathcal{B}$ and $\{V_\beta : \beta < \omega_1\}$ with the properties (1) and (2).

Since the set V_β is compact for each ordinal $\beta < \omega_1$, the property (1) implies $P = \bigcap\{V_\beta : \beta < \omega_1\} \neq \emptyset$. An immediate consequence of (2) is the equality $P = \bigcap\{U_\beta : \beta < \omega_1\} \neq \emptyset$; since all inclusions in (2) are strict, any $x \in P$ belongs to at least ω_1 different elements of $\mathcal{B}$ whence $\mathcal{B}$ is not point-countable. Fact 3 is proved.

Now it is easy to finish our solution. Assume that $f : \omega^* \to X$ is an irreducible map onto some $X \subset \Sigma(\kappa)$ for some infinite ordinal κ. The map f is closed because ω^* is compact. The space X has a point-countable π-base $\mathcal{B}$ by Fact 1. The family $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{B}\}$ is a π-base in ω^* by Fact 2; it is evident that $\mathcal{V}$ is also point-countable. Therefore ω^* has a point-countable π-base which contradicts Fact 3 and concludes our solution.

S.374. *Prove that tightness of $\beta\omega\setminus\omega$ is uncountable.*

Solution. A family $\mathcal{U}$ of subsets of a space X is called *point-countable* if the set $\{U \in \mathcal{U} : x \in U\}$ is countable for any $x \in X$. If $t(\beta\omega\setminus\omega) = \omega$ then the space $\beta\omega\setminus\omega$ has a point-countable π-base (Problem 332). However this contradicts Fact 3 of S.373.

S.375. *Prove that, for any separable compact space X, the space $C_p(X)$ embeds into $C_p(\beta\omega)$ as a closed subspace.*

Solution. Take any countable dense $D \subset X$ and any surjective map $\varphi' : \omega \to D$. The map φ' is continuous because ω is discrete. Therefore there exists a continuous map $\varphi : \beta\omega \to X$ such that $\varphi\,|\omega = \varphi'$ (Problem 257). Since $\varphi(\beta\omega) \supset \varphi(\omega) = \varphi'(\omega) = D$, the compact set $\varphi(\beta\omega)$ is dense in X and hence $\varphi(\beta\omega) = X$. Since φ is a closed continuous onto map, the mapping $\varphi^* : C_p(X) \to C_p(\beta\omega)$ defined by $\varphi^*(f) = f \circ \varphi$ for any $f \in C_p(X)$, is a closed embedding (Problem 163) so $C_p(X)$ embeds in $C_p(\beta\omega)$ as a closed subspace.

S.376. *Prove that $C_p(\beta\omega)$ embeds into $C_p(\beta\omega\setminus\omega)$ while $C_p(\beta\omega\setminus\omega)$ does not embed into $C_p(\beta\omega)$.*

Solution. Let $\omega^* = \beta\omega\setminus\omega$; if $A \subset \omega$ then $[A] = \overline{A} \cap \omega^*$ (the bar denotes the closure in $\beta\omega$).

Fact 1. (1) The space $\beta\omega$ embeds in ω^*.
(2) The space ω^* maps continuously onto $\beta\omega$.

Proof. It is easy to find a sequence $\{A_n : n \in \omega\}$ of infinite disjoint subsets of ω such that $\omega = \bigcup\{A_n : n \in \omega\}$. Choose any point $x_n \in [A_n]$ (this is possible by Fact 1 of S.370); we claim that $Y = \overline{\{x_n : n \in \omega\}}$ is homeomorphic to $\beta\omega$. To see this, take any disjoint sets $P, Q \subset X = \{x_n : n \in \omega\}$. The sets $A = \bigcup\{A_n : x_n \in P\}$ and $B = \bigcup\{A_n : x_n \in Q\}$ are disjoint and hence $\overline{P} \cap \overline{Q} \subset \overline{A} \cap \overline{B} = \emptyset$ (Fact 1 of S.371). This proves that $\overline{P} \cap \overline{Q} = \emptyset$ for any disjoint $P, Q \subset X$. Of course, it is also true if we consider the closures of P and Q in the space Y. Since Y is a compact extension of X, we can apply Fact 2 of S.286 to conclude that there is a homeomorphism $\varphi : \beta\omega \to Y$. Since Y is a subset of ω^*, we settled (1).

For any $m \in \omega$, let $h(m) = x_n$ where $n \in \omega$ is the unique natural number for which $m \in A_n$. The map $h : \omega \to X$ is continuous, so there is a continuous map $g : \beta\omega \to Y$ such that $g|\omega = f$. Given any $n \in \omega$, we have $g(A_n) = h(A_n) = \{x_n\}$ and therefore $g(\overline{A}_n) = \{x_n\}$; in particular, $g(x) = x_n$ for any x from a non-empty set $[A_n]$. This shows that $X \subset g(\omega^*)$; since $g(\omega^*)$ is a compact set dense in Y, we have $g(\omega^*) = Y$. Consequently, $f = \varphi^{-1} \circ (g|\omega^*) : \omega^* \to \beta\omega$ is a continuous onto map so Fact 1 is proved.

Now, by Fact 1 there exists a continuous onto map $f : \beta\omega\setminus\omega \to \beta\omega$; hence $f^* : C_p(\beta\omega) \to C_p(\beta\omega\setminus\omega)$ embeds $C_p(\beta\omega)$ in $C_p(\beta\omega\setminus\omega)$ (Problem 163).

Since $\beta\omega$ is separable, the space $C_p(\beta\omega)$ condenses onto a second countable space M (Problem 173). If $C_p(\beta\omega\setminus\omega)$ embeds into $C_p(\beta\omega)$ then $C_p(\beta\omega\setminus\omega)$ condenses onto a subspace of M which shows that $iw(C_p(\beta\omega\setminus\omega)) = \omega$ and hence $\beta\omega\setminus\omega$ is separable (Problem 173). However, this contradicts Problem 371 and concludes our solution.

S.377. *Prove that $C_p(\beta\omega\setminus\omega)$ does not condense onto a compact space.*

Solution. Suppose that X is a compact space and $\varphi : C_p(\beta\omega\setminus\omega) \to X$ is a condensation.

Fact 1. We have $|C_p(\beta\omega)| = \mathfrak{c}$ and $|C_p(\beta\omega\setminus\omega)| = \mathfrak{c}$.

Proof. If Z is a non-empty space then $|C_p(Z)| \geqslant \mathfrak{c}$; to see that $|C_p(\beta\omega)| \leqslant \mathfrak{c}c$ observe that $C_p(\beta\omega)$ condenses onto a second countable space M because $\beta\omega$ is separable (Problem 173). Any second countable space embeds in $\mathbb{I}^\omega$ and hence $|M| \leqslant |\mathbb{I}^\omega| \leqslant \mathfrak{c}^\omega = \mathfrak{c}$. This shows that $|C_p(\beta\omega)| \leqslant |M| \leqslant \mathfrak{c}$ and hence $|C_p(\beta\omega)| = \mathfrak{c}$.

Since $F = \beta\omega\setminus\omega$ is a closed subspace of a compact (and hence normal) space $\beta\omega$ (Fact 1 of S.370), the restriction map $\pi_F : C_p(\beta\omega) \to C_p(F) = C_p(\beta\omega\setminus\omega)$ is onto (Problem 152) and hence $|C_p(\beta\omega\setminus\omega)| \leqslant |C_p(\beta\omega)| = \mathfrak{c}$ so Fact 1 is proved.

Returning to our solution, observe that $|X| = \mathfrak{c}$ by Fact 1. If $\chi(x, X) \geqslant \mathfrak{c}$ for every $x \in X$ then $|X| \geqslant 2^{\mathfrak{c}} > \mathfrak{c}$ (Problem 330) which is a contradiction. Thus, there exists a point $x \in X$ such that $\psi(x, X) < \mathfrak{c}$; choose a family $\mathcal{B} \subset \tau(X)$ such that $\bigcap\mathcal{B} = \{x\}$ and $|\mathcal{B}| < \mathfrak{c}$. Take $f \in C_p(\beta\omega\setminus\omega)$ with $\varphi(f) = x$ and observe that the family $\mathcal{B}' = \{\varphi^{-1}(U) : U \in \mathcal{B}\}$ consists of open subsets of $C_p(\beta\omega\setminus\omega)$; it is immediate that $\{f\} = \bigcap\mathcal{B}'$ and $|\mathcal{B}'| < \mathfrak{c}$. For any $g \in C_p(\beta\omega\setminus\omega)$ there exists a homeomorphism T of the space $C_p(\beta\omega\setminus\omega)$ onto itself such that $T(f) = g$ (Problem 079) and hence $\psi(g, C_p(\beta\omega\setminus\omega)) = \psi(f, C_p(\beta\omega\setminus\omega)) < \mathfrak{c}$ for any $g \in C_p(\beta\omega\setminus\omega)$, i.e., $\psi(C_p(\beta\omega\setminus\omega)) < \mathfrak{c}$. However, this implies $\mathfrak{c} = \mathfrak{c}(\beta\omega\setminus\omega) \leqslant d(\beta\omega\setminus\omega) < \mathfrak{c}$ (see Problems 371 and 173) which is a contradiction. Our solution is complete.

S.378. *Prove that $C_p(\beta\omega)$ condenses onto a σ-compact space.*

Solution. Observe that $C_p(\omega, [-n, n]) = [-n, n]^\omega$ for each $n \in \mathbb{N}$ because ω is discrete. Thus, $C_p(\omega, [-n, n])$ is compact for each $n \in \mathbb{N}$ and therefore, the space $C_p^*(\omega) = \bigcup\{C_p(\omega, [-n, n]) : n \in \mathbb{N}\} = \bigcup\{[-n, n]^\omega : n \in \mathbb{N}\}$ is σ-compact. The restriction map $\pi : C_p(\beta\omega) \to C_p(\omega)$ is injective because ω is dense in $\beta\omega$ (Problem 152). Besides, the map π is onto due to the fact that every bounded map on ω is extendable to a continuous map defined on $\beta\omega$ (Problem 257). As a consequence $C_p(\beta\omega)$ condenses onto a σ-compact space $C_p^*(\omega)$.

S.379. *Prove that neither $C_p(\beta\omega)$ nor $C_p(\beta\omega\setminus\omega)$ has a dense σ-compact subspace.*

Solution. Suppose that $C_p(\beta\omega\setminus\omega)$ has a dense σ-compact subspace. Then there is a compact set $K \subset C_p(\beta\omega\setminus\omega)$ which separates the points of $\beta\omega\setminus\omega$ (see Fact 5 of S.310). Define a map $e_K : \beta\omega\setminus\omega \to C_p(K)$ by the formula $e_K(x)(f) = f(x)$ for any $x \in \beta\omega\setminus\omega$ and $f \in K$. Then e_K is injective (Fact 2 of S.351) and hence it embeds $\beta\omega\setminus\omega$ in $C_p(K)$. As a consequence, $t(\beta\omega\setminus\omega) \leqslant t(C_p(K)) = \omega$ (Problem 149) which contradicts Problem 374. Thus $C_p(\beta\omega\setminus\omega)$ has no dense σ-compact subspace.

Observe that $\beta\omega\setminus\omega$ is a closed subspace of $\beta\omega$ (Fact 1 of S.370) and hence the restriction map $\pi : C_p(\beta\omega) \to C_p(\beta\omega\setminus\omega)$ is continuous and onto (Problem 152). If P is a dense σ-compact subspace of $C_p(\beta\omega)$ then $\pi(P)$ is a dense σ-compact subspace of $C_p(\beta\omega\setminus\omega)$ which is a contradiction. This proves that $C_p(\beta\omega)$ does not have a dense σ-compact subspace either.

S.380. *Prove that either of the spaces $C_p(\beta\omega)$ or $C_p(\beta\omega\setminus\omega)$ maps openly and continuously onto another.*

Solution. Observe that $\beta\omega\setminus\omega$ is a closed subspace of $\beta\omega$ (Fact 1 of S.370) and hence the restriction map $\pi : C_p(\beta\omega) \to C_p(\beta\omega\setminus\omega)$ is open, continuous and onto (Problem 152). Thus $C_p(\beta\omega\setminus\omega)$ is an open continuous image of $C_p(\beta\omega)$.

On the other hand, the space $\beta\omega$ embeds in $\beta\omega\setminus\omega$ (Fact 1 of S.376) and hence we can consider that $\beta\omega$ is a closed subspace of $\beta\omega\setminus\omega$. The relevant restriction map is open, continuous and onto (Problem 152) so $C_p(\beta\omega)$ is an open continuous image of $C_p(\beta\omega\setminus\omega)$.

S.381. *Prove that neither of the spaces $C_p(\beta\omega)$ or $C_p(\beta\omega\setminus\omega)$ is normal.*

Solution. The space $C_p(\omega_1+1)$ is not normal (Problem 320); since $w(\omega_1+1)=\omega_1$, the space ω_1+1 can be considered to be a closed subspace of $\mathbb{I}^{\omega_1}$ (Problem 209). The restriction map $\pi : C_p(\mathbb{I}^{\omega_1}) \to C_p(\omega_1+1)$ is onto because $\mathbb{I}^{\omega_1}$ is normal and ω_1+1 is closed in $\mathbb{I}^{\omega_1}$ (Problem 152). This shows that we can apply Problem 291 to convince ourselves that if $C_p(\mathbb{I}^{\omega_1})$ is normal, then $C_p(\omega_1+1)$ is also normal which is a contradiction. Thus, $C_p(\mathbb{I}^{\omega_1})$ is not normal.

Take any countable dense subspace D of $\mathbb{I}^{\omega_1}$ (Problem 108) and fix any surjection $\varphi : \omega \to D$. The map φ is continuous because the space ω is discrete so there is a continuous map $\Phi : \beta\omega \to \mathbb{I}^{\omega_1}$ with $\Phi|\omega = \varphi$. It is evident that $\Phi(\beta\omega) = \mathbb{I}^{\omega_1}$ and therefore the dual map $\Phi^* : C_p(\mathbb{I}^{\omega_1}) \to C_p(\beta\omega)$ is an embedding such that $Z = \Phi^*(C_p(\mathbb{I}^{\omega_1}))$ is closed in $C_p(\beta\omega)$ (Problem 163). If $C_p(\beta\omega)$ is normal, then Z is also normal and hence $C_p(\mathbb{I}^{\omega_1})$ is normal being homeomorphic to Z. This contradiction shows that $C_p(\beta\omega)$ is not normal.

Observe finally that there exists a continuous onto map $\lambda : \beta\omega\setminus\omega \to \beta\omega$ (Fact 1 of S.376). The dual map λ^* embeds $C_p(\beta\omega)$ in $C_p(\beta\omega\setminus\omega)$ as a closed subspace (Problem 163) so if $C_p(\beta\omega\setminus\omega)$ is normal, then so is $C_p(\beta\omega)$. This contradiction shows that $C_p(\beta\omega\setminus\omega)$ is not normal so our solution is complete.

S.382. *Prove that, for any discrete space D, we have $p(C_p(\beta D)) = \omega$.*

Solution. If D is finite there is nothing to prove, so assume that D is an infinite discrete space.

Fact 1. If $A, B \subset D$ and $A \cap B = \emptyset$ then $\overline{A} \cap \overline{B} = \emptyset$ (the bar denotes the closure in the space βD).

Proof. Let $f : D \to \{0,1\}$ be defined by $f(d)=1$ if $d \in A$ and $f(d)=0$ for all $d \in D\setminus A$. Then f is a continuous map and hence there is $g \in C(\beta D, \{0,1\})$ such that $g|D = f$ (Problem 257). The sets $g^{-1}(0)$ and $g^{-1}(1)$ are closed in βD; it is immediate that $A \subset g^{-1}(1)$ and $B \subset g^{-1}(0)$ whence $\overline{A} \cap \overline{B} \subset g^{-1}(0) \cap g^{-1}(1) = \emptyset$ so Fact 1 is proved.

Fact 2. Let E be a countable discrete subspace of βD. Suppose that $A, B \subset E$ and $A \cap B = \emptyset$. Then $\overline{A} \cap \overline{B} = \emptyset$.

Proof. If E is finite then we have nothing to prove so assume that E is infinite and take some faithful enumeration $\{x_n : n \in \omega\}$ of the set E. There exists a disjoint family $\{U_n : n \in \omega\} \subset \tau(\beta D)$ such that $x_n \in U_n$ for each $n \in \omega$ (Fact 1 of S.369). Let $V_n = U_n \cap D$; since D is dense in βD, we have $\overline{V_n} = \overline{U_n}$ for each $n \in \omega$. Let $A' = \bigcup\{V_n : x_n \in A\}$ and $B' = \bigcup\{V_n : x_n \in B\}$. Observe that $x_n \in \overline{V_n}$ for each $n \in \omega$; this implies $A \subset \overline{A'}$ and $B \subset \overline{B'}$. Since $A', B' \subset D$ and $A' \cap B' = \emptyset$, we can apply Fact 1 to conclude that $\overline{A} \cap \overline{B} \subset \overline{A'} \cap \overline{B'} = \emptyset$ so Fact 2 is proved.

Fact 3. Let A and B be countable discrete disjoint subsets of βD. If A is infinite then there is an infinite $A' \subset A$ such that $\overline{A'} \cap B = \emptyset$.

Proof. Take any faithful enumeration $\{a_n : n \in \omega\}$ of the set A. The space $K = \overline{A}$ is a compact extension of A such that, for any disjoint $P, Q \subset A$, we have $\overline{P} \cap \overline{Q} = \emptyset$ by Fact 2. Therefore there exists a homeomorphism $\varphi : \beta\omega \to K$ such that $\varphi(n) = a_n$ for each $n \in \omega$ (see Fact 2 of S.286). If $P \subset K$ then $\mathrm{cl}_K(P) = \overline{P}$ so it suffices to find an infinite $A' \subset A$ such that $\overline{A'} \cap B' = \emptyset$ where $B' = B \cap K = B \cap (K \backslash A)$. This shows that we can identify K with $\beta\omega$ considering that $A = \omega$ and $B' \subset \beta\omega \backslash \omega$. Since B' is countable, the set $C = (\beta\omega \backslash \omega) \backslash B'$ is a G_δ-set in $\beta\omega \backslash \omega$; besides, C is non-empty because $\beta\omega \backslash \omega$ is uncountable (Problem 368). Therefore C has non-empty interior in $\beta\omega \backslash \omega$ (Problem 370) and hence there is a non-empty $U \in \tau(\beta\omega \backslash \omega)$ such that $U \cap B' = \emptyset$. The family $\{\overline{W} \cap (\beta\omega \backslash \omega) : W$ is an infinite subset of $\omega\}$ is a base in $\beta\omega \backslash \omega$ by Fact 2 of S.370 so there exists an infinite $A' \subset \omega$ such that $\overline{A'} \cap (\beta\omega \backslash \omega) \subset U$ which implies $\overline{A'} \cap B' = \emptyset$. Fact 3 is proved.

Fact 4. Let A be an infinite subspace of a space X. Then there is an infinite $B \subset A$ such that the subspace B is discrete.

Proof. If the set I of isolated points of A is infinite then we can take $B = I$. If I is finite then $A' = A \backslash I$ has no isolated points and hence any $U \in \tau^*(A')$ is infinite. Take any distinct $b_0, b_0' \in A'$ arbitrarily and use regularity of A' to find a set $U_0 \in \tau(b_0, A')$ such that $b_0' \notin \overline{U}_0$ (the bar denotes the closure in A'). Then $\overline{U}_0 \neq A'$; suppose that we have $b_0, \ldots, b_n \in A'$ and $U_0, \ldots, U_n \in \tau(A')$ such that $b_i \in U_i$ for all $i \leqslant n$, the family $\{\overline{U}_i : i \leqslant n\}$ is disjoint and $V = A' \setminus (\bigcup_{i \leqslant n} \overline{U}_i) \neq \emptyset$. Take any $b_{n+1} \in V$ and find $U_{n+1} \in \tau(b_{n+1}, A')$ such that $\overline{U}_{n+1} \cap (\bigcup_{i \leqslant n} \overline{U}_i) = \emptyset$ – this is possible by regularity of the space A'. It is evident that this inductive construction can be carried out for all $n \in \mathbb{N}$ and hence we obtain a disjoint family $\{U_n : n \in \omega\}$ of open subsets of A' and a set $B = \{b_n : n \in \omega\} \subset A$ with $b_n \in U_n$ for all $n \in \omega$. The set B is infinite because, for any distinct $n, m \in \mathbb{N}$, the points b_n and b_m lie in disjoint sets U_n and U_m whence $b_n \neq b_m$. Besides, B is discrete because $U_n \cap B = \{b_n\}$ for all $n \in \mathbb{N}$ and hence each set $\{b_n\}$ is open in B. Fact 4 is proved.

Fact 5. Given any $n \in \mathbb{N}$, let $\Delta_n = \{x = (x_1, \ldots, x_n) \in (\beta D)^n$: there exist $i, j \leqslant n$ such that $i \neq j$ and $x_i = x_j\}$. Then the space $D_n = (\beta D)^n \setminus \Delta_n$ is countably compact for all $n \in \mathbb{N}$.

Proof. Observe that $\Delta_1 = \emptyset$ and hence the space $D_1 = \beta D$ is even compact. Now fix any natural $n > 1$ and assume that $E \subset D_n$ is a countably infinite closed discrete subspace of D_n. Let $p_i : (\beta D)^n \to \beta D$ be the natural projection onto the ith factor for all $i \leqslant n$. Any infinite subset of E is closed and discrete in D_n so to obtain the desired contradiction, we are going to consider smaller infinite subsets of E to reduce our situation to simpler ones.

For $E_0 = E$ consider the set $p_1(E_0) \subset \beta D$. If $p_1(E_0)$ is infinite then use Fact 4 to find an infinite discrete $E' \subset p_1(E_0)$. Taking a point $x(e) \in p_1^{-1}(e) \cap E_0$ for each $e \in E'$ we obtain an infinite set $E_1 = \{x(e) : e \in E'\} \subset E_0$ such that $p_1(E_1)$ is an infinite discrete subspace of βD and $p_1|E_1$ is an injection. If $p_1(E_0)$ is finite then there is $e \in p_1(E_0)$ such that the set $E_1 = p_1^{-1}(e) \cap E_0$ is infinite. It is evident that in this case $p_1(E_1)$ is a singleton (≡consists of one point). Suppose that $i < n$ and we have infinite subsets $E_1 \supset \cdots \supset E_i$ of the set E such that, for each $j \leqslant i$, the set $p_j(E_j)$ is either a singleton or $p_j(E_j)$ is infinite, discrete and $p_j|E_j$ is an injection.

Consider the set $p_{i+1}(E_i) \subset \beta D$. If $p_{i+1}(E_i)$ is infinite then use Fact 4 to find an infinite discrete $E' \subset p_{i+1}(E_i)$. Taking a point $x(e) \in p_{i+1}^{-1}(e) \cap E_i$ for each $e \in E'$ we obtain an infinite set $E_{i+1} = \{x(e) : e \in E'\} \subset E_i$ such that $p_{i+1}(E_{i+1})$ is an infinite discrete subspace of βD and $p_{i+1}|E_{i+1}$ is an injection. If $p_{i+1}(E_i)$ is finite then there is $e \in p_{i+1}(E_i)$ such that the set $E_{i+1} = p_{i+1}^{-1}(e) \cap E_i$ is infinite. It is evident that in this case $p_{i+1}(E_{i+1})$ is a singleton. This inductive procedure can be carried out n times to obtain an infinite set $E_n \subset E$ such that, for every $j \leqslant n$, the set $p_j(E_n)$ is either a singleton or $p_j(E_n)$ is infinite, discrete and $p_j|E_n$ is an injection. Without loss of generality we consider that $E = E_n$.

Our next step is to prove that

$(*)$ For any infinite $E' \subset E$ and any distinct $i, j \leqslant n$ there exists an infinite $E'' \subset E'$ such that $\overline{p_i(E'')} \cap \overline{p_j(E'')} = \emptyset$.

Assume first that $p_i(E')$ and $p_j(E')$ are singletons. If $p_i(E') = \{e\} = p_j(E')$ then for any $d \in E'$ we have $p_j(d) = e = p_i(d)$. However, $d \in D_n$ so $p_j(d) \neq p_i(d)$ by definition of D_n. This contradiction shows that $p_i(E')$ and $p_j(E')$ are distinct singletons and hence $\overline{p_i(E')} \cap \overline{p_j(E')} = \emptyset$ so we can take $E'' = E'$.

Now, assume that $p_i(E') = \{e\}$ is a singleton and $p_j(E')$ is infinite and hence discrete. Choose infinite disjoint sets $A, B \subset E'$. The sets $p_j(A)$ and $p_j(B)$ are disjoint and $p_j(A) \cup p_j(B)$ is a discrete subset of βD; hence $\overline{p_j(A)} \cap \overline{p_j(B)} = \emptyset$ by Fact 2. As a consequence, the closure of one of the sets $p_j(A)$ and $p_j(B)$, say, $p_j(A)$, does not contain e in its closure. It is evident that we can take $E'' = A$ because in this case $\overline{p_i(E'')} \cap \overline{p_j(E'')} = \{e\} \cap \overline{p_j(A)} = \emptyset$.

The case when the set $p_j(E')$ is a singleton and $p_i(E')$ is infinite is considered analogously, so assume that both sets $p_i(E')$ and $p_j(E')$ are infinite and hence discrete. Let $A_0 = \emptyset$ and assume that, for some number $n \in \omega$, we have a set $A_n \subset E'$ such that $|A_n| = n$ and $p_i(A_n) \cap p_j(A_n) = \emptyset$. Since the maps $q_i = p_i|E'$ and $q_j = p_j|E'$ are injective, and the set $B = p_i(A_n) \cup p_j(A_n)$ is finite, we can choose a point $a \in E' \backslash \left(q_i^{-1}(B) \cup q_j^{-1}(B)\right)$. Then the set $A_{n+1} = A_n \cup \{a\}$ has exactly $n+1$ elements and we have

$$\begin{aligned} p_i(A_{n+1}) \cap p_j(A_{n+1}) = &\left(p_i(A_n) \cap p_j(A_n)\right) \cup \left(p_i(A_n) \cap \left\{p_j(a)\right\}\right) \\ &\cup \left(p_j(A_n) \cap \{p_i(a)\}\right) \cup \left(\{p_i(a)\} \cap \left\{p_j(a)\right\}\right) = \emptyset. \end{aligned}$$

Thus our inductive procedure can be continued giving us, for all $n \in \omega$, a set $A_n \subset E'$ such that $|A_n| = n$ and $p_i(A_n) \cap p_j(A_n) = \emptyset$. It is easy to see that $A = \bigcup_{n\in\omega} A_n$ is an infinite subset of E' and $p_i(A) \cap p_j(A) = \emptyset$. Since $p_i(A)$ and $p_j(A)$ are infinite countable disjoint and discrete subsets of βD, Fact 3 can be applied to obtain an infinite $A' \subset A$ such that $\overline{p_i(A')} \cap p_j(A) = \emptyset$ and hence $\overline{p_i(A')} \cap p_j(A') = \emptyset$. Applying Fact 3 again, we can find an infinite $E'' \subset A'$ such that $\overline{p_j(E'')} \cap p_i(A') = \emptyset$ and hence $\overline{p_j(E'')} \cap p_i(E'') = \emptyset$. Of course, we also have $\overline{p_i(E'')} \cap p_j(E'') \subset \overline{p_i(A')} \cap p_j(A') = \emptyset$. It is an easy exercise to see that the set $p_i(E'') \cup p_j(E'')$ is discrete so Fact 2 can be applied to conclude that $\overline{p_i(E'')} \cap \overline{p_j(E'')} = \emptyset$ which shows that $(*)$ is finally proved.

Now, let $\{(i_1, j_1), \ldots, (i_k, j_k)\}$ be some enumeration of all pairs (i, j) where $i, j \leqslant n$ and $i \neq j$. Here $k = \frac{n(n-1)}{2}$ but we do not need this; it suffices to know that k is finite. Applying k times the property $(*)$, we obtain infinite subsets $E_1 \supset \cdots \supset E_K$ of the set E such that $\overline{p_{i_m}(E_m)} \cap \overline{p_{j_m}(E_m)} = \emptyset$ for all $m \leqslant k$. It is clear that E_K is an infinite subset of E such that $\overline{p_i(E_k)} \cap \overline{p_j(E_k)} = \emptyset$ for all distinct $i, j \leqslant n$. Therefore, E_K is an infinite closed and discrete subset of D_n for which $E_k \subset K = \prod\left\{\overline{p_i(E_k)} : k \leqslant n\right\}$. It is immediate that K is compact and $K \subset D_n$ so we found an infinite closed and discrete subset in a compact space K which is a contradiction concluding the proof of Fact 5.

Returning to our solution, recall that, given $x_1, \ldots, x_n \in \beta D$ and rational intervals $O_1, \ldots, O_n$ the set $[x_1, \ldots, x_n; O_1, \ldots, O_n] = \{f \in C_p(\beta D) : f(x_i) \in U_i$ for all $i \leqslant n\}$ is called *a standard open set* of the space $C_p(\beta D)$. All possible standard open sets form a base $\mathcal{O}$ in the space $C_p(\beta D)$ (Problem 056).

Suppose that there is a point-finite family $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \subset \tau^*(C_p(\beta D))$. Since every non-empty open set contains an element of $\mathcal{O}$, we can choose $V_\alpha \in \mathcal{O}$ with $V_\alpha \neq \emptyset$ and $V_\alpha \subset U_\alpha$ for each $\alpha < \omega_1$. It is clear that the family $\{V_\alpha : \alpha < \omega_1\}$ is also point-finite and uncountable so, to obtain the desired contradiction, we can assume, without loss of generality, that $U_\alpha = V_\alpha$ for all $\alpha < \omega_1$.

Thus, we are assuming that all elements of $\mathcal{U}$ are standard, i.e., for each $\alpha < \omega_1$, we have $U_\alpha = \left[x_1^\alpha, \ldots, x_{n_\alpha}^\alpha; O_1^\alpha, \ldots, O_{n_\alpha}^\alpha\right]$ where $\left\{x_i^\alpha : i \leqslant n_\alpha\right\} \subset \beta D$, $x_i^\alpha \neq x_j^\alpha$ for all $i, j \leqslant n_\alpha$ with $i \neq j$ and O_i^α is a rational interval for each $i \leqslant n_\alpha$. Observe that if we have a family $\left\{W_i^\alpha : i \leqslant n_\alpha\right\}$ of rational intervals with $W_i^\alpha \subset O_i^\alpha$ for each $i \leqslant n_\alpha$, then $\left[x_1^\alpha, \ldots, x_{n_\alpha}^\alpha ; w_1^\alpha, \ldots, w_{n_\alpha}^\alpha\right] \subset \left[x_1^\alpha, \ldots, x_{n_\alpha}^\alpha ; O_1^\alpha, \ldots, O_{n_\alpha}^\alpha\right]$. It is easy to show that there exists a disjoint family $\left\{W_i^\alpha : i \leqslant n_\alpha\right\}$ of rational intervals such that $W_i^\alpha \subset O_i^\alpha$ for all $i \leqslant n_\alpha$. That the set $\left[x_1^\alpha, \ldots, x_{n_\alpha}^\alpha ; W_1^\alpha, \ldots, W_{n_\alpha}^\alpha\right]$ is non-empty, is an easy consequence of Problem 034.

This argument shows that, for each $\alpha < \omega_1$, we can assume that $\left\{O_i^\alpha : i \leqslant n_\alpha\right\}$ is a disjoint family of non-empty rational intervals of $\mathbb{R}$. There is an uncountable $\mathcal{U}' \subset \mathcal{U}$ such that, for some $n \in \mathbb{N}$ and an n-tuple $(O_1, \ldots, O_n)$ of disjoint rational intervals, we have $n_\alpha = n$ and $O_i^\alpha = O_i$ for all $\alpha < \omega_1$ such that $U_\alpha \in \mathcal{U}'$. If $x_\alpha = \left(x_1^\alpha, \ldots, x_n^\alpha\right)$ for all $\alpha \in B = \{\beta < \omega_1 : U_\beta \in \mathcal{U}'\}$ then $E = \{x_\alpha : \alpha \in B\}$ is an uncountable subset of D_n (see Fact 5 for the definition of D_n).

For each $x = (x_1, \ldots, x_n) \in D_n$ there exists $f \in C(\beta D)$ such that $f(x_i) \in O_i$ for each $i \leqslant n$ (Problem 034). The set $O_f = f^{-1}(O_1) \times \cdots \times f^{-1}(O_n)$ is an open neighbourhood of the point x and, if $x_\alpha \in O_f$ then $f\left(x_i^\alpha\right) \in O_i$ for all $i \leqslant n$ which shows that $f \in U_\alpha$. Since $\mathcal{U}'$ is point-finite, the function f belong to at most finitely many elements of $\mathcal{U}'$ so there are only finitely many $\alpha \in B$ such that $x_\alpha \in O_f$. Consequently, each point $x \in D_n$ has a neighbourhood which contains only finitely many elements of E which shows that E is a closed discrete subset of D_n. Since the set E is uncountable (and hence infinite), the space D_n cannot be countably compact; this contradicts Fact 5 and finishes our solution.

S.383. *Prove that, for any compact space X and any continuous surjective mapping $f: X \to \beta\omega$, there exists $F \subset X$ such that $f(F) = \beta\omega$ and $f|F$ is a homeomorphism.*

Solution. The map f is perfect so we can apply Problem 366 to find a closed $F \subset X$ such that $f(F) = \beta\omega$ and the map $f|F$ is irreducible. Since every condensation of a compact space is a homeomorphism, it suffices to prove that the mapping $g = f|F : F \to \beta\omega$ is injective.

Fact 1. Suppose that $h : Y \to Z$ is a closed irreducible map. Given any $W \in \tau^*(Y)$, let $h^\#(W) = Z \setminus h(Y \setminus W)$. Then the set $W_h = h^{-1}(h^\#(W))$ is dense in W and hence $h^\#(W)$ is dense in $h(W)$.

Proof. For an arbitrary $O \in \tau^*(Y)$, we have $Y \setminus O \neq Y$ and therefore $h(Y \setminus O) \neq Z$ because the map h is irreducible. Thus $h^\#(O) \neq \emptyset$ and hence $\emptyset \neq h^{-1}(h^\#(O)) \subset O$ for any $O \in \tau^*(Y)$. Now, if W_h is not dense in W then $O = W \setminus \overline{W_h} \in \tau^*(Y)$ and hence $O_h \subset O$ is a non-empty open subset of W with $O \cap W_h = \emptyset$. However, $O \subset W$ implies $O_h \subset W_h$; this contradiction shows that $O = \emptyset$, i.e., W_h is dense in W. Since the map h is continuous, the set $h(W_h) = h^\#(W)$ is dense in $h(W)$ so Fact 1 is proved.

Fact 2. The set $\overline{U}$ is open in $\beta\omega$ for any $U \in \tau(\beta\omega)$. As a consequence, for any $U, V \in \tau(\beta\omega)$ such that $U \cap V = \emptyset$, we have $\overline{U} \cap \overline{V} = \emptyset$.

Proof. If $U = \emptyset$ then there is nothing to prove. If $U \neq \emptyset$ then $\overline{U} = \overline{U \cap \omega}$ because ω is dense in $\beta\omega$. Now apply Fact 1 of S.370 to conclude that $\overline{U \cap \omega}$ and hence $\overline{U}$ is open in $\beta\omega$. Observe that if a set P does not intersect an open set W then $\overline{P} \cap W = \emptyset$ as well. Therefore, $U \cap V = \emptyset$ implies $\overline{U} \cap V = \emptyset$; since $\overline{U}$ is open, by the same remark, we have $\overline{U} \cap \overline{V} = \emptyset$ so Fact 2 is proved.

Returning to our solution, suppose that $x, y \in F$, $x \neq y$ and $g(x) = g(y)$. Fix $Ox \in \tau(x, F)$, $Oy \in \tau(y, F)$ such that $\overline{Ox} \cap \overline{Oy} = \emptyset$. Apply Fact 1 to convince ourselves that the set $g^\#(Ox)$ is dense in $g(Ox)$ and $g^\#(Oy)$ is dense in $g(Oy)$. Since $z = g(x) = g(y) \in g(Ox) \cap g(Oy)$, we have $z \in \overline{g^\#(Ox)} \cap \overline{g^\#(Oy)}$. However, $U = g^\#(Ox)$ and $V = g^\#(Oy)$ are disjoint open subsets of $\beta\omega$; therefore $\overline{U} \cap \overline{V} = \emptyset$ by Fact 2. This contradiction shows that g is a homeomorphism so our solution is complete.

S.384. *Let T be the two arrows space. Prove that T is a perfectly normal hereditarily separable compact space and* $\text{ext}(C_p(T)) = \mathfrak{c}$. *Deduce from this fact that $C_p(T)$ is not normal.*

Solution. We have $T = T_0 \cup T_1$, where $T_0 = (0, 1] \times \{0\}$ and $T_1 = [0,1) \times \{1\}$. If we consider T_0 as a subspace of T then, for any $z = (t, 0) \in T_0$, the family $\{(a, t] \times \{0\} : 0 < a < t\}$ is a base at the point z. An easy consequence is that the map $i : T_0 \to [-1, 0)$ defined by $i(t, 0) = -t$, is a homeomorphism if $[-1, 0)$ is considered to be a subspace of the Sorgenfrey line S (see Problem 165).

Analogously, if we consider T_1 as a subspace of T then, for any $z = (t, 1) \in T_1$, the family $\{[t, a) \times \{1\} : t < a < 1\}$ is a base at the point z, so it is immediate that the map $i : T_1 \to [0, 1)$ defined by $i(t, 1) = t$, is a homeomorphism if $[0, 1)$ is considered to be a subspace of the Sorgenfrey line S (see Problem 165).

Since the space T is introduced from scratch, we must check the Tychonoff property of T. It suffices to prove that T is Hausdorff and compact (see Problem 124). The map $\pi : T \to [0, 1]$ defined by $\pi(t, i) = t$, is continuous if $[0, 1]$ is considered with the topology induced from $\mathbb{R}$. Take any distinct $z_0, z_1 \in T$. If $\pi(z_0) \neq \pi(z_1)$ then there are sets $U_i \in \tau(\pi(z_i), [0, 1])$, $i = 0, 1$ such that $U_0 \cap U_1 = \emptyset$. Then $V_0 = \pi^{-1}(U_0)$ and $V_1 = \pi^{-1}(U_1)$ are open in T and separate the points z_0 and z_1. Now, if $z_0 = (t, 0)$ and $z_1 = (t, 1)$ for some $t \in (0, 1)$ then consider the sets $U_0 = ((0, t] \times \{0\}) \cup ((0, t) \times \{1\})$ and $U_1 = ([t, 1) \times \{1\}) \cup ((t, 1) \times \{0\})$. It is immediate that $U_i \in \tau(z_i, T)$ and $U_0 \cap U_1 = \emptyset$ so the Hausdorff property of T is verified.

We already saw that T_i is homeomorphic to a subspace of S for each $i = 0, 1$. Given any $Y \subset T$, we have $Y = Y_0 \cup Y_1$ where $Y_i = Y \cap T_i$ for $i \in \{0, 1\}$. Since S is hereditarily separable and hereditarily Lindelöf, so is T_i and hence Y_i is separable and Lindelöf for each $i = 0, 1$. A union of two separable and Lindelöf spaces is separable and Lindelöf so Y is separable and Lindelöf for any $Y \subset T$. This shows that T is hereditarily separable and hereditarily Lindelöf. Since we only know so far, that T is Hausdorff, it is necessary to observe that all properties mentioned in this paragraph hold for Hausdorff spaces.

We next prove that the map π is closed. Take any point $t \in (0, 1)$ and any set $W \in \tau(\pi^{-1}(t), T)$; if $z_i = (t, i)$ for $i \in \{0, 1\}$ then $\pi^{-1}(t) = \{z_0, z_1\}$. Since $z_0 \in W$, there is $a \in (0, t)$ such that $U_0 = ((a, t] \times \{0\}) \cup ((a, t) \times \{1\}) \subset W$. Analogously, $z_1 \in W$ implies that there is $b \in (t, 1)$ such that $U_1 = ([t, b) \times \{1\}) \cup ((t, b) \times \{0\}) \subset W$. The set $V = (a, b)$ is open in $[0, 1]$; besides, $t \in V$ and $\pi^{-1}(V) \subset W$. Therefore, for every $t \in (0, 1)$ and every $W \in \tau(\pi^{-1}(t), T)$ there is $V \in \tau(t, [0, 1])$ such that $\pi^{-1}(V) \subset W$. The proof of the same property at the points $t = 0$ and $t = 1$ is easier so we omit it. Now apply Fact 2 of S.271 (which is true for Hausdorff spaces) to conclude that the map π is closed. The set $\pi^{-1}(z)$ consists of at most two points for each $z \in T$ so $\pi^{-1}(z)$ is compact for all $z \in T$. Thus the map π is perfect; since $[0, 1]$ is a compact space, we can apply Fact 2 of S.259 (which is also true for Hausdorff spaces) to see that T is compact and hence normal.

To see that T is perfectly normal, it suffices to establish that every closed set in T is a G_δ-set. This is equivalent to saying that every open set of T is an F_σ-set. So let $U \in \tau^*(T)$. Since we already have regularity of the space T, for any $z \in U$ we can take $U_z \in \tau(z, T)$ such that $\overline{U}_z \subset U$. Since every subspace of T is Lindelöf, we have a countable $A \subset U$ such that $U = \bigcup\{U_z : z \in A\} = \bigcup\{\overline{U}_z : z \in A\}$ so U is an F_σ-subset of T which shows that we proved perfect normality of the space T.

For any $t \in (0, 1)$, let $z_i = (t, i)$, $i \in \{0, 1\}$. Define a function $f_t \in C(T)$ as follows: $f(z_i) = i$, $i \in \{0, 1\}$; $f(z) = 0$ if $\pi(z) < t$ and $f(z) = 1$ if $\pi(z) > t$. It is clear that the set $F = \{f_t : t \in (0, 1)\}$ has cardinality c. We must check that $F \subset C(T)$ and F is closed and discrete in $C_p(T)$. Each function f_t is continuous because both $f_t^{-1}(0)$ and $f_t^{-1}(1)$ are open sets; hence $F \subset C(T)$.

Now, assume that $f \in C_p(T)$ is an accumulation point of the set F. If $f(z) \notin \{0, 1\}$ for some $z \in T$ then $U = \{g \in C_p(T) : g(z) \notin \{0, 1\}\}$ is an open neighbourhood of f with $U \cap F = \emptyset$. Thus $f(T) \subset \{0, 1\}$; assume that $f(z) = 0$ for some $z \in T$. If $z' \in T$ and $\pi(z') < \pi(z)$ then $f(z') = 0$ because otherwise $f(z') = 1$ and the set $V = \{g \in C_p(T) : g(z) < 1/2$ and $g(z') > 1/2\}$ is an open neighbourhood of f which does not meet F. Analogously, if $f(z) = 1$ then $f(z') = 1$ for any $z' \in T$ with $\pi(z') > \pi(z)$. Note that $f_t(0, 1) = 0$ and $f_t(1, 0) = 1$ for all $f_t \in F$ whence $f(0, 1) = 0$ and $f(1, 0) = 1$. There exists $a \in [0,1]$ such that $f(z) = 1$ for all $z \in T$ with $\pi(z) > a$ and $f(z) = 0$ for all $z \in T$ such that $\pi(z) < a$. If $a = 0$, then $f(z) = 1$ for all $Z \in T \backslash \{(0, 1)\}$ which contradicts continuity of f at the point $y_0 = (0, 1)$. If $a = 1$ then $f(z) = 0$ for all $z \in T \backslash \{(1, 0)\}$ which contradicts continuity of f at the point $y_1 = (1, 0)$.

Thus, $0 < a < 1$; if $z_i = (a, i)$ for $i = 0, 1$ then $f(z_i) = i$. Indeed, z_0 is in the closure of the set $A = (0, a) \times \{0\}$ on which the function f is equal to zero and z_1 belongs to the closure of the set $B = (a, 1) \times \{1\}$ on which f is identically 1. Thus the set $W = \{g \in C_p(T) : g(z_0) < 1/2$ and $g(z_1) > 1/2\}$ is an open neighbourhood of f such that $W \cap F = \{f_a\}$ which demonstrates again that f is not an accumulation point for F. This contradiction shows that F is closed and discrete in $C_p(T)$. Therefore, $\text{ext}(C_p(T)) \geqslant \mathfrak{c}$; since $w(C_p(T)) = |T| = \mathfrak{c}$, we have $\text{ext}(C_p(T)) \leqslant w(C_p(T)) = \mathfrak{c}$ so $\text{ext}(C_p(T)) = \mathfrak{c}$.

Finally, observe that $C_p(T)$ is not normal because normality of $C_p(T)$ implies $\text{ext}(C_p(T)) = \omega$ (Problem 295) which is a contradiction with $\text{ext}(C_p(T)) = \mathfrak{c}$.

S.385. *Let T be the two arrows space. Show that $p(C_p(T)) = \mathfrak{c}$.*

Solution. We have $T = T_0 \cup T_1$, where $T_0 = (0, 1] \times \{0\}$ and $T_1 = [0, 1) \times \{1\}$. For any $t \in (0, 1)$ let $z_0(t) = (t, 0) \in T_0$ and $z_1(t) = (t, 1) \in T_1$.

Fact 1. Given an arbitrary function $f \in C(T)$ and any positive number ε, the set $A(f, \varepsilon) = \{t \in (0, 1) : |f(z_1(t)) - f(z_0(t))| \geqslant \varepsilon\}$ is finite.

Proof. Suppose that $S = \{t_i : i \in \omega\} \subset (0, 1)$ is a faithfully indexed set such that $|f(z_1(t_i)) - f(z_0(t_i))| \geqslant \varepsilon$ for all $i \in \varepsilon$. Passing to a smaller infinite subset if necessary, we can assume that S is a monotonous sequence which converges to a point $t \in [0, 1]$. We have two cases.

(1) The sequence S is increasing. Then, for every set $W \in \tau(z_0(t), T)$, there exists $k \in \omega$ such that $\{z_0(t_i), z_1(t_i)\} \subset W$ for all $i \geqslant k$. The function f being continuous at the point $z = z_0(t)$, there is $W \in \tau(z_0(t), T)$ such that $f(W) \subset (f(z_0(t)) - \frac{\varepsilon}{3}, f(z_0(t)) + \frac{\varepsilon}{3})$. Take an arbitrary $i \in \omega$ such that $\{z_0(t_i), z_1(t_i)\} \subset W$; then $|f(z_0(t_i)) - f(z)| < \frac{\varepsilon}{3}$ and $|f(z_1(t_i)) - f(z)| < \frac{\varepsilon}{3}$ whence $|f(z_1(t_i)) - f(z_0)(t_i)| \leqslant |f(z_1(t_i)) - f(z)| + |f(z_0(t_i)) - f(z)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$, which is a contradiction.

(2) The sequence S is decreasing. Then, for every set $W \in \tau(z_1(t), T)$, there exists $k \in \omega$ such that $\{z_0(t_i), z_1(t_i)\} \subset W$ for all $i \geqslant k$. The function f being continuous at the point $z = z_1(t)$, there is $W \in \tau(z_1(t), T)$ such that $f(W) \subset (f(z_1(t)) - \frac{\varepsilon}{3}, f(z_1(t)) + \frac{\varepsilon}{3})$. Take an arbitrary $i \in \omega$ such that $\{z_0(t_i), z_1(t_i)\} \subset W$; then $|f(z_0(t_i)) - f(z)| < \frac{\varepsilon}{3}$ and $|f(z_1(t_i)) - f(z)| < \frac{\varepsilon}{3}$ whence $|f(z_1(t_i)) - f(z_0)(t_i)| \leqslant |f(z_1(t_i)) - f(z)| + |f(z_0(t_i)) - f(z)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$, which is a contradiction. Fact 1 is proved.

If $z \in T$, let $e(z)(f) = f(z)$ for all $f \in C_p(T)$. Then $e(z) : C_p(T) \to \mathbb{R}$ is a continuous function (Problem 166). Given any $t \in (0, 1)$, let $\varphi_t(f) = f(z_1(t)) - f(z_0(t))$ for any function $f \in C_p(T)$. Then $\varphi_t : C_p(T) \to \mathbb{R}$ is continuous because $\varphi_t = e(z_1(t)) - e(z_0(t))$, i.e., φ_t is a difference of two continuous functions. Denote by φ the function which is identically zero on $C_p(T)$.

Claim. The subspace $K = \{\varphi_t : t \in (0, 1)\} \cup \{\varphi\}$ of the space $C_p(C_p(T))$ is homeomorphic to $A(\mathfrak{c}c)$.

Proof. It is evident that $|K| = \mathfrak{c}$, so it is sufficient to show that $K \setminus U$ is finite for any $U \in \tau(\varphi, C_p(C_p(T)))$. There exist functions $f_1, \ldots, f_n \in C_p(T)$ and $\varepsilon > 0$ such that $V = \{\delta \in C_p(C_p(T)) : |\delta(f_i)| < \varepsilon$ for all $i \leqslant n\} \subset U$. The set $A = \bigcup\{A(f_i, \varepsilon) : i \leqslant n\}$ is finite by Fact 1. For any $i \leqslant n$, if $t \notin A$ then $t \notin A(f_i, \varepsilon)$ and hence $|\varphi_t(f_i)| < \varepsilon$ by definition of $A(f_i, \varepsilon)$. This shows that $\varphi_t \in V \subset U$ if $t \notin A$, i.e., the set $K \setminus U \subset A$ is finite so our claim is proved.

Returning to our solution, observe that it is a straightforward consequence of our claim that $p(C_p(T)) = a(C_p(C_p(T))) \geqslant \mathfrak{c}$ (Problem 178). On the other hand, $p(C_p(T)) \leqslant nw(C_p(T)) = nw(T) \leqslant |T| = \mathfrak{c}c$ whence $p(C_p(T)) = \mathfrak{c}$ and our solution is complete.

S.386. *Consider the two arrows space T and let S be the Sorgenfrey line. Prove that $C_p(T)$ embeds into $C_p(S)$ while $C_p(S)$ does not embed into $C_p(T)$.*

Solution. We have $T = T_0 \cup T_1$, where $T_0 = (0, 1] \times \{0\}$ and $T_1 = [0, 1) \times \{1\}$. If we consider T_0 as a subspace of the space T then, for any point $z = (t, 0) \in T_0$, the family $\{(a, t] \times \{0\} : 0 < a < t\}$ is a base at the point z. An easy consequence is that the map $i : T_0 \to [-1, 0)$ defined by $i(t, 0) = -t$, is a homeomorphism if $[-1, 0)$ is considered to be a subspace of the Sorgenfrey line S (see Problem 165). Thus the map $i^{-1} : [-1, 0) \to T_0$ is also a homeomorphism and hence i^{-1} is a continuous map from $[-1, 0)$ to the space T (Problem 023).

Analogously, if we consider T_1 as a subspace of T then, for any $z = (t, 1) \in T_1$, the family $\{[t, a) \times \{1\} : t < a < 1\}$ is a base at the point z so it is immediate that the map $j : T_1 \to [0, 1)$ defined by $j(t, 1) = t$, is a homeomorphism if $[0, 1)$ is considered to be a subspace of the Sorgenfrey line S (see Problem 165). Therefore, the map j^{-1} : $[0, 1) \to T$ is also continuous (here we applied Problem 023 again).

Fact 1. Two arrows space is a continuous image of the Sorgenfrey line.

Proof. All subsets of $\mathbb{R}$ in this proof carry the topology induced from the Sorgenfrey line S. Let $\varphi(x) = i^{-1}(x)$ for all $x \in [-1, 0)$ and $\varphi(x) = j^{-1}(x)$ if $x \in [0, 1)$. It is easy to see that $\varphi : [-1, 1) \to T$ is a continuous onto map. The set $[-1, 1)$ is clopen in the Sorgenfrey line S so the map $\pi : S \to [-1, 1)$ defined by $\pi(x) = x$ if $x \in [-1, 1)$ and $\pi(x) = 0$ for all $x \in S \setminus [-1, 1)$, is continuous. It is clear that $\varphi \circ \pi$ maps S continuously onto T so Fact 1 is proved.

Returning to our solution, fix a continuous onto map $p : S \to T$. The map $p^* : C_p(T) \to C_p(S)$ defined by $p^*(f) = f \circ p$ for all $f \in C_p(T)$, is an embedding (Problem 163) so $C_p(T)$ embeds in $C_p(S)$.

To see that $C_p(S)$ does not embed in $C_p(T)$ observe that $t(C_p(T)) \leqslant \omega$ because T is compact (Problem 149). If $C_p(S)$ is homeomorphic to a subspace of $C_p(T)$ then $t(C_p(S)) \leqslant t(C_p(T)) = \omega$ (Problem 159) and hence $t(C_p(S)) = \omega$. Applying Problem 149 again we convince ourselves that $l(S \times S) \leqslant \omega$ which is a contradiction with Problem 165. Therefore $C_p(S)$ does not embed in $C_p(T)$ so our solution is complete.

S.387. *Let M_0 be the one-point compactification of the Mrowka space M. Prove that M_0 is a sequential compact space which is not a Fréchet–Urysohn space.*

Solution. Let us first prove the following simple fact that might be needed for further references.

Fact 1. The one-point compactification of any locally compact space is a compact Hausdorff (and hence Tychonoff) space.

Proof. Let X be a locally compact space; then $\alpha(X) = X \cup \{a\}$ where $a \notin X$. It is clear that any distinct $x, y \in X$ can be separated by open sets in X and hence in $\alpha(X)$. Take any $x \in X$; since X is locally compact, there is $U \in \tau(x, X)$ such that $K = \mathrm{cl}_X(U)$ is compact. Therefore $V = \{a\} \cup (X \setminus K) \in \tau(a, \alpha(X))$, $U \in \tau(x, \alpha(X))$ and $U \cap V = \emptyset$ so $\alpha(X)$ is a Hausdorff space.

To see that $\alpha(X)$ is compact, take any open cover $\mathcal{U}$ of the space $\alpha(X)$. Pick any $U \in \mathcal{U}$ with $a \in U$; then $K = X \setminus U$ is compact and hence there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that $K \subset \bigcup \mathcal{U}'$. As a consequence, the family $\mathcal{U}' \cup \{U\}$ is a finite subcover of $\alpha(X)$ and hence $\alpha(X)$ is compact. Now apply Problem 124 to conclude that $\alpha(X)$ is a normal and hence Tychonoff space. Fact 1 is proved.

The Mrowka space M is locally compact by Problem 142 so the Alexandroff compactification of M makes sense. It follows from Fact 1 that M_0 is a compact space. We have $M_0 = \mathcal{M} \cup \omega \cup \{a\}$ where $a \notin M = \mathcal{M} \cup \omega$ and the space M is Fréchet–Urysohn (Problem 142). Take any non-closed $A \subset M_0$ and choose any $x \in \overline{A} \setminus A$. Since all points of ω are isolated, we must have $x \in \mathcal{M}$ or $x = a$. If $x \in \mathcal{M}$ then x is in the closure of $A \setminus \{a\}$ in the space M. Since M is Fréchet–Urysohn, there is a sequence $\{A_n : n \in \omega\} \subset A$ with $A_n \to x \notin A$.

Next, observe that $\mathcal{M} \setminus U$ is finite for any $U \in \tau(a, M_0)$ because $\mathcal{M} \setminus U$ is closed and discrete in a compact space $M_0 \setminus U$ (see Problem 142). As a consequence, any faithfully indexed sequence $\{x_n : n \in \omega\} \subset \mathcal{M}$ converges to a.

Now assume that $x = a$; if $A \cap \mathcal{M}$ is infinite, then, by the previous remark, any faithfully indexed sequence $\{x_n : n \in \omega\} \subset A \cap \mathcal{M}$ converges to $a \in M_0 \setminus A$. Therefore, the set $A' = A \cap \omega$ is infinite, $a \in \overline{A'}$ and $A'' = A \cap \mathcal{M}$ is finite. It is easy to see that there is $W \in \tau(A'', M)$ such that $K = \mathrm{cl}_M(W)$ is compact. The set $V = M_0 \setminus K$ is an open neighbourhood of a in the space M_0 and hence $a \in \overline{V \cap A'}$. This shows that $B = V \cap A'$ is an infinite set with $A'' \cap \mathrm{cl}_M(B) = \emptyset$. Since only the points from A'' can be accumulation points of B, we conclude that the infinite set B has no accumulation points in the space M which implies that the family $\{\{y\} : y \in B\} \subset \tau(M)$ is discrete which is a contradiction with pseudocompactness of M (see Problem 142). We proved that in all possible cases there exists a sequence

$\{x_n : n \in \omega\} \subset A$ which converges to some point outside of A. Hence M_0 is a sequential space.

To see that the space M_0 is not Fréchet–Urysohn, let $A = \omega$ and observe that $a \in \overline{A}$. If $\{A_n : n \in \omega\} \subset A$ is a sequence with $a_n \to a$ then the family $\{\{A_n\} : n \in \omega\}$ is infinite, discrete in M and consists of non-empty open subsets of M which is a contradiction with pseudocompactness of M (Problem 142). Thus M_0 is not a Fréchet–Urysohn space so our solution is complete.

S.388. *Let M_0 be the one-point compactification of the Mrowka space M. Prove that $M_0 \backslash \omega$ is homeomorphic to the Alexandroff compactification of a discrete space.*

Solution. We have $M_0 = \mathcal{M} \cup \omega \cup \{a\}$, where $a \notin M = \mathcal{M} \cup \omega$ and M is the Mrowka space (Problem 142). Observe first that the space $M' = M_0 \setminus \omega$ is closed in M_0 and hence compact because M_0 is compact (Problem 387).

If D is a discrete space then D is homeomorphic to the cardinal $\kappa = |D|$ taken with the discrete topology. It is immediate that $\alpha(D)$ is homeomorphic to $A(\kappa)$ so it suffices to prove that $M' = \mathcal{M} \cup \{a\}$ is homeomorphic to $A(|\mathcal{M}|)$. It is easy to see that it is sufficient to show that, for any $U \in \tau(a, M')$, the set $M' \setminus U$ is finite.

To do this, take any $V \in \tau(M_0)$ for which $V \cap M' = U$; then $V \in \tau(a, M_0)$ and hence the set $M \setminus V$ is a compact subspace of M. The set $\mathcal{M}$ is closed and discrete in M (Problem 142) and hence $\mathcal{M} \cap (M \setminus V)$ is closed and discrete in $M \setminus V$. But the set $M \setminus V$ is compact so $M' \setminus U = \mathcal{M} \cap (M \setminus V)$ is finite, so our solution is complete.

S.389. *Let M_0 be the one-point compactification of the Mrowka space M. Prove that, for every second countable space Z and any continuous $f : M_0 \to Z$, the set $f(M_0)$ is countable.*

Solution. We have $M_0 = \mathcal{M} \cup \omega \cup \{a\}$ where $a \notin M = \mathcal{M} \cup \omega$ and M is the Mrowka space (Problem 142).

Fact 1. Let κ be any infinite cardinal. Then, for any second countable space X and any continuous map $g : A(\kappa) \to X$, the set $g(A(\kappa))$ is countable.

Proof. Let $g(b) = r$ and fix a countable local base $\{O_n : n \in \mathbb{N}\}$ at the point r in the space X (here b is the unique non-isolated point of $A(\kappa)$). For each $n \in \mathbb{N}$ there is a finite set $A_n \subset \kappa$ such that $g(A(\kappa) \backslash A_n) \subset O_n$. The set $A = \bigcup\{A_n : n \in \mathbb{N}\}$ is countable; if $x \in A(\kappa) \setminus A$ then $x \in A(\kappa) \setminus A_n$ and hence $g(x) \in O_n$ for all $n \in \mathbb{N}$. This shows that $g(x) \in \bigcap\{O_n : n \in \mathbb{N}\} = \{r\}$ for any $x \in A(\kappa) \backslash A$ and hence the set $g(A(\kappa)) = \{r\} \cup g(A)$ is countable so Fact 1 is proved.

Returning to our solution, observe that $M_0 \backslash \omega$ is homeomorphic to $A(\kappa)$ for $\kappa = |M_0|$ (388); hence we can apply Fact 1 to conclude that $f(M_0 \backslash \omega)$ is a countable subset of Z. Since ω is countable, the set $f(M_0) = f(M_0 \backslash \omega) \cup f(\omega)$ is also countable so our solution is complete.

S.390. *Let M_0 be the one-point compactification of the Mrowka space M. Prove that $C_p(M_0)$ is not Lindelöf.*

Solution. Given a set Z, a finite $K \subset Z$, a function $f \in \mathbb{R}^Z$ and $\varepsilon > 0$, we let $W_Z(f, K, \varepsilon) = \{g \in \mathbb{R}^Z : |g(x) - f(x)| < \varepsilon$ for all $x \in K\}$. It is clear that the family $\{W_Z(f, K, \varepsilon) : K$ is a finite subset of Z and $\varepsilon > 0\}$ is a local base at f in $\mathbb{R}^Z$.

Now, if X is a space and we are given a finite $K \subset Z$, a function $f \in C_p(X)$ and $\varepsilon > 0$, then we let $O_X(f, K, \varepsilon) = \{g \in C_p(X) : |g(x) - f(x)| < \varepsilon$ for all $x \in K\}$. It is clear that the family $\{O_X(f, K, \varepsilon) : K$ is a finite subset of X and $\varepsilon > 0\}$ is a local base at f in $C_p(X)$.

We have $M_0 = \mathcal{M} \cup \omega \cup \{a\}$ where $a \notin M = \mathcal{M} \cup \omega$ and M is the Mrowka space (Problem 142). We will write $\mathbb{D}$ to denote the discrete space $\{0, 1\}$. Call a space X *zero-dimensional* if it has a base of clopen sets. The expression $X \simeq Y$ says that the spaces X and Y are homeomorphic. If X is a set (or a space) and $A \subset X$ then let $\chi_A(x) = 1$ for all $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. The function χ_A is called *the characteristic function of the set A in the set\space X*.

Fact 1. If a space X is zero-dimensional then $C_p(X, \mathbb{D})$ is dense in $\mathbb{D}^X$.

Proof. Take any $f \in \mathbb{D}^X$ and any finite $K = \{x_1, \ldots, x_n\} \subset X$; since X has a base of clopen sets, there exist disjoint clopen sets $U_1, \ldots, U_n$ such that $x_i \in U_i$ for all $i \leqslant n$. Let $U_{n+1} = X \setminus \bigcup_{i \leqslant n} U_i$; then $\{U_1, \ldots, U_n, U_{n+1}\}$ is a clopen partition of the space X. Given any $i \leqslant n$, let $g(x) = f(x_i)$ for all $x \in U_i$; if $U_{n+1} \neq \emptyset$, then take any $y \in U_{n+1}$ and let $g(x) = f(y)$ for all $x \in U_{n+1}$. It is evident that $g : X \to \mathbb{D}$ and $g|K = f|K$. The function g is continuous because we must only check that $g^{-1}(0)$ and $g^{-1}(1)$ are open in X and this is indeed true because each U_i is open in X and therefore $g^{-1}(0) = \bigcup\{U_i : f(x_i) = 0\}$ and $g^{-1}(1) = \bigcup\{U_i : f(x_i) = 1\}$ are also open in X.

As a consequence, for any finite $K \subset X$ and any $\varepsilon > 0$, we have found a function $g \in C_p(X, \mathbb{D})$ such that $f|K = g|K$ and hence $|f(x) - g(x)| = 0 < \varepsilon$ for all $x \in K$. This shows that $g \in W_X(f, K, \varepsilon)$ and hence f is in the closure of $C_p(X, \mathbb{D})$. The function $f \in \mathbb{D}^X$ has been taken arbitrarily so Fact 1 is proved.

Fact 2. The space M_0 is zero-dimensional.

Proof. It is evident that a space is zero-dimensional if and only if it has a local base of clopen sets at every of its points. If $x \in \omega$ then $\{\{x\}\}$ is a base at x which consists of clopen subsets. If $x \in \mathcal{M}$ then $x \setminus A$ is a clopen set in M_0 for any finite $A \subset M_0 \setminus \{x\}$ so x also has a local base of clopen sets (recall that any $x \in \mathcal{M}$ can also be considered to be the respective subset of ω). Finally, let $x = a$; for any $U \in \tau(a, M_0)$ the set $P = M_0 \setminus U$ is a compact subset of M. For each $y \in P$ take $U_y \in \tau(X, M)$ such that $\mathrm{cl}_M(U_y)$ is compact (see Problem 142). We proved that there exists a clopen V_y such that $y \in V_y \subset U_y$. The set V_y is closed in a compact set $\overline{U}_y$ so V_y is also compact. By compactness of P, we can choose $y_1, \ldots, y_k \in P$ such that $P \subset V = V_{y_1} \cup \ldots V_{y_k}$. The set V is compact and open in M; hence $W = M_0 \setminus V$ is a clopen neighbourhood of a such that $W \subset U$. As a consequence, the point $x = a$ also has a local base which consists of clopen subsets of M_0 so Fact 2 is proved.

Fact 3. The space $C_p(M_0, \mathbb{D})$ is not countably compact. An easy consequence is that the space $C_p(M_0, \mathbb{D}) \times \omega^\omega$ embeds in $C_p(M_0)$ as a closed subspace.

Proof. Since ω is dense in M_0, the space M_0 is separable; hence $C_p(M_0)$ condenses onto a second countable space (Problem 173). If $C_p(M_0, \mathbb{D})$ is countably compact then, being a subspace of $C_p(M_0)$, it also condenses onto a second countable space Y. Therefore $C_p(M_0, \mathbb{D})$ is homeomorphic to Y (Problem 140). Any second countable countably compact space is compact so both Y and $C_p(M_0, \mathbb{D})$ are compact. Facts 1 and 2 imply that $C_p(M_0, \mathbb{D})$ is dense in $\mathbb{D}^{M_0}$ so $C_p(M_0, \mathbb{D}) = \mathbb{D}^{M_0}$, i.e., every $f: M_0 \to \mathbb{D}$ is continuous. However, it is easy to see that this is false; take, for example, the function $f = \chi\{a\}$. Since a is not an isolated point of M_0, the function f is not continuous. This proves that $C_p(M_0, \mathbb{D})$ is not countably compact.

Let $S \subset C_p(M_0, \mathbb{D})$ be a countably infinite closed discrete subset of $C_p(M_0, \mathbb{D})$; then S^ω is a closed subset of $(C_p(M_0, \mathbb{D}))^\omega$ and hence $C_p(M_0, \mathbb{D}) \times S^\omega$ is a closed subset of $C_p(M_0, \mathbb{D}) \times (C_p(M_0, \mathbb{D}))^\omega \simeq (C_p(M_0, \mathbb{D}))^\omega$. Observe first that $\mathbb{D}^\omega$ can be considered a closed subset of $\mathbb{R}$ because it is homeomorphic to the Cantor set $\mathbb{K} \subset \mathbb{R}$ (Problem 128). An immediate consequence is that $C_p(M_0, \mathbb{D}^\omega)$ embeds in $C_p(M_0)$ as a closed subspace (Problem 090). Note also that $C_p(M_0, \mathbb{D}^\omega)$ is homeomorphic to $(C_p(M_0, \mathbb{D}))^\omega$ (Problem 112). Observe finally that $C_p(M_0, \mathbb{D}^\omega)$ embeds in $C_p(M_0)$ as a closed subspace and $C_p(M_0, \mathbb{D}) \times S^\omega$ embeds in $C_p(M_0, \mathbb{D}^\omega)$ as a closed subspace so $C_p(M_0, \mathbb{D}) \times S^\omega \simeq C_p(M_0, \mathbb{D}) \times \omega^\omega$ embeds as a closed subspace in $C_p(M_0)$ so Fact 3 is proved.

Fact 4. Let X and Y be Hausdorff (not necessarily Tychonoff) spaces. If $f: X \to Y$ is a continuous map then

(1) The set $G(f) = \{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$.
(2) If $\pi_X : X \times Y \to X$ is the natural projection then $\pi_X|G(f) : G(f) \to X$ is a homeomorphism. In particular $G(f)$ is a closed subspace of $X \times Y$ homeomorphic to X. The set $G(f)$ is called *the graph* of the function f.

Proof. Since π_X is a continuous map, the mapping $\varphi = \pi_X|G(f)$ is continuous. If $z_1 = (x_1, f(x_1))$ and $z_2 = (x_2, f(x_2))$ are distinct points of $G(f)$ then it is easy to see that $x_1 = \varphi(z_1) \neq x_2 = \varphi(z_2)$ so the map φ is a condensation.

To see that $\varphi^{-1} : X \to G(f)$ is also continuous take any $z_0 = (x_0, f(x_0)) \in G(f)$ and any $W \in \tau(z_0, X \times Y)$. There exist $U \in \tau(x_0, X)$ and $V \in \tau(f(x_0), Y)$ such that $U \times V \subset W$; by continuity of f there is $U_1 \in \tau(x_0, X)$ such that $U_1 \subset U$ and $f(U_1) \subset V$. We claim that $\varphi^{-1}(U_1) \subset W$. Indeed, take any $x \in U_1$; then $f(x) \in V$ and hence $\varphi^{-1}(x) = (x, f(x)) \in U_1 \times V \subset U \times V \subset W$. Thus, U_1 witnesses continuity of φ^{-1} at the point x_0 and hence φ is a homeomorphism.

To see that $G(f)$ is closed in $X \times Y$ take any $z_0 = (x_0, y_0) \in (X \times Y)\backslash G(f)$. Then $y_0 \neq f(x_0)$ and hence there are disjoint $W \in \tau(f(x_0), Y)$ and $V \in \tau(y_0, Y)$. Since f is continuous, there exists $U \in \tau(x_0, X)$ such that $f(U) \subset W$. The set $O = U \times V$ is an open neighbourhood of z_0 in the space $X \times Y$. If $z = (x, y) \in O \cap G(f)$ then $x \in U$, $y \in V$ and $y = f(x) \in V$ whence $y \in W \cap V = \emptyset$ which is a contradiction. Hence $O \cap G(f) = \emptyset$, i.e., each point $z_0 \in (X \times Y)\backslash G(f)$ has an open neighbourhood contained in $(X \times Y)\backslash G(f)$. Therefore, the set $(X \times Y)\backslash G(f)$ is open in $X \times Y$ and hence the graph of f is closed. Fact 4 is proved.

Fact 5. Let X be a zero-dimensional Lindelöf space. Then any open F_σ-subset of X is homeomorphic to a closed subspace of $X \times \omega$.

Proof. It is straightforward that U is a Lindelöf space. Since X is zero-dimensional, for each $x \in U$ there is a clopen $U_x \in \tau(x, X)$ such that $U_x \subset U$. The open cover $\{U_x : x \in U\}$ of the Lindelöf space U has a countable subcover and therefore U is a countable union of clopen subsets of X. Since every finite union and any difference of clopen sets is a clopen set, we can find a family $\mathcal{U} = \{U_n : n \in \omega\}$ of clopen subsets of X such that $U_n \subset U_{n+1}$ for all $n \in \omega$ and $\bigcup \mathcal{U} = U$. The set $W_{n+1} = U_{n+1} \setminus U_n$ is also clopen (maybe, empty) for all $n \in \omega$; if we let $W_0 = U_0$ then the family $\{W_n : n \in \omega\}$ is a partition of U which consists of clopen subsets of X. Given any $x \in U$, let $f(x) = n$ if $x \in W_n$; it is clear that f is defined consistently. Continuity of f follows from the fact that the inverse image of any subset of ω is open being a union of some clopen subsets of U. Apply Fact 4 to conclude that $G(f) \subset U \times \omega$ is homeomorphic to U. Since $U \times \omega \subset X \times \omega$, the set $G(f)$ is also a subset of $X \times \omega$ so it suffices to show that $G(f)$ is closed in $X \times \omega$.

Take any $z_0 = (x_0, n_0) \in (X \times \omega) \setminus G(f)$; we have two cases.

(1) If $x_0 \in U$ apply Fact 4 to observe that $G(f)$ is closed in $U \times \omega$ and hence $W = (U \times \omega) \setminus G(f)$ is an open neighbourhood of z_0 in $U \times \omega$. Since $U \times \omega$ is open in $X \times \omega$, the set W is also open in $X \times \omega$ so z_0 has an open neighbourhood in $X \times \omega$ which does not meet $G(f)$.
(2) If $x_0 \in X \setminus U$ then $W = X \setminus U_{n_0} \in \tau(x_0, X)$ and hence the set $W' = W \times \{n_0\}$ is an open neighbourhood of z_0 in $X \times \omega$. Now, if $z = (x, n) \in G(f) \cap W'$ then $n = n_0$ and $f(x) = n_0$ whence $x \in U_{n_0}$. However, also $x \in W = X \setminus U_{n_0}$ which is a contradiction.

Thus, every $z \in (X \times \omega) \setminus G(f)$ has an open neighbourhood which does not meet $G(f)$ so $G(f)$ is a closed subset of $X \times \omega$ homeomorphic to U. Fact 5 is proved.

Returning to our solution, for each $x \in \mathcal{M}$ take a clopen $U_x \in \tau(x, M_0)$ such that $U_x \cap \mathcal{M} = \{x\}$; this is possible because M_0 is zero-dimensional (Fact 2) and $\mathcal{M}$ is discrete (Problem 142). Let $f_x = \chi U_x$ for all $x \in \mathcal{M}$; the set $F = \{f_x : x \in \mathcal{M}\} \subset C_p(X, \mathbb{D})$ is discrete because $O_{M_0}(f_x, \{x\}, 1) \cap F = \{f_x\}$ for every $x \in \mathcal{M}$. Consider the set $H = \{f \in C_p(M_0, \mathbb{D}) : f(\mathcal{M} \cup \{a\}) = \{0\}\}$; then $F \cup H$ is closed in $C_p(M_0, \mathbb{D})$. To see this, take any $f \in C_p(M_0, \mathbb{D}) \setminus (F \cup H)$. Since $\mathcal{M}$ is dense in $M' = \mathcal{M} \cup \{a\}$, there is $x \in \mathcal{M}$ such that $f(x) \neq 0$ and hence $f(x) = 1$; then $V = O_{M_0}(f, \{x\}, 1) \cap (F \cup H)$ contains at most the function f_x so $V \setminus \{f_x\}$ is an open neighbourhood of f which does not meet $F \cup H$.

Consider the set $E = \{f \in \mathbb{D}^\omega : f^{-1}(0) \supset U \cap \omega$ for some $U \in \tau(M', M_0)\}$. Since $M_0 \setminus M' = \omega$ is a countable set, M' is a G_δ-set in M_0 and hence $\chi(M', M_0) \leqslant \omega$ (Problem 327). Fix an external base $\mathcal{B} = \{U_n : n \in \omega\}$ of the set M' in M_0. The set $E_n = \{f \in \mathbb{D}^\omega : f(U_n \cap \omega) = \{0\}\}$ is closed in $\mathbb{D}^\omega$ for each $n \in \omega$. Of course, $E_n \subset E$ for all $n \in \omega$; if $f \in E$ then there is $W \in \tau(M', M_0)$ such that $f(W \cap \omega) = \{0\}$. There is $n \in \omega$ such that $U_n \subset W$ and hence $f(U_n \cap \omega) \subset f(W \cap \omega) = \{0\}$ which shows that we have $f \in E_n$. As a consequence, $E = \bigcup\{E_n : n \in \omega\}$ and E is an F_σ-set in the space $\mathbb{D}^\omega$; therefore $E' = \mathbb{D}^\omega \setminus E$ is a G_δ set in $\mathbb{D}^\omega$.

Let $\pi : F \cup H \to \mathbb{D}^\omega$ be the restriction map, i.e., $\pi(f) = f|\omega$ for every $f \in F \cup H$. The graph $G(\pi)$ of the map π is homeomorphic to $F \cup H$ and closed in the space $(F \cup H) \times \mathbb{D}^\omega$ by Fact 4. Therefore, the set $Q = G(\pi) \cap ((F \cup H) \times E')$ is closed in $(F \cup H) \times E'$.

Observe that, for any $f \in H$, we have $f^{-1}(0) \in \tau(M', M_0)$ and hence $\pi(f) \in E$. On the other hand, if $f = f_x \in F$ for some $x \in \mathcal{M}$ then $\pi(f) \notin E$ for otherwise there is $W \in \tau(M', M_0)$ with $f(W \cap \omega) = \{0\}$; since f is continuous and $x \in \overline{W}$, we have $f_x(x) = 0$ which is a contradiction. Therefore $\pi(f) \in E'$ for any $f \in F$. This proves that $\pi(Q) = F$ and hence Q is an uncountable closed discrete subspace of $(F \cup H) \times E'$.

There exists a family $\{O_n : n \in \omega\} \subset \tau(\mathbb{D}^\omega)$ such that $E' = \bigcap_{n\in\omega} O_n$. Therefore the space E' embeds as a closed subspace into $\prod\{O_n : n \in \omega\}$ (Fact 7 of S.271). Each $O_n \in \tau(\mathbb{D}^\omega)$ is an F_σ-set so it embeds into $\mathbb{D}^\omega \times \omega$ as a closed subset (Fact 5). Thus $\prod\{O_n : n \in \omega\}$ embeds in $(\mathbb{D}^\omega \times \omega)^\omega \simeq \mathbb{D}^\omega \times \omega^\omega$ as a closed set. Since $\mathbb{D}$ is a closed subset of ω, the space $\mathbb{D}^\omega$ embeds as a closed subset in ω^ω; as a consequence, $\mathbb{D}^\omega \times \omega^\omega$ embeds as a closed subspace in $\omega^\omega \times \omega^\omega \simeq \omega^\omega$. This shows that E' embeds in ω^ω as a closed subspace and hence $(F \cup H) \times E'$ embeds as a closed subspace in $(F \cup H) \times \omega^\omega$. Since $F \cup H$ is closed in $C_p(M_0, \mathbb{D})$, the set $(F \cup H) \times E'$ embeds in $C_p(M_0, \mathbb{D}) \times \omega^\omega$ as a closed subspace. Since Q is an uncountable closed discrete subset of $(F \cup H) \times E'$, the space $C_p(M_0, \mathbb{D}) \times \omega^\omega$ also has an uncountable closed discrete subset. Finally, $C_p(M_0, \mathbb{D}) \times \omega^\omega$ embeds in $C_p(M_0)$ as a closed subspace (Fact 3) and hence $\mathrm{ext}(C_p(M_0)) > \omega$. Thus, the space $C_p(M_0)$ cannot even be normal (Problem 295) so our solution is complete.

S.391. *Let M_0 be the one-point compactification of the Mrowka space M. Prove that $C_p(M_0)$ does not have a dense σ-compact subspace.*

Solution. If $C_p(M_0)$ has a dense σ-compact subspace then there is a compact $K \subset C_p(M_0)$ which separates the points of M_0 (Fact 3 of S.312). Such compact spaces are called *Eberlein compact spaces* (this definition was first introduced before Fact 12 of S.351). It was proved in Fact 19 of S.351 that any pseudocompact subset of an Eberlein compact space is compact so the pseudocompact subset M of the space M_0 must be compact which is false (Problem 142). This contradiction shows that $C_p(M_0)$ has no dense σ-compact subspace.

It is not, in fact, necessary to use such a sophisticated result for our solution. We can also observe that the evaluation map $e_K : M_0 \to C_p(K)$ defined by the formula $e_K(x)(f) = f(x)$ for all $x \in M_0$ and $f \in K$, is an embedding (Fact 2 of S.351 but this is simple). Therefore M_0 embeds in $C_p(K)$; since ω is dense in M_0, we can apply Fact 9 of S.351 (it is also simple) to conclude that we have $w(M_0) = nw(M_0) = \omega$ (Fact 4 of S.307) which is false because M_0 is not even Fréchet–Urysohn (Problem 387).

S.392. *Let M_0 be the one-point compactification of the Mrowka space M. Prove that $C_p(M_0)$ is a Fréchet–Urysohn space.*

Solution. Since M_0 is compact, the space $C_p(M_0)$ has countable tightness (see Problem 149). Take any set $A \subset C_p(M_0)$ and any $f \in \overline{A}$. There exists a countable

$B \subset A$ such that $f \in \overline{B}$. Define a map $\varphi : M_0 \to \mathbb{R}^B$ by $\varphi(x)(f) = f(x)$ for all $x \in M_0$ and $f \in B$. It is immediate that φ is a continuous map; if $Z = \varphi(M_0) \subset \mathbb{R}^B$ then Z is second countable and hence countable by Problem 389. The map $\varphi : M_0 \to Z$ is closed and hence the dual map $\varphi^* : C_p(Z) \to C_p(M_0)$ defined by $\varphi^*(f) = f \circ \varphi$ for all $f \in C_p(Z)$, is a closed embedding (see Problem 163). It is also immediate that we have the inclusions $B \subset \overline{B} \subset \varphi^*(C_p(Z))$ (the last inclusion holds because $\varphi^*(C_p(Z))$ is closed in $C_p(M_0)$). Since the space $C_p(Z)$ is homeomorphic to $\varphi^*(C_p(Z))$, we have $w(\overline{B}) \leqslant w(\varphi^*(C_p(Z))) = \omega$ so the space $\overline{B}$ is second countable and hence Fréchet–Urysohn. Thus there is a sequence $\{f_n : n \in \omega\} \subset B \subset A$ which converges to f. Since we have taken arbitrarily a set $A \subset C_p(M_0)$ and $f \in \overline{A}$, we proved that $C_p(M_0)$ is a Fréchet–Urysohn space.

S.393. *Prove that $\mathbb{I}^X = \beta(C_p(X, \mathbb{I}))$ if and only if every countable subset of X is closed and C^*-embedded in X.*

Solution. Assume that every countable subset of X is closed and C^*-embedded in X. Therefore, every countable $A \subset X$ is discrete so every function is continuous on A. Given any $A \subset X$, let $\pi_A : C_p(X, \mathbb{I}) \to C_p(A, \mathbb{I})$ be the restriction map defined by $\pi_A(f) = f|A$ for any $f \in C_p(X, \mathbb{I})$. It is immediate that π_A is a restriction of the relevant natural projection $p_A : \mathbb{I}^X \to \mathbb{I}^A$. Observe that $\pi_A(C_p(X, \mathbb{I}))$ consists precisely of those functions from A to $\mathbb{I}$ which can be extended to a continuous function from X to $\mathbb{I}$. Since the set A is C^*-embedded in the space X, every function $f : A \to \mathbb{I}$ extends to a continuous bounded function $g_1 : X \to \mathbb{R}$. Let $\lambda(t) = -1$ if $t < -1$, $\lambda(t) = 1$ if $t > 1$ and $\lambda(t) = t$ for all $t \in [-1,1]$. Then $\lambda : \mathbb{R} \to \mathbb{I}$ is a continuous map and $\lambda(t) = t$ for all $t \in \mathbb{I}$. We have $g = \lambda \circ g_1 : X \to \mathbb{I}$ and $g(x) = \lambda(g_1(x)) = \lambda(f(x)) = f(x)$ for all $x \in A$ because $g_1|A = f$ and $f(x) \in \mathbb{I}$ for all $x \in A$. We proved that, for any countable $A \subset X$ and any $f : A \to \mathbb{I}$, there exists $g \in C(X, \mathbb{I})$ such that $g|A = f$. In other words $\pi_A(C_p(X, \mathbb{I})) = \mathbb{I}^A$.

A trivial consequence of Problem 034 is the fact that $C_p(X, \mathbb{I})$ is dense in $\mathbb{I}^X$. Given any continuous function $\varphi : C_p(X, \mathbb{I}) \to \mathbb{I}$ there exists a countable $A \subset X$ and a continuous map $h : \pi_A(C_p(X, \mathbb{I})) \to \mathbb{I}$ such that $h \circ \pi_A = \varphi$ (see Problem 299 which is applicable because $C_p(X, \mathbb{I})$ is dense in $\mathbb{I}^X$). We saw that $\pi_A(C_p(X, \mathbb{I})) = \mathbb{I}^A$ and therefore we have a continuous map $\Phi : \mathbb{I}^X \to \mathbb{I}$ defined by $\Phi = h \circ p_A$. If $f \in C_p(X, \mathbb{I})$ then $\Phi(f) = h(p_A(f)) = h(\pi_A(f)) = \varphi(f)$, i.e., $\Phi|C_p(X, \mathbb{I}) = \varphi$. The function φ has been chosen arbitrarily so we can apply Fact 1 of S.309 to conclude that $\mathbb{I}^X = \beta(C_p(X, \mathbb{I}))$; this settles sufficiency.

Fact 1. If Z is any space and $Z \subset Y \subset \beta Z$ then $\beta Y = \beta Z$.

Proof. It is clear that βZ is a compact extension of Y. Given any compact space K and a continuous map $f : Y \to K$, the map $g = f|Z : Z \to K$ is continuous and hence there is a continuous map $h : \beta Z \to K$ such that $h|Z = g = f|Z$. The map $h|Y$ is continuous and coincides with f on a dense set Z; therefore $h|Y = f$ (Fact 0 of S.351). Now apply Problem 258 to conclude that $\beta Y = \beta Z$. Fact 1 is proved.

Assume that $\mathbb{I}^X = \beta(C_p(X, \mathbb{I}))$; fix any countable $A \subset X$ and any function $f : A \to \mathbb{I}$. Suppose first that $f \notin \pi_A(C_p(X, \mathbb{I}))$; it is easy to see that $\pi_A(C_p(X, \mathbb{I}))$ is dense in the space $\mathbb{I}^A$. The space $\mathbb{I}^A$ is second countable so there is a faithfully enumerated sequence $\{f_n : n \in \omega\} \subset \pi_A(C_p(X, \mathbb{I}))$ with $f_n \to f$. Given any $h \in \mathbb{I}^A$, let $i(h)(x) = g(x)$ for all $x \in A$; if $x \in X \setminus A$ then $i(h)(x) = 0$. It is clear that $i(h) \in \mathbb{I}^X$ for any $h \in \mathbb{I}^A$. Let $g = i(f)$ and $g_n = i(f_n)$ for all $n \in \omega$. It is straightforward that the sequence $\{g_n : n \in \omega\} \subset \mathbb{I}^X$ is faithfully enumerated and converges to g.

Since $f \notin \pi_A(C_p(X, \mathbb{I}))$, we have $C_p(X, \mathbb{I}) \subset Y = \mathbb{I}^X \setminus p_A^{-1}(f)$. Applying Fact 1, we can see that $\beta Y = \mathbb{I}^X$. The point f is a G_δ-set in $\mathbb{I}^A$ and therefore $p_A^{-1}(f)$ is a G_δ-set in $\mathbb{I}^X$. As a consequence, Y is an F_σ-set in the compact space $\mathbb{I}^X$; thus Y is σ-compact and hence normal. The sequences $P = \{g_{2n} : n \in \omega\} \subset Y$ and $S = \{g_{2n+1} : n \in \omega\} \subset Y$ are disjoint and converge to the same point $g \in \mathbb{I}^X \setminus Y$. The set $P' = P \cup \{g\}$ is compact and $P' \cap Y = P$ which implies that P is closed in Y. Analogously, the set S is also closed in Y. The space Y being normal, there exists a continuous function $r : Y \to [0, 1]$ such that $r(S) = \{0\}$ and $r(P) = \{1\}$. There exists a continuous function $R : \mathbb{I}^X = \beta Y \to [0, 1]$ such that $R|Y = r$. Since $g \in \overline{P}$ (the bar denotes the closure in $\mathbb{I}^X$) and $R(P) = r(P) = \{1\}$, we have $R(g) = 1$ by continuity of R. However, $g \in \overline{S}$ and $R(S) = r(S) = \{0\}$ whence $R(g) = 0$ which is a contradiction.

This contradiction proves that $\pi_A(C_p(X, \mathbb{I})) = \mathbb{I}^A$ for any countable $A \subset X$. In other words, every $f : A \to \mathbb{I}$ extends to a continuous function $F : X \to \mathbb{I}$. In particular, every function $f : A \to \mathbb{I}$ is continuous on A and hence any countable $A \subset X$ is discrete. If $x \in \overline{A} \setminus A$ then the set $A' = A \cup \{x\}$ is also countable and hence discrete which contradicts the fact that A is not closed in A'. This shows that there are no points in $\overline{A} \setminus A$ and hence every countable A is closed in X.

To see that A is C^*-embedded, take any bounded $f : A \to \mathbb{R}$ there is $n \in \mathbb{N}$ such that $f(A) \subset [-n, n]$. Observe that the function $r(t) = \frac{1}{n}t$ is a homeomorphism between $[-n, n]$ and $\mathbb{I}$ with the inverse function $s(t) = nt$. Let $h = r \circ f$; then h is a function on A and $h : A \to \mathbb{I}$. There exists $H \in C_p(X, \mathbb{I})$ such that $H|A = h$; consider the function $F = s \circ H$. Then $F \in C^*(X)$ and $F(x) = s(H(x)) = s(h(x)) = s(r(f(x))) = f(x)$ for all $x \in A$ which shows that A is C^*-embedded in X so our solution is complete.

S.394. *Prove that $C_p(X)$ has a dense σ-compact subspace if and only if $C_p(X, \mathbb{I})$ has a dense σ-compact subspace.*

Solution. Assume that $C_p(X)$ has a dense σ-compact subspace. Since $C_p(X)$ is homeomorphic to $C_p(X, (-1, 1))$ (Fact 1 of S.295), the space $C_p(X, (-1, 1))$ also has a dense σ-compact subspace P. We have $C_p(X, (-1, 1)) \subset C_p(X, \mathbb{I})$; it is an easy consequence of Problem 034 that $C_p(X, (-1,1))$ is dense in $C_p(X, \mathbb{I})$ so P is also dense in $C_p(X, \mathbb{I})$ which settles necessity.

Now assume that Q is a dense σ-compact subspace of $C_p(X, \mathbb{I})$. The mapping $\varphi_n : C_p(X, \mathbb{I}) \to C_p(X, [-n, n])$ defined by $\varphi_n(f) = n \cdot f$, is continuous and onto (see Problem 091) so $P_n = \varphi_n(Q) \subset C_p(X, [-n, n])$ is a dense σ-compact subspace of $C_p(X, [-n, n])$. The set $P = \bigcup \{P_n : n \in \mathbb{N}\}$ is σ-compact and dense in the space $C_p^*(X) = \bigcup \{C_p(X, [-n, n]) : n \in \mathbb{N}\}$ which in turn is dense in $C_p(X)$ (Fact 3 of S.310). Therefore P is a dense σ-compact subset of $C_p(X)$.

S.395. *Prove that $C_p(X)$ has a dense Lindelöf subspace if and only if $C_p(X, \mathbb{I})$ has a dense Lindelöf subspace.*

Solution. Assume that $C_p(X)$ has a dense Lindelöf subspace. Since $C_p(X)$ is homeomorphic to $C_p(X, (-1, 1))$ (Fact 1 of S.295), the space $C_p(X, (-1, 1))$ also has a dense Lindelöf subspace P. We have $C_p(X,(-1, 1)) \subset C_p(X, \mathbb{I})$; it is an easy consequence of Problem 034 that $C_p(X, (-1,1))$ is dense in $C_p(X, \mathbb{I})$ so P is also dense in $C_p(X, \mathbb{I})$ which settles necessity.

Now assume that Q is a dense Lindelöf subspace of $C_p(X, \mathbb{I})$. The mapping $\varphi_n : C_p(X, \mathbb{I}) \to C_p(X, [-n, n])$ defined by $\varphi_n(f) = n \cdot f$, is continuous and onto (see Problem 091) so $P_n = \varphi_n(Q) \subset C_p(X, [-n, n])$ is a dense Lindelöf subspace of $C_p(X, [-n, n])$. The set $P = \bigcup\{P_n : n \in \mathbb{N}\}$ is Lindelöf and dense in the space $C_p^*(X) = \bigcup\{C_p(X, [-n, n]) : n \in \mathbb{N}\}$ which in turn is dense in $C_p(X)$ (Fact 3 of S.310). Therefore P is a dense Lindelöf subset of $C_p(X)$.

S.396. *Suppose that $C_p(X, \mathbb{I})$ is σ-compact. Prove that X is discrete and hence the space $C_p(X, \mathbb{I})$ is compact.*

Solution. Recall that a space Z is called *a P-space* if every G_δ-subset of Z is open. It is easy to see that this is equivalent to saying that, for any $x \in Z$ and any sequence $\{F_n : n \in \omega\}$ of closed subsets of Z, if $x \notin \bigcup\{F_n : n \in \omega\}$ then $x \notin \overline{\bigcup\{F_n : n \in \omega\}}$.

Fact 1. If $C_p(Z, \mathbb{I})$ is σ-countably compact then Z is a P-space.

Proof. If not then there exists a point $x \in Z$ and closed sets $\{F_n : n \in \mathbb{N}\}$ such that $x \notin F_n$, $F_n \subset F_{n+1}$ for each $n \in \mathbb{N}$ and $x \in \overline{\bigcup\{F_n : n \in \mathbb{N}\}}$. The subspace $I_x = \{f \in C_p(Z, \mathbb{I}) : f(x) = 0\}$ is closed in $C_p(Z, \mathbb{I})$ and hence it is σ-countably compact. Let $I_x = \bigcup\{K_n : n \in \mathbb{N}\}$ where each K_n is countably compact. We claim that, for each $n \in \mathbb{N}$ and each $\varepsilon > 0$, there is $K_n \in \mathbb{N}$ such that for every function $f \in K_n$ there is $z \in F_{k_n}$ with $f(z) < \varepsilon$. If it were not true then, for each $i \in \mathbb{N}$, there is $f_i \in K_n$ such that $f_i(y) \geqslant \varepsilon$ for each $y \in f_i$. Since K_n is countably compact, the set $\{f_i : i \in \mathbb{N}\}$ has a accumulation point $f \in K_n$. If $y \in F = \bigcup\{F_i : i \in \mathbb{N}\}$ then $y \in f_m$ for some $m \in \mathbb{N}$ and hence $f_i(y) \geqslant \varepsilon$ for all $i \geqslant m$. It is immediate that $f(y) \geqslant \varepsilon$ as well. Thus, we have $f(y) \geqslant \varepsilon$ for all $y \in F$ while $f(x) = 0$ which contradicts continuity of f and the fact that $x \in \overline{F}$.

Therefore, we can fix a sequence $\{K_n : n \in \mathbb{N}\} \subset \mathbb{N}$ with the following properties:

(1) $K_{n+1} > K_n$ for each $n \in \mathbb{N}$.
(2) For each $f \in K_n$ there is $y \in F_{k_n}$ such that $f(y) < \frac{1}{2^n}$.

Apply the Tychonoff property of Z to choose a continuous $g_n : Z \to \left[0, \frac{1}{2^n}\right]$ such that $g_n(x) = 0$ and $g_n(F_{k_n}) = \left\{\frac{1}{2^n}\right\}$ for each $n \in \mathbb{N}$. The function $g = \Sigma_{n \in \mathbb{N}}\, g_n$ is a uniform limit of the sequence $\{g_1 + \cdots + g_n\}_{n \in \mathbb{N}}$ and hence $g \in C_p(Z)$; it is easy to see that $g \in C_p(Z, [0, 1]) \subset C_p(Z, \mathbb{I})$. It is evident that $g(x) = 0$ so $g \in I_x$. However, $g(y) \geqslant g_n(y) \geqslant \frac{1}{2^n}$ for each $y \in F_{k_n}$ whence $g \notin K_n$ for all $n \in \omega$. Therefore $g \in I_x \backslash (\bigcup\{K_n : n \in \omega\})$ which is a contradiction. Fact 1 is proved.

Returning to our solution, consider the map $e : X \to C_p(C_p(X, \mathbb{I}))$ defined by $e(x)(f) = f(x)$ for all $x \in X$ and $f \in C_p(X, \mathbb{I})$. Since for any $x \in X$ and any closed $F \subset X$ with $x \notin F$ there is $f \in C_p(X, \mathbb{I})$ such that $f(x) = 1$ and $f(F) = \{0\}$, we can apply Problem 166 to conclude that e embeds X in $C_p(C_p(X, \mathbb{I}))$. The space $C_p(X, \mathbb{I})$ being σ-compact, we have $t(X) \leqslant t(C_p(C_p(X, \mathbb{I}))) \leqslant \omega$ (Problem 149), i.e., $t(X) \leqslant \omega$.

Now, if $A \subset X$ and $x \in \overline{A} \setminus A$ then there is a countable $B \subset A$ such that $x \in \overline{B}$. It is evident that $X \setminus B$ is a G_δ-set; therefore $x \in X \setminus B \in \tau(x, X)$ because X is a P-space by Fact 1. Thus B is closed which is a contradiction with $x \in \overline{B} \setminus B$. This shows that every $A \subset X$ is closed and hence X is discrete. As a consequence, $C_p(X, \mathbb{I}) = \mathbb{I}^X$ is a compact space so our solution is complete.

S.397. *Prove that the following conditions are equivalent for any space* X:

(*i*) $C_p(X, \mathbb{I})$ *is countably compact.*
(*ii*) $C_p(X, \mathbb{I})$ *is* σ*-countably compact.*
(*iii*) *Every* G_δ*-subset of* X *is open in* X.

Solution. The implication (i) $\Rightarrow$ (ii) is obvious; the implication (ii) $\Rightarrow$ (iii) was proved in Fact 1 of S.396. The implication (iii) $\Rightarrow$ (i) was established in Fact 2 of S.310.

S.398. *Prove that the following conditions are equivalent for any space* X:

(*i*) $C_p(X, \mathbb{I})$ *is pseudocompact.*
(*ii*) $C_p(X, \mathbb{I})$ *is* σ*-pseudocompact.*
(*iii*) $C_p(X, \mathbb{I})$ *is* σ*-bounded.*
(*iv*) *Every countable subset* A *of* X *is closed and* C^**-embedded in* X.

Solution. Given any set $A \subset X$, denote by $p_A : \mathbb{I}^X \to \mathbb{I}^A$ the natural projection defined by $p_A(f) = f|A$ for any function $f \in \mathbb{I}^X$. The implication (i) $\Rightarrow$ (ii) is evident. Any pseudocompact subspace of any space is bounded in that space; therefore any σ-pseudocompact space is σ-bounded so (ii) $\Rightarrow$ (iii) holds. Observe also that $C_p(X, \mathbb{I})$ is dense in $\mathbb{I}^X$ so we can apply Fact 1 of S.286 to conclude that

($*$) $C_p(X, \mathbb{I})$ is pseudocompact if and only if $p_A(C_p(X, \mathbb{I})) = \mathbb{I}^A$ for any countable $A \subset X$, i.e., for every (not necessarily continuous) function $f : A \to \mathbb{I}$ there exists $g \in C_p(X, \mathbb{I})$ such that $g|A = f$.

Now assume that $C_p(X, \mathbb{I})$ is pseudocompact and take any countable $A \subset X$; it follows from ($*$) that every $f \in \mathbb{I}^A$ is a restriction to A of some $g \in C_p(X, \mathbb{I})$, so any $f \in \mathbb{I}^A$ is continuous on A, i.e., $C_p(A, \mathbb{I}) = \mathbb{I}^A$. This shows that every countable subset of X is discrete; if $A \subset X$ is countable and $x \in \overline{A} \setminus A$ then the set $A' = A \cup \{x\}$ is also countable and hence discrete which contradicts the fact that A is not closed in A'. This shows that there are no points in $\overline{A} \setminus A$ and hence every countable $A \subset X$ is closed in X.

To see that every countable $A \subset X$ is C^*-embedded, take any bounded function $f : A \to \mathbb{R}$; there is $n \in \mathbb{N}$ such that $f(A) \subset [-n, n]$. Observe that the function $r(t) = \frac{1}{n}t$ is a homeomorphism between $[-n, n]$ and $\mathbb{I}$ with the inverse function $s(t) = nt$.

Let $h = r \circ f$; then h is a function on A and $h : A \to \mathbb{I}$. It follows from (*) that there exists $H \in C_p(X, \mathbb{I})$ such that $H|A = h$; consider the function $F = s \circ H$. Then $F \in C^*(X)$ and $F(x) = s(H(x)) = s(h(x)) = s(r(f(x))) = f(x)$ for all $x \in A$ which shows that A is C^*-embedded in X, so we proved that (i) $\Rightarrow$ (iv).

To prove (iv) $\Rightarrow$ (i), assume that (iv) holds and take any countable $A \subset X$; we must first note that A is discrete. Indeed, if $B \subset A$ then B is also countable and hence closed by (iv). Since all subsets of A are closed (in X and hence in A), the set A is discrete and, in particular, all functions on A are continuous. Thus the condition (iv) says that any bounded $f \in \mathbb{R}^A$ can be extended to a continuous function on the whole X. To establish that (i) holds it suffices to prove $(*)$, i.e., to show that, for any $f \in \mathbb{I}^A$ there is $g \in C_p(X, \mathbb{I})$ such that $g|A = f$. We know that the function f extends to a continuous function $g_1 : X \to \mathbb{R}$. Let $\lambda(t) = -1$ if $t < -1$, $\lambda(t) = 1$ if $t > 1$ and $\lambda(t) = t$ for all $t \in [-1, 1]$. Then $\lambda : \mathbb{R} \to \mathbb{I}$ is a continuous map and $\lambda(t) = t$ for all $t \in \mathbb{I}$. We have $g = \lambda \circ g_1 : X \to \mathbb{I}$ and $g(x) = \lambda(g_1(x)) = \lambda(f(x)) = f(x)$ for all $x \in A$ because $g_1|A = f$ and $f(x) \in \mathbb{I}$ for all $x \in A$. This proves that $g|A = f$ and hence $(*)$ holds; therefore the implication (iv) $\Rightarrow$ (i) is settled.

At this point we have (iv) $\Leftrightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) so it suffices to show that (iii) $\Rightarrow$ (i). This implication is the most difficult one and requires a good insight into retractions, bounded sets and their properties. Given a space Z, a continuous map $r : Z \to Z$ is called *a retraction* if $r \circ r = r$. A subspace $R \subset Z$ is called *a retract of* Z if there is a retraction $r : Z \to Z$ such that $r(Z) = R$. Given a retraction $r : Z \to Z$, the set $R = r(Z)$ is closed in Z and $r(x) = x$ for any $x \in R$ (Fact 1 of S.351).

Fact 1. Let Z be an arbitrary space.

(1) If $r : Z \to Z$ is a retraction and $R = r(Z)$ then R is a retract of Y for any $Y \subset Z$ such that $R \subset Y$.
(2) If R is a retract of Z then R is C-embedded in Z.
(3) If $\varphi : Z \to Z$ is a homeomorphism and R is a retract of Z then $\varphi(R)$ is a retract of Z and $\varphi(R)$ is homeomorphic to R.

Proof. (1) If $r' = r|Y$ then $r' : Y \to Z$ and $r'(Y) = r(Y) \subset r(Z) = R \subset Y$, i.e., we can consider that $r' : Y \to Y$. Given any $y \in Y$, we have $r'(r'(y)) = r(r(y)) = r(y) = r'(y)$ which shows that $r' \circ r' = r'$ and hence r' is a retraction. Since $R \subset Y$, we have $r'(x) = r(x) = x$ for any $x \in R$ and therefore $R = r(Z) \supset r(Y) = r'(Y) \supset r'(R) = R$ whence $R = r'(Y)$.

(2) If $f \in C(R)$ and $r : Z \to R$ is a retraction, then $g = f \circ r \in C(Z)$ and $g|R = f$; indeed, if $x \in R$ then $r(x) = x$ and hence $g(x) = f(r(x)) = f(x)$ so R is C-embedded in Z.

(3) It is clear that $\varphi|R : R \to f(R)$ is a homeomorphism; if $r : Z \to R$ is a retraction then $r' = \varphi \circ r \circ \varphi^{-1} : Z \to Z$ is also a retraction; indeed, $r' \circ r' = \varphi \circ r \circ \varphi^{-1} \circ \varphi \circ r \circ \varphi^{-1} = \varphi \circ r \circ r \circ \varphi^{-1} = \varphi \circ r \circ \varphi^{-1} = r'$. Now, $r'(Z) = \varphi(r(\varphi^{-1}(Z))) = \varphi(r(Z)) = \varphi(R)$ so Fact 1 is proved.

Fact 2. Let Z be an arbitrary space

(1) If $A \subset Z$ is bounded in Z then $\overline{A}$ is also bounded in Z.
(2) If $A \subset Z$ is bounded in Z and $B \subset A$ then B is also bounded in Z.

(3) If A is C-embedded in Z and A is bounded in Z then A is pseudocompact.
(4) If R is a retract of Z and there is a bounded $A \subset Z$ with $R \subset A$ then R is pseudocompact.

Proof. (1) Take any function $f \in C(Z)$; the set $f(A)$ is bounded in $\mathbb{R}$ and hence $f(A) \subset [-n, n]$ for some $n \in \mathbb{N}$. Since $[-n, n]$ is compact, we have $f(\overline{A}) \subset \overline{f(A)} \subset [-n, n]$ (the first bar denotes the closure in Z and the second one, in $\mathbb{R}$) and therefore $f(\overline{A})$ is a bounded subset of $\mathbb{R}$. Thus $\overline{A}$ is bounded.

(2) This is evident because any subset of a bounded subset of $\mathbb{R}$ is a bounded subset of $\mathbb{R}$.

(3) If A is not pseudocompact then there is an unbounded $f \in C(A)$. Since A is C-embedded, there is $g \in C(Z)$ such that $g|A = f$. It is clear that g is not bounded on A so A is not bounded in Z which is a contradiction.

(4) Apply (2) to see that R is bounded in Z. The set R is C-embedded in Z by Fact 1 so it is pseudocompact by (3). Fact 2 is proved.

Fact 3. Given a space Z, $\varepsilon > 0$ and any $f \in C_p(Z)$, let $I(f, \varepsilon) = \{g \in C_p(Z) : |g(z) - f(z)| \leqslant \varepsilon$ for all $z \in Z\}$. Then $I(f, \varepsilon)$ is a retract of the space $C_p(Z)$ and $I(f, \varepsilon)$ is homeomorphic to $C_p(Z, \mathbb{I})$.

Proof. Denote by u the function which is identically zero on Z. It is evident that $C_p(Z, \mathbb{I}) = I(u, 1)$. It is easy to verify that the map $\varphi_\varepsilon : C_p(Z) \to C_p(Z)$ defined by $\varphi_\varepsilon(h) = \varepsilon \cdot h$ for all $h \in C_p(Z)$, is a homeomorphism. It is immediate that $\varphi_\varepsilon(C_p(Z, \mathbb{I})) = I(u, \varepsilon)$. Now define $T_f : C_p(Z) \to C_p(Z)$ by $T_f(h) = h + f$ for all $h \in C_p(Z)$. The mapping T_f is a homeomorphism by Problem 079 and it is straightforward that $T_f(I(u, \varepsilon)) = I(f, \varepsilon)$. The set $C_p(Z, \mathbb{I})$ is a retract of $C_p(Z)$ (Problem 092) and the map $\varphi = T_f \circ \varphi_\varepsilon$ is a homeomorphism of the space $C_p(Z)$ onto itself such that $\varphi(C_p(Z, \mathbb{I})) = I(f, \varepsilon)$. Now apply Fact 1 to conclude that $I(f, \varepsilon)$ is a retract of $C_p(Z)$ and $I(f, \varepsilon)$ is homeomorphic to $C_p(Z, \mathbb{I})$. Fact 3 is proved.

To finally prove (iii) $\Rightarrow$ (i) assume that $C_p(X, \mathbb{I}) = \bigcup\{B_n : n \in \omega\}$ where each B_n is bounded in $C_p(X, \mathbb{I})$. Then the set $\overline{B}_n$ is also bounded in $C_p(X, \mathbb{I})$ by Fact 2 and $C_p(X, \mathbb{I}) = \bigcup\{\overline{B}_n : n \in \omega\}$ so we can assume, without loss of generality, that each B_n is closed in $C_p(X, \mathbb{I})$. Given functions $f, g \in C(X, \mathbb{I})$, let $d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. Then d is a complete metric on $C(X, \mathbb{I})$ (Problem 248). We will denote the metric space $(C(X, \mathbb{I}), d)$ by $C_u(X, \mathbb{I})$. Given any $f \in C_u(X, \mathbb{I})$ and any $\varepsilon > 0$, let $B(f, \varepsilon) = \{g \in C_u(X, \mathbb{I}) : d(f, g) < \varepsilon\}$. It is evident that the topology of $C_u(X, \mathbb{I})$ contains the topology of $C_p(X, \mathbb{I})$ and therefore each B_n is closed in $C_u(X, \mathbb{I})$. The space $C_u(X, \mathbb{I})$ has the Baire property (see Problems 274 and 269); the set $B(u, \frac{1}{2})$ is open in $C_u(X, \mathbb{I})$ and hence it also has the Baire property (Problem 275). We have the inclusion $B(u, \frac{1}{2}) \subset \bigcup\{B_n : n \in \omega\}$ and hence all sets $B_n \cap B(u, \frac{1}{2})$ cannot be nowhere dense in $C_u(X, \mathbb{I})$; thus there is $n \in \omega$ such that $U \subset B_n \cap B(u, \frac{1}{2})$ for some open $U \subset C_u(X, \mathbb{I})$. By definition of the metric topology, there is $f \in U$ and $\varepsilon > 0$ such that $B(f, 2\varepsilon) \subset U$. It is straightforward that $I(f, \varepsilon) \subset B(f, 2\varepsilon) \subset U \subset B_n$. The set $I(f, \varepsilon)$ is a retract of the space $C_p(X)$ (Fact 3); since $I(f, \varepsilon) \subset C_p(X, \mathbb{I}) \subset C_p(X)$, the set $I(f, \varepsilon)$ is also a retract of $C_p(X, \mathbb{I})$ by Fact 1. Since $I(f, \varepsilon)$ is contained in a bounded set B_n, we can apply Fact 2 to conclude that $I(f, \varepsilon)$ is pseudocompact.

Since $C_p(X, \mathbb{I})$ is homeomorphic to $I(f, \varepsilon)$ (Fact 3), the space $C_p(X, \mathbb{I})$ is also pseudocompact. This finishes the proof of (iii) $\Rightarrow$ (i) so our solution is complete.

S.399. *Prove that the following conditions are equivalent for any space X*:

(*i*) *$C_p(X)$ is σ-pseudocompact.*
(*ii*) *$C_p(X)$ is σ-bounded.*
(*iii*) *The space X is pseudocompact and every countable subset of X is closed and C^*-embedded in X.*

Solution. Every σ-pseudocompact space is σ-bounded so (i) $\Rightarrow$ (ii).

Fact 1. Suppose that A is a bounded set in a space Z. Then, for any continuous map $f: Z \to Y$, the set $f(A)$ is bounded in the space Y. As a consequence, every continuous image of a σ-bounded space is a σ-bounded space.

Proof. If $r : Z \to \mathbb{R}$ is a continuous function such that $r(f(A))$ is not bounded in $\mathbb{R}$ then $g = r \circ f \in C(Z)$ and $g(A) = r(f(A))$ is not bounded in $\mathbb{R}$ which is a contradiction. Therefore $f(A)$ is bounded in Y.

Now, assume that $Z = \bigcup\{B_n : n \in \omega\}$ where each B_n is bounded in Z. If $f: Z \to Y$ is a continuous onto map then $Y = \bigcup\{f(B_n) : n \in \omega\}$ and each $f(B_n)$ is bounded in Y so Y is σ-bounded. Fact 1 is proved.

Fact 2. The space $\mathbb{R}^\omega$ is not σ-bounded.

Proof. Assume that $\mathbb{R}^\omega = \bigcup\{B_n : n \in \omega\}$ where each B_n is bounded in $\mathbb{R}^\omega$. We can assume, without loss of generality, that each B_n is closed in $\mathbb{R}^\omega$ (see Fact 2 of S.398). The space $\mathbb{R}^\omega$ is normal and hence each B_n is C-embedded in $\mathbb{R}^\omega$; apply Fact 2 of S.398 to conclude that each B_n is pseudocompact. The space $\mathbb{R}^\omega$ being second countable so is B_n for each $n \in \omega$; therefore B_n is compact for all $n \in \omega$. It turns out that $\mathbb{R}^\omega$ is σ-compact which is false (Fact 2 of S.186). The obtained contradiction proves Fact 2.

To prove (ii) $\Rightarrow$ (iii) assume that $C_p(X)$ is σ-bounded. Since $C_p(X, \mathbb{I})$ is a continuous image of $C_p(X)$ (Problem 092), the space $C_p(X, \mathbb{I})$ is also σ-bounded by Fact 1. Therefore, every countable $A \subset X$ is closed and C^*-embedded in X (Problem 398). If X is not pseudocompact then $C_p(X)$ maps continuously onto $\mathbb{R}^\omega$ (Fact 1 of S.186) so $\mathbb{R}^\omega$ has to be σ-bounded by Fact 1. However, Fact 2 says that $\mathbb{R}^\omega$ is not σ-bounded; this contradiction completes the proof of (ii) $\Rightarrow$ (iii).

Now, if (iii) holds then $C_p(X, \mathbb{I})$ is pseudocompact by Problem 398. Since $[-n, n]$ is homeomorphic to $\mathbb{I}$, the space $C_p(X,[-n, n])$ is homeomorphic to $C_p(X, \mathbb{I})$ for each $n \in \mathbb{N}$; as a consequence, $C_p(X,[-n, n])$ is pseudocompact for each $n \in \mathbb{N}$ so the space $C_p(X) = \bigcup\{C_p(X, [-n, n]) : n \in \mathbb{N}\}$ is σ-pseudocompact. This settles (iii) $\Rightarrow$ (i) and completes our solution.

S.400. *Prove that there exists a dense pseudocompact subspace X of the cube $\mathbb{I}^{\mathfrak{c}}$ such that every countable subspace of X is closed and C^*-embedded in X. Observe that $C_p(X)$ is σ-pseudocompact. Hence $C_p(X)$ can be σ-pseudocompact for an infinite space X.*

Solution. In Fact 4 of S.286 it was proved that there exists a dense pseudocompact subspace X of the space $\mathbb{I}^{\mathfrak{c}}$ such that $C_p(X, \mathbb{I})$ is pseudocompact. Applying Problem 398 we can conclude that every countable $A \subset X$ is closed and C^*-embedded in X. Since X is also pseudocompact, Problem 399 is applicable to convince ourselves that $C_p(X)$ is σ-pseudocompact. Of course, X has to be infinite to be dense in $\mathbb{I}^{\mathfrak{c}}$.

S.401. *Prove that the following conditions are equivalent*:

(*i*) *X is a realcompact space.*
(*ii*) *X embeds as a closed subspace into $\mathbb{R}^\kappa$ for some κ.*
(*iii*) *If X is a dense subspace of a space Y and $Y \neq X$ then there exists a continuous function $f : X \to \mathbb{R}$ which does not extend to Y continuously.*
(*iv*) *If $z \in \beta X \backslash X$ then there exists a G_δ-set H in βX such that $z \in H \subset \beta X \backslash X$.*

Solution. If X is realcompact then $i(X)$ is closed in $\mathbb{R}^{C(X)}$ which is homeomorphic to $\mathbb{R}^\kappa$ for $\kappa = |C(X)|$. Since i is an embedding, we proved that (i) $\Rightarrow$ (ii).

Assume that X is a closed subspace of $\mathbb{R}^\kappa$ for some cardinal κ. Let $p_\alpha : \mathbb{R}^\kappa \to \mathbb{R}$ be the natural projection to the αth factor for all $\alpha < \kappa$. Suppose that X is a dense subspace of a space Y; fix any $y \in Y \backslash X$ and observe that if we find $f \in C(X)$ which is not continuously extendable over $X \cup \{y\}$ then f is not continuously extendable over Y so, for the proof of (ii) $\Rightarrow$ (iii), we can assume that $Y = X \cup \{y\}$.

Suppose that each function $f_\alpha = p_\alpha|X$ is extendable to a continuous function $g_\alpha : Y \to \mathbb{R}$ on the space Y. The function $g = \Delta\{g_\alpha : \alpha < \kappa\}$ maps Y to $\mathbb{R}^\kappa$ in such a way that $g|X$ is the identity, i.e., $g(x) = x$ for each $x \in X$. Since X is dense in Y, we have $g(Y) \subset \overline{g(X)} = \overline{X} = X$. This means $g(y) = z$ for some $z \in X$. There are $U \in \tau(y, Y)$, $V \in \tau(z, Y)$ such that $U \cap V = \emptyset$. Since $V \in \tau(g(y), X)$, by continuity of g there is $W \in \tau(y, Y)$ such that $W \subset U$ and $g(W) \subset V$. The set X is dense in Y so we can take a point $x \in W \cap X$. Then $g(x) = x \in V$ and hence $x \in W \cap V \subset U \cap V = \emptyset$ which is a contradiction. This shows that one of the functions f_α is not extendable over Y so (ii) $\Rightarrow$ (iii) is proved.

Fact 1. Let X be a dense subspace of a space Y. If every function $f \in C(X)$ such that $f(x) \geqslant 1$ for all $x \in X$, can be extended continuously over Y then every $f \in C(X)$ can be extended continuously over Y.

Proof. Let u be the function with $u(x) = 0$ for all $x \in X$. Given any $f \in C(X)$, let $f_1 = \max(f, u) + 1$ and $f_0 = \max(-f, u) + 1$. It is straightforward that $f = f_1 - f_0$ and, for all $i = 1, 2$, we have $f_i(x) \geqslant 1$ for all $x \in X$. Take any $g_0, g_1 \in C(Y)$ such that $g_i|X = f_i$, $i = 1, 2$; then $g = g_1 - g_0 \in C(Y)$ and $g|X = f$ so Fact 1 is proved.

Returning to our solution fix any $z \in \beta X \backslash X$. By Fact 1 applied to $Y = X \cup \{z\}$, there exists $f \in C(X)$ such that $f(x) \geqslant 1$ for all $x \in X$ and f does not extend to Y. The function $g = \frac{1}{f}$ is continuous on X and bounded by 1 so there is $G \in C(\beta X)$ such that $G|X = g$. If $G(z) \neq 0$ then the function $\frac{1}{G|Y}$ is a continuous extension of the function f over the space Y, a contradiction. Thus, $G(z) = 0$ and $G(x) = \frac{1}{f(x)} \neq 0$ for all $x \in X$. Therefore $H = G^{-1}(0)$ is a G_δ-subset of βX such that $z \in H \subset \beta X \backslash X$ so (iii) $\Rightarrow$ (iv) is proved.

Fact 2. Let $i = \Delta C(X) : X \to \mathbb{R}^{C(X)}$; for the open interval $J = (0,1) \subset \mathbb{R}$ take any homeomorphism $\varphi : J \to \mathbb{R}$ and let $j = \Delta C(X, J) : X \to J^{C(X, J)}$. There exists a homeomorphism $\Phi : J^{C(X, J)} \to \mathbb{R}^{C(X)}$ such that $\Phi(j(x)) = i(x)$ for any $x \in X$. In particular, $\Phi(j(X)) = i(X)$ so the map j embeds X in $J^{C(X, J)}$ and $j(X)$ is closed in $J^{C(X, J)}$ if and only if $i(X)$ is closed in $\mathbb{R}^{C(X)}$.

Proof. Let $\Phi(v)(f) = \varphi(v(\varphi^{-1} \circ f))$ for any $v \in J^{C(X, J)}$ and $f \in C(X)$. This gives us a map $\Phi : J^{C(X, J)} \to \mathbb{R}^{C(X)}$. Given any $f \in C(X)$, let $p_f : \mathbb{R}^{C(X)} \to \mathbb{R}$ be the natural projection onto the fth factor. Analogously, if $g \in C(X, J)$ then $q_g : J^{C(X, J)} \to J$ is the natural projection onto the gth factor. Observe that we have the equalities $p_f \circ \Phi(v) = \varphi(v(\varphi^{-1} \circ f)) = \varphi(q_g(v))$ where $g = \varphi^{-1} \circ f$. Therefore $p_f \circ \Phi = \varphi \circ q_g$ is a continuous map for each $f \in C(X)$ and hence Φ is continuous (Problem 102). If $w \in \mathbb{R}^{C(X)}$ then let $v(g) = \varphi^{-1}(w(\varphi \circ g))$ for each $g \in C(X, J)$. Then $v \in J^{C(X, J)}$ and $\Phi(v) = w$ because $\Phi(v)(f) = \varphi(v(\varphi^{-1} \circ f)) = \varphi(\varphi^{-1}(w(\varphi \circ \varphi^{-1} \circ f))) = w(f)$ for each $f \in \mathbb{R}^{C(X)}$. Therefore the map Φ is surjective.

Now, if $v_1, v_2 \in J^{C(X, J)}$ and $v_1 \neq v_2$ then there is $g \in C(X,J)$ such that $v_1(g) \neq v_2(g)$. For the function $f = \varphi \circ g \in C(X)$, we have

$$\Phi(v_1(f) = \varphi(v_1(\varphi^{-1} \circ f)) = \varphi(v_1(g)) \neq \varphi(v_2(g)) = \Phi(v_2)(f)$$

which shows that $\Phi(v_1) \neq \Phi(v_2)$ and hence the map Φ is a bijection. To check continuity of Φ^{-1}, observe that $\Phi^{-1}(w)(g) = \varphi^{-1}(w(\varphi \circ g))$ for any $w \in \mathbb{R}^{C(X)}$ and any $g \in C(X, J)$. Indeed, letting $v = \Phi^{-1}(w)$ and $f = \varphi \circ g$, we obtain $\Phi(v) = w$ and therefore $w(f) = \varphi(v(g))$ whence $v(g) = \varphi^{-1}(w(f)) = \varphi^{-1}(w(\varphi \circ g))$ and this is what was promised.

Take any $g \in C(X,J)$; then, for any $w \in \mathbb{R}^{C(X)}$, we have $q_g \circ \Phi^{-1}(w) = \Phi^{-1}(w)(g) = \varphi^{-1}(w(\varphi \circ g)) = \varphi^{-1}(p_f(w))$, where $f = \varphi \circ g$. As a consequence, the function $q_g \circ \Phi^{-1} = \varphi^{-1} \circ p_f$ is continuous for every $g \in C(X, J)$ so the map Φ^{-1} is continuous (Problem 102). This shows that Φ is a homeomorphism.

The last thing we need to know about Φ is that $\Phi(j(X)) = i(X)$. Indeed, for any $x \in X$, we have $\Phi(j(x))(f) = \varphi(j(x)(\varphi^{-1} \circ f)) = \varphi(\varphi^{-1}(f(x)) = f(x) = i(x)(f)$ for any $f \in C(X)$ and therefore $\Phi(j(x)) = i(x)$ for every $x \in X$.

Finally, observe that the map j is an embedding because $j = \Phi^{-1} \circ i$; since any homeomorphism and its inverse are closed maps, the set $i(X) = \Phi(j(X))$ is closed in $\mathbb{R}^{C(X)}$ if and only if $j(X) = \Phi^{-1}(i(X))$ is closed in $J^{C(X, J)}$. Fact 2 is proved.

To establish (iv) $\Rightarrow$ (i) assume that (iv) holds; we must prove that $i(X)$ is closed in $\mathbb{R}^{C(X)}$. By Fact 2 this is equivalent to the set $j(X)$ being closed in $J^{C(X, J)}$. Every $g \in C(X, J)$ is a continuous bounded function from X to $[0, 1]$, so there exists a unique continuous function $s(g) : \beta X \to [0, 1]$ such that $s(g)|X = g$. The map

$$s = \Delta\{s(g) : g \in C(X,J)\} : \beta X \to [0,1]^{C(X,J)}$$

is continuous and $s|X = j$; considering that $J^{C(X,J)} \subset [0, 1]^{C(X, J)}$, let us check that $j(X) = s(\beta X) \cap J^{C(X, J)}$. The inclusion $j(X) \subset s(\beta X) \cap J^{C(X, J)}$ follows from $s|X = j$. Take any $z \in \beta X \backslash X$; by (iv) there is a G_δ-set H in βX such that $z \in H \subset \beta X \backslash X$. By Fact 2 of S.328 we can find a closed G_δ-set G in βX such that $x \in G \subset H$. by Fact 1 of S.358 there exists $f \in C(\beta X)$ such that $G = f^{-1}(0)$. Since βX is compact, there is $n \in \mathbb{N}$ with $f(\beta X) \subset [-n, n]$. If $h = \frac{1}{2n}|f|$ then $h \in C(\beta X, [0, \frac{1}{2}])$ and $h^{-1}(0) = G$.

The function $g = h|X$ maps X into $(0, \frac{1}{2}] \subset (0,1)$ because all zeros of h are in $\beta X \setminus X$; it is evident that $h = s(g)$ and $g \in C(X, J)$. Therefore $s(z)(g) = s(g)(z) = h(z) = 0 \notin J$ which shows that $s(z) \notin J^{C(X,\,J)}$. This proves that $j(X) = s(\beta X) \cap J^{C(X,\,J)}$ and hence $j(X)$ is closed in $J^{C(X,\,J)}$ because $s(\beta X)$ is compact and hence closed in $[0,1]^{C(X,\,J)}$. This settles (iv) $\Rightarrow$ (i) and makes our solution complete.

S.402. *Prove that an arbitrary product of realcompact spaces is a realcompact space.*

Solution. Assume that X_t is realcompact for all $t \in T$. Apply problem 401 to find a set A_t such that X_t is homeomorphic to a closed subset F_t of the space $\mathbb{R}^{A_t}$. We lose no generality if we assume that the space $A_t \cap A_s = \emptyset$ if $s, t \in T$, $s \neq t$. Then $F = \Pi\{F_t : t \in T\}$ is a closed subset of $\Pi\{\mathbb{R}^{A_t} : t \in T\}$ and this product is homeomorphic to $\mathbb{R}^A$ where $A = \bigcup\{A_t : t \in T\}$ (Problem 103). It is evident that F is homeomorphic to $X = \Pi\{X_t : t \in T\}$ and therefore X embeds as a closed subspace in $\mathbb{R}^\kappa$ for $\kappa = |A|$. Applying Problem 401 again we convince ourselves that X is realcompact.

S.403. *Prove that a closed subset of a realcompact space is a realcompact space.*

Solution. If X is realcompact then we can consider that X is a closed subset of $\mathbb{R}^\kappa$ for some cardinal κ (Problem 401). If F is a closed subset of X then F is also a closed subset of the same space $\mathbb{R}^\kappa$. Applying Problem 401 again we conclude that F is realcompact.

S.404. *Prove that an open subset of a realcompact space is not necessarily realcompact.*

Solution. The space ω_1 is an open subspace of $\omega_1 + 1$. Every $f \in C(\omega_1)$ is bounded because ω_1 is countably compact (Problem 314). Since $\omega_1 + 1 = \beta\omega_1$ (Problem 314), every continuous $f : \omega_1 \to \mathbb{R}$ extends to a continuous $g : (\omega_1 + 1) \to \mathbb{R}$ which shows that ω_1 is not realcompact (Problem 401).

S.405. *Let X be an arbitrary space. Suppose that X_t is a realcompact subspace of X for any $t \in T$. Prove that $\bigcap\{X_t : t \in T\}$ is a realcompact subspace of X.*

Solution. The space $Y = \bigcap\{X_t : t \in T\}$ is homeomorphic to a closed subspace of the product $\Pi\{X_t : t \in T\}$ (Fact 7 of S.271) so realcompactness of Y follows from Problems 402 and 403.

S.406. *Prove that any Lindelöf space is a realcompact space.*

Solution. Let X be a Lindelöf space. It suffices to prove that, for any $z \in \beta X \setminus X$, there is a G_δ-set H in βX such that $z \in H \subset \beta X \setminus X$ (Problem 401). For any $x \in X$ take $U_x \in \tau(x, \beta X)$ such that $z \notin \overline{U}_x$ (the bar denotes the closure in βX). The open cover $\{U_x : x \in X\}$ of the Lindelöf space X has a countable subcover; let $A \subset X$ be a countable set such that $X \subset \bigcup\{U_x : x \in A\}$. The set $H = \beta X \setminus (\bigcup\{\overline{U}_x : x \in A\})$ is a G_δ in βX and $z \in H \subset \beta X \setminus X$. This proves that X is realcompact.

S.407. *Prove that any pseudocompact realcompact space is compact.*

Solution. Let X be a pseudocompact realcompact space. We can consider that X is a closed subset of $\mathbb{R}^\kappa$ for some κ (Problem 401). We need the map $\pi_\alpha : \mathbb{R}^\kappa \to \mathbb{R}$ which is the natural projection onto the αth factor for all $\alpha < \kappa$. The set $X_\alpha = \pi_\alpha(X)$ is a pseudocompact subset of $\mathbb{R}$ for each α; thus X_α is compact. It is evident that we have $X \subset \Pi\{X_\alpha : \alpha < \kappa\}$; besides, X is a closed subset of the space $\mathbb{R}^\kappa$ which contains $\Pi\{X_\alpha : \alpha < \kappa\}$. Therefore X is also closed in the compact space $\Pi\{X_\alpha : \alpha < \kappa\}$ so X is compact.

S.408. *Let X be a realcompact space. Suppose that $Y \subset X$ can be represented as a union of G_δ-subsets of X. Prove that $X \backslash Y$ is realcompact. In particular, any F_σ-subspace of a realcompact space is realcompact.*

Solution. Call a set $U \subset X$ *functionally open in* X if there exists a function $f \in C(X)$ and $V \in \tau(\mathbb{R})$ such that $U = f^{-1}(V)$.

Fact 1. Let R be a realcompact space. Suppose that Z is an arbitrary space and $f : R \to Z$ is a continuous map. Then $f^{-1}(B)$ is realcompact for any realcompact $B \subset Z$.

Proof. The graph $G(f) = \{(y, f(y)) : y \in R\}$ of the mapping f is closed in the space $R \times Z$ (see Fact 4 of S.390). If $f_B = f|B : f^{-1}(B) \to B$ then, for the graph $G(f_B) = \{(y, f(y)) : y \in f^{-1}(B)\}$ of the function f_B we have $G(f_B) = G(f) \cap (R \times B)$. Since $G(f)$ is closed in $R \times Z$, the set $G(f_B)$ is closed in a realcompact space $R \times B$ (402) so $G(f_B)$ is realcompact. Applying Fact 4 of S.390 again we observe that $G(f_B)$ is homeomorphic to $f^{-1}(B)$ so $f^{-1}(B)$ is realcompact and Fact 1 is proved.

Fact 2. Any F_σ-subset P of any space Z is the intersection of functionally open subsets of Z.

Proof. It suffices to show that, for any $y \in Z \setminus P$ there exists a functionally open $U \subset Z$ such that $P \subset U$ and $y \notin U$. We have $P = \bigcup\{F_n : n \in \omega\}$, where each F_n is closed in Z. Since $y \notin F_n$, there exists $f_n \in C(Z, [0, 2^{-n}])$ such that $f(y) = 0$ and $f(F_n) \subset \{2^{-n}\}$ for all $n \in \omega$. If $g_n = f_0 + \cdots + f_n$ for all $n \in \omega$ then the sequence $\{g_n : n \in \omega\}$ converges uniformly to a function $g \in C(Z)$. It is evident that $g(z) \geqslant 0$ for any $z \in Z$; besides, $g(y) = 0$ and $g(z) \geqslant f_n(z) = 2^{-n} > 0$ for any $z \in F_n$, $n \in \omega$. Thus $g(y) \notin V = (0, +\infty)$ and therefore $y \notin U = g^{-1}(V)$. Since $g(y) > 0$ for all $y \in P$, we have $P \subset U$ so Fact 2 is proved.

Returning to our solution observe that any $V \subset \mathbb{R}$ is realcompact because V is second countable and hence Lindelöf (Problem 406). It follows from Fact 1 and this observation that any functionally open subset of X is realcompact. This, together with Fact 2, proves that any F_σ-subset of X is realcompact (see Problem 405). Since $X \setminus Y$ is the intersection of F_σ-subsets of X, we can apply Problem 405 again to conclude that $X \setminus Y$ is realcompact.

S.409. *Prove that $C_p(X)$ is a realcompact space if and only if it is a locally realcompact space.*

Solution. It is evident that any realcompact space is locally realcompact, so let us prove the converse for $C_p(X)$. Denote by u the function which is identically zero on X. Given any $g \in C_p(X)$, let $T_g(f) = f - g$ for all $f \in C_p(X)$. The map T_g is a homeomorphism (Problem 079). Suppose that $P \subset C_p(X)$ is realcompact and $\mathrm{Int}(P) \neq \emptyset$ (this is what we get from local realcompactness). Take any $g \in \mathrm{Int}(P)$; then $P' = T_g(P)$ is a realcompact subspace of $C_p(X)$ with $u \in \mathrm{Int}(P')$. Thus there are $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that $O(x_1, \ldots, x_n, \varepsilon) = \{f \in C_p(X) : |f(x_i)| < \varepsilon$ for all $i \leqslant n\} \subset P'$. Consequently, the space $F = \{f \in C_p(X) : f(x_i) = 0$ for all $i \leqslant n\}$ is realcompact being a closed subset of $C_p(X)$ and hence of P'.

Fact 1. Take an arbitrary space Z and $m \in \mathbb{N}$; denote by o_m the point of $\mathbb{R}^m$ with all coordinates equal to zero. Given a non-empty finite $K \subset Z$ consider the set $H_K = \{f \in C_p(Z, \mathbb{R}^m) : f(z) = o_m$ for any $z \in K\}$. Then the space $C_p(Z, \mathbb{R}^m)$ is linearly homeomorphic to $H_K \times (\mathbb{R}^m)^K$.

Proof. For every $z \in Z$ the natural projection $\pi_z : (\mathbb{R}^m)^Z \to \mathbb{R}^m$ of $(\mathbb{R}^m)^Z$ onto the factor determined by z is continuous; clearly, $\pi_z(f) = f(z)$ for any $f \in (\mathbb{R}^m)^Z$. We will also need the map $e_z = \pi_z|C_p(Z, \mathbb{R}^m)$ for every $z \in Z$. Let $K = \{z_1, \ldots, z_l\}$; fix $U_i \in \tau(z_i, Z)$, $i = 1, \ldots, l$ such that the family $\{U_1, \ldots, U_l\}$ is disjoint. There exists $u_i \in C(Z)$ such that $u_i(z_i) = 1$ and $u_i|(X \setminus U_i) \equiv 0$ for all $i \leqslant l$. Given a function $f \in C_p(Z, \mathbb{R}^m)$, let $r(f) = f(z_1) \cdot u_1 + \cdots + f(z_l) \cdot u_l$ and $\varphi(f) = (f - r(f), f|K)$. It is immediate that $r(f) : Z \to \mathbb{R}^m$ is a continuous map and $r(f)(z_i) = f(z_i)$ for all $i \leqslant m$ so $f - r(f) \in H_K$. As a consequence $\varphi(f) \in H_K \times (\mathbb{R}^m)^K$ for each $f \in C_p(Z, \mathbb{R}^m)$, i.e., $\varphi : C_p(Z, \mathbb{R}^m) \to H_K \times (\mathbb{R}^m)^K$.

Given any $i \leqslant m$, the map e_{zi} is linear and continuous. It is easy to deduce from this fact that the map $f \mapsto f(z_i) \cdot u_i = e_{zi}(f) \cdot u_i$ is also linear and continuous for any $i \leqslant m$. This shows that the map r is linear and continuous and hence so is the map $f \mapsto f - r(f)$; an immediate consequence is that the map φ is linear and continuous.

Now, if $g \in H_K$ and $h \in (\mathbb{R}^m)^K$ then letting $f = \delta(g, h) = g + \sum_{i=1}^{l} h(z_i) \cdot u_i$ we obtain a function $f \in C_p(Z, \mathbb{R}^m)$ such that $\varphi(f) = (g, h)$, i.e., the map φ is onto. Now, suppose that $f, f' \in C_p(Z, \mathbb{R}^m)$ and $f \neq f'$. If $f|K \neq f'|K$ then $\varphi(f) \neq \varphi(f')$ because the second coordinates of $\varphi(f)$ and $\varphi(f')$ are distinct. If $f|K = f'|K$ then $r(f) = r(f')$ and therefore $f - r(f) \neq f' - r(f')$ so again $\varphi(f) \neq \varphi(f')$.

Thus the map φ is a bijection and $\delta : H_K \times (\mathbb{R}^m)^K \to C_p(Z, \mathbb{R}^m)$ is the inverse of φ. To see that δ is continuous, observe that it maps $H_K \times (\mathbb{R}^m)^K$ into a product, namely $(\mathbb{R}^m)^Z$, so it suffices to verify that, for any $z \in Z$, the map $\delta_z = \pi_z \circ \delta$ is continuous. Note first that the map $(g, h) \to g \to g(z) = \pi_z(g)$ is continuous being the composition of two natural projections. Since $u_i(z)$ is a constant, every map $h \to h(z_i) \cdot u_i(z)$ is a natural projection multiplied by a constant; hence the map δ is continuous being the composition of arithmetical operations with natural projections. This shows that $\varphi : C_p(Z, \mathbb{R}^m) \to H_K \times (\mathbb{R}^m)^K$ is a linear homeomorphism so Fact 1 is proved.

Returning to our solution apply Fact 1 to $m = 1$ and the set $K = \{x_1, \ldots, x_n\}$; then $H_K = F$ and hence $C_p(X)$ is homeomorphic to the product $F \times \mathbb{R}^K$. We saw that the first factor is realcompact; the second one is also realcompact because it is second countable and hence Lindelöf (Problem 406). Now apply Problem 402 to conclude that $C_p(X)$ is realcompact.

S.410. *Prove that $C_p(X)$ is a realcompact space if and only if $C_p(X, \mathbb{I})$ is realcompact.*

Solution. If $C_p(X)$ is realcompact then $C_p(X, \mathbb{I})$ is also realcompact because it is a closed subspace $C_p(X)$ (Problem 403). Now, suppose that $C_p(X, \mathbb{I})$ is realcompact. Observe that $C_p(X)$ is homeomorphic to $C_p(X,(-1,1))$ (Fact 1 of S.295). Given any $x \in X$, let $e_x(f) = f(x)$ for all $f \in C_p(X)$. The map $e_x : C_p(X) \to \mathbb{R}$ is continuous (Problem 166) and therefore the set $U_x = \{f \in C_p(X, \mathbb{I}) : |f(x)| < 1\} = e_x^{-1}((-1,1)) \cap C_p(X, \mathbb{I})$ is an F_σ-set in $C_p(X, \mathbb{I})$; it follows from Problem 408 that U_x is realcompact for every $x \in X$. Finally, apply Problem 405 to conclude that $C_p(X,(-1, 1)) = \bigcap\{U_x : x \in X\}$ is realcompact and hence so is $C_p(X)$.

S.411. *Give an example of a locally realcompact non-realcompact space.*

Solution. The space ω_1 is even locally compact being an open subspace of a compact space $\omega_1 + 1$ (Problem 314). However, ω_1 is not realcompact because it is a countably compact non-compact space (see Problems 314 and 407).

S.412. *Let X be any space. Prove that, for any realcompact space Y and any continuous map $\varphi : X \to Y$, there exists a continuous map $\Phi : \upsilon X \to Y$ such that $\Phi|X = f$.*

Solution. By Problem 401, there is a set B such that Y embeds as a closed subspace in $\mathbb{R}^B$ and hence we can assume that $Y \subset \mathbb{R}^B$. For the set $A = C(X)$ we can identify X with the subset $\tilde{X} = \{\beta_x : x \in X\} \subset \mathbb{R}^A$, where $\beta_x(f) = f(x)$ for any $x \in X$ and $f \in A$ (Problem 167). By definition, υX is the closure of $\tilde{X}$ in the space $\mathbb{R}^A$, so we consider that $X \subset \upsilon X = \overline{X} \subset \mathbb{R}^A$. Given a coordinate $b \in B$, denote by $p_b : \mathbb{R}^B \to \mathbb{R}$ the natural projection onto the bth factor. Analogously, the map $q_f : \mathbb{R}^A \to \mathbb{R}$ is the natural projection to the fth factor. Observe that $q_f|X = f$ for any $f \in A = C(X)$.

For any $b \in B$, the map $p_b \circ \varphi$ belongs to $C(X) = A$ so fix $f_b \in A$ with $p_b \circ \varphi = f_b$. It is clear that $q_{fb}|X = f_b$ and therefore $q_{fb}|vX : \upsilon X \to \mathbb{R}$ is an extension of the map f_b to υX. For any $x \in \upsilon X$ let $\Phi(x)(b) = q_{fb}(x) \in \mathbb{R}$; this defines a point $\Phi(x) \in \mathbb{R}^B$ so we have a map $\Phi : \upsilon X \to \mathbb{R}^B$. We claim that the map Φ is continuous, $\Phi|X = \varphi$ and $\Phi(\upsilon X) \subset Y$, i.e., $\Phi : \upsilon X \to Y$ is a continuous extension of the map φ.

The map Φ is continuous because $p_b \circ \Phi = q_{fb}$ is continuous for any $b \in B$ (see Problem 102). If $x \in X$ then $\Phi(x)(b) = q_{fb}(x) = f_b(x) = p_b \circ \varphi\ (x) = \varphi\ (x)(b)$ for every $b \in B$; this shows that $\Phi|X = \varphi$. Finally, X is dense in υX implies that $\varphi\ (X)$ is dense in $\Phi(\upsilon X)$ so $\Phi(vX) \subset \overline{\varphi(X)} \subset \overline{Y} = Y$ (the closure is taken in $\mathbb{R}^B$ and the last equality holds because Y is closed in $\mathbb{R}^B$). We proved that the map $\Phi : \beta X \to Y$ is an extension of φ so our solution is complete.

S.413. *Let rX be a realcompact extension of a space X. Prove that the following properties are equivalent*:

(*i*) *For any realcompact space Y and any continuous map $f : X \to Y$, there exists a continuous map $F : rX \to Y$ such that $F|X = f$.*

(*ii*) *For any realcompact extension sX of the space X, there exists a continuous map $\pi : rX \to sX$ such that $\pi(x) = x$ for all $x \in X$.*

(*iii*) *There is a homeomorphism* $\varphi : rX \to \upsilon X$ *such that* $\varphi(x) = x$ *for any* $x \in X$, *i.e.,* rX *is canonically homeomorphic to* υX.

Solution. Take $Y = sX$ and $f : X \to Y$ defined by $f(x) = x$ for any $x \in X$. If $F : rX \to Y$ is the extension of f whose existence is guaranteed by (i), then $\pi = F$ satisfies (ii) so (i) $\Rightarrow$ (ii) is established.

(ii) $\Rightarrow$ (iii). Fix a continuous map $\pi : rX \to \upsilon X$ such that $\pi(x) = x$ for any $x \in X$. It suffices to prove that $\varphi = \pi$ is the required homeomorphism. The map $f : X \to rX$ defined by $f(x) = x$ for all $x \in X$, has a continuous extension $F : \upsilon X \to rX$ by Problem 412. Let $i : \upsilon X \to \upsilon X$ be the identity, i.e., $i(z) = z$ for all $z \in \upsilon X$. For the continuous maps $\pi \circ F : \upsilon X \to \upsilon X$ and $i : \upsilon X \to \upsilon X$, we have $i|X = (\pi \circ f)|X$ for a dense subset X of the space υX. This makes it possible to apply Fact 0 of S.351 to conclude that we have $\pi \circ F = i$.

Analogously, if $j : rX \to rX$ is the identity defined by $j(x) = x$ for all $x \in rX$ then the continuous maps $F \circ \pi$ and j coincide of a dense set X of the space rX. Applying Fact 0 of S.351 again, we can conclude that $F \circ \pi = j$. This shows that F and π are bijections and the map $\pi^{-1} = F$ is continuous, i.e., π is the promised homeomorphism.

(iii) $\Rightarrow$ (i). Let $f : X \to Y$ be a continuous map of X to a realcompact space Y. By Problem 412 there exists a continuous $F_1 : \upsilon X \to Y$ such that $F_1|X = f$. Then $F = F_1 \circ \varphi$ maps rX continuously into Y and if $x \in X$ then $F(x) = F_1(\varphi(x)) = F_1(x) = f(x)$ and therefore $F|X = f$.

S.414. *Let* X *be an arbitrary space and suppose that* $X \subset Y \subset \upsilon X$. *Prove that* υY *is canonically homeomorphic to* υX.

Solution. It is clear that υX is a realcompact extension of the space Y; take any continuous map $f : Y \to Z$ of Y to a realcompact space Z. The mapping $f_1 = f|X : X \to Z$ is also continuous so there exists a continuous map $F : \upsilon X \to Z$ such that $F|X = f_1$. We have two continuous maps $F_1 = F|Y : Y \to Z$ and $f : Y \to Z$ such that $F_1|X = F|X = f_1 = f|X$. As a consequence, $F_1 = f$ (Fact 0 of S.351), i.e., F is a continuous extension of the map f. Now apply Problem 413 to conclude that vY is canonically homeomorphic to υX.

S.415. *Prove that* Y *is a bounded subset of* X *if and only if* $\text{cl}_{\upsilon X}(Y)$ *is compact.*

Solution. Let Y be a bounded subset of X; denote the set $\text{cl}_{\upsilon X}(Y)$ by Z. Given any function $f : \upsilon X \to \mathbb{R}$ the function $f|X$ is bounded on Y and hence f is bounded on Y. Thus Y is bounded in υX and therefore Z is also bounded in υX (Fact 2 of S.398).

Fact 1. Any closed and bounded subset of a realcompact space is compact.

Proof. Let F be a closed and bounded subset of a realcompact space T. We can consider that T is a closed subset of $\mathbb{R}^\kappa$ for some κ (Problem 401). Then F is also a closed subspace of $\mathbb{R}^\kappa$. Let $\pi_\alpha : \mathbb{R}^\kappa \to \mathbb{R}$ be the natural projection onto the αth factor for all $\alpha < \kappa$. Since the function $\pi_\alpha|T$ is bounded on F, the function π_α is bounded on F for all $\alpha < \kappa$. This means $\pi_\alpha(f) \subset K_\alpha$ where K_α is a compact subset of $\mathbb{R}$ for each $\alpha < \kappa$. As a consequence, $F \subset K = \prod\{K_\alpha : \alpha < \kappa\}$. Since F is closed in $\mathbb{R}^\kappa$, it is also closed in a smaller space K which shows that F is compact. Fact 1 is proved.

Returning to our solution, observe that υX is realcompact and Z is closed and bounded in υX; thus Z is compact and we proved necessity.

Now, assume that $Z = \mathrm{cl}_{\upsilon X}(Y)$ is compact and take any continuous map $f : X \to \mathbb{R}$. There exists a continuous map $F : \upsilon X \to \mathbb{R}$ such that $F|X = f$ (Problem 412). Since Z is compact, the set $F(Z)$ is bounded in $\mathbb{R}$. Therefore $f(Y) = F(Y) \subset F(Z)$ is bounded in $\mathbb{R}$; this shows that Y is bounded in X so our solution is complete.

S.416. *Prove that X is σ-bounded if and only if υX is σ-compact.*

Solution. Assume that $X = \bigcup\{X_n : n \in \omega\}$ where X_n is bounded in X. The set $Y_n = \overline{X}_n$ is a compact subspace of υX (the bar denotes the closure in υX) by Problem 415. Let $Y = \bigcup\{Y_n : n \in \omega\}$ and suppose that there exists $y \in \upsilon X \setminus Y$. Since $X \subset Y \subset \upsilon X$, we can apply Problem 414 to conclude that υY is canonically homeomorphic to υX. In particular, every $f \in C(Y)$ is extendable continuously to υX. However, the space Y is σ-compact and hence realcompact (Problem 406); since υX is an extension of Y and $\upsilon X \neq Y$, there exists $f \in C(Y)$ which does not extend to υX continuously (Problem 401). This contradiction shows that $\upsilon X = Y$, i.e., υX is σ-compact.

To prove sufficiency, assume that $\upsilon X = \bigcup\{K_n : n \in \omega\}$ where K_n is compact for each $n \in \omega$. Then $X = \bigcup\{X_n : n \in \omega\}$ where $X_n = K_n \cap X$ for all $n \in \omega$. Observe that $\overline{X}_n \subset K_n$ and hence $\overline{X}_n$ is compact for all $n \in \omega$. Finally, apply Problem 415 to convince ourselves that each X_n is bounded in X and hence X is σ-bounded.

S.417. *Prove that, for any space X, the space υX is canonically homeomorphic to the space $\{x \in \beta X : H \cap X \neq \emptyset$ for every G_δ-set $H \subset \beta X$ with $x \in H\}$.*

Solution. It is clear that the space $rX = \{y \in \beta X : H \cap X \neq \emptyset$ for any non-empty G_δ-set $H \subset \beta X\}$ is an extension of X. The space βX is realcompact and $\beta X \setminus rX$ is a union of G_δ-subsets of βX. Indeed, if $z \in \beta X \setminus rX$ then there is a non-empty $H \subset \beta X$ such that H is a G_δ-set in βX and $H \cap X = \emptyset$. It is evident that $z \in H \subset \beta X \setminus rX$ which proves that $\beta X \setminus rX$ is a union of G_δ-subsets of βX. Now apply Problem 408 to conclude that rX is a realcompact extension of the space X.

Take any realcompact space Y and any continuous map $f : X \to Y$. There exists a continuous map $F : \beta X \to \beta Y$ such that $F\,|\,X = f$ (Problem 258). Suppose that $y \in rX$ and $F(y) \notin Y$. Apply Problem 401 to find a G_δ-set P in βY such that $F(y) \in P \subset \beta Y \setminus Y$. Then $H = F^{-1}(P)$ is a G_δ-set in βX and $y \in H \subset \beta X \setminus X$ which is a contradiction with $y \in rX$. Thus $F(y) \in Y$ for any $y \in rX$ and therefore $F\,|rX : rX \to Y$ is a continuous extension of f. Now apply Problem 413 to conclude that rX is canonically homeomorphic to υX.

S.418. *Prove that $t_m(X) \leqslant t_0(X) \leqslant d(X)$ for any space X. In particular, functional tightness of a separable space is countable.*

Solution. Assume that $t_0(X) \leqslant \kappa$, i.e., that every κ-continuous function on X is continuous on X. Every strictly κ-continuous function f on X is κ-continuous on X so f is continuous. This proves that $t_m(X) \leqslant t_0(X)$.

Fix any dense set $D \subset X$ with $|D| = \kappa = d(X)$ and take any κ-continuous function $f: X \to \mathbb{R}$. If f is not continuous then there is $x \in X$ and $A \subset X$ such that $x \in \overline{A}$ and $f(x) \notin \text{cl}(f(A))$ (the bar denotes the closure in X and $\text{cl}(P)$ stands for the closure of the set P in $\mathbb{R}$). There exist disjoint $U, V \in \tau(\mathbb{R})$ such that $f(x) \in U$ and $\text{cl}(f(A)) \subset V$. Evidently, $A \subset V' = f^{-1}(V)$; besides, $A \subset \overline{V' \cap D}$. Indeed, if $a \in A \setminus (\overline{V' \cap D})$ then $a \in \overline{D \setminus V'}$ because D is dense in X. The function f is continuous on the set $\{a\} \cup (D \setminus V')$ so $f(a) \in \text{cl}(f(D \setminus V')) \subset \text{cl}(\mathbb{R} \setminus V) = \mathbb{R} \setminus V$ while $f(a) \in f(A) \subset V$ which is a contradiction.

Therefore $x \in \overline{A} \subset \overline{V' \cap D}$; the function f is continuous on $\{x\} \cup (V' \cap D)$ so $f(x) \in \text{cl}(f(V' \cap D)) \subset \text{cl}(V) \subset \text{cl}(\mathbb{R} \setminus U) = \mathbb{R} \setminus U$ while $f(x) \in U$. This contradiction proves that f is continuous and hence $t_0(X) \leqslant \kappa$. We established that $t_0(X) \leqslant d(X)$ so our solution is complete.

S.419. *Prove that $ht_m(X) = ht_0(X) = t(X)$ and hence $t_0(X) \leqslant t(X)$ for any space X. Give an example of a space X for which $t_0(X) < t(X)$.*

Solution. Assume first that $t(X) \leqslant \kappa$. If $f: X \to \mathbb{R}$ is a κ-continuous function, then take any $A \subset X$ and any $x \in \overline{A}$. There exists $B \subset A$ with $x \in \overline{B}$ and $|B| \leqslant \kappa$. Since f is continuous on the set $\{x\} \cup B$, we have $f(x) \in \text{cl}(f(B)) \subset \text{cl}(f(A))$ which shows that f is continuous (the bar denotes the closure in X and $\text{cl}(P)$ stands for the closure of the set P in $\mathbb{R}$). Thus $t_0(X) \leqslant \kappa$ and therefore $t_0(X) \leqslant t(X)$. Since tightness is a hereditary cardinal function, we have $ht_0(X) \leqslant t(X)$. Applying Problem 418, we can see that $ht_m(X) \leqslant ht_0(X) \leqslant t(X)$ so, to establish the equality $ht_m(X) = ht_0(X) = t(X)$, it suffices to prove that $t(X) \leqslant ht_m(X)$.

Fact 1. For an arbitrary space Z with $t_m(Z) \leqslant \kappa$, if $z \in Z$ is not an isolated point of Z then there is $A \subset Z \setminus \{z\}$ such that $|A| \leqslant \kappa$ and $z \in \overline{A}$.

Proof. Suppose that this is not true and fix a point $z \in \overline{Z \setminus \{z\}}$ such that $z \notin \overline{A}$ for any $A \subset Z \setminus \{z\}$ with $|A| \leqslant \kappa$. The function $f: Z \to \mathbb{R}$ defined by $f(z) = 1$ and $f|(Z \setminus \{z\}) \equiv 0$, is discontinuous. However, if $B \subset X$ and $|B| \leqslant \kappa$ then we have $f|B \in \pi_B(C(Z))$. Indeed, if $z \notin B$ then $f|B$ is the restriction of $g \equiv 0$ which is continuous on Z. If $z \in B$ then $z \notin \overline{B \setminus \{z\}}$ so there is $g \in C(Z)$ such that $g(z) = 1$ and $g|\overline{B \setminus \{z\}} \equiv 0$. It is clear that $g|B = f|B$ so f is strictly κ-continuous discontinuous function. This contradiction shows that such a point z cannot exist so Fact 1 is proved.

Returning to our solution assume that $ht_m(X) \leqslant \kappa$. If $t(X) > \kappa$ then there exists $A \subset X$ and $x \in \overline{A}$ such that $x \notin \overline{B}$ for any $B \subset A$ with $|B| \leqslant \kappa$. For the space $Z = A \cup \{x\}$ we have $t_m(Z) \leqslant \kappa$ so, by Fact 1, there must exist $B \subset A = Z \setminus \{x\}$ such that $|B| \leqslant \kappa$ and $x \in \text{cl}_Z(B)$. Then $x \in \overline{B}$ which is a contradiction. Thus $t(X) \leqslant t_m(Z) \leqslant ht_m(X)$, so we proved that $ht_m(X) = ht_0(X) = t(X)$.

Now let $X = \{0, 1\}$; then $t_0(X) \leqslant d(X) = \omega$ (see Problems 418 and 108). However, $t(X) > \omega$ (Problem 359). Therefore X is a compact space with $t_0(X) < t(X)$ so our solution is complete.

S.420. *Let Y be an $\mathbb{R}$-quotient image of X. Prove that $t_m(Y) \leqslant t_m(X)$.*

Solution. Let $\varphi : X \to Y$ be an $\mathbb{R}$-quotient map. For $\kappa = t_m(X)$, take any strictly κ-continuous function $f : Y \to \mathbb{R}$. We claim that the function $f \circ \varphi$ is strictly

κ-continuous on X. Indeed, if $A \subset X$ and $|A| \leqslant \kappa$ then $B = \varphi(A)$ has cardinality $\leqslant \kappa$ and therefore there is $g \in C(Y)$ such that $g|B = f|B$. It is evident that $h = g \circ \varphi \in C(X)$ and $h|A = (f \circ \varphi)|A$ so the function $f \circ \varphi$ is κ-continuous on X. Since $t_m(X) \leqslant \kappa$, the function $f \circ \varphi$ is continuous, and hence f is continuous by the definition of $\mathbb{R}$-quotient map. This proves that $t_m(Y) \leqslant \kappa = t_m(X)$.

S.421. *Prove that, for any infinite cardinal κ, every κ-continuous function on a normal space is strictly κ-continuous. As a consequence, $t_0(X) = t_m(X)$ for any normal space X.*

Solution. Assume that a space X is normal and take any κ-continuous function $f : X \to \mathbb{R}$. Given any $A \subset X$ with $|A| \leqslant \kappa$, we have $t_0(\overline{A}) \leqslant d(\overline{A}) \leqslant \kappa$. Since f is κ-continuous on $\overline{A}$, it is continuous on $\overline{A}$. By normality of X, there exists $g \in C(X)$ such that $g|\overline{A} = f|\overline{A}$ and therefore $g|A = f|A$. This shows that the function f is strictly κ-continuous, i.e., every κ-continuous function on X is strictly κ-continuous.

To show that $t_0(X) = t_m(X)$, it suffices to prove that $t_0(X) \leqslant t_m(X)$ (Problem 418). Assume that $t_m(X) = \kappa$ and take any κ-continuous function $f : X \to \mathbb{R}$. Since X is normal, the function f is strictly κ-continuous and hence continuous. Thus, $t_0(X) \leqslant \kappa = t_m(X)$ whence $t_0(X) = t_m(X)$.

S.422. *Prove that, for an arbitrary space X and any closed $Y \subset X$, we have $q(Y) \leqslant q(X)$.*

Solution. Let $\kappa = q(X)$. The set $cY = \mathrm{cl}_{\beta X}(Y)$ is a compact extension of Y. There exists a continuous map $f : \beta Y \to cY$ such that $f(y) = y$ for all $y \in Y$ (Problem 258). It is evident that f is onto; besides, $f(\beta Y \setminus Y) \subset cY \setminus Y \subset \beta X \setminus X$ (Fact 1 of S.259). Given any $z \in \beta Y \setminus Y$, the point $y = f(z)$ belongs to $\beta X \setminus X$. Since X is κ-placed in βX, there exists and G_κ-set H of βX such that $y \in H \subset \beta X \setminus X$. Then $H' = f^{-1}(H)$ is a G_κ-set in βY and $z \in H' \subset \beta Y \setminus Y$. Therefore, Y is κ-placed in βY and hence $q(Y) \leqslant \kappa = q(X)$.

S.423. *Prove that a dense subspace of a Moscow space is a Moscow space.*

Solution. Let Y be a dense subspace of a Moscow space X. Given an arbitrary set $U \in \tau^*(Y)$, fix any $V \in \tau(X)$ with $V \cap Y = U$. It is straightforward that $\mathrm{cl}_X(U) = \mathrm{cl}_X(V)$ and therefore $\mathrm{cl}_Y(U) = \mathrm{cl}_X(V) \cap Y$. Given any $y \in \mathrm{cl}_Y(U)$, we have $y \in \mathrm{cl}_X(V)$ so there is a G_δ-set H in the space X such that $y \in H \subset \mathrm{cl}_X(V)$. It is obvious that $H' = H \cap Y$ is a G_δ-set in Y and $y \in H' \subset \mathrm{cl}_X(V) \cap Y = \mathrm{cl}_Y(U)$. We proved that, for any $U \in \tau^*(Y)$ and any point $y \in \mathrm{cl}_Y(U)$ there exists a G_δ-set H' in the space Y such that $y \in H' \subset \mathrm{cl}_Y(U)$; hence Y is a Moscow space.

S.424. *Prove that $C_p(X)$ is a Moscow space for any space X.*

Solution. Given any $A \subset X$, let $\pi_A : \mathbb{R}^X \to \mathbb{R}^A$ be the natural projection onto the face $\mathbb{R}^A$. If we have points $x_1, \ldots, x_n \in X$ and sets $O_1, \ldots, O_n \in \tau^*(\mathbb{R})$ then the set $[x_1, \ldots, x_n; O_1, \ldots, O_n] = \{f \in \mathbb{R}^X : f(x_i) \in O_i \text{ for all } i \leqslant n\}$ is called *a standard open subset of* $\mathbb{R}^X$. The family $\mathcal{B}$ of standard open subsets is a base in $\mathbb{R}^X$ by definition of the pointwise convergence topology. If $V = [x_1, \ldots, O_1, \ldots, O_n] \in \mathcal{B}$ then $\mathrm{supp}(V) = \{x_1, \ldots, x_n\}$.

Take any $U \in \tau^*(\mathbb{R}^X)$. Denote by $\mathcal{U}$ a maximal disjoint family of standard open sets contained in U. Since $c(\mathbb{R}^X) = \omega$ (Problem 109), the family $\mathcal{U}$ is countable; by maximality of $\mathcal{U}$ the set $U' = \bigcup\mathcal{U}$ is dense in U. Let $A = \bigcup\{\text{supp}(V) : V \in \mathcal{U}\}$; then A is a countable subset of X. The set $P = \text{cl}(\pi_A(U))$ (the closure is taken in $\mathbb{R}^A$) is a G_δ-set in $\mathbb{R}^A$ because $\mathbb{R}^A$ is second countable. We claim that $\overline{U} = \pi_A^{-1}(P)$ (the bar denotes the closure in $\mathbb{R}^X$).

Fact 1. Let Y and Z be any spaces; given an open continuous onto map $f : Y \to Z$, we have $\text{cl}_Y(f^{-1}(A)) = f^{-1}(\text{cl}_Z(A))$ for each $A \subset Z$.

Proof. The inclusion $\text{cl}_Y(f^{-1}(A)) \subset f^{-1}(\text{cl}_Z(A))$ is an immediate consequence of continuity of f. The set $W = Y \setminus \text{cl}_Y(f^{-1}(A))$ is open in Y so $f(W)$ is open in Z and does not intersect A. Therefore, $\text{cl}_Z(A) \cap f(W) = \emptyset$. This shows that $f^{-1}(\text{cl}_Z(A)) \subset Y \setminus W = \text{cl}_Y(f^{-1}(A))$; hence $\text{cl}_Y(f^{-1}(A)) = f^{-1}(\text{cl}_Z(A))$ and Fact 1 is proved.

Observe that $\overline{U} = \overline{U}'$ and $\text{cl}(\pi_A(U')) = \text{cl}(\pi_A(U)) = P$ because U' is dense in U. Since $\text{supp}(V) \subset A$ for each $V \in \mathcal{U}$, we have $\pi_A^{-1}(\pi_A(V)) = V$ for all $V \in \mathcal{U}$. An immediate consequence is that $\pi_A^{-1}(\pi_A(U')) = U'$. The map π_A is open so $\overline{U}' = \pi_A^{-1}(\text{cl}(\pi_A(U'))) = \pi_A^{-1}(P)$ by Fact 1. Therefore, $\overline{U} = \overline{U'} = \pi_A^{-1}(P)$ is a G_δ-set in $\mathbb{R}^X$ because the inverse image of any G_δ-set is, trivially, a G_δ-set. Of course, this implies that $\mathbb{R}^X$ is a Moscow space and hence $C_p(X)$ is also a Moscow space being dense in $\mathbb{R}^X$ (Problem 423) so our solution is complete.

S.425. *Let Y be any space with $m(Y) \leqslant \kappa$. Suppose that $X \subset Y = \overline{X}$ and $q(X) \leqslant \kappa$. Prove that X is κ-placed in Y.*

Solution. The space βY is a compact extension of X so there is a continuous map $f : \beta X \to \beta Y$ such that $f(x) = x$ for any $x \in X$ (Problem 258). Take any $y \in Y \setminus X$; we have $f(\beta X \setminus X) \subset \beta Y \setminus X$ (Fact 1 of S.259). Since $f(X) = X$ and $f(\beta X) = \beta Y$, we have $f(\beta X \setminus X) = \beta Y \setminus Y$ so $f^{-1}(y) \subset \beta X \setminus X$.

If $|f^{-1}(y)| = 1$, i.e., $f^{-1}(y) = \{z\}$ for some $z \in \beta X \setminus X$, then there exists a G_κ-set H in βX such that $z \in H \subset \beta X \setminus X$ because X is κ-placed in βX. Take any family $\mathcal{U} \subset \tau(\beta X)$ with $|\mathcal{U}| \leqslant \kappa$ and $\bigcap\mathcal{U} = H$. The map f is closed and, for each $U \in \mathcal{U}$, we have $U \supset f^{-1}(y)$ so there is $O_U \in \tau(y, \beta Y)$ such that $f^{-1}(O_U) \subset U$ (Fact 2 of S.271). If $\mathcal{V} = \{O_U : U \in \mathcal{U}\}$ and $G = \bigcap\mathcal{V}$ then $y \in G$ and $f^{-1}(G) = \bigcap\{f^{-1}(O_U) : U \in \mathcal{U}\} \subset \bigcap\mathcal{U} = H \subset \beta X \setminus X$ so $G \cap X = \emptyset$, i.e., $y \in G \subset \beta Y \setminus X$. It is evident that $G' = G \cap Y$ is a G_κ-set in Y such that $y \in G' \subset Y \setminus X$.

Now, suppose that there exist distinct points $z, t \in f^{-1}(y)$ and choose sets $U_z \in \tau(z, \beta X)$ and $U_t \in \tau(t, \beta X)$ such that $\text{cl}_{\beta X}(U_z) \cap \text{cl}_{\beta X}(U_t) = \emptyset$; consider the sets $V_z = U_z \cap X$, $V_t = U_t \cap X$. We will also need the sets $F_z = \text{cl}_Y(V_z)$ and $F_t = \text{cl}_Y(V_t)$. There exist $W_z, W_t \in \tau(Y)$ such that $W_z \cap X = V_z$ and $W_t \cap X = V_t$; it is clear that $F_z = \text{cl}_Y(V_z) = \text{cl}_Y(W_z)$ and $F_t = \text{cl}_Y(V_t) = \text{cl}_Y(W_t)$. Observe also that $F_z \cap F_t \cap X = \text{cl}_X(V_z) \cap \text{cl}_X(V_t) \subset \text{cl}_{\beta X}(U_z) \cap \text{cl}_{\beta X}(U_t) = \emptyset$ and therefore $F_z \cap F_t \subset Y \setminus X$. Since $z \in \text{cl}_{\beta X}(V_z)$, $t \in \text{cl}_{\beta X}(V_t)$ and $f(z) = f(t) = y$, we have

$$y \in \text{cl}_{\beta Y}(f(V_z)) \cap \text{cl}_{\beta Y}(f(V_t)) \cap Y = \text{cl}_Y(f(V_z)) \cap \text{cl}_Y(f(V_t)) = F_z \cap F_t,$$

because $f(V_z) = V_z$ and $f(V_t) = V_t$.

We saw that both sets F_z and F_t are closures of open subsets of Y so there are G_κ-sets P_z and P_t of the space Y such that $y \in P_z \subset F_z$ and $y \in P_t \subset F_t$. Of course, $P = P_z \cap P_t$ is a G_κ-subset of Y and $y \in P \subset F_z \cap F_t \subset Y \setminus X$ so our solution is complete.

S.426. *Prove that $t_m(X) \leqslant \kappa$ if and only if $C_p(X)$ is κ-placed in $\mathbb{R}^X$.*

Solution. If Z is an arbitrary space, then a set $W \in \tau(C_p(Z))$ is called *a standard open subset of* $C_p(Z)$ if there are $z_1, \ldots, z_n \in Z$ and $O_1, \ldots, O_n \in \tau(\mathbb{R})$ such that $W = [z_1, \ldots, z_n; O_1, \ldots, O_n] = \{f \in C_p(Z) : f(z_i) \in O_i \text{ for all } i \leqslant n\}$. Standard open sets form a base $\mathcal{B}$ in the space $C_p(Z)$. If $W = [z_1, \ldots, z_n; O_1, \ldots, O_n] \in \mathcal{B}$ then supp $(W) = \{z_1, \ldots, z_n\}$.

Fact 1. Let Z be an arbitrary space; given a cardinal κ, call a set $H \subset C_p(Z)$ *a standard G_κ-set in the space $C_p(Z)$* if there exist $A \subset Z$ and $f \in C_p(Z)$ such that $|A| \leqslant \kappa$ and $H = G(f, A) = \{g \in C_p(Z) : g|A = f|A\}$. Then

(1) Every standard G_κ-set is a G_κ-set in $C_p(Z)$.
(2) For any $f \in C_p(Z)$, the family $\{G(f, A) : A \subset Z \text{ and } |A| \leqslant \kappa\}$ forms a base at f in the family of all G_κ-sets in $C_p(X)$ in the sense that, for any G_κ-set $H \ni f$ there is $A \subset Z$ such that $|A| \leqslant \kappa$ and $f \in G(f, A) \subset H$.

Proof. For any point $z \in Z$ and any function $f \in C_p(Z)$, it is immediate that $G(f, \{z\}) = \bigcap\{[z; (f(z) - \frac{1}{n}, f(z) + \frac{1}{n})] : n \in \mathbb{N}\}$ is a G_δ-set in $C_p(Z)$ so the set $G(f, A) = \bigcap\{G(f, \{z\}) : z \in A\}$ is the intersection $\leqslant \kappa$ of G_δ-subsets of $C_p(Z)$. Therefore, G(f, A) is a G_κ-set whenever $|A| \leqslant \kappa$ so (1) is proved.

To prove (2), assume that we have $f \in H = \bigcap \mathcal{U}$ where $\mathcal{U} \subset \tau(C_p(Z))$ and $|\mathcal{U}| \leqslant \kappa$. For each $U \in \mathcal{U}$ fix a standard open $O_U \in \tau(f, C_p(Z))$ and consider the set $A = \bigcup\{\text{supp}(O_U) : U \in \mathcal{U}\}$. It is evident that $|A| \leqslant \kappa$; if $g \in G(f, A)$, $U \in \mathcal{U}$ and $O_U = [z_1, \ldots, z_n; O_1, \ldots, O_n]$ then $g(z_i) = f(z_i) \in O_i$ for all $i \leqslant n$ because supp$(U) \subset A$ and $g|A = f|A$. Thus, $g \in O_U \subset U$ for all $U \in \mathcal{U}$, i.e., $g \in H$ which shows that $G(f, A) \subset H$ so Fact 1 is proved.

Returning to our solution, let $\pi_A : \mathbb{R}^X \to \mathbb{R}^A$ be the natural projection onto the face $\mathbb{R}^A$. Evidently, $\pi_A|C_p(X)$ coincides with the relevant restriction map. Let Z be the set X with the discrete topology; then the standard G_κ-sets in $C_p(Z)$ coincide with the G_κ-sets in $\mathbb{R}^X$ which have the form $G(f, A) = \{g \in \mathbb{R}^X : g|A = f|A\}$ for some $f \in \mathbb{R}^X$ and $A \subset X$ with $|A| \leqslant \kappa$.

Assume that $t_m(X) \leqslant \kappa$; given any $f \in \mathbb{R}^X \setminus C_p(X)$, the discontinuous function f cannot be strictly κ-continuous because $t_m(X) \leqslant \kappa$. Therefore, there is a set $A \subset X$ such that $|A| \leqslant \kappa$ and $f|A \notin \pi_A(C_p(X))$. This, evidently, implies that $f \in G(f, A) \subset \mathbb{R}^X \setminus C_p(X)$; since $G(f, A)$ is a G_κ-set in $\mathbb{R}^X$ (Fact 1), this proves that $C_p(X)$ is κ-placed in $\mathbb{R}^X$.

Now, if $C_p(X)$ is κ-placed in $\mathbb{R}^X$, take any strictly κ-continuous discontinuous $f \in \mathbb{R}^X$. since $C_p(X)$ is κ-placed in $\mathbb{R}^X$, there exists a G_κ-set H in $\mathbb{R}^X$ such that $f \in H \subset \mathbb{R}^X \setminus C_p(X)$. Apply Fact 1 to find a set $A \subset X$ such that $|A| \leqslant \kappa$ and $G(f, A) \subset H$. Then $G(f, A) \subset \mathbb{R}^X \setminus C_p(X)$ which implies $f|A \notin \pi_A(C_p(X))$, i.e., f is not strictly κ-continuous, a contradiction. Therefore, every strictly κ-continuous function on X is continuous, i.e., $t_m(X) \leqslant \kappa$ so our solution us complete.

S.427. *Prove that $C_p(X)$ is realcompact if and only if it is ω-placed in $\mathbb{R}^X$.*

Solution. Assume that $C_p(X)$ is ω-placed in $\mathbb{R}^X$. This means that $\mathbb{R}^X \backslash C_p(X)$ is a union of G_δ-subsets of $\mathbb{R}^X$. Since the space $\mathbb{R}^X$ is realcompact, we can apply Problem 408 to conclude that $C_p(X)$ is also realcompact.

On the other hand, if $C_p(X)$ is realcompact then $q\,(C_p(X)) = \omega$ (Problem 401) and $\mathbb{R}^X$ is a Moscow space (424). Since $C_p(X)$ is dense in $\mathbb{R}^X$, we can apply Problem 425 to convince ourselves that $C_p(X)$ is ω-placed in $\mathbb{R}^X$.

S.428. *Suppose that there exists a non-empty G_δ-subspace $H \subset C_p\,(X)$ such that H is realcompact. Prove that $C_p(X)$ is realcompact.*

Solution. Take any $f_0 \in H$; the map $T_{f_0} : C_p(X) \to C_p(X)$ defined by the formula $T_{f_0}(f) = f - f_0$ for all $f \in C_p(X)$, is a homeomorphism (Problem 079). Therefore $H' = T_{f_0}(H)$ is also a realcompact G_δ-subset of $C_p(X)$ such that the function $u \equiv 0$ belongs to H'. Apply Fact 1 of S.426 conclude that there is a countable $A \subset X$ such that $I(A) = \{\, f \in C_p(X) : f \mid A \equiv 0 \,\} \subset H'$. The set $I(A)$ is closed in the whole of $C_p(X)$ and hence in H' so $I(A)$ is realcompact. It is evident that $I(A) = I(\overline{A}) = \{f \in C_p(X) : f|\overline{A} \equiv 0\}$. For the set $Y = X\backslash\overline{A}$, we have $(*)$ the map $\pi = \pi_Y|I(A) : I(A) \to \pi_Y\,(I(A)) \subset C_p(Y)$ is a homeomorphism.

The map π_Y is continuous (Problem 152) so continuity of π is clear. If $f, g \in I(A)$ and $f \neq g$ then there is $x \in X$ such that $f(x) \neq g(x)$; since $f|\overline{A} = g|\overline{A}$, the point x has to belong to Y and hence $\pi(f) = \pi_Y(f) \neq \pi_Y(g) = \pi(g)$, i.e., the map π is a condensation.

To see that the map π^{-1} is continuous, take any $g \in Z = \pi_Y\,(I(A))$ and let $\pi^{-1}\,(g) = f$. Given $x_1, \dots, x_n \in X$ and $\varepsilon > 0$, assume, without loss of generality, that $x_1, \dots, x_k \in \overline{A}$ and $x_{k+1}, \dots, x_n \in Y$. Let $W = \{h \in Z : |\, h(x_i) - g(x_i)\,| < \varepsilon$ for all $i = k + 1, \dots, n\}$. Then $W \in \tau(g, Z)$; if $h \in W$ and $h' = \pi^{-1}(h)$ then $h'(x_i) = h(x_i)$ and hence $|\, h'(x_i) - f(x_i)\,| = |\, h'(x_i) - g(x_i)\,| < \varepsilon$ for all indices $i = k + 1, \dots, n$. Besides, $h' \in I(A)$ and therefore $h'(x_i) = 0 = f(x_i)$ for all $i \leqslant k$. As a consequence, $|\, h'(x_i) - f(x_i)\,| < \varepsilon$ for all $i \leqslant n$ which shows that we have $\pi^{-1}\,(W) \subset O\,(f, x_1, \dots, x_n, \varepsilon) = \{f' \in C_p(X) : |\,f'(x_i) - f(x_i)\,| < \varepsilon$ for all $i \leqslant n\}$. Since the sets $O\,(f, x_1, \dots, x_n, \varepsilon)$ from a local base at f in $C_p(X)$, we proved that π^{-1} is a continuous map at every $g \in Z$ so $(*)$ is settled.

By $(*)$, the set Z is realcompact being homeomorphic to a realcompact space $I(A)$; observe that Z is dense in $C_p(Y)$. Indeed, if $g \in C_p(Y)$ and $y_1, \dots, y_n \in Y$ then there are functions $f_i \in C(X)$, $i = 1, \dots, n$, such that $f_i(y_i) = 1$ and $f_i|(\overline{A} \cup (\{y_1, \dots, y_n\}\backslash\{y_i\})) \equiv 0$. Then $f = \sum_{i=1}^{n} g(y_i).\ f_i$ belongs to $I(A)$ and we have $\pi_Y(f)\,(y_i) = g\,(y_i)$ for all $i \leqslant n$. Since $\pi_Y(f) \in Z$, this shows that Z intersects every basic neighbourhood of g, i.e., g is in the closure of Z.

The space $\mathbb{R}^Y$ is a Moscow space (Problem 424) and Z is a realcompact dense subspace of $C_p(Y)$ and hence of $\mathbb{R}^Y$; this implies, by Problem 425, that Z is ω-placed in $\mathbb{R}^Y$. Now suppose that $C_p(X)$ is not realcompact. Then, it is not ω-placed in $\mathbb{R}^X$ (Problem 427) and hence there is $f \in \mathbb{R}^X \backslash C_p(X)$ such that $f \mid B \in \pi_B\,(C_p(X))$ for every countable $B \subset X$. In particular, there is $g \in C_p(X)$ such that $g|A = f|A$. Observe that f is ω-continuous on $\overline{A}$ and hence $f|\overline{A}$ is continuous (see Problem 418). The function $h = f - g$ is still discontinuous and strictly ω-continuous; besides, $h|\overline{A} \equiv 0$ because h is continuous on $\overline{A}$ and $h|A \equiv 0$ (see Fact 0 of S.351).

Our next step is to observe that $\pi_Y(h) \in \mathbb{R}^Y \backslash Z$. Indeed, if not, then there is $w \in I(A)$ such that $w|Y = h|Y$; since $w|\overline{A} = h|\overline{A} \equiv 0$, we have $h = w$ is a continuous function which is a contradication. Since Z is ω -placed in $\mathbb{R}^Y$, there is a countable $B \subset Y$ such that $h|B \notin \pi_B(Z) = \pi_B(I(A))$. However, h is strictly ω-continuous so there is $\upsilon \in C_p(X)$ such that $\upsilon \mid (A \cup B) = h \mid (A \cup B)$. As a consequence, $\upsilon \mid A \equiv 0$ so $\upsilon|\overline{A} \equiv 0$ by continuity of υ. Hence $\upsilon \in I(A)$ and $h|B = \nu \mid B \in \pi_B(I(A)) = \pi_B(Z)$ which is a contradiction showing that $C_p(X)$ is realcompact.

S.429. *Prove that $t_m(X) = q(C_p(X))$ for any space X. In particular, $C_p(X)$ is realcompact if any only if $t_m(X) = \omega$.*

Solution. Assume first that $t_m(X) \leqslant \kappa$ for some cardinal κ. Then the space $C_p(X)$ is κ-placed in $\mathbb{R}^X$ (Problem 426). The space $K = \beta(\mathbb{R}^X)$ is a compact extension of $C_p(X)$; given any $z \in K \backslash C_p(X)$, we have two cases:

(1) $z \in \mathbb{R}^X$; then $z \in \mathbb{R}^X \backslash C_p(X)$ and therefore we can find a G_κ-set H in the space $\mathbb{R}^X$ such that $z \in H \subset \mathbb{R}^X \backslash C_p(X)$ because $C_p(X)$ is κ-placed in $\mathbb{R}^X$. It is easy to see that there exists a G_κ-set H' in the space K such that $H' \cap \mathbb{R}^X = H$; evidently, $z \in H' \subset K \backslash C_p(X)$.
(2) $z \in K \backslash \mathbb{R}^X$. The set $\mathbb{R}^X$ is ω-placed in K because $\mathbb{R}^X$ is realcompact so there exists a G_δ-set H in the space K such that $z \in H \subset K \backslash \mathbb{R}^X \subset K \backslash C_p(X)$.

This shows that $C_p(X)$ is κ-placed in K; take a map $\varphi : \beta(C_p(X)) \to K$ such that $\varphi(f) = f$ for any $f \in C_p(X)$. Given any point $z \in \beta(C_p(X)) \backslash C_p(X)$, we have $\varphi(f) \in K \backslash C_p(X)$. (Fact 1 of S.259) and therefore there is a G_κ-set H in the space K such that $\varphi(z) \in H \subset K \backslash C_p(X)$. Then $H' = \varphi^{-1}(H)$ is a G_κ-set in $\beta(C_p(X))$ and $z \in H' \subset \beta(C_p(X)) \backslash C_p(X)$ which proves that $C_p(X)$ is κ-placed in $\beta(C_p(X))$, i.e., $q(C_p(X)) \leqslant \kappa$. Thus, we proved that $q(C_p(X)) \leqslant t_m(X)$.

Finally, observe that if $q(C_p(X)) \leqslant \kappa$ then $C_p(X)$ is κ-placed in $\mathbb{R}^X$ by Problem 425. As a consequence, $t_m(X) \leqslant \kappa$ (Problem 426) which shows that $t_m(X) \leqslant q(C_p(X))$ so $t_m(X) = q(C_p(X))$ and our solution is complete.

S.430. *Prove that $t_m(\beta\omega \backslash \omega) > \omega$. As a consequence, the space $C_p(\beta\omega \backslash \omega)$ is not realcompact.*

Solution. Let $\mathcal{B}$ be the family of all non-empty clopen subsets of $\omega^* = \beta\omega \backslash \omega$. By Facts 1 and 2 of S.370, the family $\mathcal{B}$ is a base in ω^*. Take $U_0 \in \mathcal{B}$ arbitrarily; assume that α is a countable ordinal and we have a family $\{U_\beta : \beta < \alpha\} \subset \mathcal{B}$ such that $U_{\beta'} \subset U_\beta$ (the inclusion is strict) for all $\beta < \beta' < \alpha$.

Since U_β is compact for each $\beta < \alpha$, the set $U'_\alpha = \bigcap\{U_\beta : \beta < \alpha\}$ is a non-empty G_δ-subset of ω^*. Therefore $\text{Int}(U'_\alpha) \neq \emptyset$ by Problem 370 and hence there exists $U_\alpha \in \mathcal{B}$ with $U_\alpha \subset V'_\alpha$ (the inclusion is strict). This concludes our inductive construction giving us a family $\{U_\beta : \beta < \omega_1\} \subset \mathcal{B}$ such that $U_{\beta'} \subset U_\beta$ (the inclusion is strict) for all $\beta < \beta' < \omega_1$. Since the set U_β is compact for each ordinal $\beta < \omega_1$, we have $P = \bigcap\{U_\beta : \beta < \omega_1\} \neq \emptyset$.

Let $f(x) = 1$ for all $x \in P$ and $f(x) = 0$ if $x \in \omega^* \setminus P$. Observe that the set P is not open in ω^* because otherwise $K = \omega^* \setminus P$ is compact while $\{\omega^* \setminus U_\alpha : \alpha < \omega_1\}$ is an open cover of K which has no finite subcover. Thus $P = f^{-1}((-\frac{1}{2}, \frac{1}{2}))$ is not open in ω^* whence f is a discontinuous function.

However, f is strictly ω-continuous. To see it, take any countable $A \subset \omega^*$. There exists $\alpha < \omega_1$ such that $U_\alpha \cap A = P \cap A$; the function $g : X \to \{0, 1\}$ defined by $g(x) = 0$ if $x \in U_\alpha$ and $g(x) = 1$ for all $x \in \omega^* \setminus U_\alpha$, is continuous and $g|A = f|A$. This proves that there exists a strictly ω-continuous discontinuous function on $\beta\omega \setminus \omega$ and hence $t_m(\beta\omega \setminus \omega) > \omega$. Applying Problem 429, we conclude that $C_p(\beta\omega \setminus \omega)$ is not realcompact.

S.431. *Give an example of a space X for which $C_p(X)$ is realcompact while $C_p(Y)$ is not realcompact for some closed $Y \subset X$.*

Solution. If $X = \beta\omega$ then $t_m(X) \leqslant d(X) = \omega$ (Problem 418), so the space $C_p(X)$ is realcompact (Problem 429). However, $Y = \beta\omega \setminus \omega$ is a closed subset of X such that $C_p(Y)$ is not realcompact (Problem 430).

S.432. *Prove that an open continuous image of a realcompact space is not necessarily realcompact.*

Solution. If $X = \beta\omega$ then $t_m(X) \leqslant d(X) = \omega$ (Problem 418), so the space $C_p(X)$ is realcompact (Problem 429). However, $Y = \beta\omega \setminus \omega$ is a closed subset of X such that $C_p(Y)$ is not realcompact (Problem 430). The space X is compact and hence normal, so $\pi_Y(C_p(X)) = C_p(Y)$ (Problem 152). The map π_Y is open (Problem 152) so $C_p(Y)$ is not realcompact while being an open continuous image of a realcompact space $C_p(X)$.

S.433. *Give an example of a space X with $t_0(X) \neq t_m(X)$.*

Solution. Given an arbitrary space Z, let $Z^* = \bigcup\{\mathrm{cl}_{\beta Z}(A) : A$ is a countable subset of $Z\}$. If B is a compact extension of Z, canonically homeomorphic to βZ then $Z^* = \bigcup\{\mathrm{cl}_B(A) : A$ is a countable subset of $Z\}$.

Fact 1. Let Z be a pseudocompact space. Then, for every strictly ω-continuous function $f : Z \to \mathbb{R}$, there exists a strictly ω-continuous function $f^* : Z^* \to \mathbb{R}$ such that $f^*|Z = f$.

Proof. Take any $x \in Z^*$ and any countable $A \subset Z$ such that $x \in \overline{A}$ (the bar denotes the closure in βZ). Since f is strictly ω-continuous, there is $g \in C(Z)$ such that $g|A = f|A$. The space Z being pseudocompact, the function g is bounded so there exists $h \in C(\beta Z)$ such that $h|Z = g$; we let $f^*(x) = h(x)$. Of course, we must prove that the choice of $f^*(x)$ does not depend on the choice of the set $A \subset Z$ and the function g (note that g determines a unique h by Fact 0 of S.351).

So, assume that we have some countable $B \subset Z$ such that $x \in \overline{B}$ and some function $g' \in C(Z)$ such that $g'|B = f|B$; let $h' \in C(\beta Z)$ be the extension of g' to βZ. The set $A \cup B$ is countable so there is $g'' \in C(Z)$ such that $g''|(A \cup B) = f|(A \cup B)$; if h'' is the extension of g'' to βZ then $h''|A = g''|A = f|A = g|A = h|A$ implies

$h''|\overline{A} = h|\overline{A}$ (Fact 0 of S.351); analogously, $h''|B = g''|B = f|B = g'|B = h'|B$ implies $h''|\overline{B} = h'|\overline{B}$. Since $x \in \overline{A} \cap \overline{B}$, we have $h(x) = h''(x) = h'(x)$, i.e., $h'(x) = h(x)$ so the definition of f^* is consistent.

To see that f^* is strictly ω-continuous, take any countable $C \subset Z^*$. For any $c \in C$ there is a countable $A_c \subset Z$ with $c \in \overline{A}_c$; the set $A = \bigcup\{A_c : c \in C\}$ is countable and $C \subset \overline{A}$. Choose any $g \in C(Z)$ such that $g|A = f|A$; there is $h \in C(\beta Z)$ with $h|Z = g$. By definition of f^*, we have $f^*(x) = h(x)$ for any $x \in \overline{A}$; therefore $f^*|C = h|C$ so Fact 1 is proved.

Fact 2. Suppose that M_t is a second countable compact space for each $t \in T$. A dense set D of the product $M = \prod\{M_t : t \in T\}$ is pseudocompact if and only if $p_S(D) = M_S = \prod_{t\in S} M_t$ for any countable $S \subset T$. Here $p_S : M \to M_S$ is the natural projection onto the face M_S.

Proof. If D is pseudocompact then, given any countable $S \subset T$, the set $p_S(D)$ is a dense pseudocompact subspace of the second countable space M_S. Any second countable pseudocompact space is compact (Problem 138) so $p_S(D) = M_S$ and we proved necessity. Now, assume that $p_S(D) = M_S$ for any countable $S \subset T$. Given any continuous function $f : D \to \mathbb{R}$, we can apply Problem 299 to conclude that there exists a countable $S \subset T$ and a continuous map $h : p_S(D) \to \mathbb{R}$ such that $h \circ p_S = f$. But $p_S(D) = M_S$ is a compact space. Hence $f(D) = h(M_S)$ is a compact and hence bounded subset of $\mathbb{R}$. It turns out that every $f \in C(D)$ is bounded so D is pseudocompact and Fact 2 is proved.

We denote by I the closed interval [0, 1] with the natural topology. Let A be any set with $|A| = \mathfrak{c}$. Fix any disjoint family $\{A_\alpha : \alpha < \mathfrak{c}\} \subset \exp(A)$ such that $\bigcup\{A_\alpha : \alpha < \mathfrak{c}\} = A$ and $|A_\alpha| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$. Let $u \in I^A$ be defined by $u(a) = 0$ for all $a \in A$. Given any $B \subset A$, we let $u_B = u|B \in I^B$. We will also need the sets $\Sigma(B) = \{x \in I^B : |\{b\in B : x(b) \neq 0\}| \leqslant \omega\} \subset I^B$ and $R(B) = I^B \setminus \Sigma(B)$ for each $B \subset A$. The map $\pi_B : I^A \to I^B$ is the natural projection onto the face I^B.

Fact 3. The set $R(B)$ is pseudocompact for any uncountable $B \subset A$; therefore I^B is canonically homeomorphic to $\beta(R(B))$. If we identify $\beta(R(B))$ and I^B then $R(B)^* = I^B$.

Proof. Take any countable $B' \subset B$ and any $x \in I^{B'}$. Letting $q(b) = x(b)$ for all $b \in B'$ and $q(b) = 1$ for all $b \in B \setminus B'$, we obtain a point $q \in R(\mathrm{B})$ such that $q|B' = x$. This proves that $\pi_{B'}(R(B)) = I^{B'}$ for any countable $B' \subset B$. Applying Fact 2 of S.309 and Fact 2 of this solution, we can see that I^B is canonically homeomorphic to $\beta(R(B))$.

To show that $R(B)^* = I^B$, take any $x \in I^B$. It is easy to find uncountable sets $L_n \subset B$, $n \in \omega$ such that the family $\{L_n : n \in \omega\}$ is disjoint; let $y_n(b) = 1$ for all $b \in L_n$ and $y_n(b) = x(b)$ for all $b \in B \setminus L_n$. Then $E = \{y_n : n \in \omega\} \subset R(B)$ and $x \in \overline{E}$; to see this, take any finite $K \subset B$. There is $n \in \omega$ such that $L_n \cap K = \emptyset$ and therefore $y_n|K = x|K$. This proves that, for any finite $K \subset B$ there is $n \in \omega$ such that $y_n|K = x|K$; an easy consequence is that $x \in \overline{E}$. Since $x \in I^B$ has been chosen arbitrarily and E is countable, we showed that $R(B)^* = I^B$ so Fact 3 is proved.

We will need the sets $\Sigma_\alpha = \Sigma(A\backslash A_\alpha)$ for each $\alpha < \mathfrak{cc}$ and the set $\Sigma = \Sigma(A)$. The set $Q = \bigcup\{I^B : B$ is a countable subset of $A\}$ has cardinality $\mathfrak{cc}$ so we can choose an enumeration $Q = \{q_\alpha : \alpha < \mathfrak{c}\}$; for each $\alpha < \mathfrak{c}$, we have $q_\alpha \in I^{B_\alpha}$ for some countable $B_\alpha \subset A$.

We first construct, by transfinite induction, an injective map $\gamma : \mathfrak{c} \to \mathfrak{c}$ such that $B_\alpha \cap A_{\gamma(\alpha)} = \emptyset$ for all $\alpha < \mathfrak{c}$. Observe that countability of B_0 implies that there are only countably many $\alpha < \mathfrak{c}$ such that $A_\alpha \cap B_0 \neq \emptyset$. Therefore we can choose $\gamma(0) < \mathfrak{c}$ such that $A_{\gamma(0)} \cap B_0 = \emptyset$. Assume that $\alpha < \mathfrak{c}$ and we have a set $S_\alpha = \{\gamma(\beta) : \beta < \alpha\} \subset \mathfrak{c}$ such that $A_{\gamma(\beta)} \cap B_\beta = \emptyset$ and $\gamma(\beta) \neq \gamma(\beta')$ for any distinct $\beta, \beta' < \alpha$. The set $T_\alpha = S_\alpha \cup \{\beta : A_\beta \cap B_\alpha \neq \emptyset\}$ has cardinality $< \mathfrak{c}$ because $|S_\alpha| < \mathfrak{c}$ and the set $\{\beta : A_\beta \cap B_\alpha \neq \emptyset\}$ is countable. Taking any $\gamma(\alpha) \in \mathfrak{c}\backslash T_\alpha$, we obtain a set $S_{\alpha+1} = \{\gamma(\beta) : \beta \leqslant \alpha\} \subset \mathfrak{c}$ such that $A_{\gamma(\beta)} \cap B_\beta = \emptyset$ and $\gamma(\beta) \neq \gamma(\beta')$ for any distinct $\beta, \beta' \leqslant \alpha$.

Consequently, our inductive construction can be carried on for all $\alpha < \mathfrak{c}$ giving us a set $\{\gamma(\beta) : \beta < \mathfrak{c}\}$ such that $\beta \neq \beta'$ implies $\gamma(\beta) \neq \gamma(\beta')$ for all $\beta, \beta' < \mathfrak{c}$ and, besides, $A_{\gamma(\beta)} \cap B_\beta = \emptyset$ for all $\beta < \mathfrak{c}$.

For any $\alpha < \mathfrak{c}$, let $h_{\gamma(\alpha)}(a) = q_\alpha(a)$ for all $a \in B_\alpha$; if $\alpha \in \mathfrak{c}\backslash(A_{\gamma(\alpha)} \cup B_\alpha)$ then $h_{\gamma(\alpha)}(a) = 0$. This gives us $h_{\gamma(\alpha)} \in \Sigma_{\gamma(\alpha)}$ for each $\alpha < \mathfrak{c}$. For any $\beta \in \mathfrak{c}\backslash\gamma(\mathfrak{c})$, let $h_\beta = u_{A\backslash A_\beta}$. It is straightforward that $h_{\gamma(\beta)}|B_\beta = q_\beta$ for all $\beta < \mathfrak{c}$ so we have obtained a set $\{h_\alpha : \alpha < \mathfrak{c}\}$ such that

(∗) $h_\alpha \in \Sigma_\alpha$ for any $\alpha < \mathfrak{c}$; besides, for any countable $B \subset A$ and any $q \in I^B$, there exists $\alpha < \mathfrak{c}$ such that $A_\alpha \cap B = \emptyset$ and $h_\alpha|B = q$.

Indeed, there is $\beta < \mathfrak{c}$ such that $B = B_\beta$ and $q = q_\beta$. For $\alpha = \gamma(\beta)$, we have $A_\alpha \cap B = A_\alpha \cap B_\beta = \emptyset$ and $h_\alpha|B = q_\beta = q$ so (∗) is proved.

For each $\alpha < \mathfrak{c}$, let $X_\alpha = \{h_\alpha\} \times R(A_\alpha) \subset I^A$. Now, let $X = \bigcup\{X\alpha : \alpha < \mathfrak{c}\}$. The topology on X is induced from the cube I^A.

Let us prove that $t_m(X) = \omega$. It follows from (∗) that $\pi_B(X) = I^B$ for any countable $B \subset A$; therefore X is pseudocompact (Fact 2) and βX is canonically homeomorphic to I^A (Fact 2 of S.309), so we will identify I^A with βX. Take any strictly ω-continuous function $f : X \to \mathbb{R}$. There exists a strictly ω-continuous function $f^* : X^* \to \mathbb{R}$ such that $f^*|X = f$ (Fact 1). We next prove that $\Sigma \subset X^*$. To see this, take any $x \in \Sigma$; the set $B = x^{-1}((0, 1])$ is countable so we can find a sequence $\{\alpha_n : n \in \omega\} \subset \mathfrak{c}$ with the following properties:

(1) $\alpha_n \neq \beta_m$ for all $m, n \in \omega, m \neq n$.
(2) $h_{\gamma(\alpha_n)} = x|B_{\alpha_n}$ for all $n \in \omega$.
(3) $B_{\alpha_n} \supset B$ for all $n \in \omega$.

Take any $x_n \in X_{\gamma(\alpha_n)}$ for all $n \in \omega$. Then $P = \{x_n : n \in \omega\} \subset X$ and $x \in \overline{P}$ (the closure is taken in I^A). Indeed, take any finite $K \subset A$; since the family $\{A_{\gamma(\propto_n)} : n \in \omega\}$ is disjoint, there is $n \in \omega$ such that $A_{\gamma(\alpha_n)} \cap K = \emptyset$. This implies that $x_n|K = h_{\gamma(\alpha_n)}|K = x|K$ because $x|(K \cap B_{\alpha_n}) = h_{\gamma(\alpha_n)}|(K \cap B_{\alpha_n})$ and $h_{\gamma(\alpha_n)}(a) = 0 = x(a)$ for all $a \in K\backslash B_{\alpha_n}$. We proved that, for any finite $K \subset A$ there is $n \in \omega$ such that $x_n|K = x|K$; an evident consequence is that $x \in \overline{P}$.

Since $\Sigma \subset X^*$, the function f^* is defined at all points of Σ. Therefore, a strictly ω-continuous function $f^*|\Sigma$ is continuous because the space Σ is Fréchet–Urysohn (it is a subspace of a Fréchet–Urysohn space defined in Problem 135 and also denoted there by Σ) and hence has countable tightness (Problem 419). It is easy to see that $\pi_B(\Sigma) = I^B$ for any countable $B \subset A$. Therefore the space Σ is pseudocompact (Fact 2) and hence I^A is canonically homeomorphic to $\beta(\Sigma)$ (Fact 2 of S.309). Thus, there exists a continuous function $k : I^A \to \mathbb{R}$ such that $k|\Sigma = f^*|\Sigma$. If we prove that $k|X = f$, then we will prove continuity of the function f.

Take any $\alpha < \mathbf{c}$. It is clear that $X_\alpha \subset Z_\alpha = \{h_\alpha\} \times I^{A_\alpha}$. Observe that the map $\pi = \pi_{A_\alpha}|Z_\alpha : Z_\alpha \to I^{A_\alpha}$ is a homeomorphism such that $\pi(X_\alpha) = R(A_\alpha)$. Applying Fact 3 we can consider that $Z_\alpha = \beta X_\alpha$; since Z_α is compact, the closure in Z_α coincides with the closure in I^A so $X_\alpha^* = Z_\alpha$ and hence $Z_\alpha \subset X^*$. Therefore the function f^* is also defined on Z_α which is separable being homeomorphic to $I^{\mathbf{c}}$. As a consequence, $f^*|Z_\alpha$ is continuous on Z_α (Problem 418); besides, $k_\alpha = k|Z_\alpha$ is also continuous and coincides with $f^*|Z_\alpha$ on a dense set $\Sigma \cap Z_\alpha$ which implies $f^*|X_\alpha = f|X_\alpha = k|X_\alpha$ (Fact 0 of S.351), i.e., $k|X_\alpha = f|X_\alpha$. The ordinal α has been chosen arbitrarily so $k|X = f$ and hence f is continuous which proves that $t_m(X) \leqslant \omega$.

To show that $t_0(X) > \omega$, we will produce an ω-continuous discontinuous function $g : X \to \mathbb{R}$. Let $g(x) = 1$ if $x \in X_0$ and $g(x) = 0$ for all $x \in X \backslash X_0$. It follows from (∗) that $X \backslash X_0$ is dense in X, so the set $X \backslash X_0 = g^{-1}(0)$ is not closed in X whence the function g is not continuous.

Now take any countable $M \subset X$; then $M = M_0 \cup M_1$ where $M_0 = M \cap X_0$ and $M_1 = M \cap (X \backslash X_0)$. The function $g|M$ is continuous if and only if M_i is closed in M for each $i = 0, 1$; thus it suffices to show that $\overline{M}_0 \cap M_1 = \emptyset$ and $\overline{M}_1 \cap M_0 = \emptyset$ (the bar denotes the closure in X). The first equality holds because X_0 is closed in X and hence $\overline{M}_0 \cap M_1 \subset X_0 \cap M_1 = \emptyset$.

Observe that $\pi_{A_0}(X \backslash X_0) \subset \Sigma(A_0)$ and hence $\pi_{A_0}(M_1) \subset \Sigma(A_0)$. Since the closure in I^{A_0} of any countable subset of $\Sigma(A_0)$ is contained in $\Sigma(A_0)$ (Fact 3 of S.307), we have $\pi_{A_0}(\overline{M}_1) \subset [\pi_{A_0}(M_1)] \subset \sum(A_0)$ (the brackets denote the closure in the space I^{A_0}) while $\pi_{A_0}(M_0) \subset \pi_{A_0}(X_0) = R(A_0) \subset I^{A_0} \backslash \Sigma(A_0)$. It follows from $\pi_{A_0}(\overline{M}_1) \cap \pi_{A_0}(M_0) = \emptyset$ that $\overline{M}_1 \cap M_0 = \emptyset$ and therefore the function $g|M$ is continuous. Since there exists an ω-continuous discontinuous function on X, we have $t_0(X) > \omega = t_m(X)$ so our solution is complete.

S.434. *(Uspenskij's theorem) Prove that $q(X) = t_0(C_p(X)) = t_m(C_p(X))$ for any space X. In particular, X is a realcompact space if and only if functional tightness of $C_p(X)$ is countable.*

Solution. Assume that $t_m(C_p(X)) \leq \kappa$; then $q(C_p(C_p(X))) = t_m(C_p(X)) \leqslant \kappa$ (Problem 429). Since X embeds in $C_p(C_p(X))$ as a closed subspace (Problem 167), we have $q(X) \leqslant q(C_p(C_p(X))) \leqslant \kappa$ (Problem 422) which proves that $q(X) \leqslant t_m(C_p(X))$.

Given a space Z, a cardinal κ and a set $A \subset Z$, let $[A_k] = \bigcup\{\overline{B} : B \subset A$ and $|B| \leq \kappa\}$. The set $[A]_\kappa$ is called *the κ-closure of A in Z.*

Fact 1. Let $\varphi : Y \to Z$ be a continuous onto map. Suppose that Y has a base $\mathcal{B}$ such that, for any $U \in \mathcal{B}$, there exists $V \subset \tau(Z)$ such that $\varphi(U) \subset V \subset [\varphi(U)]_\kappa$. Then $t_0(Y) \leqslant \kappa$ implies $t_0(Z) \leqslant \kappa$.

Proof. Take any κ-continuous function $f : Z \to \mathbb{R}$. It is evident that $f \circ \varphi$ is κ-continuous on Y; since $t_0(Y) \leqslant \kappa$, the function $f \circ \varphi$ is continuous. Fix any point $z \in Z$ and any $\varepsilon > 0$; there is $y \in Y$ such that $\varphi(y) = z$. The function $f \circ \varphi$ being continuous, there is $U' \in \tau(y, Y)$ such that $|f(\varphi(y')) - f(\varphi(y))| < \frac{\varepsilon}{2}$ for all $y' \in U'$. Pick any $U \in \mathcal{B}$ such that $y \in U \subset U'$.

There exists a set $V \in \tau(Z)$ such that $f(U) \subset V \subset [f(U)]_\kappa$. Since $z \in f(U)$, we have $V \in \tau(z, Z)$. Observe that $f(\varphi(U)) = (f \circ \varphi)(U) \subset (f(z) - \frac{\varepsilon}{2}, f(z) + \frac{\varepsilon}{2})$. Since $V \subset [f(U)]_\kappa$, for any $z' \in V$ there is $B \subset f(U)$ such that $|B'| \leqslant \kappa$ and $z' \in \overline{B}'$. The set $B = B' \cup \{z'\}$ has cardinality $\leqslant \kappa$ so $f|B$ is continuous. As a consequence, $f(z') \in \overline{f(B)} \subset \overline{(f(z) - \frac{\varepsilon}{2}, f(z) + \frac{\varepsilon}{2})} = \left[f(z) - \frac{\varepsilon}{2}, f(z) + \frac{\varepsilon}{2}\right] \subset (f(z) - \varepsilon, f(z) + \varepsilon)$. This shows that $|f(z') - f(z)| < \varepsilon$ for all $z' \in V$, i.e., V witnesses continuity of f at the point z. Since z has been chosen arbitrarily, we proved that any κ-continuous function on Z is continuous, i.e., $t_0(Z) \leqslant \kappa$. Fact 1 is proved.

Returning to our solution suppose that $q(X) \leqslant \kappa$. The space $C_p(X)$ is homeomorphic to $C_p(X, (-1, 1)) \subset C_p(X, \mathbb{I})$. (Fact 1 of S.295). The restriction map $\pi : C_p(\beta X, \mathbb{I}) \to C_p(X, \mathbb{I})$ is a condensation (it is continuous and injective because it is a restriction of a continuous injective map $\pi_X : C_p(\beta X) \to C_p(X)$ (Problem 152) and it is onto because every $f \in C(X, \mathbb{I})$ extends to a continuous function over the whole βX (Problem 257)). Let $T = C_p(X, (-1, 1))$ and $S = \pi^{-1}(Z) \subset C_p(\beta X, \mathbb{I})$. Then $\varphi = \pi|S : S \to T$ is a condensation. We claim that S has a base with the property introduced in Fact 1.

Given points $y_1, \dots, y_n \in \beta X$ and sets $O_1, \dots, O_n \in \tau^*(\mathbb{I})$, let $W(y_1, \dots, y_n; O_1, \dots, O_n) = \{f \in S : f(y_i) \in O_i \text{ for all } i \leqslant n\}$. If $x_1, \dots, x_k \in X$, and $G_1, \dots, G_k \in \tau^*((-1, 1))$, let $B(x_1, \dots, x_k; G_1, \dots, G_k) = \{f \in T : f(x_i) \in G_i \text{ for all } i \leqslant k\}$. It is clear that the family $\mathcal{B} = \{W(y_1, \dots, y_n; O_1, \dots, O_n) : n \in \mathbb{N}, y_i \in \beta X \text{ and } O_i \in \tau^*(\mathbb{I}) \text{ for all } i \leqslant n\}$ is a base in S. To prove that $\mathcal{B}$ is as in Fact 1, it suffices to show that $\varphi(U) \subset V \subset [\varphi(U)]_\kappa$ for any $U = W(y_1, \dots, y_n; O_1, \dots, O_n) \in \mathcal{B}$ and $V = B(y_{k_1}, \dots, y_{k_m}; O'_{k_1}, \dots, O'_{k_m})$, where $\{y_{k_1}, \dots, y_{k_m}\} = \{y_1, \dots, y_n\} \cap X$ and $O'_{k_i} = O_{k_i} \cap (-1, 1)$ for all $i \leqslant m$.

Since it is evident that $\varphi(U) \subset V$, let us prove that $V \subset [\varphi(U)]_\kappa$. Take any $f \in V$. To avoid heavy indexing, we assume, without loss of generality, that $U = W(y_1, \dots, y_m, z_1, \dots, z_p; O_1, \dots, O_m, Q_1, \dots, Q_p)$ where $y_1, \dots, y_m \in X$ and $z_1, \dots, z_p \in \beta X \setminus X$; then $V = B(y_1, \dots, y_m; O_1, \dots, O_m)$. Since $q(X) \leqslant \kappa$, there exists a family $\mathcal{H} = \{H_\alpha : \alpha < \kappa\} \subset \tau(\beta X)$ with the following properties:

(1) $K = \{z_1, \dots, z_p\} \subset \bigcap \mathcal{H}$.
(2) For any finite $F \subset X$, there is $\alpha < \kappa$ such that $F \cap H_\alpha = \emptyset$.
(3) $L = \{y_1, \dots, y_m\} \subset X \setminus H_\alpha = \emptyset$ for all $\alpha < \kappa$.

Given $n \in \mathbb{N}$ and $\alpha < \kappa$, let $P_{n,\alpha} = \left\{x \in X \setminus H_\alpha : f(x) \in I_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]\right\}$. The set $P_{n,\alpha}$ is closed in X and $K \cap \overline{P}_{n,\alpha} = \emptyset$ for all $n \in \mathbb{N}$, $\alpha < \kappa$ (the bar denotes the closure in βX). The inclusion $f(X) \subset (-1, 1)$ implies that, for each point $x \in X$, there is $n \in \mathbb{N}$ such that $f(x) \in I_n$. Recalling that $\bigcap \mathcal{H} \subset \beta X \setminus X$, we convince ourselves that $\bigcup \{P_{n,\alpha} : n \in \mathbb{N}, \alpha < \kappa\} = X$. Take a function $g \in C_p(\beta X)$ such that $g|X = f$. Choose $r_i \in Q_i \cap (-1, 1)$ for all $i \leqslant p$. There exists $l \in \mathbb{N}$, such that $\{r_1, \dots, r_p\} \cup f(L) \subset I_l$.

It is evident that $g(\overline{P}_{n,a}) \subset I_n$ for all $\alpha < \kappa$. By normality of βX, for each $\alpha < \kappa$ and $n > l$ there exists $g_{n,\alpha} \in C(\beta X, I_n)$, such that $g_{n,\alpha}(z_i) = r_i$ for all $i \leqslant p$ and $g_{n,\alpha}|\overline{P}_{n,\alpha} = g|\overline{P}_{n,\alpha}$. In particular, $g_{n.\alpha}(z_i) = r_i \in Q_i$ for all i $\leqslant$ p and $g_{n,\alpha}(y_j) = g(y_j) = f(y_j) \in O_j$ for all j $\leqslant$ m; besides, we have $g_{n,\alpha}(\beta X) \subset I_n \subset (-1, 1)$ so $g_{n,\alpha} \in S$ for all $n > l$ and $\alpha < \kappa$. Therefore $g_{n,\alpha} \in U$ for all $n > l$ and $\alpha < \kappa$.

We have $L \subset P_{n,\alpha}$ for all $n > l$ and $\alpha < \kappa$. If $f_{n,\alpha} = g_{n,\alpha}|X = \varphi(g_{n,\alpha}) \in \varphi(U)$ then the set $B = \{f_{n,\alpha} : \alpha < \kappa, n > l\}$ is contained in $\varphi(U)$ and $|B| \leqslant \kappa$. By the property (2), given any finite $F \subset X$, there exists $\alpha < \kappa$ such that $F \cap H_\alpha = \emptyset$. We can find $n > l$ such that $f(F) \subset I_n$; then $f_{n,\alpha}|F = f|F$. This shows that, for any finite $F \subset X$ there is $h \in B$ such that $h|F = f|F$; thus $f \in \overline{B} \subset [\varphi(U)]_\kappa$. Since the function $f \in V$ has been chosen arbitrarily, we proved that $V \subset [\varphi(U)]_\kappa$. Thus the base $\mathcal{B}$ is as in Fact 1; since $t_0(S) \leqslant t(S) \leqslant t(C_p(\beta X)) = \omega$ (Problem 419), we have $t_0(C_p(X)) = t_0(T) \leqslant \kappa$. This shows that $t_0(C_p(X)) \leqslant q(X)$ and we finally have $t_m(C_p(X)) \leqslant t_0(C_p(X)) \leqslant q(X) \leqslant t_m(C_p(X))$. An immediate consequence is the equality $q(X) = t_m(C_p(X)) = t_0(C_p(X))$. To finish our solution observe that X is realcompact if and only if $q(X) \leqslant \omega$ (Problem 401) which is equivalent to $t_m(C_p(X)) = \omega$.

S.435. *Prove that $q(X) = q(C_p(C_p(X)))$ for any space X. In particular, X is realcompact if and only if so is $C_p(C_p(X))$.*

Solution. We have $q(X) = t_m(C_p(X)) = q(C_p(C_p(X)))$ by Problems 429 and 434. Now, X is realcompact if and only if $q(X) = q(C_p(C_p(X))) = \omega$ (Problem 401).

S.436. *For any space X, consider the restriction map $\pi : C_p(\upsilon X) \to C_p(X)$ defined by $\pi(f) = f|X$. Prove that π is a condensation and $\pi|A : A \to \pi(A)$ is a homeomorphism for any countable $A \subset C_p(\upsilon X)$.*

Solution. Given any $g \in C_p(\upsilon X)$, a set $\{u_1, \dots, u_n\} \subset \upsilon X$ and $\varepsilon > 0$, we let $O(g, u_1, \dots, y_n, \varepsilon) = \{h \in C_p(\upsilon X) : |g(u_i) - h(u_i)| < \varepsilon$ for all $i \leqslant n\}$. The sets $O(g, u_1, \dots, y_n, \varepsilon)$ form a local base at the point g in the space $C_p(\upsilon X)$.

The map π is injective because X is dense in υX (Problem 152). Since $\mathbb{R}$ is a realcompact space, for any $f \in C(X)$ there exists $g \in C(\upsilon X)$ such that $g|X = f$ (Problem 412). Therefore π is onto so it is a condensation.

Now fix any countable $A \subset C_p(\upsilon X)$; let $B = \pi(A)$ and $\varphi = (\pi|A)^{-1}$. To prove that $\varphi : B \to A$ is continuous, take any $C \subset B$ and any $f \in$ (the closure is taken in $C_p(X)$). Assume that the function $\varphi(f)$ is not in the closure of $\varphi(C)$ in the space $C_p(\upsilon X)$. Then there are $y_1, \dots, y_m \in X, z_1, \dots, z_k \in \upsilon X \setminus X$ and $\varepsilon > 0$, such that $W \cap \varphi(C) = \emptyset$ where $W = O(\varphi(f), y_1, \dots, y_m, z_1, \dots, z_k, \varepsilon)$.

For every $g \in A \cup \{\varphi(f)\}$ the set $H_i(g) = g^{-1}(g(z_i))$ is a G_δ -set in υX for all $i \leqslant \kappa$; therefore $P_i = \bigcap\{H_i(g) : g \in A \cup \{\varphi(f)\}\}$ is also a G_δ -subset of υX. It follows easily from Problem 417 that $P_i \cap X \neq \emptyset$; fix a point $x_i \in P_i \cap X$ for each $i \leqslant \kappa$. It is straightforward that $g(x_i) = g(z_i)$ for each $g \in A \cup \{\varphi(f)\}$ and $i \leqslant \kappa$.

Since $f \in$, we have $W' \cap C \neq \emptyset$ where $W' = O(\varphi(f), y_1, \dots, y_m, x_1, \dots, x_k, \varepsilon)$; pick any $g' \in W' \cap C$. Observe that $g = \varphi(g')$ is an extension of g to υX; hence $g(x_i) = g'(x_i)$ for each $i \leqslant \kappa$ and $g(y_i) = g'(y_i)$ for all $i \leqslant m$. Since $\varphi(f)(y_i) = f(y_i)$ for all $i \leqslant m$, we have $|g(y_i) - \varphi(f)(y_i)| = |g'(y_i) - f(y_i)| < \varepsilon$ for all $i \leqslant m$. Besides,

$|g(z_i) - \varphi(f)(z_i)| = |g(x_i) - f(x_i)| = |g'(x_i) - f(x_i)| < \varepsilon$ for all $i \leqslant \kappa$; therefore we have $g \in W \cap \varphi(C)$ which is a contradiction. The function $f \in \overline{C}$ has been chosen arbitrarily, so we proved that $\varphi(\text{cl}_B(C)) \subset \text{cl}_A(\varphi(C))$ for each $C \subset B$. Thus $\varphi = (\pi|A)^{-1}$ is continuous so $\pi|A$ is a homeomorphism.

S.437. *Let* X *be an arbitrary space. Let* II: $C_p(\upsilon X) \to C_p(X)$ *be the restriction map. Prove that the topology of* $C_p(\upsilon X)$ *is the strongest one on* $C(\upsilon X)$ *such that* $\pi|A : A \to \pi(A)$ *is a homeomorphism for each countable* $A \subset C_p(\upsilon X)$.

Solution. Given a topology τ on $C(\upsilon X)$ and $A \subset C(\upsilon X)$, let τ_A $\{U \cap A : U \in \tau\}$, i.e., τ_A is the topology of a subspace of the space $(C(\upsilon X),\tau)$ induced on A. We will also need the map $\pi_A = \pi|A : A \to \pi(A) \subset C_p(X)$ and the topology r_A induced on $\pi(A)$ by the topology of $C_p(X)$.

Let $\mathcal{A}$ be the family of all topologies μ on the set $C(\upsilon X)$ such that the mapping π_A : $(A, \mu_A) \to (\pi(A), \upsilon_A)$ is a homeomorphism for any countable set $A \subset C(\upsilon X)$. Denote by τ the topology of $C_p(\upsilon X)$ and take some $\mu \in \mathcal{A}$. Our aim is to prove that $\mu \subset \tau$. We know that $\tau \in \mathcal{A}$ (Problem 436) and hence $\pi_A : (A,\tau_A) \to (\pi(A),\nu_A)$ is also a homeomorphism; an immediate consequence is that $\tau_a = \mu_a$ for any countable $A \subset C(\upsilon X)$.

Fact 1. In any space Z the family $\mathcal{B}(Z) = \{f^{-1}(O) : O \in \tau(\mathbb{R}), f \in C(Z)\}$ is a base of the space Z.

Proof. Indeed, if $z \in U \in \tau(Z)$ then there is a function $f \in C(Z)$ such that $f(z) = 1$ and $f|(Z \setminus U) = 0$. If $O = \mathbb{R} \setminus \{0\} \in \tau(\mathbb{R})$ then $f^{-1}(O) \in \mathcal{B}(Z)$ and $z \in f^{-1}(O) \subset U$ which proves that $\mathcal{B}(Z)$ is a base in Z so Fact 1 is proved.

Let D be the space $(C(\upsilon X),\mu)$; the family $\mathcal{B}(D)$ is a base in D (Fact 1) so $\mathcal{B}(D) \subset \tau$ implies $\mu \subset \tau$ because μ consists of all possible unions of the elements of $\mathcal{B}(D)$ and these unions belong to τ because τ is a topology.

Now suppose that μ is not contained in τ; then the family $\mathcal{B}(D)$ is not contained in τ and hence there is $f \in C(D)$ such that f is not continuous on $C_p(\upsilon X)$. Since $t_0(C_p(\upsilon X)) = q(\upsilon X) = \omega$ (Problem 434), there is a countable $A \subset C(\upsilon X)$ such that $f|A$ is discontinuous on (A, τ_A). However, $\tau_A = \mu_A$ and $f|A$ is continuous on (A, μ_A) which is a contradiction. This proves that any topology $\mu \in \mathcal{A}$ is contained in τ. Since $\tau \in \mathcal{A}$, the topology τ is precisely the strongest one in the family $\mathcal{A}$ so our solution is complete.

S.438. *Prove that, for any* X, *the space* $\upsilon(C_p(X))$ *is canonically homeomorphic to the subspace* $S_X = \{f \in \mathbb{R}^X : f$ *is strictly* ω-*continuous*$\}$ *of the space* $\mathbb{R}^X$.

Solution. Given any set Z and any $A \subset Z$, we denote by $\pi_A : \mathbb{R}^Z \to \mathbb{R}^A$ the restriction map which coincides with the natural projection of $\mathbb{R}^A$ onto the face $\mathbb{R}^A$. Observe that, when we use for $\mathbb{R}^Z$ some results proved for C_p-spaces, we are thinking of $\mathbb{R}^A$ as of $C_p(D)$ where D is the set Z with the discrete topology.

If $g \in \mathbb{R}^X \setminus S_X$ then g is not a strictly ω-continuous function so there is a countable set $A \subset X$ such that $\pi_A(g) \notin \pi_A(C_p(X))$. An immediate consequence is that the set

$H = \{h \in \mathbb{R}^X : \pi_A(h) = \pi A(g)\}$ is a G_δ-subset of $\mathbb{R}^X$ (Fact 1 of S.426) such that $h \in H \subset \mathbb{R}^X \setminus S_X$. This shows that $\mathbb{R}^X \setminus S_X$ is a union of G_δ-subsets of $\mathbb{R}^X$. The space $\mathbb{R}^X$ is realcompact so we can apply Problem 408 to conclude that S_x is a realcompact extension of the space $C_p(X)$.

Fact 1. Let Y be any space; if rY is a realcompact extension of Y such that, for any function $f \in C(Y)$ there is $g \in C(rY)$ such that $g^Y = f$ then rY is canonically homeomorphic to υY.

Proof. Take any realcompact space P and any continuous map $\varphi : Y \to P$. We can consider that P is a closed subset of $\mathbb{R}^k$ for some cardinal κ (Problem 401); let $\pi_\alpha : \mathbb{R}^k \to \mathbb{R}$ be the natural projection of $\mathbb{R}^k$ to its αth factor. Then the map $\pi_a \circ \varphi$ is continuous so we can find $g_\alpha \in C(rY)$ such that $g_\alpha|Y = \pi_\alpha \,|\, \varphi$ for all $\alpha < \kappa$. The map $\Phi = \Delta\ \{g_\alpha : \alpha < k\}: rY \to \mathbb{R}^k$ is continuous; if $y \in Y$ then

$$\Phi(y)(\alpha) = \pi_\alpha(\Phi(y)) = g_\alpha(y) = \pi_\alpha(\varphi(y)) = \varphi(y)(\alpha) \text{ for each } \alpha < \kappa$$

and therefore $\Phi(y) = \varphi(y)$ for all $y \in Y$. The set $\Phi(Y) = \varphi(Y) \subset P$ is dense in $\Phi(rY)$ so $\Phi(rY) \subset \overline{\varphi(Y)} \subset P$ because P is closed in $\mathbb{R}^\kappa$ Thus any continuous map φ of the space Y to a realcompact space P can be extended to a continuous map $\Phi : rY \to P$ so we can apply Problem 413 to conclude that rY is canonically homeomorphic to υY. Fact 1 is proved.

Now take any function $\varphi \in C(C_p(X))$; there exists a countable set $A \subset X$ and a continuous map $\delta : \pi_A(C_p(X)) \to \mathbb{R}$ such that $\delta \circ \pi_A = \varphi$ (Problem 300). Observe that $\pi_A(S_X) \subset \pi_A(C_p(X))$ and hence the map $\Phi = \delta \circ \pi_A : S_X \to \mathbb{R}$ is well defined on S_X. It is evident that $\Phi \in C(S_X)$ and $\Phi|C_p(X) = \varphi$ so we can apply Fact 1 to conclude that S_X is canonically homeomorphic to $\upsilon(C_p(X))$.

S.439. *Prove that, for any normal space X, there exists a space Y such that $C_p(Y)$ is homeomorphic to $\upsilon(C_p(X))$.*

Solution. If μ is a topology on X and $A \subset X$, let $\mu_A = \{U \cap A : U \in \mu\}$. In other words μ_A is the topology of subspace of (X, μ) induced on A. We denote the topology $\tau(X)$ by τ. The set $\upsilon(C_p(X))$ consists of all strictly ω-continuous functions on X (Problem 438). For any $f \in \upsilon(C_p(X))$, let $\tau(f) = \{f^{-1}(O) : O \in \tau(\mathbb{R})\}$. The family $\tau(f)$ is a completely regular (but, maybe, not T_1-) topology on X for each function $f \in \upsilon(C_p(X))$ (see Problem 098). It is easy to see that $\mathcal{N} = \bigcup\{\tau(f) : f \in \upsilon(C_p(X))\}$ generates a topology μ on X as a subbase (see Problem 008); apply Problem 099 to see that μ is a Tychonoff topology on X and let $Y = (X, \mu)$. The family $\mathcal{N}' = \bigcup\{\tau(f) : f \in C(X)\}$ is a base of τ(Fact 1 of S.437) and therefore $\tau \subset \mu$.

Given any countable $A \subset X$, and any $f \in \upsilon(C_p(X))$, the function $f|A$ is continuous on A; an easy consequence is that $\tau(f)_A \subset \tau_A$ and hence $\mu_A \subset \tau_A$ so $\mu_A = \tau_A$. Since $C_p(Y) = \{f : X \to \mathbb{R} : f \text{ is continuous with respect to } \mu\} \subset \mathbb{R}^X$, it suffices to show that $C_p(Y)$ coincides with $\upsilon(C_p(X))$. If $f \in \upsilon(C_p(X))$ then $\tau(f) \subset \mathcal{N}$ so $f^{-1}(O) \in \mathcal{N} \subset \mu$ for each $O \in \tau(\mathbb{R})$, i.e., f is continuous on Y. Thus $\upsilon(C_p(X)) \subset C_p(Y)$.

On the other hand, if $f \in C(Y)$ then $f|A$ is continuous on (A, μ_A) for each countable $A \subset X$. Since $\mu_A = \tau_A$, the map $f|A$ is continuous on (A, τ_A) which shows that f is ω-continuous on X. Now apply Problem 421 to conclude that f is strictly ω-continuous on X, i.e., $f \in \upsilon(C_p(X))$ (Problem 438) and hence $C_p(Y)$ coincides with $\upsilon(C_p(X))$.

S.440. *Give an example of a (non-normal) space X such that there is no space Y for which $\upsilon(C_p(X))$ is homeomorphic to $C_p(Y)$.*

Solution. There exists an infinite pseudocompact non-compact space X such that $C_p(X)$ is σ-pseudocompact and hence σ-bounded (see Problem 400). Then the space $\upsilon(C_p(X))$ is σ-compact (Problem 416), so if $C_p(Y)$ is homeomorphic to $\upsilon(C_p(X))$ then Y is finite (Problem 186). Therefore $C_p(Y)$ is metrizable as well as $C_p(X) \subset C_p(Y)$. This shows that X is countable (Problem 210) and hence compact (Problem 138) which is a contradiction. Thus there is no space Y such that $C_p(Y)$ is homeomorphic to $\upsilon(C_p(X))$.

S.441. *Suppose that $C_p(X)$ is a normal space. Prove that $\upsilon(C_p(C_p(X)))$ is homeomorphic to $C_p(C_p(\upsilon X))$.*

Solution. Let $\pi : C_p(\upsilon X) \to C_p(X)$ be the restriction map, i.e., $\pi(f) = f|X$ for every $f \in C_p(\upsilon X)$. Since X is dense and C-embedded in υX (Problem 413), the map π is a condensation (Problem 152). Observe that any bijection $b : P \to Q$ generates a homeomorphism $b^* : \mathbb{R}^Q \to \mathbb{R}^P$ defined by $b^*(f) = f \circ b$ for any $f \in \mathbb{R}^Q$. In fact, b^* coincides with the dual map for the map b if we consider P and Q as discrete spaces (Problem 163) for then $\mathbb{R}^P = C_p(P)$, $\mathbb{R}^Q = C_p(Q)$ and $b : P \to Q$ is a homeomorphism. We saw already that $\pi^{-1} : C_p(X) \to C_p(\upsilon X)$ is a bijection so $(\pi^{-1})^* : \mathbb{R}^{C_p(\upsilon X)} \to \mathbb{R}^{C_p(X)}$ is a homeomorphism; let $\Gamma = (\pi^{-1})^*|C_p(C_p(\upsilon X))$.

Given any function $\varphi \in C_p(C_p(\upsilon X))$, the map $\Gamma(\varphi) = \varphi \circ \pi^{-1} : C_p(X) \to \mathbb{R}$ is ω-continuous. Indeed, if $A \subset C_p(X)$ is countable then $\pi|\pi^{-1}(A) : \pi^{-1}(A) \to A$ is a homeomorphism (Problem 436). Therefore the map $\Gamma(\varphi)|A$ is continuous being a composition of continuous maps $\pi^{-1}|A$ and $\varphi|\pi^{-1}(A)$. Any ω-continuous function on a normal space is strictly ω-continuous (Problem 421), so $\Gamma(\varphi)$ is strictly ω-continuous, i.e., $\Gamma(\varphi) \in \upsilon(C_p(C_p(X))$ for every $\varphi \in C_p(C_p(\upsilon X))$ (Problem 438). Since the map $(\pi^{-1})^*$ is a homeomorphism, the map $\Gamma : C_p(C_p(\upsilon X)) \to \Gamma(C_p(C_p(\upsilon X)))$ is also a homeomorphism; we showed that $\Gamma(C_p(C_p(\upsilon X))) \subset \upsilon(C_p(C_p(X)))$.

Given any $\delta \in C_p(C_p(X))$, let $\varphi = \delta \circ \pi$. Then $\varphi \in C_p(C_p(\upsilon X))$ and $\Gamma(\varphi) = \varphi \circ \pi^{-1} = \delta \circ \pi \circ \pi^{-1} = \delta$. Thus $C_pC_p(X) \subset \Gamma(C_p(C_p(\upsilon X))) \subset \upsilon(C_p(C_p(X)))$; since $C_pC_p(\upsilon X)$ is a realcompact space (Problem 435), an easy application of 414 shows that $\Gamma(C_p(C_p(\upsilon X))) = \upsilon(C_p(C_p(X)))$ and therefore Γ is a homeomorphism between $C_p(C_p(\upsilon X))$ and $\upsilon(C_p(C_p(X)))$.

S.442. *Give an example of a realcompact space which is not hereditarily realcompact.*

Solution. The space ω_1 is not realcompact because it is a countably compact non-compact space (see Problem 407). Besides, $w(\omega_1) = \omega_1$ and hence ω_1 can be considered a subspace of $\mathbb{I}^{\omega_1}$ (Problem 209). The space $\mathbb{I}^{\omega_1}$ is compact and hence realcompact. However, it is not hereditarily realcompact because it has a subspace homeomorphic to ω_1 which is not realcompact.

S.443. *Prove that a space X is hereditarily realcompact if and only if $X \setminus \{x\}$ is realcompact for any $x \in X$.*

Solution. Of course, if X is hereditarily realcompact then $X \setminus \{x\}$ is realcompact for all $x \in X$. On the other hand, if $X \setminus \{x\}$ is realcompact for all $x \in X$ then, for any $Y \subset X$, we have $Y = \bigcap\{X\setminus\{x\} : x \in X\setminus Y\}$. Therefore, Y is realcompact being the intersection of realcompact subspaces of X (Problem 405).

S.444. *Prove that any realcompact space of countable pseudocharacter is hereditarily realcompact.*

Solution. Let X be a realcompact space with $\psi(X) \leqslant \omega$. Every $x \in X$ is a G_δ-set in X so $X \setminus \{x\}$ is an F_σ-subset of X; apply Problem 408 to conclude that $X \setminus \{x\}$ is realcompact for all $x \in X$. As a consequence, the space X is hereditarily realcompact by Problem 443.

S.445. *Give an example of a hereditarily realcompact space X with $\psi(X) > \omega$.*

Solution. The space $A(\omega_1)$ has uncountable pseudocharacter at its unique non-isolated point a. To prove that it is hereditarily realcompact take any point $x \in A(\omega_1)$. If $x \neq a$ then $A(\omega_1) \setminus \{x\}$ is compact and hence realcompact. If $x = a$ then $D = A(\omega_1) \setminus \{x\}$ is a discrete space of cardinality ω_1. We proved that $\mathrm{ext}(\mathbb{R}^{\omega_1}) = \omega_1$ (Fact 3 of S.215) and this means exactly that D embeds in $\mathbb{R}^{\omega_1}$ as a closed subspace. Hence the space D is also realcompact (Problem 401); we established that the space $A(\omega_1) \setminus \{x\}$ is realcompact for every $x \in A(\omega_1)$. Therefore, $A(\omega_1)$ is hereditarily realcompact by Problem 443.

S.446. *Prove that a space which condenses onto a second countable one is hereditarily realcompact.*

Solution. It follows from $iw(X) \leqslant \omega$ that $\psi(X) \leqslant \omega$ (Problem 156); observe also that $d(C_p(X)) = iw(X) \leqslant \omega$ (Problem 174). It is an immediate consequence of Problem 418 that $t_m(C_p(X)) \leqslant d(C_p(X)) \leqslant \omega$; now Problem 434 yields $q(X) = t_m(C_p(X)) \leqslant \omega$, i.e., X is a realcompact space of countable pseudocharacter. Finally, apply Problem 444 to conclude that X is hereditarily realcompact.

S.447. *Prove that $C_p(X)$ is hereditarily realcompact if and only if X is separable (and hence $\psi(C_p(X)) = iw(C_p(X)) = \omega$).*

Solution. Suppose that $C_p(X)$ is hereditarily realcompact. Given any $f \in C_p(X)$ the space $Z = C_p(X) \setminus \{f\}$ is dense in $C_p(X)$ and realcompact by our hypothesis. Since $C_p(X)$ is a Moscow space (Problem 424), the set Z is ω-placed in $C_p(X)$ (Problem 425). This means, there is a G_δ-set H in the space $C_p(X)$ such that $f \in H \subset C_p(X) \setminus Z = \{f\}$. Thus $H = \{f\}$ so $\{f\}$ is a G_δ-set in $C_p(X)$. This proves that $d(X) = \psi(C_p(X)) = iw(C_p(X)) = \omega$ (Problem 173).

On the other hand, if X is separable then $iw(C_p(X)) = \omega$ (Problem 173) so $C_p(X)$ is hereditarily realcompact by Problem 446.

S.448. *Let D be a discrete space. Prove that D is realcompact if and only if every ω_1-complete ultrafilter on the set D has a non-empty intersection.*

Solution. Suppose that D is realcompact and ξ is an ω_1-complete ultrafilter on D with $\bigcap \xi = \emptyset$. Since βD is compact, there is $z \in \bigcap\{\overline{A} : A \in \xi\}$ (the bar denotes the closure in βD). Since every $A \in \xi$ is closed in D, we have $\overline{A} \cap D = A$ for each $A \in \xi$. This shows that $z \notin D$ for otherwise $z \notin \overline{A} \cap D = A$ for each $A \in \xi$ and therefore $z \in \bigcap\{A : A \in \xi\} = \emptyset$ which is contradiction. Thus, $z \in \beta D \backslash D$; the space D being realcompact, there is a G_δ-set H in the space βD such that $z \in H \subset \beta D \backslash D$. Fix a sequence $\mathcal{S} = \{U_n : n \in \omega\} \subset \tau(\beta D)$ such that $H = \bigcap \mathcal{S}$. Let $V_n = U_n \cap D$ for all $n \in \omega$. Suppose that $V_n \notin \xi$ for some $n \in \omega$. Then $A = D \backslash V_n \in \xi$ (Problem 117), so $A \cap U_n = A \cap U_n \cap D = A \cap V_n = \emptyset$ which is a contradiction because $U_n \in \tau(z, \beta Z)$ and $z \in \overline{A}$. As a consequence, $V_n \in \xi$ for all $n \in \omega$ and we have $\bigcap\{V_n : n \in \omega\} = \bigcap\{U_n \cap D : n \in \omega\} = H \cap D = \emptyset$, a contradiction with the fact that ξ is ω_1-complete. This contradiction shows that for any realcompact discrete space D, any ω_1-complete ultrafilter on D has a non-empty intersection.

Fact 1. An ultrafilter ξ on a set D is ω_1-complete if and only if $\bigcap \gamma \neq \emptyset$ for any countable $\gamma \subset \xi$.

Proof. Necessity is trivial. To prove sufficiency, take any ultrafilter ξ on the set D such that $\bigcap \gamma \neq \emptyset$ for any countable $\gamma \subset \xi$. If $\gamma \subset \xi$ is countable and $A = \bigcap \gamma \notin \xi$ then $B = D \backslash A \in \xi$ (Problem 117) and hence $\gamma' = \gamma \cup \{B\}$ is a countable subfamily of ξ with $\bigcap \gamma' = \emptyset$ which is a contradiction. Fact 1 is proved.

Now assume that any ω_1-complete ultrafilter on D has a non-empty intersection. Given any $z \in \beta D \backslash D$, let us show that $\xi = \{U \cap D : U \in \tau(z, \beta Z)\}$ is an ultrafilter on D. Since any finite intersection of neighbourhoods of z is a neighbourhood of z and D is dense in βD, the family ξ is centered. Take any $A \subset D$ such that $A \notin \xi$. For $B = D \backslash A$ we have $\overline{A} \cap \overline{B} = \emptyset$ (Fact 1 of S.382) and $\beta D = \overline{A} \cup \overline{B}$. Therefore $\overline{A}$ and $\overline{B}$ are complementary closed sets and hence they are both open. If $z \in \overline{A}$ then $A = \overline{A} \cap D \in \xi$ which is a contradiction. Thus $z \in \overline{B}$ and therefore $D \backslash A = B = \overline{B} \cap D \in \xi$. We proved that ξ is a centered family of subsets of D such that $A \in \xi$ or $D \backslash A \in \xi$ for every $A \subset D$. Thus ξ is an ultrafilter by Problem 117. By our hypothesis, ξ cannot be ω_1-complete so there is a sequence $\{U_n : n \in \omega\} \subset \tau(z, \beta Z)$ such that $\bigcap\{U_n \cap D : n \in \omega\} = \emptyset$ (Fact 1). Thus $H = \bigcap\{U_n : n \in \omega\}$ is a G_δ-subset of βD and $z \in H \subset \beta D \backslash D$. This proves that D is ω-placed in βD and hence D is realcompact by Problem 401.

S.449. *Prove that any ω_1-complete ultrafilter on a set D has a non-empty intersection if and only if $t_m(\mathbb{R}^D) = \omega$.*

Solution. Every ω_1-complete ultrafilter on D has a non-empty intersection if and only if D is realcompact (Problem 448). The space D is realcompact if and only if $t_m(C_p(D)) = \omega$ (Problem 434). Since D is discrete, $C_p(D) = \mathbb{R}^D$ so $t_m(\mathbb{R}^D) = \omega$ if and only if every ω_1-complete ultrafilter on D has a non-empty intersection.

S.450. *Let D be a set of cardinality* $\leqslant$ **c**. *Prove that every ω_1-complete ultrafilter on the set D has a non-empty intersection.*

Solution. There exists a bijection of D onto a subset of $\mathbb{R}$. Since the properties of ultrafilters on a set are preserved by bijections, we can consider D to be a subset of $\mathbb{R}$. Give it the topology of a subspace of $\mathbb{R}$; then D is a second countable and hence Lindelöf space. Let ξ be an ω_1-complete ultrafilter on the set D. It is immediate that any countable subfamily of the family $\mathcal{F} = \{\overline{A} : A \in \xi\}$ has a non-empty intersection (the bar denotes the closure in D).

If $\bigcap \mathcal{F} = \emptyset$ then the family $\mathcal{U} = \{D\backslash\overline{A} : A \in \xi\}$ is an open cover of the Lindelöf space D. Consequently, there is a sequence $\{A_n : n \in \omega\} \subset \xi$ such that $\bigcup\{D\backslash\overline{A}_n : n \in \omega\} = D$ and hence $\bigcap\{A_n : n \in \omega\} \subset \bigcap\{\overline{A}_n : n \in \omega\} = \emptyset$ which contradicts ω_1-completeness of ξ. Thus, we can choose $x \in \bigcap \mathcal{F}$; let $\{W_n : n \in \omega\}$ be a local base at x (we can find one because D is even second countable).

If $W_n \notin \xi$ for some $n \in \omega$ then $A = X\backslash W_n \in \xi$ (Problem 117) and hence $x \notin \overline{A}$ which is a contradiction. Thus, $W_n \in \xi$ for all $n \in \omega$ and therefore $\{x\} = \bigcap \{W_n : n \in \omega\} \in \xi$. As a consequence, $x \in A$ for any $A \in \xi$ because otherwise $\{x\} \cap A = \emptyset$ so ξ is not even a centered family. This shows that $x \in \cap\ \xi$ and makes our solution complete.

S.451. *Suppose that a non-empty space X_t is realcompact for any $t \in T$. Prove that the space $X = \bigoplus\{X_t : t \in T\}$ is realcompact if and only if every ω_1-complete ultrafilter on the set T has a non-empty intersection.*

Solution. If X is realcompact choose a point $x_t \in X_t$ for every $t \in T$. The set $D = \{x_t : t \in T\}$ is realcompact being closed in X (Problem 403). Since D is discrete, every ω_1-complete ultrafilter on D has a non-empty intersection (Problem 448). Since $t \mapsto x_t$ is a bijection between D and T, every ω_1-complete ultrafilter on T also has a non-empty intersection. This settles necessity.

Fact 1. If Z_t is an arbitrary space for any $t \in T$, let $Z = \bigoplus\{Z_t : t \in T\}$. We consider that each Z_t is a clopen subset of Z (see Problem 113). For an arbitrary $A \subset T$, let $Z_A = \bigcup \{Z_t : t \in A\}$. Suppose that $A, B \subset T$ and $A \cap B = \emptyset$. Then $\mathrm{cl}_{\beta Z}(Z_A) \cap \mathrm{cl}_{\beta Z}(Z_B) = \emptyset$. As a consequence, $\mathrm{cl}_{\beta Z}(Z_A)$ is open in βZ for any $A \subset T$.

Proof. Let $f(x) = 1$ for all $x \in Z_A$ and $f(x) = 0$ if $x \in Z\backslash Z_A$. It is clear that the function $f : Z \to \{0, 1\}$ is continuous, and hence there is $g \in C(\beta Z, \{0, 1\})$ with $g|Z = f$. The sets $g^{-1}(0)$ and $g^{-1}(1)$ are closed in βZ and disjoint; since $Z_A \subset f^{-1}(1)$ and $Z_B \subset f^{-1}(0)$, we have

$$\mathrm{cl}_{\beta Z}(Z_A) \cap \mathrm{cl}_{\beta Z}(Z_B) \subset \mathrm{cl}_{\beta Z}(f^{-1}(1)) \cap \mathrm{cl}_{\beta Z}(f^{-1}(0)) \subset g^{-1}(1) \cap g^{-1}(0) = \emptyset.$$

Finally, $\beta Z = \mathrm{cl}_{\beta Z}(Z_A) \cup \mathrm{cl}_{\beta Z}(Z_{T \setminus A})$ so $\beta Z \setminus \mathrm{cl}_{\beta Z}(Z_A) = \mathrm{cl}_{\beta Z}(Z_{T \setminus A})$ is a closed set whence $\mathrm{cl}_{\beta Z}(Z_A) \in \tau(\beta Z)$. Fact 1 is proved.

Fact 2. Let Z be an arbitrary space. If F is C^*-embedded in Z then the space $\mathrm{cl}_{\beta Z}(F)$ is canonically homeomorphic to βF. In particular, if Z is a normal space and F is a closed subset of Z then the space $\mathrm{cl}_{\beta Z}(F)$ is canonically homeomorphic to βF, which we will write as $\mathrm{cl}_{\beta Z}(F) = \beta F$.

Proof. Take any $f \in C(F, \mathbb{I})$; since F is C^*-embedded in Z, there exists a function $g \in C(Z, \mathbb{I})$ such that $g|F = f$. There is $h \in C(\beta Z, \mathbb{I})$ such that $h|Z = g$. It is evident that $f_1 = h|\mathrm{cl}_{\beta Z}(F)$ is a continuous extension of f over $\mathrm{cl}_{\beta Z}(F)$. Therefore $\mathrm{cl}_{\beta Z}(F)$ is a compact extension of F such that every function $f \in C(F, \mathbb{I})$ extends to $f_1 \in C(\mathrm{cl}_{\beta Z}(F), \mathbb{I})$. Applying Fact 1 of S.309, we conclude that $\mathrm{cl}_{\beta Z}(F) = \beta F$. Now, if Z is normal then every closed $F \subset Z$ is even C-embedded in Z so again $\mathrm{cl}_{\beta Z}(F) = \beta F$ and Fact 2 is proved.

Now assume that every ω_1-complete ultrafilter on T has a non-empty intersection and take an arbitrary point $z \in \beta X \setminus X$. For each index $t \in T$, we identify the set X_t with the respective open subspace of X (Problem 113). Given any $A \subset$ T, let $X_A = \bigcup\{X_t : t \in A\}$. We claim that the family $\xi = \{A \subset T : z \in \overline{X}_A\}$ is an ultrafilter on T (the bar denotes the closure in βX). To see that ξ is centered, take $A_1, \dots, A_n \in \xi$ and let $A = A_1 \cap \dots \cap A_n$. Clearly, $z \in \overline{X} = \overline{X}_A \bigcup \overline{X}_{T \setminus A}$ so if $z \notin \overline{X}_A$ then $z \in \overline{X}_{T \setminus A}$. However, $X_{T \setminus A} = X_{T \setminus A_1} \cup \dots \cup X_{T \setminus A_n}$ which implies $z \in \overline{X}_{T \setminus A_i}$ for some $i \leqslant n$. It turns out that $z \in \overline{X}_{T \setminus A_i} \cap \overline{X}_{A_i}$ which contradicts Fact 1. This shows that $z \in \overline{X}_A$ so $A \neq \emptyset$ and hence ξ is a centered family.

Given any $A \subset T$, we have $\beta X = \overline{X} = \overline{X}_A \cup \overline{X}_{T \setminus A}$ so if $z \notin \overline{X}_A$ then $z \in \overline{X}_{T \setminus A}$. This proves that, for any $A \subset T$, we have $A \in \xi$ or $T \setminus A \in \xi$. Thus ξ is an ultrafilter on T (Problem 117). Suppose first that $t_0 \in \bigcap \xi$; if $\{t_0\} \notin \xi$, then $A = T \setminus \{t_0\} \in \xi$ (Problem 117) and therefore $t_0 \notin A \in \xi$ which is a contradiction. Thus $\{t_0\} \in \xi$, i.e., $z \in \overline{X}_{t_0}$ and $z \notin \overline{X}_{T \setminus \{t_0\}}$ (Fact 1). Note that every clopen subspace of any space is C-embedded in that space, so, in particular, X_{t_0} is C-embedded in X. Thus Fact 2 is applicable to conclude that $\beta X_{t_0} = \overline{X}_{t_0}$.

The space X_{t_0} is realcompact, so there is a G_δ-set H in the space $\beta X_{t_0} = \overline{X}_{t_0}$ such that $z \in H \subset \overline{X}_{t_0} \setminus X_{t_0}$ (Problem 401). It is immediate that any G_δ-set in a clopen subspace $\overline{X}_{t_0}$ (Fact 1) is a G_δ-set in the whole space βX so H is G_δ-set in βX such that $z \in H \subset \beta X \setminus X$.

Now if $\bigcap \xi = \emptyset$, then ξ cannot be ω_1-complete by our hypothesis so there is a sequence $\{A_n : n \in \omega\} \subset \xi$ such that $\bigcap\{A_n : n \in \omega\} = \emptyset$ (Fact 1 of S.448) which implies $\bigcap\{X_{A_n} : n \in \omega\} = \emptyset$ and hence $\bigcup\{X_{T \setminus A_n} : n \in \omega\} = X$. Since $z \in \overline{X}_{A_n}$, we have $z \notin \overline{X}_{T \setminus A_n}$ for each $n \in \omega$ (Fact 1). Thus $U_n = \beta X \setminus X_{T \setminus A_n} \in \tau(z, \beta Z)$ for each n and

$$H = \bigcap_{n \in \omega} U_n = \beta X \setminus \left(\bigcup\{\overline{X}_{T \setminus A_n} : n \in \omega\}\right) \subset \beta X \setminus \left(\bigcup\{X_{T \setminus A_n} : n \in \omega\}\right) = \beta X \setminus X.$$

Thus for any $z \in \beta X \setminus X$, we found a G_δ-subset H of the space βX such that $z \in H \subset \beta X \setminus X$. Therefore X is realcompact by 401 so our solution is complete.

S.452. *Prove that a paracompact space X is realcompact if and only if every discrete closed subspace of X is realcompact. In particular, a metrizable space M is realcompact if and only if every closed discrete subspace of M is realcompact.*

Solution. If X is realcompact then any closed (not necessarily discrete) subspace of X is realcompact (Problem 403) so necessity is clear.

Now assume that all closed discrete subspaces of X are realcompact. Fix a point $z \in \beta X \setminus X$ and consider the family $\mathcal{U} = \{U \in \tau(X) : z \notin \mathrm{cl}_{\beta X}(U)\}$. It is clear that $\mathcal{U}$ is an open cover of X so take any σ-discrete open refinement $\mathcal{V}$ of the cover $\mathcal{U}$ (Problem 230). Then $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \omega\}$, where $\mathcal{V}_n$ is a discrete family for all numbers $n \in \omega$. For each $n \in \omega$, the family $\mathcal{F}_n = \{\overline{U} : U \in \mathcal{V}_n\}$ is also discrete (the bar denotes the closure in X) and $z \notin \mathrm{cl}_{\beta X}(F)$ for any $F \in \mathcal{F}_n$.

Observe that $\Phi_n = \bigcup_n$ is closed in X and homeomorphic to $\bigoplus\{F : F \in \mathcal{F}_n\}$ for each $n \in \omega$ because every $F \in \mathcal{F}_n$ is clopen in Φ_n (see Problem 113). Choose some faithful indexing $\{F_t^n : t \in T_n\}$ of the family $\mathcal{F}_n$; we lose no generality assuming that $F_t^n \neq \emptyset$ for each $t \in T_n$. If we take $x_t^n \in F_t^n$ for all $t \in T_n$, then we obtain a bijection $t \mapsto x_t^n$ between the set T_n and a closed discrete subset $D_n = \{x_t^n : t \in T_n\}$ of the space X. By our hypothesis, the space D_n is realcompact so every ω_1-complete ultrafilter on D_n has a non-empty intersection (Problem 448); since ultrafilter properties of a set are preserved by bijections, every ω_1-complete ultrafilter on T_n also has a non-empty intersection. Given any $A \subset T_n$, let $\Phi_n^A = \bigcup\{F_t^n : t \in A\}$.

Any paracompact space is normal (Problem 231) so we can apply Fact 2 of S.451 to conclude that we have $\mathrm{cl}_{\beta X}(\Phi_n) = \beta(\Phi_n)$ for all $n \in \omega$. If $z \notin \mathrm{cl}_{\beta X}(\Phi_n)$ then let $H_n = \beta X \setminus \mathrm{cl}_{\beta X}(\Phi_n)$; then H_n is an open set (and hence a G_δ-set) of the space βX such that $z \in H_n \subset \beta X \setminus \Phi_n$.

There is more job to do if $z \in \mathrm{cl}_{\beta X}(\Phi_n) = \beta(\Phi_n)$; we claim that, in this case, the family $\xi = \{A \subset T_n : z \in \mathrm{cl}_{\beta X}(\Phi_n^A)\}$ is an ultrafilter on T_n. To establish first that ξ is centered, take any $A_1, \dots, A_k \in \xi$ and let $A = A_1 \cap \dots \cap A_k$. Clearly, $z \in \mathrm{cl}_{\beta X}(\Phi_n) = \mathrm{cl}_{\beta X}(\Phi_n^A) \cup \mathrm{cl}_{\beta X}(\Phi_n^{T_n \setminus A})$ so if $z \notin \mathrm{cl}_{\beta X}(\Phi_n^A)$ then $z \in \mathrm{cl}_{\beta X}(\Phi_n^{T_n \setminus A})$. However, $\Phi_n^{T_n \setminus A} = \Phi_n^{T_n \setminus A_1} \cup \dots \cup \Phi_n^{T_n \setminus A_k}$ which implies $z \in \mathrm{cl}_{\beta X}(\Phi_n^{T_n \setminus A_i})$ for some $i \leqslant k$. It turns out that $z \in \mathrm{cl}_{\beta X}(\Phi_n^{T_n \setminus A_i}) \cap \mathrm{cl}_{\beta X}(\Phi_n^{A_i})$ which contradicts Fact 1 of S.451. This shows that $z \in \mathrm{cl}_{\beta X}(\Phi_n^A)$ so $A \neq \emptyset$ and hence ξ is a centered family.

Given any $A \subset T_n$, we have $\beta(\Phi_n) = \mathrm{cl}_{\beta X}(\Phi_n) = \mathrm{cl}_{\beta X}(\Phi_n^A) \cup \mathrm{cl}_{\beta X}(\Phi_n^{T_n \setminus A})$ so if $z \notin \mathrm{cl}_{\beta X}(\Phi_n^A)$ then $z \in \mathrm{cl}_{\beta X}(\Phi_n^{T_n \setminus A})$. This proves that, for any $A \subset T_n$, we have $A \in \xi$ or $T_n \setminus A \in \xi$. Thus ξ is an ultrafilter on T_n (Problem 117). Since $z \notin \mathrm{cl}_{\beta X}(F_t^n)$ for any $t \in T_n$, we have $\bigcap \xi = \emptyset$ and therefore the ultrafilter ξ cannot be ω_1-complete. Hence there is a sequence $\{A_k : k \in \omega\} \subset \xi$ such that $\bigcap\{A_k : k \in \omega\} = \emptyset$ (Fact 1 of S.448) which implies $\cap\{\Phi_n^{A_k} : k \in \omega\} = \emptyset$ and hence $\bigcup\{\Phi_n^{T_n \setminus A_k} : k \in \omega\} = \Phi_n$. Since $z \in \mathrm{cl}_{\beta X}(\Phi_n^{A_k})$, we have $z \notin \mathrm{cl}_{\beta X}(\Phi_n^{T_n \setminus A_k})$ for each $k \in \omega$ (Fact 1 of S.451). As a consequence, $U_k = \beta(\Phi_n) \setminus \mathrm{cl}_{\beta X}(\Phi_n^{T_n \setminus A_k}) \in \tau(z, \beta Z)$ for each $k \in \omega$ and, for the set $H_n' = \bigcap\{U_k : k \in \omega\}$, we have

$$H_n' = \beta(\Phi_n) \setminus (\bigcup_{k \in \omega} \mathrm{cl}_{\beta X}(\Phi_n^{T_n \setminus A_k})) \subset \beta(\Phi_n) \setminus (\bigcup_{k \in \omega} \Phi_n^{T_n \setminus A_k}) = \beta(\Phi_n) \setminus \Phi_n.$$

Thus, we found a G_δ-subset H_n' of the space $\beta(\Phi_n)$ such that $z \in H_n' \subset \beta(\Phi_n) \setminus \Phi_n$. It is an easy exercise that there exists a G_δ-set H_n in the space βX such that $H_n \cap \mathrm{cl}_{\beta X}(\Phi_n) = H_n'$; thus $z \in H_n \subset \beta X \setminus \Phi_n$.

Having constructed the sets H_n for all $n \in \omega$, let $H = \bigcap \{H_n : n \in \omega\}$. It is clear that H is a G_δ-set in the space βX; besides, $z \in H$ and $H \cap \Phi_n = \emptyset$ for all numbers $n \in \omega$. An immediate consequence is that $H \cap (\bigcup_{n \in \omega} \Phi_n) = \emptyset$; since $\bigcup_{n \in \omega} \Phi_n = X$, we have $z \in H \subset \beta X \setminus X$.

Summing up, we constructed, for an arbitrary $z \in \beta X \setminus X$, a G_δ-set H in the space βX such that $z \in H \subset \beta X \setminus X$. Therefore, X is realcompact by Problem 401 and hence sufficiency is proved. Finally observe that every metrizable space is paracompact (Problem 218) so our solution is complete.

S.453. *Let X be a realcompact space. Suppose that Y is a paracompact space and $f : X \to Y$ is a continuous onto map. Prove that Y is realcompact.*

Solution. Take any closed discrete $D \subset Y$; for each $d \in D$ pick a point $x_d \in f^{-1}(d)$.

It is immediate that the set D' is closed and discrete in X and $f|D' : D' \to D$ is a bijection. The space D' is realcompact because it is closed in X (Problem 403). Therefore, every ω_1-complete ultrafilter on the set D' has a non-empty intersection (see Problem 448). The ultrafilter properties on a set are preserved by bijections so every ω_1-complete ultrafilter on D also has a non-empty intersection. Applying Problem 448 again, we convince ourselves that D is realcompact. The set D has been chosen arbitrarily, so we proved that all closed discrete subspaces of Y are realcompact. By Problem 452 the space Y is realcompact.

S.454. *Observe that any realcompact space is Dieudonné complete. Prove that a Dieudonné complete space X is realcompact if and only if all closed discrete subspaces of X are realcompact.*

Solution. Any realcompact space X is homeomorphic to a closed subspace of a product of real lines (Problem 401); since every factor of this product is metrizable, the space X is Dieudonné complete. Now assume that X is Dieudonné complete and all closed discrete subspaces of X are realcompact.

We can consider that X is a closed subspace of a space $M = \Pi\{M_t : t \in T\}$ where each M_t is a metrizable space. Let $\pi_t : M \to M_t$ be the natural projection; denote by Y_t the set $\pi_t(X)$ for all $t \in T$. Fix any $t \in T$ and take any closed discrete $D \subset Y_t$; for each $d \in D$ pick a point $x_d \in \pi_t^{-1}(d) \cap X$. It is immediate that the set D' is closed and discrete in X and $\pi_t|D' : D' \to D$ is a bijection. The space D' is realcompact by our hypothesis and therefore every ω_1-complete ultrafilter on the set D' has a non-empty intersection (see Problem 448). The ultrafilter properties on a set are preserved by bijections so every ω_1-complete ultrafilter on D also has a non-empty intersection. Applying Problem 448 again, we convince ourselves that D is realcompact. The set D has been chosen arbitrarily, so we proved that all closed discrete subspaces of a metrizable space Y_t are realcompact. By Problem 452 the space Y_t is realcompact for all $t \in T$. Evidently, X is a closed subspace of $\Pi\{Y_t : t \in T\}$ so X is realcompact by Problems 402 and 403.

S.455. *Prove that any pseudocompact Dieudonné complete space is compact.*

Solution. Let X be a pseudocompact Dieudonné complete space. We can consider that X is a closed subspace of a space $M = \Pi\{M_t : t \in T\}$ where each M_t is a metrizable space. Given any $t \in T$, let $\pi_t : M \to M_t$ be the natural projection; denote by Y_t the set $\pi_t(X)$. The space Y_t is compact for all $t \in T$ (Problem 212) and X is a closed subspace of a compact space $Y = \Pi\{Y_t : t \in T\}$. Thus X is compact.

S.456. *Observe that any closed subspace of a Dieudonné complete space is a Dieudonné complete space; prove that any product of Dieudonné complete spaces is a Dieudonné complete space. Show that an open subspace of a Dieudonné complete space may fail to be Dieudonné complete.*

Solution. If X is Dieudonné complete space then X is a closed subspace of a space $M = \Pi\{M_t : t \in T\}$ where each M_t is a metrizable space. Of course, any closed subspace X is a closed subspace of the same product so every closed subspace of a Dieudonné complete space is Dieudonné complete.

Now assume that X_α is a Dieudonné complete space for reach $\alpha < \kappa$. This means that X_α is a closed subset of a product $M_\alpha = \prod\{M_t^\alpha : t \in T_\alpha\}$ where M_t^α is metrizable for all $t \in T_\alpha$. We lose no generality if we assume that $T_\alpha \cap T_\beta = \emptyset$ when $\alpha \neq \beta$. The space $\prod_{\alpha<\kappa} M_\alpha = \prod\{M_t^a : \alpha < \kappa, t \in T_\alpha\}$ (Problem 103) is also a product of metrizable spaces and $X = \prod\{X\alpha : \alpha < \kappa\}$ is a closed subspace of M. Thus X is also Dieudonné complete. This proves that any product of Dieudonné complete spaces is Dieudonné complete.

Finally, observe that ω_1 is an open subspace of a compact (and hence Dieudonné complete) space $\omega_1 + 1$. However, ω_1 is not Dieudonné complete being a pseudocompact non-compact space (Problem 455).

S.457. *Let X be an arbitrary space. Suppose that X_t is a Dieudonné complete subspace of X for any $t \in T$. Prove that $\cap\{X_t : t \in T\}$ is a Dieudonné complete subspace of X.*

Solution. The space $Y = \cap\{X_t : t \in T\}$ embeds in $\prod\{X_t : t \in T\}$ as a closed subspace (Fact 7 of S.271) so the Dieudonné completeness of Y follows from Problem 456.

S.458. *Let Y be a Dieudonné complete space with the Souslin property. Prove that Y is realcompact. Deduce from this fact that $C_p(X)$ is Dieudonné complete if and only if it is realcompact.*

Solution. Let Y be a Dieudonné complete space with $c(Y) = \omega$. We can consider that Y is a closed subspace of a space $M = \prod\{M_t : t \in T\}$ where each M_t is a metrizable space. Given any $t \in T$, let $\pi_t : M \to M_t$ be the natural projection; denote by Y_t the set $\pi_t(Y)$. We have $c(Y_t) = \omega$ for all $t \in T$; thus the space Y_t is second countable (Problem 214) and hence realcompact (Problem 406). The space Y is a closed subspace of a realcompact space $Z = \prod\{Y_t : t \in T\}$ (Problem 402). As a consequence, Y is realcompact (Problem 403). Finally observe that $C_p(X)$ has the Souslin property for any space X (Problem 111) so it is realcompact if and only if it is Dieudonné complete by Problem 454 and the preceding argument.

S.459. *Prove that X is Dieudonné complete if and only if it embeds as a closed subspace into a product of completely metrizable spaces.*

Solution. If X embeds as a closed subspace into a product of complete metric spaces, then it is trivially Dieudonné complete. Now assume that X is Dieudonné complete. We can consider that X is a closed subspace of $M = \prod\{M_t : t \in T\}$ where each M_t is a metrizable space.

For any $t \in T$ we can assume that M_t is a subspace of a complete metric space N_t (Problem 237); then $M_t = \bigcap\{N_t\backslash\{z\} : z \in N_t\backslash M_t\}$. The space $N_t\backslash\{z\}$ is open in N_t and hence completely metrizable for all $z \in N_t\backslash M_t$ (see Problems 269 and 260). Apply Fact 7 of S.271 to conclude that M_t is homeomorphic to a closed subspace of the space $M'_t = \prod\{N_t\backslash\{z\} : z \in N_t\backslash M_t\}$. As a consequence, the space M is homeomorphic to a closed subspace of the space $M' = \prod\{M'_t : t \in T\}$ which in turn is a product of complete metric spaces. Since X is closed in M and M is closed in M', the set X is also closed in M' which shows that any Dieudonné complete space is homeomorphic to a closed subspace of a product of complete metric spaces.

S.460. *Let X be a Dieudonné complete space. Prove that any F_σ-subspace of X is also Dieudonné complete.*

Solution. Call a set $U \subset X$ *functionally open in* X, if there exists $f \in C(X)$ and $V \in \tau(\mathbb{R})$ such that $U = f^{-1}(V)$.

Fact 1. Let R be a Dieudonné complete space. Suppose that Z is an arbitrary space and $f : R \to Z$ is a continuous map. Then $f^{-1}(B)$ is Dieudonné complete for any Dieudonné complete $B \subset Z$.

Proof. Recall that the graph $G(f) = \{(y, f(y)) : y \in R\}$ of the mapping f is closed in the space $R \times Z$ (see Fact 4 of S.390). If $f_B = f|B : f^{-1}(B) \to B$ then, for the graph $G(f_B) = \{(y, f(y)) : y \in f^{-1}(B)\}$ of the function f_B we have the equality $G(f_B) = G(f) \cap (R \times B)$. Since $G(f)$ is closed in $R \times Z$, the set $G(f_B)$ is closed in a Dieudonné complete space $R \times B$ (Problem 456) so $G(f_B)$ is Dieudonné complete. Applying Fact 4 of S.390 again we observe that $G(f_B)$ is homeomorphic to $f^{-1}(B)$ so $f^{-1}(B)$ is Dieudonné complete and Fact 1 is proved.

Returning to our solution observe that any $V \subset \mathbb{R}$ is realcompact and hence Dieudonné complete. It follows from Fact 1 and this observation that any functionally open subset of X is Dieudonné complete. Finally observe that any F_σ-subspace of X is an intersection of functionally open subsets of X (Fact 2 of S.408) and apply Problem 457 to conclude that any F_σ-subspace of X is Dieudonné complete.

S.461. *Suppose that a space X can be condensed onto a first countable Dieudonné complete space. Prove that X is Dieudonné complete.*

Solution. Suppose that $f : X \to Y$ is a condensation onto a first countable Dieudonné complete space Y. Fix a continuous map $g : \beta X \to \beta Y$ such that $g|X = f$ (Problem 258). The space βX is Dieudonné complete, so we can apply Fact 1 of S.460 to conclude that the set $Y' = g^{-1}(Y)$ is Dieudonné complete. It is evident that $X \subset Y'$; given any $z \in Y'\backslash X$, consider the set $F_z = g^{-1}(g(z)) \subset Y'$. The set $\{g(z)\}$ is a G_δ-set in Y because $\chi(Y) \leqslant \omega$. Therefore, F_z is a G_δ-set in Y'. Since f is a condensation, $F_z \cap X$ consists of exactly one point x_z. As a consequence, $H_z = F_z\backslash\{x_z\} \subset Y'\backslash X$ is also a G_δ-set in Y' so the set $G_z = Y'\backslash H_z$ is Dieudonné complete (Problem 460).

Since $z \notin G_z$ and $X \subset G_z$, we proved that, for any $z \in Y'$ there exists a Dieudonné complete space G_z such that $X \subset G_z \subset Y' \setminus \{z\}$. Therefore $X = \bigcap \{G_z : z \in Y' \setminus X\}$ so X is Dieudonné complete by Problem 457.

S.462. *Prove that every paracompact space is Dieudonné complete.*

Solution. Given spaces X and Y and a continuous map $f : X \to Y$, denote by $\hat{f}$ the unique continuous map from βX to βY such that $\hat{f}\,|X = f$ (the existence of $\hat{f}$ follows from Problem 257 and the uniqueness from Fact 0 of S.351).

Fact 1. For any paracompact space X and any $p \in \beta X \setminus X$ there exists a metrizable space M and a continuous map $\varphi : X \to M$ such that $\hat{\phi}(p) \in \beta M \setminus M$.

Proof. Let $\mathcal{U} = \{U \in \tau(X) : p \notin \mathrm{cl}_{\beta X}(U)\}$; it is evident that $\mathcal{U}$ is an open cover of X so there is a σ-discrete open refinement $\mathcal{V}$ of the cover $\mathcal{U}$. We have $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \omega\}$ where $\mathcal{V}_n$ is discrete for all $n \in \omega$. For each $n \in \omega$, the family $\mathcal{F}_n = \{\overline{V} : V \in \mathcal{V}_n\}$ is also discrete and $p \notin \mathrm{cl}_{\beta X}(F)$ for any $F \in \mathcal{F}_n$ (the bar denotes the closure in X).

For every $n \in \omega$, choose any faithful enumeration $\{F_t : t \in T_n\}$ of the family $\mathcal{F}_n$. We can assume, without loss of generality, that $T_n \cap T_m = \emptyset$ if $n \neq m$. Any paracompact space is collectionwise normal (Problem 231) so, for each $n \in \omega$, there exists a discrete family $\omega_n = \{O_t : t \in T_n\} \subset \tau(X)$ such that $F_t \subset O_t$ for each index $t \in T_n$. Making each set O_t smaller if necessary, we can consider that $p \notin \mathrm{cl}_{\beta X}(O_t)$ for all $t \in T_n$.

Apply normality of X to find $f_t \in C(X, [0, 1])$ such that $f_t(F_t) \subset \{1\}$ and $f_t(X \setminus O_t) \subset \{0\}$ for each $t \in T = \bigcup\{T_n : n \in \omega\}$.

We consider each T_n a space with the discrete topology. Given $f, g \in C^*(T_n)$ let $d_n(f, g) = \sup\{|f(t) - g(t)| : t \in T_n\}$; then d_n is a metric on $C^*(T_n)$ (see Problem 248) so we will consider $C^*(T_n)$ to be a metric space with the metric d_n. For any function $f \in C^*(T_n)$ and $r > 0$, let $B_n(f, r) = \{g \in C^*(T_n) : d_n(f, g) < r\}$.

Denote by u_n the function which is identically zero on T_n. Define a map $\varphi_n : X \to C^*(T_n)$ as follows: $\varphi_n(x) = u_n$ if $x \in X \setminus (\bigcup \mathcal{W}_n)$; if $x \in O_s$ for some $s \in T_n$ then let $\varphi_n(x)(s) = f_s(x)$ and $\varphi_n(x)(t) = 0$ for all $t \neq s$.

To see that φ_n is continuous, take any $x \in X$ and number $\varepsilon > 0$. Assume first that $x \in X \setminus (\bigcup \mathcal{W}_n)$ and hence $\varphi_n(x) = u_n$. There exists $U \in \tau(x, X)$ such that U meets at most one element of ω_n, say O_s. Since the map f_s is continuous, the set $P_s = \{y \in X : f_s(y) \geqslant \varepsilon\}$ is closed in X and contained in O_s whence $V = X \setminus P_s$ is an open neighbourhood of the point x. The set $W = U \cap V$ can only meet O_s so, for any $z \in W$, we have $\varphi_n(z)(t) = 0 = u_n(t)$ for all $t \neq s$. Since $z \notin Ps$, we have $\varphi_n(z)(s) < \varepsilon$; thus $d_n(\varphi_n(z), u_n) = |f_s(z)| = f_s(z) < \varepsilon$ which shows that $\varphi_n(W) \subset B_n(u_n, \varepsilon)$, i.e., W witnesses continuity of φ_n at the point x.

Now if $x \in O_s$ for some $s \in T$ then, by continuity of f_s, there is $U \in \tau(x, X)$ such that $|f_s(z) - f_s(x)| < \varepsilon$ for all $z \in U$. Let $W = U \cap O_s$; then for any $z \in W$ we have $\varphi_n(z)(t) = 0$ if $t \neq s$ and $\varphi_n(z)(s) = f_s(z)$ so $d_n(\varphi_n(z), \varphi_n(x)) = |f_s(x) - f_s(z)| < \varepsilon$ which shows that $\varphi_n(W) \subset B_n(\varphi_n(x), \varepsilon)$ so again the set W witnesses continuity of φ_n at the point x.

We proved that the map φ_n is continuous for each $n \in \omega$ and therefore the map $\varphi = \Delta_{n\in\omega}\varphi_n : X \to C = \Pi_{n\in\omega} C^*(T_n)$ is also continuous. We claim that the space $M = \varphi$ (X) and the map φ: $X \to M$ are as promised. Note first that M is metrizable because any countable product of metrizable spaces is a metrizable space (Problem 207) and any subspace of a metrizable space is also metrizable (Problem 203). Let $\pi_n : C \to C^*(T_n)$ be the natural projection for each $n \in \omega$.

Denote by $u \in \Pi_{n\in\omega} C^*(T_n)$ the function with $u(n) = u_n$ for each $n \in \omega$; our first observation is that $u \notin M$. Indeed, if $x \in X$ then there is $n \in \omega$ such that $x \in \bigcup \mathcal{F}_n$ because $\bigcup\{\mathcal{F}_n : n \in w\}$ is a cover of the space X. This means that $x \in F_t$ for some $t \in T_n$ and therefore $f_t(x) = 1$. As a consequence, $\varphi(x)(n) \neq u_n$ because $\varphi(x)(n)(t) = 1 \neq u_n(t) = 0$. Thus $\varphi(x) \neq u$ for all $x \in X$, i.e., $u \notin \varphi(X) = M$.

To show that $\hat{\varphi}(p) \in \beta M \backslash M$ suppose not. Then there is $x \in X$ such that $\hat{\varphi}(p) = \varphi(x)$. As a consequence $\varphi' = \hat{\varphi}|(X \cup \{p\} : X \cup \{p\} \to M$ is a continuous map. Take any $n \in \omega$ with $x \in \bigcup \mathcal{F}_n$; then $x \in F_t$ for some $t \in T_n$. We have $\pi_n(\hat{\varphi}(p)) = \pi_n(\varphi(x)) = \varphi_n(x)$ so the map $\varphi_n : X \to M_n = \varphi_n$ (X) extends to a continuous map $\varphi' : X \cup \{p\} \to M_n$ in such a way that $\varphi'(p)(t) = 1$ and $\varphi'(p)(s) = 0$ for all $s \neq t$.

Let $H = \{f \in M_n : f(t) = 0\}$; it is immediate that $d_n(f, \varphi'(p)) = 1$ for any function $f \in H$ and therefore $\varphi'(p) \notin \text{cl}_{Mn}(H)$. However, $p \notin \text{cl}_{\beta X}(Ot)$ which implies that $p \in \text{cl}_{\beta X}(X \setminus O_t)$; by continuity of φ', we have $\varphi'(p) \in \text{cl}_{M_n}(\varphi'(X \setminus O_t))$. It is evident that $\varphi'(X \setminus O_t) = \varphi_n(X \setminus O_t) \subset H$ so $\varphi'(p) \in \text{cl}_{M_n}(H)$ which is a contradiction. Thus, $\hat{\varphi}(p) \notin M$ and Fact 1 is proved.

Returning to our solution take any paracompact space X; there exists a set A such that βX embeds in $\mathbb{I}^A$ (Problem 209) so we consider βX to be a subspace of $\mathbb{I}^A$. For every $p \in \beta X \setminus X$ fix a metrizable space M_p and a continuous map $\varphi_p : X \to M_p$ such that $\hat{\varphi}_p(p) \in \beta M_p \backslash M_p$ (see Fact 1). Then $\varphi = \Delta\{\varphi_p : p \in \beta X \setminus X\}$ maps X continuously to the space $M = \prod\{M_p : p \in \beta X \setminus X\}$. It is clear that the space $K = \prod\{\beta M_p : p \in \beta X \setminus X\}$ is a compact extension of M and we have a continuous map $\Phi : \beta X \to K$ defined by $\Phi = \Delta\{\hat{\varphi}_p : p \in \beta X \backslash X\}$; it is immediate that $\Phi|X = \varphi$. For any point $p \in \beta X \setminus X$, denote by $\pi_p : M \to M_p$ the natural projection.

The graph $G(\Phi) = \{(z, \Phi(z)) : z \in \beta X\}$ is a closed subspace of $\beta X \times K$ (Fact 4 of S.390) so $G(\Phi)$ is compact. Therefore $G(\Phi)$ is closed in $\mathbb{I}^A \times K$. We claim that $G(\Phi) \cap (\mathbb{I}^A \times M) = G(\varphi) = \{(x, \varphi(x)) : x \in X\}$. Indeed, $G(\varphi) \subset G(\Phi)$ and $G(\varphi) \subset X \times M \subset \mathbb{I}^A \times M$ which implies $G(\varphi) \subset G(\Phi) \cap (\mathbb{I}^A \times M)$. On the other hand, if $p \in \beta X \setminus X$ then $\hat{\varphi}_p(p) \in \beta M_p \backslash M_p$; as a consequence, $\pi_p(\Phi(p)) = \hat{\varphi}_p(p) \in \beta M_p \backslash M_p$ which shows that $\Phi(p) \notin M$ so the point $(p, \Phi(p))$ does not belong to $\mathbb{I}^A \times$ M. Therefore, $(p, \Phi(p)) \notin G(\Phi) \cap (\mathbb{I}^A \times M)$ for all $p \in \beta X \setminus X$ and hence $G(\Phi) \cap (\mathbb{I}^A \times M) \subset G(\varphi)$, i.e., $G(\Phi) \cap (\mathbb{I}^A \times M) = G(\varphi)$. Thus, $G(\varphi)$ is a closed subspace of $\mathbb{I}^A \times M$ which is a product of metric spaces. Observe finally that $G(\varphi)$ is homeomorphic to X (Fact 4 of S.390) and hence X is embeddable in $\mathbb{I}^A \times M$ which is a product of metric spaces. Our solution is complete.

S.463. *Prove that X is Dieudonné complete if and only if, for any $z \in \beta X \setminus X$, there exists a paracompact $Y \subset \beta X$ such that $X \subset Y \subset \beta X \setminus \{z\}$.*

Solution. Assume that, for every $z \in \beta X \setminus X$ there exists a paracompact Y_z such that $X \subset Y_z \subset \beta X \setminus X$. It is evident that $X = \cap\{Y_z : z \in \beta X \setminus X\}$; each Y_z is Dieudonné complete (Problem 462) so we can apply Problem 457 to conclude that X is Dieudonné complete. This proves sufficiency.

Fact 1. Let $f: X \to Y$ be a perfect map. If Y is paracompact, then X is also paracompact.

Proof. Take any open cover $\mathcal{U}$ of the space X. Given any $y \in Y$, the set $f^{-1}(y)$ is compact so there is a finite $\mathcal{U}(y) \subset \mathcal{U}$ such that $f^{-1}(y) \subset \bigcup \mathcal{U}(y)$. By Fact 1 of S.226 there is $V_y \in \tau(y, Y)$ such that $f^{-1}(V_y) \subset \bigcup \mathcal{U}(y)$. The family $\{V_y : y \in Y\}$ is an open cover of Y so there exists a σ-discrete refinement $\mathcal{V} \subset \tau(Y)$ of the cover $\{V_y : y \in Y\}$ (Problem 230). We have $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \omega\}$ where each $\mathcal{V}_n$ is discrete in Y. It is clear that the family $\mathcal{W}_n = \{f^{-1}(V) : V \in \mathcal{V}_n\}$ is σ-discrete in X and $\mathcal{W} = \bigcup\{\mathcal{W}_n : n \in \omega\}$ is an open cover of the space X. Let $\{W_t^n : t \in T_n\}$ be a faithful enumeration of the family $\mathcal{W}_n$ for all $n \in \omega$. For each $t \in T_n$ there is $y_t \in Y$ such that $W_t^n \subset \bigcup \mathcal{U}(y_t)$; let $\mathcal{W}'_n = \{W_n^t \cap U : t \in T_n, U \in \mathcal{U}(y_t)\}$.

Given any point $x \in X$ there is $U \in \tau(x, X)$ such that U meets at most one element of $\mathcal{W}_n$, say W_t^n. Since the family $\mathcal{U}(y_t)$ is finite, the set U meets at most the elements of the family $\mathcal{W}(n, t) = \{W_t^n \cap U : U \in \mathcal{U}(y_t)\}$ which is finite. This shows that the family $\mathcal{W}'_n$ is locally finite for every $n \in \omega$. Since $W_t^n = \bigcup \mathcal{W}(n, t)$ for all $t \in T_n$, we have $\bigcup \mathcal{W}_n = \bigcup \mathcal{W}'_n$ and hence the family $\mathcal{W}' = \bigcup\{\mathcal{W}'_n : n \in \omega\}$ is a σ-locally finite refinement of the cover $\mathcal{U}$. Applying Problem 230 once more we conclude that X is paracompact. Fact 1 is proved.

Returning to our solution, assume that X is Dieudonné complete and take any point $z \in \beta X \setminus X$. We can consider that the space X is a closed subspace of a product $M = \Pi\{M_t : t \in T\}$ where each M_t is a metric space. We will need the space $K = \Pi\{\beta M_t : t \in T\}$ which is compact and contains M; given any $t \in T$, let $\pi_t : K \to \beta M_t$ be the natural projection. Evidently, $cX = \mathrm{cl}_K(X)$ is a compact extension of the space X. There exists a continuous map $f: \beta X \to cX$ such that $f(x) = x$ for all $x \in X$ (Problem 258). Then $f(\beta X \setminus X) \subset cX \setminus X$ (Fact 1 of S.259) and, in particular, $f(z) \in cX \setminus X$. Since $\mathrm{cl}_M(X) = X = \mathrm{cl}_K(X) \cap M$ the point $f(z)$ cannot belong to M for otherwise $f(z) \in \mathrm{cl}_M(X) \setminus X$ which is a contradiction. Thus there exists $t \in T$ such that $\pi_t(f(z)) \in \beta M_t \setminus M_t$. We have $h = \pi_t \circ f: \beta X \to \beta M_t$ and $h(z) \in \beta M_t \setminus M_t$. The map h is perfect (see Problem 122) and therefore the map $h_1 = h|h^{-1}(M_t) : h^{-1}(M_t) \to M_t$ is also perfect (Fact 2 of S.261). The space M_t is metric and hence paracompact; thus $Y = h^{-1}(M_t)$ is also paracompact by Fact 1. Finally observe that $h(X) \subset M_t$ and $h(z) \in \beta M_t \setminus M_t$ whence $X \subset Y \subset \beta X \setminus \{z\}$. This proves necessity and makes our solution complete.

S.464. *Prove that any pseudocomplete space has the Baire property.*

Solution. Let X be a pseudocomplete space and fix a pseudocomplete sequence $\{\mathcal{B}_n : n \in \omega\}$ of π-bases in X. Suppose that some $U \in \tau^*(X)$ is of first category, i.e., there is a family $\{P_n : n \in \omega\}$ of closed nowhere dense subsets of X such that $U \subset \bigcup_{n \in \omega} P_n$. Since P_0 is nowhere dense, the open set $U \setminus P_0$ is non-empty, so there is $W_0 \in \mathcal{B}_0$ such that $W_0 \subset U \setminus P_0$. Assume that we have sets W_i, $i \leqslant n$ with the following properties:

(1) $\overline{W}_{i+1} \subset W_i$ for all $i = 0, \dots n-1$.
(2) $W_i \subset U$ and $W_i \in \mathcal{B}_i$ for all $i \leqslant n$.
(3) $Wi \cap P_i = \emptyset$ for every $i \leqslant n$.

The open set $W_n \setminus P_{n+1} \subset W_n \subset U$ is non-empty because P_{n+1} is nowhere dense; it follows easily from regularity of the space X and from the fact that $\mathcal{B}_{n+1}$ is a π-base in X that there exists $W_{n+1} \in \mathcal{B}_{n+1}$ such that $\overline{W}_{n+1} \subset W_n \backslash P_{n+1}$. It is straightforward that the sets $W_0, \dots, W_n, W_{n+1}$ have the properties (1)–(3).

Thus, our inductive construction can be continued giving us a sequence $\{W_i : i \in \omega\}$ with the properties (1)–(3). Since our sequence of π-bases is pseudocomplete, we have $F = \bigcap\{W_i : i \in \omega\} \neq \emptyset$. If $x \in F$ then $x \in U$ by (2); the property (3) implies $x \notin P_i$ for all $i \in \omega$. Thus, $x \in U \backslash \bigcup_{i \in \omega} P_i$ which is a contradiction. Therefore, every non-empty open $U \subset X$ is of second category, i.e., X has the Baire property.

S.465. *Prove that any Čech-complete space is pseudocomplete.*

Solution. Call a space Z *strongly pseudocomplete* if it has a pseudocomplete sequence of bases. We are going to prove the following fact for future references.

Fact 1. Any Čech-complete space is strongly pseudocomplete.

Proof. Take a Čech-complete space X; we can choose $\{O_n : n \in \omega\} \subset \tau(\beta X)$ such that $X = \bigcap\{O_n : n \in \omega\}$. Observe that $\mathcal{B}_n = \{U \in \tau^*(X) : \mathrm{cl}_{\beta X}(U) \subset O_n\}$ is a base in X for each $n \in \omega$. Indeed, if $x \in X$ and $V \in \tau(x, X)$ then take any $W \in \tau(\beta X)$ with $W \cap X = V$ and observe that $x \in W \cap O_n \in \tau(x, \beta X)$. Choose any $W' \in \tau(x, \beta X)$ with $\mathrm{cl}_{\beta X}(W') \subset W \cap O_n$; then $W' \cap X \in \mathcal{B}_n$ and $x \in W' \cap X \subset V$.

Therefore, each $\mathcal{B}_n$ is a base in X so it suffices to prove that the sequence $\{\mathcal{B}_n : n \in \omega\}$ is pseudocomplete. Take any family $\{U_i : i \in \omega\}$ such that $U_i \in \mathcal{B}_i$ and $\overline{U}_{i+1} \subset U_i$ for all $i \in \omega$ (the bar denotes the closure in X). Since βX is compact, there is $z \in \bigcap\{\mathrm{cl}_{\beta X}(U_i) : i \in \omega\}$. Since $\mathrm{cl}_{\beta X}(U_i) \subset O_i$, the point z belongs to O_i for each $i \in \omega$ and therefore $z \in \bigcap\{O_i : i \in \omega\} = X$. As a consequence, $z \in \mathrm{cl}_{\beta X}(U_i) \cap X = \overline{U}_i$ for all $i \in \omega$. Thus $z \in \bigcap\{\overline{U}_i : i \in \omega\} = \cap\{U_i : i \in \omega\}$ and hence $\bigcap\{U_i : i \in \omega\} \neq \emptyset$. The pseudocompleteness of the sequence $\{\mathcal{B}_n : n \in \omega\}$ being established, we proved that X is strongly pseudocomplete so Fact 1 is proved.

Now, to finish our solution it suffices to apply Fact 1 and observe that any strongly pseudocomplete space is pseudocomplete.

S.466. *Prove that any non-empty open subspace of a pseudocomplete space is pseudocomplete.*

Solution. Let X be a pseudocomplete space; fix a sequence $\{\mathcal{B}_n : n \in \omega\}$ of π-bases of X witnessing this. Given any $U \in \tau^*(X)$, let $\mathcal{B}'_n = \{V \cap U : V \in \mathcal{B}_n\}$ for each number $n \in \omega$. It is immediate that the family $\left\{\mathcal{B}'_n : n \in \omega\right\}$ is a pseudocomplete sequence of π-bases in U.

S.467. *Suppose that X has a dense pseudocomplete subspace. Prove that X is pseudocomplete. In particular, if X has a dense Čech-complete subspace then X is pseudocomplete.*

Solution. Let Y be a dense pseudocomplete subspace of X; take any pseudocomplete sequence $\{\mathcal{B}_n : n \in \omega\}$ of π-bases of Y. For any $U \in \tau^*(Y)$ fix $U' \in \tau(X)$ such that $U' \cap Y = U$. Let $\mathcal{B}'_n = \{U' : U \in \mathcal{B}_n\}$ for each $n \in \omega$.

Each $\mathcal{B}'_n$ is a π-base in X; to see this, take any $W \in \tau^*(X)$. Pick any set $V \in \tau^*(X)$ with $\overline{V} \subset W$; the family $\mathcal{B}_n$ being a π-base in Y, there is $U \in \mathcal{B}_n$ such that $U \subset V \cap Y$. Then $U' \subset \overline{U}' = \overline{U} \subset \overline{V} \subset W$ so $U' \subset W$ and $U' \in \mathcal{B}'_n$ which shows that $\mathcal{B}'_n$ is a π-base in X for all $n \in \omega$. Finally suppose that $U_i \in \mathcal{B}'_i$ and $\overline{U}_{i+1} \subset U_i$ for each $i \in \omega$. Then $V_i = U_i \cap Y \in \mathcal{B}_i$ and $\mathrm{cl}_Y(V_{i+1}) = \overline{V}_{i+1} \cap Y = \overline{U}_{i+1} \cap Y \subset U_i \cap Y = V_i$ for each $i \in \omega$. Since $\{\mathcal{B}_i : i \in \omega\}$ is a pseudocomplete sequence, we have $\bigcap\{V_i : i \in \omega\} \neq \emptyset$ and therefore $\bigcap\{U_i : i \in \omega\} \supset \bigcap\{V_i : i \in \omega\} \neq \emptyset$ so X is pseudocomplete.

Finally suppose that X has a dense Čech-complete subspace Y; then Y is pseudocomplete (Problem 465). We proved that any space with a dense pseudocomplete subspace is pseudocomplete so X is pseudocomplete.

S.468. *Prove that a metrizable space is pseudocomplete if and only if it has a dense Čech-complete subspace.*

Solution. If any space has a dense Čech-complete subspace then it is pseudocomplete (Problem 467) so sufficiency is clear.

Fact 1. Let (X, d) be a metric space. Suppose that we have a sequence $\{\mathcal{U}_n : n \in \omega\}$ of families of non-empty open subsets of X with the following properties:

(1) $\mathcal{U}_n$ is disjoint and $\bigcup \mathcal{U}_n$ is dense in X for any $n \in \omega$.
(2) diam $(U) \leqslant \frac{1}{n+1}$ for all $U \in \mathcal{U}_n$ and all $n \in \omega$.
(3) For any $U \in \mathcal{U}_{n+1}$ there is $V \in \mathcal{U}_n$ such that $\overline{U} \subset V$.
(4) The sequence $\{\mathcal{U}_n : n \in \omega\}$ is pseudocomplete.

Then $D = \bigcap\{\bigcup \mathcal{U}_n : n \in \omega\}$ is a dense Čech-complete subspace of X.

Proof. To see that D is dense, take any $O \in \tau^*(X)$; fix any $x \in O$ and $r > 0$ such that $B(x, r) \subset O$. Pick any $n \in \omega$ with $\frac{1}{n} < r/2$ and observe that, by density of $\bigcup \mathcal{U}_n$, there is $U \in \mathcal{U}_n$ such that $U \cap B(x, 1/n) \neq \emptyset$. Choose any $z \in U \cap B(x, 1/n)$ and note that if $y \in U$ then

$$d(y,x) \leqslant d(y,z) + d(z,x) \leqslant \mathrm{diam}(U) + 1/n \leqslant 1/n + 1/n = 2/n < r.$$

This shows that $U \subset B(x, 2/n) \subset B(x, r) \subset O$. Apply (3) to find $U_i \in \mathcal{U}_i, i \leqslant n$ such that $U_0 \supset \cdots \supset U_n = U$.

By (1), there exists $U_{n+1} \in \mathcal{U}_{n+1}$ such that $U_{n+1} \cap U_n \neq \emptyset$; the properties (1) and (3) imply $\overline{U}_{n+1} \subset U_n$. This construction can be continued inductively giving us a sequence $\{U_i : i \in \omega\}$ such that $U_i \in \mathcal{U}_i, U_n = U$ and $\overline{U}_{i+1} \subset U_i$ for all $i \in \omega$. The property (4) says that there exists $x \in \bigcap\{U_i : i \in \omega\}$; it is clear that $x \in D \cap U \subset D \cap O$ so $D \cap O \neq \emptyset$ which proves that D is dense in X.

To see that D is Čech-complete, let $\mathcal{V}_n = \{U \cap D : U \in \mathcal{U}_n\}$ for all $n \in \omega$. Since $D \subset \bigcup \mathcal{U}_n$, we have $D = \bigcup \mathcal{V}_n$, i.e., $\mathcal{V}_n$ is an open cover of the space D for each $n \in \omega$.

Assume that $\mathcal{F}$ is a filter on D dominated by the sequence $\{\mathcal{V}_n : n \in \omega\}$, i.e., for any $n \in \omega$ there is $V_n \in \mathcal{V}_n$ such that $F_n \subset V_n$ for some $F_n \in \mathcal{F}$. Take $U_n \in \mathcal{U}_n$ such that $U_n \cap D = V_n$ for all $n \in \omega$. We have $U_{n+1} \cap U_n \supset V_{n+1} \cap V_n \supset F_{n+1} \cap F_n \neq \emptyset$ because $\mathcal{F}$ is a filter. Thus, $U_{n+1} \cap U_n \neq \emptyset$ whence $\overline{U}_{n+1} \subset U_n$ for all $n \in \omega$ by (1) and (3). The property (4) implies that there is $x \in \bigcap\{U_n : n \in \omega\}$. Clearly, $x \in D$; we claim that $x \in \bigcap\{\overline{F} : F \in \mathcal{F}\}$. Indeed, suppose that there is $F \in \mathcal{F}$ such that $x \notin F$. There is a number $r > 0$ such that $B(x, r) \cap F = \emptyset$; choose any number $n \in \omega$ with $\frac{1}{n} < r$. Pick any $y \in U_n$; then $d(y, x) \leqslant \text{diam}\,(U_n) \leqslant 1/n < r$ (note that $x, y \in U_n$ so $d(x, y) \leqslant \text{diam}(U_n)$). Thus, $U_n \subset B(x, r)$ and therefore $U_n \cap F = \emptyset$. However, there is $F_n \in \mathcal{F}$ with $F_n \subset V_n \subset U_n$ and hence $F_n \cap F = \emptyset$ which is a contradiction because $\mathcal{F}$ is a filter. An easy consequence is that $x \in \bigcap\{\text{cl}_D(F) : F \in \mathcal{F}\}$.

We proved that, for every filter $\mathcal{F}$ dominated by the sequence $\{\mathcal{V}_n : n \in \omega\}$, we have $\bigcap\{\text{cl}_D(F) : F \in \mathcal{F}\} \neq \emptyset$. Applying Problem 268, we conclude that D is Čech-complete so Fact 1 is proved.

Returning to our solution, suppose that (X, d) is a pseudocomplete metric space; fix a pseudocomplete sequence $\{B_n : n \in \omega\}$ of π-bases of X. For any $n \in \omega$ consider the family $\mathcal{B}'_n = \left\{U \in \mathcal{B}_n : \text{diam}(U) < \frac{1}{n+1}\right\}$. It is immediate that $\mathcal{B}'_n$ is also a π-base in X for all $n \in \omega$ and the sequence $\{\mathcal{B}'_n : n \in \omega\}$ is pseudocomplete. Therefore we lose no generality if we assume that $\text{diam}(U) < \frac{1}{n+1}$ for every $U \in \mathcal{B}_n$ and $n \in \omega$.

Let $\mathcal{U}_0$ be a maximal disjoint subfamily of $\mathcal{B}_0$; it is an easy exercise that $\bigcup \mathcal{U}_0$ is dense in X. Suppose that we have families $\mathcal{U}_0, \ldots, \mathcal{U}_n$ with the following properties:

(a) $\mathcal{U}_i$ is disjoint and $\mathcal{U}_i \subset \mathcal{B}_i$ for all $i \leqslant n$.
(b) For any $i < n$ and $U \in \mathcal{U}_{i+1}$ there is $V \in \mathcal{U}_i$ such that $\overline{U} \subset V$.
(c) The set $\bigcup \mathcal{U}_i$ is dense in X for all $i \leqslant n$.

For a fixed $U \in \mathcal{U}_n$ call a family $\mathcal{C} \subset \mathcal{B}_{n+1}$ strongly inscribed in U if $\overline{V} \subset U$ for any $V \in \mathcal{C}$. Let $\mathcal{U}^U_{n+1}$ be a maximal disjoint subfamily of $\mathcal{B}_{n+1}$ strongly inscribed in U. It is straightforward that $\bigcup \mathcal{U}^U_{n+1}$ is dense in U; let $\mathcal{U}_{n+1} = \bigcup\{\mathcal{U}^U_{n+1} : U \in \mathcal{U}_n\}$. We skip an easy checking that the properties (a)–(c) are fulfilled for the families $\mathcal{U}_0, \ldots, \mathcal{U}_n, \mathcal{U}_{n+1}$. Thus our inductive construction can be continued to provide a sequence $\{\mathcal{U}_n : n \in \omega\}$ with the properties (a)–(c). Recalling that $\text{diam}(U) < \frac{1}{n+1}$ for any $U \in \mathcal{U}_n$, we can convince ourselves that properties (1)–(4) from Fact 1 hold for the sequence $\{\mathcal{U}_n : n \in \omega\}$. Thus Fact 1 is applicable to conclude that we have a dense Čech-complete subspace $D \subset X$. This settles necessity so our solution is complete.

S.469. *Give an example*

(*a*) *Of a Baire space which is not pseudocomplete.*
(*b*) *Of a pseudocomplete space which has no dense Čech-complete subspace. Observe that it is an immediate consequence of (b) that there exist pseudocomplete non-Čech- complete spaces.*

Solution. (a) Let N be a countably infinite set. Take an arbitrary ultrafilter ξ on N such that $\bigcap \xi = \emptyset$. Denote by N_ξ the set $N \cup \{\xi\}$ with the topology $\tau_\xi = \{A :$

$A \subset N\} \cup \{B : \xi \in B$ and $N \backslash B \notin \xi\}$. Then τ_ξ is a Tychonoff topology on N_ξ and $C_p(N_\xi)$ is a Baire space (Problem 279). Assume that $C_p(N_\xi)$ is pseudocomplete; since it is metrizable (Problem 210), it has a dense Čech-complete subspace (Problem 468) so N_ξ has to be discrete (Problem 265) which it is not. This contradiction shows that $C_p(N_\xi)$ is a Baire space that is not pseudocomplete.

(b) It is easy to see that any pseudocompact space Z is pseudocomplete (take $\mathcal{B}_n = \tau^*(X)$ for all $n \in \omega$) so it suffices to give an example of a pseudocompact space that has no dense Čech-complete subspace. There exists an infinite pseudocompac space X such that $C_p(X, \mathbb{I})$ is pseudocompact and hence pseudocomplete (see Problems 398 and 400). If $C_p(X, \mathbb{I})$ has a dense Čech-complete subspace then X has to be discrete (Problem 287) which it is not. Therefore $C_p(X, \mathbb{I})$ is an example of a pseudocomplete space that has no dense Čech-complete subspace.

S.470. *Prove that any product of pseudocomplete spaces is a pseudocomplete space.*

Solution. Given any space Z and any π-bases $\mathcal{B}$ and $\mathcal{C}$ in Z, let $\mathcal{B}[\mathcal{C}] = \{U \in \mathcal{B} :$ there exists $V \in \mathcal{C}$ such that $\overline{U} \subset V\} \subset \mathcal{B}$. It is straightforward that $\mathcal{B}[\mathcal{C}]$ is a π-base in Z. We write $\mathcal{B} < \mathcal{C}$ if for any $U \in \mathcal{B}$ there is $V \in \mathcal{C}$ such that $\overline{U} \subset V$. This relation is clearly transitive, i.e., for any π-bases $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in the space Z, if $\mathcal{A} < \mathcal{B}$ and $\mathcal{B} < \mathcal{C}$, then $\mathcal{A} < \mathcal{C}$. Another trivial observation is that $\mathcal{A}[\mathcal{B}] < \mathcal{B}$ for any π-bases $\mathcal{A}$ and $\mathcal{B}$ in the space Z.

Fact 1. Let Z be a pseudocomplete space. Then there exists a pseudocomplete sequence $\{\mathcal{B}_n : n \in \omega\}$ of π-bases in Z such that

(1) $\mathcal{B}_{n+1} < \mathcal{B}_n$ for each $n \in \omega$.
(2) For each $m \in \omega$, if we have a family $\{U_i : i \geqslant m\}$ such that $U_i \in \mathcal{B}_i$ and $\overline{U}_{i+1} \subset U_i$ for all $I \geqslant m$ then $\bigcap\{U_i : i \geqslant m\} \neq \emptyset$.

Proof. Let $\{\mathcal{C}_n : n \in \omega\}$ be a pseudocomplete sequence of π-bases of Z. Note first that if we take any π-base $\mathcal{C}'_n \subset \mathcal{C}_n$ for all $n \in \omega$ then the sequence $\{\mathcal{C}'_n : n \in \omega\}$ is also pseudocomplete. If we let $\mathcal{B}_0 = \mathcal{C}_0$ and $\mathcal{B}_{n+1} = \mathcal{C}_{n+1}[\mathcal{B}_n]$ for each $n \in \omega$ then we get a pseudocomplete sequence $\{\mathcal{B}_n : n \in \omega\}$ of π-bases in Z such that $\mathcal{B}_{n+1} < \mathcal{B}_n$ for all $n \in \omega$ so (1) is proved.

Suppose that $\{U_i : i \geqslant m\}$ is a family as in (2). Since $\mathcal{B}_m < \mathcal{B}_{m-1}$, there exists $U_{m-1} \in \mathcal{B}_{m-1}$ such that $\overline{U}_m \subset U_{m-1}$. Going "backwards" in this manner, we obtain sets $U_{m-1}, \ldots, U_1, U_0$ such that $U_i \in \mathcal{B}_i$ and $\overline{U}_{i+1} \subset U_i$ for all $i < m$. Since we have the same property for all $i \geqslant m$, we have obtained a sequence $\{\mathcal{B}_i : i \in \omega\}$ such that $U_i \in \mathcal{B}_i$ and $\overline{U}_{i+1} \subset U_i$ for all $i \in \omega$. The sequence $\{\mathcal{B}_i : i \in \omega\}$ being pseudocomplete, we have $\bigcap\{U_i : i \geqslant m\} = \bigcap\{U_i : i \in \omega\} \neq \emptyset$ so Fact 1 is proved.

Returning to our solution let X_t be a pseudocomplete non-empty space for each $t \in T$; fix a pseudocomplete sequence $\{\mathcal{B}^t_n : n \in \omega\}$ of π-bases in X_t for each $t \in T$. Fact 1 shows that we can assume, without loss of generality, that the sequence $\{\mathcal{B}^t_n : n \in \omega\}$ has the properties (1) and (2) for all $t \in T$. If $O_t \in \tau(X_t)$ for all $t \in T$, the set $\Pi_{t \in T}\, O_t$ is called *standard* if $O_t \neq X_t$ for only finitely many t. The family of all standard sets is a base in the space $X = \Pi\{X_t : t \in T\}$ (see Problem 101). If $O = \prod_{t\in T} O_t$ is a standard set then supp $(O) = \{t \in T : O_t \neq X_t\}$. Let $p_t : X \to X_t$ be the natural projection for all $t \in T$.

Let $\mathcal{B}_n = \left\{O = \prod_{t\in T} O_t \in \mathcal{S} : O_t \in \mathcal{B}_n^t \text{ for all } t \in \text{supp}(O)\right\}$ for each $n \in \omega$. We claim that $\mathcal{B}_n$ is a π-base in X for each $n \in \omega$. Indeed, if $U \in \tau^*(X)$ then there is $O = \prod_{t\in T} O_t \in \mathcal{S}$ with $O \subset U$. Since $\mathcal{B}_n^t$ is a π-base in X_t for all $t \in \text{supp}(O)$, there is $V_t \in \mathcal{B}_n^t$ with $V_t \subset O_t$ for each $t \in \text{supp}(O)$. Let $V_t = X_t$ for all $t \in T\setminus\text{supp}(O)$; then $V = \prod\{V_t : t \in T\} \in \mathcal{B}_n$ and $V \subset O \subset U$ which proves that each $\mathcal{B}_n$ is a π-base in the space X.

We show next that the sequence $\{\mathcal{B}_n : n \in \omega\}$ is pseudocomplete. Take any family $\{U_n : n \in \omega\}$ such that $U_n \in \mathcal{B}_n$ and $\overline{U}_{n+1} \subset U_n$ for all $n \in \omega$ We have $U_n = \prod_{t\in T} U_t^n$ where $U_t^n \in B_n^t$ for all $n \in \omega$ and $t \in \text{supp}(U_n)$. Fix any $s \in T$; we claim that $\bigcap\{U_s^n : n \in \omega\} \neq \emptyset$. Indeed, if $s \notin \bigcup\{\text{supp}(U_n) : n \in \omega\}$ then $U_s^n = X_s$ for all $n \in \omega$ and hence $\bigcap\{U_s^n : n \in \omega\} = X_s \neq \emptyset$.

Observe that $\text{supp}(U_{n+1}) \supset \text{supp}(U_n)$ for all $n \in \omega$. Indeed, $U_{n+1} \subset U_n$ implies that $U_t^{n+1} = p_t(U_{n+1}) \subset p_t(U_n) = U_n^t$ for all $t \in T$. Consequently, if $U_t^n \neq X_t$ then $U_t^{n+1} \neq X_t$, i.e., $t \in \text{supp}(U_n)$ implies $t \in \text{supp}(U_{n+1})$.

Now assume that $s \in \text{supp}(U_n)$ for some $n \in \omega$ and let $m \in \omega$ be the smallest of the numbers $n \in \omega$ such that $s \in \text{supp}(U_n)$. Thus, $U_s^n = X_s$ for all $n < m$ and $s \in \text{supp}$ (U_n) for all $n \geqslant m$ by our previous remark. Therefore $U_s^n \in \mathcal{B}_n^s$ for all $n \geqslant m$. Given any $n \geqslant m$, observe that $\overline{U}_{n+1} = \prod\{\overline{U_t^{n+1}} : t \in T\}$ and therefore $\overline{U_s^{n+1}} = p_s(\overline{U}_{n+1})$ $\subset p_s(U_n) = U_s^n$ for all $n \geqslant m$ (all closures are denoted by a bar; we hope there is no confusion as to in which space the closure is taken). By property (2) for the sequence $\{\mathcal{B}_n^s : n \in \omega\}$, we have $\bigcap\{U_s^n : n \geqslant m\} \neq \emptyset$. However, $U_s^n = X_s$ for all $n < m$ so $\bigcap\{U_s^n : n \in \omega\} = \bigcap\{U_s^n : n \geqslant m\} \neq \emptyset$.

We proved that there is $x_t \in \bigcap\{U_t^n : n \in \omega\}$ for each $t \in T$. Letting $x(t) = x_t$ for all $t \in T$ we obtain a point $x \in X$ such that $x \in \bigcap\{U_n : n \in \omega\}$ which shows that the sequence $\{U_n : n \in \omega\}$ of π-bases in X is pseudocomplete.

S.471. *Prove that an open metrizable image of a pseudocomplete space is a pseudocomplete space.*

Solution. Call a map $f : Y \to Z$ *almost open* if, for any $U \in \tau^*(Y)$, there is $V \in \tau^*(Z)$ such that $f(U) \subset V \subset \overline{f(U)}$. The map f is *weakly open* if for any $U \in \tau^*(Y)$ there is $V \in \tau^*(Z)$ such that $\overline{V} = \overline{f(U)}$. It is evident that any almost open map is weakly open. A family $\mathcal{U}$ of subsets of Y is called *strongly disjoint* if $\overline{U} \cap \overline{V} = \emptyset$ for any distinct $U, V \in \mathcal{U}$. For further references, we will prove the following fact which gives more than we need for this solution.

Fact 1. Let $f : X \to M$ be a weakly open continuous map of a pseudocomplete space X onto a metrizable space M. Then M is pseudocomplete. As a consequence, an almost open continuous metrizable image of a pseudocomplete space is pseudocomplete.

Proof. Fix an arbitrary metric d on the space M such that $\tau(d) = \tau(M)$. For any $U \in \tau^*(X)$, let $\Phi(U) = \text{Int}(\overline{f(U)})$; this gives us a map $\Phi : \tau^*(X) \to \tau^*(M)$. Since

$\overline{V} = \overline{f(U)}$ for some $V \in \tau^*(X)$, we have $V \subset \mathrm{Int}(\overline{f(U)})$ and therefore $\Phi(U) \subset \overline{f(U)} = \overline{\Phi(U)}$ for any $U \in \tau^*(X)$.

Let $\{\mathcal{B}_n : n \in \omega\}$ be a pseudocomplete sequence of π-bases in X. It is straightforward that the family $\mathcal{C}_n = \{\Phi(U) : U \in \mathcal{B}_n\}$ is a π-base in M for all $n \in \omega$. Let $\mathcal{U}_0$ be a maximal strongly disjoint subfamily of the family $\{V \in \mathcal{C}_0 : \mathrm{diam}(V)<1\}$. For each $V \in \mathcal{U}_0$ take any $U \in \mathcal{B}_0$ such that $\Phi(U) = V$ and let $q_0(V) = U$. Since $q_0(V) \subset f^{-1}(\overline{V})$ for each $V \in \mathcal{U}_0$, the family $\mathcal{B}'_0 = q_0(\mathcal{U}_0)$ is also strongly disjoint.

It is easy to check that the set $\bigcup \mathcal{U}_0$ is dense in M and $q_0 : \mathcal{U}_0 \to \mathcal{B}'_0$ is a bijection. Suppose that we have constructed strongly disjoint families $\mathcal{U}_0, \dots, \mathcal{U}_n, \mathcal{B}'_0, \dots, \mathcal{B}'_n$ and bijections $q_i : \mathcal{U}_i \to \mathcal{B}'_i, i = 0, \dots, n$ with the following properties:

(1) $\mathcal{U}_i \subset \tau^*(M)$ and $\cup \mathcal{U}_i$ is dense in M for all $i \leqslant n$.
(2) For every $i \leqslant n$, we have $\mathcal{B}'_i \subset \mathcal{B}_i$ and $\Phi(q_i(V)) = V$ for each $V \in \mathcal{U}_i$.
(3) diam $(V)<\frac{1}{i+1}$ for all $V \in \mathcal{U}_i$ and all $i \leqslant n$.
(4) For any $i \in \{1, \dots, n\}$ and any $W \in \mathcal{U}_i$ there is $V \in \mathcal{U}_{i-1}$ such that $\overline{W} \subset V$.
(5) For all $i < n$, if $V \in \mathcal{U}_i$, $W \in \mathcal{U}_{i+1}$ and $W \subset V$ then $q_{i+1}(W) \subset q_i(V)$.

For any $V \in \mathcal{U}_n$ consider the family $\mathcal{B}_{n+1}(V) = \{U \in \mathcal{B}_{n+1} : \overline{U} \subset q_n(V)\}$; then $\mathcal{B}_{n+1}(V)$ is a π-base in $q_n(V)$. Take a maximal strongly disjoint subfamily $\mathcal{U}_{n+1}(V)$ of the family $\{W \in \Phi(\mathcal{B}_{n+1}(V)) : \overline{W} \subset V$ and diam $(W)<\frac{1}{n+2}\}$. It is straightforward that $\bigcup \mathcal{U}_{n+1}(V)$ is dense in V so if $\mathcal{U}_{n+1} = \bigcup\{\mathcal{U}_{n+1}(V) : V \in \mathcal{U}_n\}$ then $\mathcal{U}_{n+1}$ is strongly disjoint and $\bigcup \mathcal{U}_{n+1}$ is dense in M. For each $W \in \mathcal{U}_{n+1}(V)$ take any $\mathcal{U} \in \mathcal{B}_{n+1}(V)$ such that $\Phi(U) = W$ and let $q_{n+1}(W) = U$. If $\mathcal{B}'_{n+1}$ then the map $q_{n+1} : \mathcal{U}_{n+1} \to \mathcal{B}'_{n+1}$ is a bijection; it is easy to see that the family $\mathcal{B}'_{n+1}$ is also strongly disjoint. It is immediate from our construction that the properties (1)–(4) hold for the families $\mathcal{U}_0, \dots, \mathcal{U}_n, \mathcal{U}_{n+1}, \mathcal{B}'_0, \dots, \mathcal{B}'_n, \mathcal{B}'_{n+1}$, and bijections $q_0, \dots, q_n, q_{n+1}$. To see that (5) also holds it suffices to check it for $i = n$. Observe that if $V \in \mathcal{U}_n, W \in \mathcal{U}_{n+1}$ and $W \subset V$ then $W \in \mathcal{U}_{n+1}(V)$ and therefore $q_{n+1}(W) \in \mathcal{B}_{n+1}(V)$ which implies $\overline{q_{n+1}(W)} \subset q_n(V)$ so (5) is verified.

Thus, we can inductively construct sequences $\{\mathcal{U}_i : i \in \omega\}$ and $\{\mathcal{B}'_i : i \in \omega\}$ as well as a family $\{q_i : i \in \omega\}$ of bijections such that the properties (1)–(5) hold for all $n \in \omega$. Observe that it is immediate that the sequence $\{\mathcal{U}_i : i \in \omega\}$ has the properties (1)–(3) of Fact 1 of S.468. To see that (4) of the same Fact also holds, i.e., the sequence $\{\mathcal{U}_i : i \in \omega\}$ is pseudocomplete, take any family $\{V_i : i \in \omega\}$ such that $\overline{V}_{i+1} \subset V_i$ and $V_i \in \mathcal{U}_i$ for all $i \in \omega$. If $U_i = q_i(V_i) \in \mathcal{B}_i$, then it follows from (5) that $\overline{U}_{i+1} \subset U_i$ for all $i \in \omega$. The sequence being $\{\mathcal{B}_i : i \in \omega\}$ pseudocomplete, there is $x \in \bigcap\{U_i : i \in \omega\}$. We have $f(U_i) \subset \overline{V}_i$ for all $i \in \omega$ which implies $f(x) \in \cap\{\overline{V}_i : i \in \omega\} = \{V_i : i \in \omega\}$ so the sequence $\{\mathcal{U}_i : i \in \omega\}$ is pseudocomplete. Therefore, the property (4) of Fact 1 of S.468 holds as well for the sequence $\{\mathcal{U}_i : i \in \omega\}$ and hence the conclusion of the same Fact can be applied to this sequence to deduce that $D = \bigcap\{\bigcup \mathcal{U}_i : i \in \omega\}$ is a dense Čech-complete subspace of M. Thus M is pseudocomplete by Problem 468 so Fact 1 is proved.

To finish our solution observe that any open map is weakly open, so any open metrizable image of a pseudocomplete space is pseudocomplete by Fact 1.

S.472. *Prove that a space X is pseudocompact if and only if any continuous image of X is pseudocomplete.*

Solution. If a space X is pseudocompact then every continuous image of X is also pseudocompact and hence pseudocomplete. Now assume that every continuous image of X is pseudocomplete and X is not pseudocompact. Fix a discrete family $\mathcal{O} = \{O_n : n \in \omega\}$ of non-empty open subsets of X (Problem 136) and take a point $x_n \in O_n$ for all $n \in \omega$.

Fact 1. Given arbitrary spaces Y and Z, a map $f : Y \to Z$ is continuous if and only if, for any $y \in Y$, there is $U \in \tau(y, Y)$ such that $f|U : U \to Y$ is continuous.

Proof. If f is continuous then $U = Y$ does for all $y \in Y$. Now assume that the hypothesis of our Fact holds and take any $y \in Y$ and any $V \in \tau\,(f(y), Z)$. There is a set $U \in \tau(y, Y)$ such that $f|U$ is continuous and therefore there is $W \in \tau(y, U)$ such that $f(W) \subset V$. It is clear that W is an open neighbourhood of y in Y so W witnesses continuity of f at the point y. Fact 1 is proved.

Returning to our solution fix a dense set $D = \{f_n : n \in \omega\}$, $C_p(\mathbb{I})$; define $u \in C_p(\mathbb{I})$ by $u(t) = 0$ for all $t \in \mathbb{I}$ and let $I_n = \{tf_n : t \in [0, 1]\}$ for all $n \in \omega$. Then the map $\varphi_n : [0, 1] \to I_n$ defined by $\varphi_n(t) = tf_n$, is a homeomorphism such that $\varphi_n(0) = u$ and $\varphi_n(1) = f_n$ for all $n \in \omega$ (Fact 1 of S.301). The Tychonoff property of X implies that there is a continuous map $p_n : X \to I_n$ such that $p(x_n) = f_n$ and $p(X \setminus O_n) \subset \{u\}$ for all $n \in \omega$. We can consider that $p_n : X \to C_p(\mathbb{I})$; then p_n is still continuous (Problem 023).

If $x \in X / \bigcup \mathcal{O}$ then let $p(x) = u$; if $x \in O_{\mathrm{n}}$ for some $n \in \omega$ then let $p(x) = p_n(x)$. This defines a map $p : X \to C_p(\mathbb{I})$. Given any $x \in X$ there is $U \in \tau(x, X)$ such that U meets at most one element of $\mathcal{O}$, say O_n. It is immediate that $p|U = p_n|U$ is a continuous map because $p_n|U$ is continuous. Applying Fact 1, we conclude that the map $p : X \to C_p(\mathbb{I})$ is continuous. According to our hypothesis, the space $Y = p(X)$ is pseudocomplete. However, $D \subset Y$ so $C_p(\mathbb{I})$ contains a dense pseudocomplete subspace Y. This implies that $C_p(\mathbb{I})$ is pseudocomplete (Problem 467) and hence Baire (Problem 464) which it is not by Problem 284. This contradiction shows that X is pseudocompact so our solution is complete.

S.473. *Prove that a dense G_δ-subspace of a pseudocompact space is pseudocomplete.*

Solution. Take an arbitrary pseudocompact space P and a dense G_δ-set X of the space P; fix any sequence $\{O_n : n \in \omega\} \subset \tau(P)$ such that $X = \cap\{O_n : n \in \omega\}$. Observe that the family $\mathcal{B}_n = \{U \in \tau^*(X) : \mathrm{cl}_P(U) \subset O_n\}$ is a base in X for each $n \in \omega$. Indeed, if $x \in X$ and $V \in \tau(x, X)$ then take any $W \in \tau(P)$ with $W \cap X = V$ and observe that $x \in W \cap O_n \in \tau(x, P)$. Choose any $W' \in \tau(x,P)$ with $\mathrm{cl}_P(W')$, $W \cap O_n$; then $W' \cap X \in \mathcal{B}_n$ and $x \in W' \cap X$, V.

Therefore each $\mathcal{B}_n$ is a base in X so it suffices to prove that the sequence $\{\mathcal{B}_n : n \in \omega\}$ is pseudocomplete. Take any family $\{U_i : i \in \omega\}$ such that $U_i \in \mathcal{B}_i$

and $\overline{U}_{i+1} \subset U_i$ for all $i \in \omega$ (the bar denotes the closure in X). Choose any $V_i \in \tau(P)$ such that $V_i \cap X = U_i$ for all $i \in \omega$; it is easy to check that the family $\{V_i : i \in \omega\}$ is centered.

Since P is pseudocompact, there is $z \in \bigcap\{\mathrm{cl}_P(V_i) : i \in \omega\} = \bigcap\{\mathrm{cl}_P(U_i) : i \in \omega\}$ (Problem 136). Since $\mathrm{cl}_P(U_i) \subset O_i$, the point z belongs to O_i for each $i \in \omega$ and therefore $z \in \bigcap\{O_i : i \in \omega\} = X$. As a consequence $z \in \mathrm{cl}_P(U_i) \cap X = \overline{U}_i$, for all $i \in \omega$. Thus, $z \in \bigcap\{\overline{U}_i : i \in \omega\} = \bigcap\{U_i : i \in \omega\}$ and hence $\bigcap\{U_i : i \in \omega\} \neq \emptyset$. The pseudocompleteness of the sequence $\{\mathcal{B}_n : n \in \omega\}$ being established, we proved that X is pseudocomplete.

S.474. *Prove that a dense G_δ-subspace of a metrizable pseudocomplete space is pseudocomplete.*

Solution. Let M be a metrizable pseudocomplete space. Then there is a dense Čech-complete $D \subset M$ (Problem 468). Suppose that X is a dense G_δ-subset of the space M; fix a family $\mathcal{O} = \{O_n : n \in \omega\} \subset \tau(M)$ such that $X = \bigcap \mathcal{O}$. The set $U_i = O_i \cap D$ is dense and open in D for each $i \in \omega$; therefore $D' = \bigcap\{U_i : i \in \omega\}$ is a dense G_δ-subset of D because D has the Baire property (Problem 274). Besides, D' is Čech-complete (Problem 260). The set D' is dense in D and hence in M; since $D' = \bigcap_{i\in\omega} U_i \subset \bigcap_{i\in\omega} O_i = X$, the set D' is also dense in X. As a consequence X has a dense Čech-complete subspace so X is pseudocomplete by Problem 468.

S.475. *Prove that, if $C_p(X)$ is an open image of a pseudocomplete space then it is pseudocomplete.*

Solution. Call $f : Y \to Z$ *almost open* if, for any $U \in \tau^*(Y)$, there is $V \in \tau^*(Z)$ such that $f(U) \subset V \subset \overline{f(U)}$. Given points $y_1, \ldots, y_n \in Y$ and sets $O_1, \ldots, O_n \in \tau(\mathbb{R})$, let $[y_1, \ldots, y_n; O_1, \ldots, On]\ Y = \{f \in C_p(Y) : f(y_i) \in O_i$ for all $i \leqslant n\}$. All possible sets $[y_1, \ldots, y_n; O_1, \ldots, O_n]\ Y$ are called *standard open subsets of* $C_p(Y)$; they form a base in $C_p(Y)$. If Z is a space and Y, Z then $C_p(Y\,|Z)$ is the set $\pi_Y(C_p(Z))$ taken with the topology of subspace it inherits from $C_p(Y)$.

Fact 1. For any space Z and any Y, Z, the restriction map $\pi_Y : C_p(Z) \to C_p(Y\,|Z)$ is almost open.

Proof. Take any standard open set $U = [z_1, \ldots, z_n; O_1, \ldots, O_n]_Z$ of the space $C_p(Z)$. Without loss of generality, we can assume that $z_1, \ldots, z_k \in Y$ and $z_{k+1}, \ldots, z_n \in Z \backslash Y$. If $V = [z_1, \ldots, z_k; O_1, \ldots, O_k]Y \cap C_p(Y\,|Z)$ then V is open in $C_p(Y\,|Z)$ and $\pi_Y(U)$, V. To see that $\pi_Y(U)$ is dense in V take any $f \in V$ and any points $r_i \in O_i$ for all $i = k + 1, \ldots, n$. Given any finite set K, Y, the set $L = K\,\mathcal{U}$ $\{z_1, \ldots, z_k\}$ is also finite so there exists $g \in C_p(Z)$ such that $g|L = f|L$ and $g(z_i) = r_i$ for all $i = k + 1, \ldots, n$ (Problem 034). We have $g(z_i) \in O_i$ for all $i \leqslant n$ and hence $g \in U$; besides, $\pi_Y(g)|K = f|K$. This shows that, for any finite K, Y, there is $g' = \pi_Y(g) \in \pi_Y(U)$ such that $g'|K = f|K$. An evident consequence is that $f \in \overline{\pi_Y(U)}$. The function f has been chosen arbitrarily so $\pi_Y(U)$ is dense in V. Therefore, for any standard open U in $C_p(Z)$ there is an open V_U in the space $C_p(Y\,|Z)$ such that $\pi_Y(U) \subset V_U \subset \pi_Y(U)$.

Now, take any $W \in \tau^*(C_p(Z))$; since the standard sets form a base in $C_p(Z)$, there exists a family $\mathcal{U}$ of standard sets such that $W = \bigcup \mathcal{U}$. For any $U \in \mathcal{U}$ fix a set $V_U \in \tau(C_p(Y\,|Z))$ such that $\pi_Y(U) \subset V_U \subset \overline{\pi_Y(U)}$ and let $V = \bigcup\{V_U : U \in \mathcal{U}\}$. Then $\pi_Y(W) = \bigcup\{\pi_Y(U) : U \in \mathcal{U}\} \subset \bigcup\{V_U : U \in \mathcal{U}\} = V$. Since $\pi_Y(U)$ is dense in V_U for each $U \in \mathcal{U}$, the set $\pi_Y(W)$ is dense in V so Fact 1 is proved.

Fact 2. Suppose that $f: Y \to Z$ and $g : Z \to T$ are continuous almost open onto maps. Then $h = g \circ f: Y \to T$ is almost open. In other words, a composition of almost open maps is an almost open map.

Proof. Given any $U \in \tau^*(Y)$ there is $V \in \tau(Z)$ such that $f(U) \subset V \subset \overline{f(U)}$. Since g is also almost open, there is $W \in \tau(T)$ such that $g(V) \subset W \subset \overline{g(V)}$. It is evident that $h(U) \subset W \subset \overline{h(U)}$ so Fact 2 is proved.

Fact 3. Suppose that every countable subset of a space Z is closed and C-embedded in Z. Then $C_p(Z)$ is pseudocomplete.

Proof. Let $Q_0 = \mathbb{Q}$; suppose that we have countable disjoint subsets $Q_0, \ldots, Q_n$ which are dense in $\mathbb{R}$. Since $Q_n' = Q_0 \cup \cdots \cup Q_n$ is countable, the set $\mathbb{R}\backslash Q_n'$ is dense in $\mathbb{R}$ so we can choose a countable dense subset $Q_{n+1} \subset \mathbb{R}\backslash Q_n'$. It is clear that this inductive procedure gives us a family $\{Q_n : n \in \omega\}$ of disjoint dense subsets of $\mathbb{R}$. Let $\mathcal{O}_n = \left\{(a,b) : a,b \in Q_n \text{ and } a<b<a+\frac{1}{n+1}\right\} \subset \tau(\mathbb{R})$.

Given an arbitrary $n \in \omega$, a finite set K, $C\ Z$ and any map $u : K \to \mathcal{O}_n$, let $[u, K] = \{f \in C_p(Z) : f(z) \in u(z) \text{ for all } z \in K\}$. It is easy to see that the family $\mathcal{B}_n = \{[u, K] :$ K is a finite subset of Z and u is a map from K to $\mathcal{O}_n\}$ is a base in $C_p(Z)$ for all $n \in \omega$ (see Problem 056). To prove that the sequence $\{\mathcal{B}_n : n \in \omega\}$ is pseudocomplete we will establish the following general property.

$(*)$ Suppose that $U = [u, K] \in \mathcal{B}_n$ and $V = [v, L] \in \mathcal{B}_m$ where $m \neq n$. If $U \subset V$ then $L \subset K$ and $\overline{u(z)} \subset v(z)$ for every $z \in L$.

Suppose that there is $t \in L\backslash K$; choose $r_z \in u(z)$ for all $z \in K$ and any $r \in \mathbb{R}\backslash v(t)$ (this is possible because all intervals we consider are non-empty and not equal to $\mathbb{R}$). There exists $f \in C_p(Z)$ such that $f(z) = r_z$ for all $z \in K$ and $f(t) = r$ (Problem 034). Then $f \in U\backslash V$; this contradiction shows that $L \subset K$.

Now suppose that $r \in u(z)\backslash \upsilon(z)$ for some $z \in L$; choose any $r_y \in u(y)$ for all points $y \in K\backslash\{z\}$. There is a function $g \in C_p(Z)$ such that $f(z) = r$ and $f(y) = r_y$ for all $y \in K\backslash\{z\}$. It is clear that $g \in U\backslash V$ which again gives us a contradiction. As a consequence, $u(z)$, $\upsilon(z)$ for all $z \in L$. Given any $z \in L$, observe that $u(z)$ is an open interval with its endpoints lying in Q_n while the endpoints of $\upsilon(z)$ belong to Q_m which is disjoint from Q_n. As a consequence, the endpoints of $u(z)$ have to be inside $\upsilon(z)$ and hence $\overline{u(z)} \subset \upsilon(z)$. The property $(*)$ is proved.

Now take any family $\{U_n : n \in \omega\}$ such that $U_n \in \mathcal{B}_n$ and $\overline{U}_{n+1} \subset U_n$ for all $n \in \omega$. We have $U_n = [u_n, K_n]$; the property $(*)$ implies that $K_n \subset K_{n+1}$ for all $n \in \omega$. Fix any point $z \in P = \bigcup\{K_n : n \in \omega\}$ and the minimal $m \in \omega$ such that $z \in K_m$; then $z \in K_n$ for all $n \geqslant m$. Applying $(*)$ again, we conclude that the family $\{u_n(z) : n \geqslant m\}$ consists of decreasing intervals whose lengths tend to zero. By completeness of $\mathbb{R}$ we have $\bigcap\{\overline{u_n(z)} : n \geqslant m\} = \bigcap\{u_n(z) : n \geqslant m\} \neq \emptyset$ (the last equality holds because $\overline{u_{n+1}(z)} \subset u_n(z)$ for all $n \geqslant m$ by $(*)$.

Note that, for any countable A, Z any subset of A is countable and hence closed in Z. This shows that any subset of A is closed in A, i.e., A is discrete. Consequently, any countable subset of Z is closed and discrete.

Take $r_z \in \bigcap \{u_n(z) : n \geqslant m\}$; we have a function $h : P \to \mathbb{R}$ defined by $h(z) = r_z$ for all $z \in P$. Since P is discrete, the function h is continuous so there is $g \in C_p(Z)$ such that $g|P = h$. For any $n \in \omega$ and any $z \in K_n$, we have $g(z) = h(z) = r_z \in u_n(z)$ and therefore $g \in [u_n, K_n] = U_n$. This shows that $g \in \bigcap\{U_n : n \in \omega\}$ so $\bigcap\{U_n : n \in \omega\} \neq \emptyset$. Fact 3 is proved.

Returning to our solution, suppose that Z is a pseudocomplete space and we have an open continuous onto map $\varphi : Z \to C_p(X)$. Given any countable $A, \subset X$ the map $\pi_A : C_p(X) \to C_p(A|X)$ is almost open by Fact 1; hence the map $\pi_A \circ \varphi$ is also almost open by Fact 2. Since $w(C_p(A|X)) \leqslant \omega$, we can apply Fact 1 of S.471 to conclude that $C_p(A|X)$ is pseudocomplete and hence there is a Čech-complete $D, C_p(A|X)$ that is dense in $C_p(A|X)$ and hence in $C_p(A)$ (Problem 152). Thus A is discrete by Problem 265; this proves that every countable subset of X is discrete. As a consequence, every countable subset of X is also closed for if $A, \subset X$ is countable and $x \in \overline{A} \backslash A$ then the set $A \cup \{x\}$ is also countable and hence discrete which is a contradiction.

Finally, assume that $C_p(A|X) \neq \mathbb{R}^A$ for some countable $A, \subset X$. We saw that there is a dense Čech-complete $D \subset C_p(A|X)$. Observe first that $C_p(A|X)$ is an algebra and, in particular, $f - g \in C_p(A|X)$ for any $f, g \in C_p(A|X)$. Take any $h \in \mathbb{R}^A \backslash C_p(A|X)$; the map $T_h : \mathbb{R}^A \to \mathbb{R}^A$ defined by $T_h(g) = h + g$, is a homeomorphism (Problem 079) and $T_h(C_p(A|X)) \cap C_p(A|X) = \emptyset$. Indeed, if $g \in C_p(A|X)$ and $f = h + g \in C_p(A|X)$ then $h = f - g \in C_p(A|X)$ which is a contradiction. Therefore $\mathbb{R}^A$ has two disjoint dense Čech-complete subspaces D and $T_h(D)$ which contradicts Problem 264. Thus $C_p(A|X) = \mathbb{R}^A$ for any countable $A \subset X$. This means that every countable A, X is closed and C-embedded in X. Applying Fact 3, we conclude that $C_p(X)$ is pseudocomplete.

S.476. *Prove that, if $C_p(X)$ is pseudocomplete then $C_p(X, \mathbb{I})$ is pseudocomplete.*

Solution. The space $C_p(X)$ is homeomorphic to $C_p(X, (-1, 1))$ (Fact 1 of S.295) which is dense in $C_p(X, \mathbb{I})$. As a consequence, if $C_p(X)$ is pseudocomplete then the set $C_p(X, (-1, 1))$ is a dense pseudocomplete subspace of $C_p(X, \mathbb{I})$. Therefore $C_p(X, \mathbb{I})$ is also pseudocomplete (Problem 467).

S.477. *Give an example of a space X for which $C_p(X, \mathbb{I})$ is pseudocomplete but $C_p(X)$ is not pseudocomplete.*

Solution. In Fact 4 of S.286 it was proved that there exists a dense pseudocompact subspace X of the space $\mathbb{I}^{\mathfrak{c}}$ such that $C_p(X, \mathbb{I})$ is pseudocompact and hence pseudocomplete. The space $C_p(X)$ cannot be pseudocomplete because it does not have the Baire property (see Problems 464 and 284).

S.478. *Let X be a normal space. Prove that $C_p(X, \mathbb{I})$ is pseudocomplete if and only if it is pseudocompact.*

Solution. Call a map $f : Y \to Z$ *almost open* if, for any $U \in \tau^*(Y)$, there is $V \in \tau^*(Z)$ such that $f(U) \subset V \subset \overline{f(U)}$. Given $y_1, \ldots, y_n \in Y$ and sets $O_1, \ldots, O_n \in \tau^*(\mathbb{I})$, let $[y_1, \dots, y_n; O_1, \dots, O_n]\ Y = \{f \in C_p(Y, \mathbb{I}) : f(y_i) \in O_i$ for all $i \leqslant n\}$. All possible sets $[y_1, \dots, y_n; O_1, \dots, O_n]_{\ Y}$ are called *standard open subsets of* $C_p(Y, \mathbb{I})$; they form a base in $C_p(Y, \mathbb{I})$. If Z is a space and $Y \subset Z$ then $C_p(Y\,|Z, \mathbb{I})$ is the set $\pi_Y(C_p(Z, \mathbb{I}))$ taken with the topology of subspace it inherits from $C_p(Y, \mathbb{I})$.

Fact 1. For an arbitrary space Z and any set Y, Z, the restriction mapping $\pi_Y : C_p(Z, \mathbb{I}) \to C_p(Y\,|Z, \mathbb{I})$ is almost open.

Proof. Take any standard open set $U = [z_1, \dots, z_n; O_1, \dots, O_n]_Z$ of the space $C_p(Z, \mathbb{I})$. Without loss of generality, we can assume that $z_1, \dots, z_k \in Y$ and we have $z_{k+1}, \dots, z_n \in Z \backslash Y$. If $V = [z_1, \dots, z_k; O_1, \dots, O_k]_Y \cap C_p(Y\,|Z, \mathbb{I})$, then V is open in $C_p(Y\,|Z, \mathbb{I})$ and $\pi_Y(U) \subset V$. To see that $\pi_Y(U)$ is dense in V take any $f \in V$ and any points $r_i \in O_i$ for all $i = k + 1, \dots, n$. Given any finite set $K \subset Y$, the set $L = K \cup \{z_1, \dots, z_k\}$ is also finite so there exists $g' \in C_p(Z)$ such that $g'|L = f\,|L$ and $g'(z_i) = r_i$ for all $i = k + 1, \dots, n$ (Problem 034).

Define a function $w : \mathbb{R} \to \mathbb{I}$ as follows: $w(t) = -1$ for all $t < -1$; if $t \in \mathbb{I}$ then $w(t) = t$ and if $t > 1$ then $w(t) = 1$. It is clear that w is continuous and hence $g = w \circ g'$, is also continuous. It is immediate that $g \in C_p(Z, \mathbb{I})$; besides $g|L = g'|L = f\,|L$ and $g(z_i) = r_i$ for all $i = k + 1, \dots, n$.

We have $g(z_i) \in O_i$ for all $i \leqslant n$ and hence $g \in U$; besides, $\pi_Y(g)|K = f\,|K$. This shows that, for any finite set $K \subset Y$, there is $h = \pi_Y(g) \in \pi_Y(U)$ such that $h|K = f\,|K$. An evident consequence is that $f \in \overline{\pi_Y(U)}$. The function f has been chosen arbitrarily so the set $\pi_Y(U)$ is dense in V. Therefore, for any standard open U in $C_p(Z, \mathbb{I})$ there is an open V_U in the space $C_p(Y\,|Z, \mathbb{I})$ such that $\pi_Y(U) \subset V_U \subset \overline{\pi_Y(U)}$.

Now take any $W \in \tau^*(C_p(Z, \mathbb{I}))$; since the standard sets form a base in $C_p(Z, \mathbb{I})$, there exists a family $\mathcal{U}$ of standard sets such that $W = \bigcup \mathcal{U}$. For any $U \in \mathcal{U}$ fix a set $V_U \in \tau(C_p(Y\,|Z, \mathbb{I}))$ such that $\pi_Y(U) \subset V_U \subset \overline{\pi_Y(U)}$ and let $V = \bigcup\{V_U : U \in \mathcal{U}\}$. Then $\pi_Y(W) = \bigcup\{\pi_Y(U) : U \in \mathcal{U}\} \subset \bigcup\{V_U : U \in \mathcal{U}\} = V$. Since $\pi_Y(U)$ is dense in V_U for each $U \in \mathcal{U}$, the set $\pi_Y(W)$ is dense in V so Fact 1 is proved.

Returning to our solution, observe that if $C_p(X, \mathbb{I})$ is pseudocompact then it is pseudocomplete because every pseudocompact space is pseudocomplete. Now assume that $C_p(X, \mathbb{I})$ is pseudocomplete. Given any countable $A \subset X$, the restriction map $\pi_A : C_p(X, \mathbb{I}) \to C_p(A|X, \mathbb{I})$ is almost open by Fact 1. The space $C_p(A|X, \mathbb{I})$ being second countable, it is pseudocomplete (Fact 1 of S.471); applying Problem 468 we conclude that $C_p(A|X, \mathbb{I})$ has a dense Čech-complete subspace D. Since $C_p(A|X, \mathbb{I})$ is dense in $C_p(A, \mathbb{I})$, the space $C_p(A, \mathbb{I})$ has a dense Čech-complete subspace so A is discrete by S.287. This proves that every countable subspace of X is discrete. As a consequence, every countable subset of X is also closed for if $A \subset$ X is countable and $x \in \overline{A} \backslash A$ then the set $A < \{x\}$ is also countable and hence discrete which is a contradiction. Therefore any countable subset of X is closed C-embedded in X by normality of X. Thus $C_p(X, \mathbb{I})$ is pseudocompact by Problem 398 so our solution is complete.

S.479. *Prove that, if* $C_p(X, \mathbb{I})$ *is countably compact then the space* $C_p(X)$ *is pseudocomplete.*

Solution. If $C_p(X, \mathbb{I})$ is countably compact then any G_δ-subset of X is open in X (Problem 397), i.e., X is a P-space.

Fact 1. For any P-space Z, any countable $D \subset Z$ is closed and C-embedded in Z.

Proof. Since the complement of any subset of D is a G_δ-subset of Z, every subset of D is closed in Z. This proves that every countable subset of Z is closed and discrete in Z.

Again, let $D = \{x_n : n \in \omega\}$ be a countable subset of Z. We saw that D has to be discrete; therefore there exists a disjoint family $\{U_n : n \in \omega\} \subset \tau(Z)$ with $x_n \in U_n$ for all $n \in \omega$. By Fact 2 of S.328, for any $n \in \omega$, there is a closed G_δ-set H_n such that $x_n \in H_n \subset U_n$. Each H_n is open in Z because Z is a P-space. It is clear that $H = Z \setminus (\bigcup\{H_n : n \in \omega\})$ is also open in Z being a G_δ-subset of Z. As a consequence, the family $\mathcal{H} = \{H_n : n \in \omega\}$ is discrete in Z.

Let $f_n : X \to \{0, 1\}$ be defined by $f_n(x) = 1$ for all $x \in H_n$ and $f_n(x) = 0$ for all points $x \in X \setminus H_n$. Since H_n is a clopen subset of Z, the function f_n is continuous for all $n \in \omega$. Now take any function $f : D \to \mathbb{R}$; we claim that $g = \sum_{n\in\omega} f(x_n) \cdot f_n$ is continuous on Z. Indeed, if $x \in Z$ then there is $U \in \tau(x, Z)$ which meets at most one element of $\mathcal{H}$, say, H_k. Then $g|U = (f(x_k) \cdot f_k)|U$ is a continuous function. Therefore g is continuous by Fact 1 of S.472; it is clear that $g|D = f$, i.e., the function f can be continuously extended over the space Z. The function $f : D \to \mathbb{R}$ has been chosen arbitrarily, so we proved that any countable $D \subset Z$ is closed and C-embedded in Z, i.e., Fact 1 is proved.

Now apply Fact 1 of this solution and Fact 3 of S.475 to conclude that $C_p(X)$ is pseudocomplete.

S.480. *Give an example of a space X such that $C_p(X)$ is pseudocomplete but $C_p(X, \mathbb{I})$ is not countably compact.*

Solution. A space X is called *a P-space* if every G_δ-subset of X is open in X. The space X is a P-space if and only if $C_p(X, \mathbb{I})$ is countably compact (Problem 397).

Fact 1. There exists a disjoint family $\mathcal{E} = \{E_\alpha : \alpha < \mathfrak{c}\}$ of subsets of $\mathbb{R}$ such that $|E_\alpha \cap O| = \mathfrak{c}$ for any $O \in \tau^*(\mathbb{R})$ and any $\alpha < \mathfrak{c}$.

Proof. Let $Q_0 = \mathbb{Q}$; assume that $\beta < \mathfrak{c}$ and we have a family $\{Q_\alpha : \alpha < \beta\}$ of disjoint countable dense subsets of $\mathbb{R}$. The set $Q'_\beta = \bigcup\{Q_\alpha : \alpha < \beta\}$ has cardinality strictly less than $\mathfrak{c}$; since every $O \in \tau^*(\mathbb{R})$ has cardinality $\mathfrak{c}$, we have $\text{Int}\left(Q'_\beta\right) = \emptyset$ and therefore $\mathbb{R} \setminus Q'_\beta$ is dense in $\mathbb{R}$. The space $\mathbb{R} \setminus Q'_\beta$ being second countable, there exists a countable dense $Q_\beta \subset \mathbb{R} \setminus Q'_\beta$. It is clear that the family $\{Q_\alpha : \alpha \leqslant \beta\}$ consists of disjoint countable dense subsets of $\mathbb{R}$. Thus, this inductive construction can be carried out for all $\beta < \mathfrak{c}$ giving us a disjoint family $\{Q_\alpha : \alpha < \mathfrak{c}\}$ of countable dense subsets of $\mathbb{R}$. It is easy to find a disjoint family $\{P_\alpha : \alpha < \mathfrak{c}\}$ such that $\mathfrak{c} = \bigcup\{P_\alpha : \alpha < \mathfrak{c}\}$ and $|P_\alpha| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$. Letting $E_\alpha = \bigcup\{Q_\beta : \beta \in P_\alpha\}$ for each $\alpha < \mathfrak{c}$, we obtain the promised family $\mathcal{E} = \{E_\alpha : \alpha < \mathfrak{c}\}$. Fact 1 is proved.

Fact 2. There exists a space X with the following properties:

(a) X can be condensed onto a P-space.
(b) X can be condensed onto $\mathbb{R}$ and any open subset of X has cardinality $\mathfrak{c}$. In particular, X has no isolated points.

Proof. Take any set A of cardinality $\mathfrak{c}$; given an arbitrary point $p \in \mathbb{R}^A$, we let $\Sigma(p) = \{x \in \mathbb{R}^A : |\{a \in A : x(a) \neq p(a)\}| \leqslant \omega\}$. Denote by $\mathcal{C}(A)$ the family of all countable subsets of A. Given any $x \in \Sigma(p)$, let $\text{supp}(x) = \{a \in A : x(a) \neq p(a)\}$; for any $B \subset A$, denote by $p_B : \mathbb{R}^A \to \mathbb{R}^B$ the natural projection onto the face $\mathbb{R}^B$. Given any $B \in \mathcal{C}(A)$, consider the set $\Sigma_B(p) = \{x \in \Sigma(p) : \text{supp}(x) \subset B\}$. It is straightforward that the map $p_B|\Sigma_B(p) : \Sigma_B(p) \to \mathbb{R}^B$ is a bijection and hence $|\Sigma_B(p)| \leqslant |\mathbb{R}^B| = \mathfrak{c}$. Observing that $\Sigma(p) = \bigcup\{\Sigma_B(p) : B \in \mathcal{C}(A)\}$, we obtain $|\Sigma(p)| \leqslant \mathfrak{c} \cdot |\mathcal{C}(A)| = \mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$. Thus, we have

(1) $|\Sigma(p)| = \mathfrak{c}$ for any $p \in \mathbb{R}^A$.

Let τ be a topology on $\mathbb{R}^A$ generated by all G_δ-subsets of $\mathbb{R}^A$. We omit a simple verification of the fact that $R = (\mathbb{R}^A, \tau)$ is a Tychonoff P-space. Let us show that

(2) The set $\Sigma(p)$ is dense in R for any $p \in \mathbb{R}^A$.

The family $\mathcal{G}$ of all G_δ-subsets of $\mathbb{R}^A$ is a base in R so it suffices to show that every non-empty $H \in \mathcal{G}$ contains a point from $\Sigma(p)$. Recall that a set $U = \prod_{a \in A} U_a$ is called *standard in* $\mathbb{R}^A$ if $U_a \in \tau(\mathbb{R})$ for all $a \in A$ and the set $\text{supp}(U) = \{a \in A : U_a \neq \mathbb{R}\}$ is finite. Standard sets form a base in $\mathbb{R}^A$ (Problem 101).

Take any point $x \in H$; it is easy to see that there are standard sets U_n, $n \in \omega$ such that $x \in G = \bigcap\{U_n : n \in \omega\} \subset H$. If $B = \bigcup\{\text{supp}(U_n) : n \in \omega\}$ then B is a countable subset of A and hence the point $y \in \mathbb{R}^A$ defined by $y|B = x|B$ and $y(a) = p(a)$ for all $a \in A \setminus B$, belongs to $\Sigma(p)$. It is immediate that $y \in G \subset H$ so $y \in \Sigma(p) \cap H$ which proves that $\Sigma(p)$ is dense in R, so (2) is established.

Our next important step is to prove that

(3) $w(R) \leqslant \mathfrak{c}$.

Given any $B \in \mathcal{C}(A)$ and any $z \in \mathbb{R}^B$, let $O(z, B) = \{y \in R : y|B = z\}$. It is straightforward that $O(p_B(x), B) \in \tau(x, R)$ for any $x \in \mathbb{R}^A$ and any countable $B \subset A$. If $x \in U \in \tau(R)$, then there is $H \in \mathcal{G}$ such that $x \in H \subset U$. Find standard sets U_n, $n \in \omega$ such that $x \in G = \bigcap\{U_n : n \in \omega\} \subset H$. If $B = \bigcup\{\text{supp}(U_n) : n \in \omega\}$ then $B \in \mathcal{C}(A)$ and it is immediate that $O(p_B(x), B) \subset G \subset H \subset U$ which shows that the family $\{O(p_B(x), B) : B \in \mathcal{C}(A)\}$ is a local base at x in the space R. As a consequence, the family $\mathcal{B} = \{O(z, B) : B \in \mathcal{C}(A), z \in \mathbb{R}^B\}$ is base in R. Since $|\mathcal{B}| \leqslant |\mathcal{C}(A)| \cdot |\mathbb{R}^\omega| = \mathfrak{c}$, we proved that $w(R) \leqslant \mathfrak{c}$, i.e., (3) holds.

Choose a family $\{A_\alpha : \alpha < \mathfrak{c}\}$ of disjoint subsets of A such that $|A_\alpha| = \mathfrak{c}$ for any $\alpha < \mathfrak{c}$. Let $q_\alpha(a) = 1$ if $a \in A_\alpha$ and $q_\alpha(a) = 0$ for all $a \in A \setminus A_\alpha$. Then

(4) $\Sigma(q_\alpha) \cap \Sigma(q_\beta) = \emptyset$ if $\alpha \neq \beta$.

Indeed, if $x \in \Sigma(q_\alpha)$ then $p_{A_\beta}(x)$ can take only countably many values distinct from zero because $p_{A_\beta}(q_\alpha)$ is identically zero on A_β. Since every $y \in \Sigma(q_\beta)$ takes the value 1 at all but countably many $a \in A_\beta$, it is impossible that $x = y$. This settles (4).

Let $Y = \bigcup\{\Sigma(q_\alpha) : \alpha < \mathfrak{c}\} \subset R$; it is immediate that $|Y| = \mathfrak{c}$. Choose an enumeration $\{U_\alpha : 0 < \alpha < \mathfrak{c}\}$ of the base $\mathcal{U} = \{U \cap Y : U \in \mathcal{B}\}$ of the space Y. It is easy to see that $\Sigma(q_\alpha) \cap U$ has cardinality $\mathfrak{c}$ for any $\alpha < \mathfrak{c}$ and $U \in \mathcal{B}$. Thus the set $S_\alpha = \Sigma(q_\alpha) \cap U_\alpha$ has cardinality $\mathfrak{c}$ for all $\alpha < \mathfrak{c}$. The set $Y' = Y \setminus \bigcup_{0<\alpha<\mathfrak{c}} S_\alpha$ also has cardinality $\mathfrak{c}$; since the family $\{S_\alpha : 0 < \alpha < \mathfrak{c}\}$ is disjoint, we can construct a bijection $\varphi : Y \to \mathbb{R}$ such that $\varphi(Y') = E_0$ and $\varphi(S_\alpha) = E_\alpha$ whenever $0 < \alpha < \mathfrak{c}$ (see Fact 1 for the definition of the sets E_α).

We finally have our space $X = \{(y, \varphi(y)) : y \in Y\} \subset Y \times \mathbb{R}$. Let $\pi_1 : X \to Y$ and $\pi_2 : X \to \mathbb{R}$ be the restrictions of the respective natural projections. Since φ is a bijection, both maps π_1 and π_2 are condensations. Thus X condenses onto $\mathbb{R}$ and onto a P-space Y. Take any $W \in \tau^*(X)$ and any $x = (y, \varphi(y)) \in W$. There exist $U \in \tau(y, Y)$ and $V \in \tau(\varphi(y), \mathbb{R})$ such that $(U \times V) \cap X \subset W$. There is $\alpha < \mathfrak{c}$ such that $y \in U_\alpha \subset U$. Observe that $S_\alpha \subset U_\alpha$ and $E_\alpha = \varphi(S_\alpha)$; besides, the set $Q = E_\alpha \cap V$ has cardinality continuum (Fact 1) and hence $Q' = \varphi^{-1}(Q) \subset S_\alpha \subset U_\alpha$ also has cardinality $\mathfrak{c}$. We have $\{(y,\varphi(y)) : y \in Q'\} \subset (U \times V) \cap X \subset W$ and therefore $|W| \geqslant |Q'| = \mathfrak{c}$ so Fact 2 is proved.

Returning to our solution, let us show that the space X constructed in Fact 2 is as promised. The space X condenses onto a P-space Y; since every countable subset of Y is closed and C-embedded (Fact 1 of S.479), the space $C_p(Y)$ is pseudocomplete (Fact 3 of S.475). If $\pi_1 : X \to Y$ is a condensation then the dual map π_1^* embeds $C_p(Y)$ in $C_p(X)$ as a dense subspace (Problem 163). Therefore $C_p(X)$ is pseudocomplete because it has a dense pseudocomplete subspace (Problem 467).

However, $C_p(X, \mathbb{I})$ is not countably compact because X is not a P-space. Indeed, the map π_2 condenses X onto $\mathbb{R}$ so the set $\{x\} = \pi_2^{-1}(\pi_2(x))$ is G_δ for each $x \in X$. If X were a P-space, then all points of X would be isolated which is false by Fact 2. This contradiction shows that $C_p(X, \mathbb{I})$ is not countably compact (Problem 397) so our solution is complete.

S.481. *Prove that, if $C_p(X, \mathbb{I})$ is pseudocompact then the space $(C_p(X, \mathbb{I}))^\kappa$ is pseudocompact for any cardinal κ.*

Solution. Let $X_\alpha = X$ for all $\alpha < \kappa$; then $C_p(X_\alpha, \mathbb{I})$ is pseudocompact and hence every countable subset of X_α is closed and C^*-embedded in X_α (Problem 398). The space $(C_p(X, \mathbb{I}))^\kappa$ is homeomorphic to $C_p(\oplus\{X_\alpha : \alpha < \kappa\}, \mathbb{I})$ by Problem 114. This shows that pseudocompactness of $(C_p(X, \mathbb{I}))^\kappa$ is equivalent to pseudocompactness of $C_p(Y, \mathbb{I})$, where $Y = \oplus\{X_\alpha : \alpha < \kappa\}$. By Problem 398, the space $C_p(Y, \mathbb{I})$ is pseudocompact if and only if all countable subsets of Y are closed and C^*-embedded in Y. We identify each X_α with the respective clopen subspace of Y (Problem 113).

Take any countable $A \subset Y$; the set $A_\alpha = A \cap X_\alpha$ is countable and hence closed in X_α for each $\alpha < \kappa$. By definition of the topology on Y, a set U is open in Y if and only if $U \cap X_\alpha \in \tau(X_\alpha)$ for all $\alpha < \kappa$ (Problem 113). This is equivalent to saying that a set F is closed in Y if and only if every $F \cap X_\alpha$ is closed in X_α. Since $A_\alpha = A \cap X_\alpha$ is indeed closed in X_α, the set A is closed in Y, i.e., every countable subset of Y is closed in Y. This implies that all countable subsets of Y are discrete and hence every function is continuous on any countable subset of Y.

To show that A is C^*-embedded, take any bounded function $f : A \to \mathbb{R}$. The function $f_\alpha = f|A_\alpha$ can be extended to a continuous function $g_\alpha : X_\alpha \to \mathbb{R}$ for each $\alpha < \kappa$ such that $A_\alpha \neq \emptyset$. Now, if $x \in X_\alpha$ and $A_\alpha \neq \emptyset$ then let $g(x) = g_\alpha(x)$ and put $g(x) = 0$ for all $x \in Y \setminus (\bigcup\{X_\alpha : A_\alpha \neq \emptyset\})$.

The function g is continuous on Y because, for each $x \in Y$, we have $x \in X_\alpha$ for some $\alpha < \kappa$ so X_α is an open neighbourhood of x on which $g(x)$ is either constant or coincides with a continuous function g_α (see Fact 1 of S.472). We proved that every countable subset of Y is closed and C^*-embedded in Y so $C_p(Y, \mathbb{I})$ is pseudocompact (Problem 398). We already observed that $C_p(Y, \mathbb{I})$ is homeomorphic to $(C_p(X, \mathbb{I}))^\kappa$ so $(C_p(X, \mathbb{I}))^\kappa$ is also pseudocompact.

S.482. *Prove that, if $C_p(X, \mathbb{I})$ is countably compact then so is$(C_p(X, \mathbb{I}))^\kappa$ for any cardinal κ.*

Solution. Call a space Z *a P-space* if every G_δ-subset is open in Z. Let $X_\alpha = X$ for all $\alpha < \kappa$; then the space $C_p(X_\alpha, \mathbb{I})$ is countably compact and hence every X_α is a P-space. (Problem 397). The space $(C_p(X, \mathbb{I}))^\kappa$ is homeomorphic to $C_p(\oplus\{X_\alpha : \alpha < \kappa\}, \mathbb{I})$ by Problem 114. This shows that countable compactness of $(C_p(X, \mathbb{I}))^\kappa$ is equivalent to countable compactness of $C_p(Y, \mathbb{I})$, where $Y = \oplus\{X_\alpha : \alpha < \kappa\}$. By Problem 397, the space $C_p(Y, \mathbb{I})$ is countably compact if and only if Y is a P-space. We identify each X_α with the respective clopen subspace of Y (Problem 113).

Take any G_δ-set H is the space Y; then $H \cap X_\alpha$ is a G_δ-subset of X_α and hence $H \cap X_\alpha$ is open in X_α. By definition of the topology on Y, the set H is open in Y. It turns out that every G_δ-subset of Y is open in Y, i.e., Y is a P-space. Applying Problem 397 again we conclude that $C_p(Y, \mathbb{I})$ is countably compact; therefore the space $(C_p(X, \mathbb{I}))^\kappa$ is also countably compact being homeomorphic to $C_p(Y, \mathbb{I})$.

S.483. *Give an example of a countably compact space X such that $X \times X$ is not pseudocompact.*

Solution. Given a set A, denote by $\mathcal{C}(A)$ the family of all countably infinite subsets of A. We will construct our space using certain subspaces of $\beta\omega$.

Fact 1. If A is an infinite subset of $\beta\omega$ then $\overline{A} = 2^{\mathfrak{c}}$.

Proof. It is clear that $|\overline{A}| \leqslant |\beta\omega| = 2^{\mathfrak{c}}$ (Problem 368). There exists an infinite discrete $B \subset A$ (Fact 4 of S.382). The space $K = \overline{B}$ is a compact extension of B such that $\overline{D} \cap \overline{E} = \emptyset$ for all $D, E \subset B$ with $D \cap E = \emptyset$ (Fact 2 of S.382). As a consequence, the space K is homeomorphic to $\beta\omega$ (Fact 2 of S.286) and therefore $|K| = 2^{\mathfrak{c}}$. Thus $|\overline{A}| \geqslant |\overline{B}| = |K| = 2^{\mathfrak{c}}$ so Fact 1 is proved.

Fact 2. There exist countably compact spaces Y and Z such that $Y \times Z$ is not pseudocompact.

Proof. Given any $A \in \mathcal{C}(\beta\omega)$, denote by $p(A)$ some accumulation point of A (which always exists because $\beta\omega$ is compact). Let $Y_0 = \omega \subset \beta\omega$; if we have subsets $\{Y_\alpha : \alpha < \beta\}$ of the space $\beta\omega$ for some $\beta < \omega_1$, let $Y'_\beta = \bigcup\{Y_\alpha : \alpha < \beta\}$ and $Y_\beta = Y'_\beta \cup \{p(A) : A \in \mathcal{C}(Y'_\beta)\}$. This shows that we can define the sets Y_α for all $\alpha < \omega_1$. The space $Y = \bigcup \{Y_\alpha : \alpha < \omega_1\}$ is countably compact. Indeed, if A is a countably infinite subset of Y then $A \subset Y_\alpha$ for some $\alpha < \omega_1$. Therefore $p(A)$ is an accumulation point of A which belongs to $Y_{\alpha+1} \subset Y$. Since every $A \in C(Y)$ has an accumulation point in Y, the space Y is countably compact (Problem 132).

Let us prove by transfinite induction that $|Y_\alpha| \leqslant \mathfrak{c}$ for all $\alpha < \omega_1$. First of all $|Y_0| = \omega \leqslant \mathfrak{c}$. If we assume that $\beta < \omega_1$ and $|Y_\alpha| \leqslant \mathfrak{c}$ for each $\alpha < \beta$ then $\left|Y'_\beta\right| = |\bigcup\{Y_\alpha : \alpha<\beta\}| \leqslant \mathfrak{c}$. As a consequence, $|Y_\beta| \leqslant \left|Y'_\beta\right| + \left|\mathcal{C}(Y'_\beta)\right| \leqslant \mathfrak{c} + \mathfrak{c}^\omega = \mathfrak{c}$; thus $|Y| \leqslant \mathfrak{c} \cdot \omega_1 = \mathfrak{c}$.

We claim that the space $Z = \omega \cup (\beta_\omega \backslash Y)$ is also countably compact. Indeed, any countably infinite $A \subset Z$ has $2^\mathfrak{c}$ accumulation points in $\beta\omega$ by Fact 1; since we only have $\mathfrak{c} < 2^\mathfrak{c}$ points in Y, some accumulation point of A belongs to $\beta\omega \backslash Y \subset Z$.

In the space $Y \times Z \subset \beta\omega \times \beta\omega$ consider the set $\Delta = \{(n, n) : n \in \omega\}$. Observe that $Y \cap Z = \omega$ so if $x \in \beta\omega$ and $(x, x) \in Y \times Z$ then $x \in Y \cap Z = \omega$. This shows that $\Delta = \Phi \cap (Y \times Z)$ where $\Phi = \{(x, x) : x \in \beta\omega\}$. The function $f : \beta\omega \to \beta\omega$ defined by $f(x) = x$ for all $x \in \beta\omega$, is continuous, so its graph Φ is closed in $\beta\omega \times \beta\omega$ (Fact 4 of S.390). Therefore the set Δ is closed in $Y \times Z$. Each $n \in \omega$ is an isolated point of $\beta\omega$, so every $U_n = \{(n, n)\}$ is open in $Y \times Z$; thus $\{U_n : n \in \omega\}$ is a discrete family of non-empty open subsets of $Y \times Z$. Hence $Y \times Z$ is not pseudocompact (Problem 136) so Fact 2 is proved.

To finish our proof, let $X = Y \oplus Z$, where Y and Z are the spaces constructed in Fact 2. Any finite union of countably compact spaces is countably compact so X is a countably compact space. If $X \times X$ is pseudocompact then $Y \times Z$ is a non-pseudocompact clopen subspace of $X \times X$ which contradicts Observation two of S.140. Therefore $X \times X$ is not pseudocompact so our solution is complete.

S.484. *Give an example of a space X such that $C_p(X, \mathbb{I})$ is not countably compact but has a dense countably compact subspace.*

Solution. In Fact 2 of S.480 we constructed a space X without isolated points which can be condensed onto a P-space and onto $\mathbb{R}$. The space $C_p(X, \mathbb{I})$ is not countably compact because X is not a P-space. Indeed, if a map φ condenses X onto $\mathbb{R}$ then $\{x\} = \varphi^{-1}(\varphi(x))$ is a G_δ-set for each $x \in X$. If X were a P-space, then all points of X would be isolated which is false. This contradiction shows that $C_p(X, \mathbb{I})$ is not countably compact (Problem 397).

Let Y be a P-space such that there exists a condensation $p : X \to Y$. The dual map p^* embeds the space $C_p(Y)$ into $C_p(X)$ (Problem 163). It is evident that $p^*(C_p(Y, \mathbb{I})) \subset C_p(X, \mathbb{I})$; we claim that $p^*(C_p(Y, \mathbb{I}))$ is dense in $C_p(X, \mathbb{I})$. To see this take any function $f \in C_p(X, \mathbb{I})$ and any finite $K \subset X$; since p is a condensation, there exists a function $g \in C_p(Y)$ such that $g|p(K) = (f \circ p^{-1})|p(K)$ (Problem 034). Define a function $w : \mathbb{R} \to \mathbb{I}$ as follows: $w(t) = -1$ for all $t < -1$; if $t \in \mathbb{I}$ then $w(t) = t$ and if $t > 1$ then $w(t) = 1$. It is clear that w is continuous; besides, $h = w \circ g \in C_p(Y, \mathbb{I})$ and $h|p(K) = g|p(K) = (f \circ p^{-1})|p(K)$.

For the function $h' = p^*(h) = h \circ p \in p^*(C_p(Y, \mathbb{I}))$, we have the equalities $h'(x) = h(p(x)) = f(p^{-1}(p(x))) = f(x)$ for any $x \in K$. This shows that, for any finite $K \subset X$, there is $h' \in p^*(C_p(Y, \mathbb{I}))$ such that $h'|K = f|K$. An evident consequence is that f is in the closure of the set $p^*(C_p(Y, \mathbb{I}))$. The function f has been chosen arbitrarily so $p^*(C_p(Y, \mathbb{I}))$ is dense in $C_p(X, \mathbb{I})$. Finally observe that $p^*(C_p(Y, \mathbb{I}))$ is countably compact being homeomorphic to a countably compact space $C_p(Y, \mathbb{I})$.

Thus $C_p(X, \mathbb{I})$ has a dense countably compact subspace without being countably compact.

S.485. *Prove that the following are equivalent:*

(*i*) *The space $C_p(X)$ is pseudocomplete.*
(*ii*) *$v(C_p(X)) = \mathbb{R}^X$, i.e., $\mathbb{R}^X$ is canonically homeomorphic to $v(C_p(X))$.*
(*iii*) *Any countable subset of X is closed and C-embedded in X.*

Solution. Call a map $f : Y \to Z$ *almost open* if, for any $U \in \tau^*(Y)$, there is $V \in \tau^*(Z)$ such that $f(U) \subset V \subset \overline{f(U)}$. Assume that $C_p(X)$ is pseudocomplete. Given a countable $A \subset X$, the restriction map $\pi_A : C_p(X) \to \pi_A(C_p(X)) \subset C_p(A)$ is almost open (Fact 1 of S.475). Therefore, the second countable space $\pi_A(C_p(X))$ is pseudocomplete by Fact 1 of S.471. As a consequence, there is a dense Čechcomplete $D \subset \pi_A(C_p(X))$ (Problem 468). The set $C_p(A|X) = \pi_A(C_p(X))$ is dense in $C_p(A)$ and hence in $\mathbb{R}^A$ (Problem 152). Assume that $C_p(A|X) \neq \mathbb{R}^A$; we saw that there is a dense Čech-complete $D \subset C_p(A|X)$. Observe first that $C_p(A|X)$ is an algebra and, in particular, $f - g \in C_p(A|X)$ for any $f, g \in C_p(A|X)$. Take any function $h \in \mathbb{R}^A \backslash C_p(A|X)$; the map $T_h : \mathbb{R}^A \to \mathbb{R}^A$ defined by $T_h(g) = h + g$, is a homeomorphism (Problem 079) and $T_h(C(A\backslash X)) \cap C_p(A\backslash X) = \emptyset$. Indeed, if $g \in C_p(A|X)$ and $f = h + g \in C_p(A|X)$ then $h = f - g \in C_p(A|X)$ which is a contradiction. Therefore the space $\mathbb{R}^A$ has two disjoint dense Čech-complete subspaces D and $T_h(D)$ which contradicts Problem 264. Thus $C_p(A|X) = \mathbb{R}^A$ for any countable $A \subset X$. This means that every countable $A \subset X$ is closed and C-embedded in X so we proved that (i) $\Rightarrow$ (iii); since (iii) $\Rightarrow$ (i) by Fact 3 of S.475, we showed that (i) $\Leftrightarrow$ (iii).

Now, if (iii) holds then, for any $f \in \mathbb{R}^X$ and any countable $A \subset X$, there is a function $g \in C_p(X)$ such that $g|A = f|A$. Therefore every $f \in \mathbb{R}^X$ is strictly ω-continuous on X; applying Problem 438 we conclude that $\mathbb{R}^X$ is canonically homeomorphic to $\upsilon(C_p(X))$. This settles (iii) $\Rightarrow$ (ii).

Finally, assume that (ii) holds. The set S of strictly ω-continuous functions on X is contained in $\mathbb{R}^X$; since S is realcompact (Problem 438), we have $S = \mathbb{R}^X$ (see Problem 414). Thus every $f \in \mathbb{R}^X$ is strictly ω-continuous on X. Given any countable $A \subset X$ and any $g : A \to \mathbb{R}$, define $g' \in \mathbb{R}^X$ by $g'|A = g$ and $g'(x) = 0$ for all $x \in X \backslash A$. Since g' is strictly ω-continuous, there is $h \in C_p(X)$ such that $h|A = g'|A = g$. This proves that every countable $A \subset X$ is C-embedded in X. If $x \in \overline{A} \backslash A$ then let $f(x) = 1$ and $f(y) = 0$ for all $y \in A$. Since $A \cup \{x\}$ is C-embedded, there is $g \in C_p(X)$ such that $g|(A \cup \{x\}) = f$. By continuity of g, we have $1 = g(x) \in \overline{g(A)} = \overline{\{0\}} = \{0\}$ which is a contradiction. This shows that every countable $A \subset X$ is closed in X. Therefore (ii) $\Rightarrow$ (iii) is established so our solution is complete.

S.486. *Prove that X is discrete if and only if $C_p(X)$ is pseudocomplete and realcompact.*

Solution. If X is discrete then $C_p(X) = \mathbb{R}^X$ is realcompact by Problem 401 and pseudocomplete by Problem 470.

Now assume that $C_p(X)$ is realcompact and pseudocomplete. Then $t_m(X) \leqslant \omega$ by Problem 429. Suppose that some point $x \in X$ is not isolated in X; apply Fact 1 of S.419 to find a countable set $A \subset X \setminus \{x\}$ such that $x \in \overline{A}$. However, all countable subsets of X are closed in X by Problem 485 which shows that $x \in \overline{A} = A$ which is a contradiction. Hence all points of X are isolated, i.e., the space X is discrete.

S.487. *Prove that, if $C_p(X)$ is homeomorphic to $\mathbb{R}^\kappa$ for some κ then X is discrete.*

Solution. Any $\mathbb{R}^\kappa$ is realcompact by Problem 401 and pseudocomplete by Problem 470. Thus, if $C_p(X)$ is homeomorphic to $\mathbb{R}^\kappa$ then it is also realcompact and pseudocomplete. As a consequence, X is discrete by Problem 486.

S.488. *Suppose that there is an open subspace $Y \subset C_p(X)$ homeomorphic to $\mathbb{R}^\kappa$ for some κ. Is it true that X is discrete?*

Solution. Yes, it is true; we will need a couple of facts to prove this.

Fact 1. Suppose that Z_t is a pseudocomplete space for every $t \in T$. Then the space $Z = \oplus\,\{Z_t : t \in T\}$ is also pseudocomplete.

Proof. We identify each Z_t with the respective clopen subspace of the space Z. Let $\{\mathcal{B}_t^n : n \in \omega\}$ be a pseudocomplete sequence of π-bases in Z_t for each $t \in T$. It is straightforward that $\mathcal{B}_n = \cup\{\mathcal{B}_t^n : t \in T\}$ is a π-base in Z for each $n \in \omega$. To see that the sequence $\{\mathcal{B}_n : n \in \omega\}$ is pseudocomplete, take any family $\{Un : n \in \omega\}$ such that $\overline{U}_{n+1} \subset U_n$ and $U_n \in \mathcal{B}_n$ for all $n \in \omega$. There is $t \in T$ such that $U_0 \in \mathcal{B}_t^n$; we claim that $U_{\hat{n}} \in \mathcal{B}_t^n$ for all $n \in \omega$. Indeed, if $U_n \in \mathcal{B}_s^n$ for some $s \neq t$ then $U_n \subset Z_s$ and therefore $U_n \cap Z_t = \emptyset$ whence $U_n \cap U_0 = \emptyset$ which is a contradiction with $U_n \subset U_0$. Since the sequence $\{\mathcal{B}_t^n : n \in \omega\}$ is pseudocomplete, we have $\bigcap\{U_n : n \in \omega\} \neq \emptyset$ so Fact 1 is proved.

Fact 2. Any locally pseudocomplete space is pseudocomplete. In other words, if Z is a space and each $z \in Z$ has a pseudocomplete neighbourhood then Z is pseudocomplete.

Proof. Since any open subspace of a pseudocomplete space is pseudocomplete (Problem 466), any $z \in Z$ has an open pseudocomplete neighbourhood. Consider the family $\mathcal{U} = \{U \in \tau^*(Z) : U \text{ is pseudocomplete}\}$. We saw that $\bigcup \mathcal{U} = Z$; let $\mathcal{V}$ be a maximal disjoint subfamily of $\mathcal{U}$. It is easy to see that $\bigcup \mathcal{V}$ is dense in Z; besides, $Y = \bigcup \mathcal{V} = \oplus\{V : V \in \mathcal{V}\}$ because each $V \in \mathcal{V}$ is a clopen subspace of Y (see Problem 113). Apply Fact 1 to conclude that Y is pseudocomplete. Since Z has a dense pseudocomplete subspace, it is pseudocomplete by Problem 467 so Fact 2 is proved.

Returning to our solution assume that some open $U \subset C_p(X)$ is homeomorphic to $\mathbb{R}^\kappa$. Then U is pseudocomplete by Problem 470 and realcompact by Problem 401. Apply Problem 428 to see that $C_p(X)$ is also realcompact. Given any $f \in C_p(X)$, the map $T_f : C_p(X) \to C_p(X)$ defined by $T_f(g) = f + g$ for all $g \in C_p(X)$, is a homeomorphism (Problem 079). Fix any $h \in U$; if $g \in C_p(X)$ then $V = T_{g-h}(U)$ is homeomorphic to U and $g = g - h + h \in V$. Since U is pseudocomplete, so is V and therefore every $g \in C_p(X)$ has a pseudocomplete open neighbourhood in $C_p(X)$.

Finally, apply Fact 2 to conclude that $C_p(X)$ is pseudocomplete. Since $C_p(X)$ is realcompact and pseudocomplete, the space X is discrete by Problem 486.

S.489. *Given an arbitrary space X, suppose that, for some cardinal* κ*, there exists a continuous onto map* $\varphi : \mathbb{R}^\kappa \to C_p(X)$ *such that* $\varphi(af + bg) = a\varphi(f) + b\varphi(g)$ *for all* $f, g \in \mathbb{R}^\kappa$ *and* $a, b \in \mathbb{R}$ *(such maps are called linear). Prove that X is discrete.*

Solution. To avoid going into a general theory, let us define *a linear space* as a subspace L of some $\mathbb{R}^T$ such that $f, g \in L$ implies $af + bg \in L$ for any $a, b \in \mathbb{R}$. The set T will be always clear from the context; we will also consider L to be a topological space with the topology inherited from $\mathbb{R}^T$. Elements of a linear space will be often called *vectors*. Observe that any linear space contains the function $\mathbf{0}$ which is identically zero on T. Given a linear space L and vectors $f_1, \ldots, f_n \in L$, call a vector $f \in L$ *a linear combination of the vectors* $f_1, \ldots, f_n$ if there exist $a_1, \ldots, a_n \in \mathbb{R}$ such that $f = a_1 f_1 + \cdots + a_n f_n$.

Call a set $A \subset L$ *linearly independent* if, for any $n \in \mathbb{N}$, any distinct vectors $f_1, \ldots, f_n \in A$ and any $a_1, \ldots, a_n \in \mathbb{R}$, if $a_1 f_1 + \cdots + a_n f_n = \mathbf{0}$ then $a_i = 0$ for all $i \leqslant n$. A subspace $L' \subset L$ is called *a linear subspace* of L if $af + bg \in L'$ for any $f, g \in L'$ and $a, b \in \mathbb{R}$. Given an arbitrary set P in a linear space L, let $\langle P \rangle$ be the set of all linear combinations of elements of P. The set $\langle P \rangle$ is called *the linear hull* of P; it is easy to see that the linear hull of any set is a linear subspace.

For any linear space L, call a set $B \subset L$ *a Hamel basis* of L if B is a linearly independent set and $\langle B \rangle = L$. A function $p : L \to \mathbb{R}$ is called *a linear functional* if $p(af + bg) = ap(f) + bp(g)$ for any $f, g \in L$ and $a, b \in \mathbb{R}$. We denote by L^* the set of all continuous linear functionals on L with the topology and arithmetic operations inherited from $C_p(L)$. It is evident that L^* is also a linear space. For any linear spaces L and M, a map $u : L \to M$ is called *linear* if $u(af + bg) = au(f) + bu(g)$ for any $f, g \in L$ and $a, b \in \mathbb{R}$.

Fact 1. If L is a linear space then, for any linearly independent set $A \subset L$, there exists a Hamel basis B of the space L such that $A \subset B$.

Proof. Since linear independence of a set is defined in terms of finite subsets of this set, the union of any increasing chain of linearly independent sets is a linearly independent set. This makes it possible to apply Zorn's lemma to find a maximal linearly independent set B such that $A \subset B$. Let us establish that $\langle B \rangle = L$.

To obtain contradiction suppose not and fix any vector $f \in L \setminus \langle B \rangle$. We claim that the set $B \cup \{f\}$ is linearly independent. Indeed, take any distinct vectors $f_1, \ldots, f_n \in L \cup \{f\}$ and suppose that we have $a_1 f_1 + \cdots + a_n f_n = \mathbf{0}$ for some $a_1, \ldots, a_n \in \mathbb{R}$. If $\{f_1, \ldots, f_n\} \subset B$ then $a_i = 0$ for $i \leqslant n$ because B is independent. If $f \in \{f_1, \ldots, f_n\}$ we can assume, without loss of generality, that $f = f_n$. If $a_n = 0$ then $a_1 f_1 + \cdot + a_n f_n = a_1 f_1 + \cdots + a_{n-1} f_{n-1} = \mathbf{0}$ so $a_1, \ldots, a_{n-1} = 0$ because B is linearly independent so again $a_i = 0$ for all $i \leqslant n$. Finally, if $a_n \neq 0$ then the equality $f = f_n = (-\frac{a_1}{a_n}) f_1 + \cdots + (-\frac{a_{n-1}}{a_n}) f_{n-1}$ shows that $f \in \langle B \rangle$ which is again a contradiction. We proved that $B \cup \{f\}$ is linearly independent; since this contradicts maximality of B, we have $L\backslash\langle B\rangle = \emptyset$, i.e., $L = \langle B\rangle$ so B is a Hamel basis of L and Fact 1 is proved.

Fact 2. Assume that L is a linear space and B is a Hamel basis in L. Fix any map $\xi : B \to \mathbb{R}$ and let $R(L,B) = \bigcup\{\mathbb{R}^n \times B^n : n \in \mathbb{N}\}$. If $(a, f) \in R(L, B)$ where $a = (a_1, \dots, a_n)$ and $f = (f_1, \dots, f_n)$, let $s(a, f) = a_1 f_1 + \cdots + a_n f_n$ and $\Xi(a, f) = a_1\xi(f_1) + \cdots + a_n\xi(f_n)$. Then $s(a, f) = \mathbf{0}$ implies $\Xi(a, f) = 0$ for any $(a, f) \in R(L, B)$.

Proof. We use induction on n. If $n = 1$ then $a_1 f_1 = \mathbf{0}$ for some $f_1 \in B$ so $a_1 = 0$ because B is linearly independent. Therefore $\Xi(a, f) = 0 \cdot \xi(f_1) = 0$. Now assume that we proved our fact for all $k \leqslant n$ and consider $a = (a_1, \dots, a_{n+1}) \in \mathbb{R}^{n+1}$ and $f = (f_1, \dots, f_{n+1}) \in B^{n+1}$ such that $s(a, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1} = \mathbf{0}.$ If the functions $f_1, \dots, f_{n+1}$ are distinct then $a_1 = \cdots = a_{n+1} = 0$ by linear independence of B so $\Xi(a, f) = 0$. If $f_i = f_j$ for some $i < j$ then $\mathbf{0} = a_1 f_1 + \cdots + a_n f_n = a_1 f_1 + \cdots + \big(a_i + a_j\big) f_i + \cdots + a_{j-1} f_{j-1} + a_{j+1} f_{j+1} + \cdots + a_n f_n$ so we can apply the induction hypothesis to the $(n - 1)$-tuples $b = (a_1, \dots, (a_i + a_j), \dots, a_{j-1}, a_{j+1}, \dots, a_n)$ and $g = (f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_n)$ to conclude that $\Xi(a, f) = a_1\xi(f_1) + \cdots + a_n\xi(f_n) = a_1\xi(f_1) + \cdot + (a_i + a_j)\xi(f_i) + \cdots + a_{j-1}\xi(f_{j-1}) + a_{j+1}\xi(f_{j+1}) + \cdots + a_n\xi(f_n) = \Xi(b, g) = 0$ so Fact 2 is proved.

Fact 3. Let L be a linear space. If B is a Hamel basis in L and $\xi : B \to \mathbb{R}$ then there is a unique linear functional $\Xi : L \to \mathbb{R}$ such that $\Xi|B = \xi$.

Proof. Given any $g \in L$ there exist $f_1, \dots, f_n \in B$ and $a_1, \dots, a_n \in \mathbb{R}$ such that $g = a_1 f_1 + \cdots + a_n f_n$; let $\Xi(g) = a_1\xi(f_1) + \cdots + a_n\xi(f_n)$. Of course, we must prove consistency of this definition, i.e., that we have the equality $b_1\xi(h_1) + \cdots b_k\xi(h_k) = a_1\xi(f_1) + \cdots + a_n\xi(f_n)$ if $g = b_1 h_1 + \cdots + b_k h_k$ for some $h_1, \dots, h_k \in B$ and $b_1, \dots, b_k \in \mathbb{R}$. To do this, note that $a_1 f_1 + \cdots + a_n f_n + (-b_1)h_1 + \cdots + (-b_k)h_k = \mathbf{0}$, so we have $a_1\xi(f_1) + \cdots + a_n\xi(f_n) - b_1\xi(h_1) - \cdots - b_k\xi(h_k) = 0$ by Fact 2 applied to the $(n + k)$-tuples $(a_1, \dots, a_n, -b_1, \dots, -b_k)$ and $(f_1, \dots, f_n, g_1, \dots, G_k)$. Therefore $a_1\xi(f_1) + \cdots + a_n\xi(f_n) = b_1\xi(h_1) + \cdots + b_k\xi(h_k)$, i.e., Ξ is well defined. It is immediate from the definition that Ξ is a linear functional and $\Xi|B = \xi$.

To see the uniqueness, suppose that $\Psi : L \to \mathbb{R}$ is any linear functional such that $\Psi|B = \xi$. The linearity of Ψ implies that if $g \in L$ and $g = a_1 f_1 + \cdots + a_n f_n$ for some $f_1, \dots, f_n \in B$ and $a_1, \dots, a_n \in \mathbb{R}$ then $\Psi(g) = a_1\xi(f_1) + \cdots + a_n\xi(f_n) = \Xi(g)$ so $\Psi(g) = \Xi(g)$ for any $g \in L$, i.e., $\Psi = \Xi$ and Fact 3 is proved.

Fact 4. Suppose that L and M are linear spaces and $u : L \to M$ is a continuous linear onto map. Then the dual map $u^* : C_p(M) \to C_p(L)$ restricted to M^* is a linear embedding of M^* in L^*; in particular, $u^*(M^*)$ is a linear subspace of L^*.

Proof. It is evident that u^* is a linear map and it follows from Problem 163 that u^* is an embedding. Given any $p \in M^*$, we have

$$u^*(p)(af + bg) = p(au(f) + bu(g)) = ap(u(f) + bp(u(g)) = au^*(f) + bu^*(g)$$

for any $f, g \in L$ and $a, b \in \mathbb{R}$ which proves that $u^*(p)$ is a linear functional. The fact that $u^*(M^*)$ is a linear subspace of L^* is an immediate consequence of linearity of u^* so Fact 4 is proved.

Given a space Z and $z \in Z$, let $e_z(f) = f(z)$ for any $f \in C_p(Z)$. Then e_z is a continuous linear functional on $C_p(Z)$ for any $z \in Z$ (Problem 196) and it follows immediately from Problems 196 and 197 that $(C_p(Z))^* = L_p(Z) = \{\lambda_1 e_{z1} + \cdots + \lambda_n e_{zn} : n \in \mathbb{N}, z_1, \ldots, z_n \in Z \text{ and } \lambda_i \in \mathbb{R}, \text{ for all } i = 1, \ldots, n\}$. Observe also that the space $E(Z) = \{e_z : z \in Z\} \subset C_p(C_p(Z))$ is homeomorphic to Z (Problem 167).

Fact 5. For any space Z, the set $E(Z)$ is a Hamel basis in $L_p(Z)$.

Proof. It is immediate from the definition of $L_p(Z)$ that $\langle E(Z)\rangle = L_p(Z)$. To prove that $E(Z)$ is linearly independent, take any distinct $z_1, \ldots, z_n \in Z$ such that $w = \lambda_1 e_{z1} + \cdots + \lambda_n e_{zn} = \mathbf{0}$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. For each $i \leqslant n$ fix a function $f_i \in C(Z)$ such that $f_i(z_i) = 1$ and $f_i(z_j) = 0$ for all $i \neq j$ (034). Then $0 = \mathbf{0}(f_i) = w(f_i) = \lambda_i$ for each $i \leqslant n$ so $E(Z)$ is linearly independent. Fact 5 is proved.

Fact 6. If Z is any space and $f : E(Z) \to \mathbb{R}$ is a continuous function then there exists a unique continuous linear functional $\hat{f} : L_p(Z) \to \mathbb{R}$ such that $\hat{f}|E(Z) = f$.

Proof. Since $E(Z)$ is a Hamel basis of $L_p(Z)$ (Fact 5), there exists a unique linear functional $\hat{f} : L_p(Z) \to \mathbb{R}$ such that $\hat{f}|E(Z) = f$ (Fact 3). Let us prove that $\hat{f}$ is continuous. It follows from linearity of $\hat{f}$ that

$$\hat{f}(w) = \hat{f}(\lambda_1 e_{z_1} + \cdots \lambda_n e_{z_n}) = \lambda_1 f(e_{z_1}) + \cdots \lambda_n f(e_{z_n})$$

for each $w = \lambda_1 e_{z_1} + \cdots \lambda_n e_{z_n} \in L_p(Z)$. The function $e : Z \to E(Z)$ defined by $e(z) = e_z$ for all $z \in Z$, is a homeomorphism (Problem 167) so the function $g = f \circ e$ is continuous on Z and hence the map $i_g : C_p(C_p(Z)) \to \mathbb{R}$ defined by $i_g(u) = u(g)$ for all $u \in C_p(C_p(X))$, is continuous (Problem 166). Finally, observe that

$$\begin{aligned} i_g(w) = w(g) = w(f \circ e) &= \lambda_1 f(e(z_1) + \cdots + \lambda_n f(e(z_n)) \\ &= \lambda_1 f(e_{z_1}) + \cdots + \lambda_n f(e_{z_n}) = \hat{f}(w) \end{aligned}$$

for each $w = \lambda_1 e_{z_1} + \cdots + \lambda_n e_{z_n} \in L_p(X)$. Thus the functional $\hat{f}$ is continuous because it coincides with the restriction of a continuous function i_g to $L_p(Z)$. Fact 6 is proved.

Returning to our solution observe that $\mathbb{R}^\kappa$ coincides with $C_p(D(\kappa))$; denote the discrete space $D(\kappa)$ by D to simplify notation. We have a continuous linear onto map $\varphi : C_p(D) \to C_p(X)$ so φ^* embeds $L_p(X)$ in $L_p(D)$ as a linear subspace (Fact 4). The set $\varphi^*(E(X))$ is linearly independent because φ^* is a linear bijection between $L_p(X)$ and $\varphi^*(L_p(X))$. Therefore there exists a Hamel basis H in the linear space $L_p(D)$ such that $\varphi^*(E(X)) \subset H$ (see Facts 1 and 5). Given any map $p : \varphi^*(E(X)) \to \mathbb{R}$, take a map $q : H \to \mathbb{R}$ defined by $q(f) = p(f)$ for any $f \in \varphi^*(E(X))$ and $q(h) = 0$ for any $h \in H \setminus \varphi^*(E(X))$. Then $q : H \to \mathbb{R}$ so there is a linear functional $u : L_p(D) \to \mathbb{R}$ such that $u|H = q$ (Fact 3). The map $v = u|E(D)$ is continuous because $E(D)$ is a discrete space (recall that $E(D)$ is homeomorphic to D!). Therefore there is a unique continuous functional $\hat{\upsilon} : L_p(D) \to \mathbb{R}$ such that $\hat{\upsilon}|E(D) = \upsilon$. But u is also a linear functional on $L_p(D)$ with $u|E(D) = \upsilon$ so the functional $u = \hat{\upsilon}$ is continuous. As a consequence, the map $p = u|\varphi^*(E(X))$ is continuous.

We proved that any map $p : \varphi^*(E(X)) \to \mathbb{R}$ is continuous and therefore $\varphi^*(E(X))$ is a discrete space. Since X is homeomorphic to $\varphi^*(E(X))$, the space X is also discrete so our solution is complete.

S.490. *Prove that any open image of a projectively complete space is projectively complete.*

Solution. Suppose that X is projectively complete and $f : X \to Y$ is an open continuous onto map. If M is second countable and $g : Y \to M$ is a continuous open onto map then $g \circ f$ is an open continuous map of X onto M. Since X is projectively complete, the space M has to be Čech-complete so Y is projectively complete.

S.491. *Prove that any product of Čech-complete spaces is projectively complete.*

Solution. Call a space X *strongly pseudocomplete* if it has a pseudocomplete sequence of bases.

Fact 1. Let X be a Čech-complete space. Suppose that $f : X \to Y$ is an open continuous map of X onto a paracompact space Y. Then f is *inductively perfect*, i.e., there exists a closed subset $F \subset X$ such that $f(f) = Y$ and $f|F : F \to Y$ is a perfect map.

Proof. There exists a continuous map $g : \beta X \to \beta Y$ such that $g|X = f$ (Problem 257). It is clear that $g(\beta X) = \beta Y$ so the map g is perfect (Problem 122). As a consequence, the map $h = g|g^{-1}(Y) : g^{-1}(Y) \to Y$ is also perfect (Fact 2 of S.261). Let $X' = g^{-1}(Y)$; we are going to prove that

(1) For any $W \in \tau(X')$ such that $f(W \cap X) = Y$ there is $W' \in \tau(X')$ such that $\overline{W'} \subset W$ and $f(W' \cap W) = Y$ (the bar denotes the closure in X').

To prove (1), for any point $y \in Y$, choose a set $V(y) \in \tau(X')$ such that $V(y) \cap f^{-1}(y) \neq \emptyset$ and $\overline{V(y)} \subset W$. The family $\mathcal{U} = \{f(V(y) \cap X) : y \in Y\}$ is an open cover of Y so there is an open locally finite refinement $\{U_s : s \in S\}$ of the family $\mathcal{U}$. For any $s \in S$ choose $y_s \in Y$ such that $U_s \subset f(V(y_s) \cap X)$ and consider the family $\mathcal{V} = \{V(y_s) \cap h^{-1}(U_s) : s \in S\}$. It is locally finite in X' because the family $\{h^{-1}(U_s) : s \in S\}$ is locally finite and $V(y_s) \cap h^{-1}(U_s) \subset h^{-1}(U_s)$ for each $s \in S$. Every locally finite family is closure-preserving (Fact 2 of S.221), so we have $\overline{\bigcup \mathcal{V}} = \bigcup\{\overline{V(y_s) \cap h^{-1}(U_s)} : s \in S\} \subset W$ because $\overline{V(y_s)} \subset W$ for every $s \in S$. This shows that $\overline{W'} \subset W$ if $W' = \bigcup \mathcal{V}$. Besides, for any $y \in Y$ there is $s \in S$ such that $y \in U_s$; hence $y \in f(V \cap X)$ where $V = h^{-1}(U_s) \cap V(y_s) \in \mathcal{V}$. As a consequence $y \in f(V \cap X) \subset f(W' \cap X)$ so (1) is proved.

Since X is Čech-complete, there exists a family $\{O_n : n \in \omega\} \subset \tau(X')$ such that $X = \bigcap\{O_n : n \in \omega\}$ (Problem 259). Apply property (1) to construct a sequence $\{G_n : n \in \omega\} \subset \tau(X')$ such that $f(G_n \cap X) = Y$ and $\overline{G}_{n-1} \subset G_n \cap \left(\bigcap_{i=1}^n O_i\right)$ for each $n \in \omega$. The set $F = \bigcap(G_n : n \in \omega\} = \bigcap\{\overline{G}_n : n \in \omega)$ is closed in X' so $h|F$:

$F \to h(f)$ is a perfect map (it is an easy exercise that the restriction of a perfect map to a closed subset is a perfect map). Since $G_n \subset O_n$ for each $n \in \omega$, the set F is contained in X so $h|F = g|F = f|F$. Given $y \in Y$, we have $g(\overline{G}_n) \supset f(G_n \cap X) = Y$ and hence $P_n = h^{-1}(y) \cap \overline{G}_n \neq \emptyset$ for any $n \in \omega$. Since the set $h^{-1}(y) = g^{-1}(y)$ is compact, the decreasing sequence $\{P_n : n \in \omega\}$ consists of compact sets so $P = \bigcap\{P_n : n \in \omega\} \neq \emptyset$. It is clear that, for any $x \in P$, we have $x \in F$ and $f(x) = h(x) = y$ which gives $f(f) = Y$ showing that Fact 1 is proved.

Fact 2. Let Y and Z be any spaces. Suppose that $\varphi : Y \to Z$ is an onto map and, for every $y \in Y$ there is a local base $\mathcal{B}_y$ of Y at y such that $\mathcal{C}_y = \{\varphi(U) : U \in \mathcal{B}_y\}$ is a local base of Z at $\varphi(y)$. Then φ is an open continuous map.

Proof. Given any $y \in Y$ and any $W \in \tau(\varphi(y), Z)$ there is $V \in \mathcal{B}_y$ such that $\varphi(y) \in \varphi(V) \subset W$ because $\mathcal{C}_y$ is a local base at $\varphi(y)$. This proves continuity of φ at every $y \in Y$. Now, if $U \in \tau(Y)$ and $z \in \varphi(U)$ then take any $y \in U$ such that $\varphi(y) = z$. There is a set $V \in \mathcal{B}_y$ such that $V \subset U$; it is clear that $\varphi(V) \in \tau(z,Z)$ and $\varphi(V) \subset \varphi(U)$, i.e., each $z \in \varphi(U)$ has a neighbourhood contained in $\varphi(U)$. Thus $\varphi(U)$ is open in Z and therefore the map φ is open. Fact 2 is proved.

To avoid confusion in what follows recall that we identify any ordinal with the set of its predecessors; in particular, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$ and, in general, $n = \{0,\dots, n-1\}$ for any $n \in \mathbb{N}$. It is convenient to consider that $\omega^0 = 1 = \{\emptyset\}$; let $\mathcal{O} = \{\omega^n : n \in \omega\}$ and $\mathcal{O}_n = \bigcup\{\omega^k : k \leqslant n\}$. Given any $s \in \mathcal{O}$, let $l(s) = n$ if $s \in \omega^n$. In other words, $\mathcal{O}$ is the set of all finite sequences of elements of ω including the empty sequence; clearly, $l(s)$ is the length of the sequence s. Given a separable metrizable space X, call a family $\mathcal{U} = \{U_s : s \in \mathcal{O}\} \subset \tau(X)$ *an A-system in X* if

(A1) $U_\emptyset = X$ and $U_s = \bigcup\{U_t : t \in \omega^{n+1}$ and $t|n = s\}$ for any $n \in \omega$ and $s \in \omega^n$.
(A2) If $s \in \mathcal{O}$ and $l(s) = n \in \mathbb{N}$ then $\mathrm{diam}(U_s) < \frac{1}{n}$.
(A3) For any $f \in \omega^\omega$, we have $\bigcap\{U_{f|n} : n \in \omega\} \neq \emptyset$.

Fact 3. A separable metrizable space X is Čech-complete if and only if there exists an A-system in X.

Proof. Assume that X is Čech-complete and fix a complete metric d in X (see 269). It follows from the Lindelöf property of X that we can choose an open cover $\{W_n : n \in \omega\}$ of the space X such that $\mathrm{diam}(W_n) < 1$ for all $n \in \omega$. Letting $U_\emptyset = X$ and $U_s = W_{s(0)}$ for each $s \in \omega^1$ we obtain a family $\{U_s : s \in \mathcal{O}_1\} \subset \tau(X)$. Suppose that $n \in \mathbb{N}$ and we have a family $\{U_s : s \in \mathcal{O}_n\} \subset \tau(X)$ with the following properties:

(B1) $U_\emptyset = X$.
(B2) If $k < n$ and $s \in \omega^k$ then $\overline{U}_t \subset U_s$ for any $t \in \omega^{k+1}$ with $t|k = s$.
(B3) If $k < n$ and $s \in \omega^k$ then $U_s = \{U_t : t \in \omega^{k+1}$ and $t|k = s\}$.
(B4) If $s \in \mathcal{O}_n$ and $l(s) = k \geqslant 1$ then $\mathrm{diam}(Us) < \frac{1}{k}$.

Take any $s \in \omega^n$ and find an open cover $\{W^s_k : k \in \omega\}$ of the set U_s such that $\overline{W^s_k} \subset U_s$ and $\mathrm{diam}(W^s_k) < \frac{1}{n+1}$ for each $k \in \omega$. For any $t \in \omega^{n+1}$, let $U_t = W^s_k$ where $s = t|n$ and $k = t(n)$. It is immediate that we also have the properties (B1)–(B4) for the family $\{U_s : s \in \mathcal{O}_{n+1}\}$. Therefore, we can continue this inductive construction which provides us with a family $\mathcal{U} = \{U_s : s \in \mathcal{O}\}$ with the properties (B1)–(B4) for all $n \in \mathbb{N}$. Observe that (B1) and (B3) imply (A1), and (A2) follows from (B4). Let us show that $\mathcal{U}$ also has (A3). Indeed, if $f : \omega \to \omega$ then $\overline{U_{f|n}} \subset U_{f|(n-1)}$ and $\operatorname{diam}(U_{f|n}) = \operatorname{diam}\left(\overline{U_{f|n}}\right) < \frac{1}{n}$ for all $n \in \mathbb{N}$. This shows that $\bigcap\left\{U_{f|n} : n \in \omega\right\} = \bigcap\left\{\overline{U_{f|n}} : n \in \omega\right\} \neq \emptyset$ (Problem 236) and therefore $\mathcal{U}$ is an A-system in X so necessity is proved.

Now assume that a family $\mathcal{U} = \{U_s : s \in \mathcal{O}\}$ is an A-system in X. For any function $f \in \omega^\omega$, observe that the set $P_f = \bigcap\{U_{f|n} : n \in \omega\}$ consists of precisely one point because $P_f \neq \emptyset$ by (A3) and the diameters of the sets $U_{f|n}$ tend to zero. Let $\varphi(f) = x$ where $x \in X$ is the point for which $P_f = \{x\}$. We claim that $\varphi : \omega^\omega \to X$ is a continuous open onto map.

Fix $m \in \omega$ and $s \in \omega^m$ and let $W_s = \{f \in \omega^\omega : f|m = s\}$. It is straightforward from the definition of φ that $\varphi(Ws) \subset U_s$. Take any $x \in U_s$ and let $s_m = s$. Assume that $m \leqslant n \in \omega$ and we have constructed functions $\{s_k : m \leqslant k \leqslant n\}$ with the following properties:

(i) $sk \in \omega^k$ for each $k \in \{m, \ldots, n\}$.
(ii) $x \in U_{s_k}$ for each $k \in \{m, \ldots, n\}$.
(iii) $s_{k+1}|k = s_k$ for all $k \in \{m, \ldots, n-1\}$.

Since $x \in U_{s_n} = \bigcup\{U_t : t \in \omega^{n+1}$ and $t|n = s_n\}$, there is $s_{n+1} \in \omega^{n+1}$ such that $s_{n+1}|n = s_n$ and $x \in U_{s_{n+1}}$. It is evident that (i)–(iii) hold for the sequence $\{s_k : k \leqslant n+1\}$ so our inductive construction can be continued giving us a sequence $\{s_n : m \leqslant n < \omega\}$ with the properties (i)–(iii). It follows from (iii) that there exists $f \in \omega^\omega$ such that $f|n = s_n$ for each $n \geqslant m$. Condition (ii) implies that $x \in \bigcap\{U_{f|n} : n \in \omega\}$ and therefore $\varphi(f) = x$ so $\varphi(Ws) = U_s$. Since $s \in \mathcal{O}$ has been chosen arbitrarily, we proved that

($*$) $\varphi(W_s) = U_s$ for any $s \in \mathcal{O}$.

If we take s $= \emptyset$ then we obtain $\varphi(\omega^\omega) = X$, i.e., the map φ is onto. For any point $x \in X$ take any $f \in \omega^\omega$ such that $\varphi(f) = x$. Observe that the family $\mathcal{C}_x = \{U_{f|n} : n \in \omega\}$ is a local base of X at x. Indeed, all elements of $\mathcal{C}_x$ are open in X and contain x. Given any $\varepsilon > 0$, choose $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$ and note that, for any $y \in U_{f|n}$, we have $d(x, y) \leqslant \operatorname{diam}(U_{f|n}) < \frac{1}{n}$ so $x \in U_{f|n} \subset B(x, \varepsilon)$ which proves that $\mathcal{C}_x$ is a local base at x. Besides, the family $\mathcal{B}_f = \{\{g \in \omega^\omega : g|n = f|n\} : n \in \omega\}$ is a local base of ω^ω at f and $\{\varphi(U) : U \in \mathcal{B}_f\} = \mathcal{C}_x$ by ($*$), so we can apply Fact 2 to conclude that the map φ is continuous and open.

Since the space ω^ω is Čech-complete (see Problems 204 and 207) and X is paracompact, we can apply Fact 1 to conclude that there is a closed $F \subset \omega^\omega$ such that $\varphi(f) = X$ and $\varphi|F$ is a perfect map. The set F is Čech-complete (Problem 260) and a perfect image of a Čech-complete space is Čech-complete (Problem 261) so X is Čech-complete and Fact 3 is proved.

Fact 4. Any strongly pseudocomplete space is projectively complete.

Proof. Let Z be a strongly pseudocomplete space; fix a pseudocomplete sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of bases in Z. Take any open map $\varphi : Z \to M$ onto a separable metric space (M, d). Let $V_\emptyset = Z$, $U_\emptyset = M$ and suppose that $n \in \omega$ and we constructed sets U_s and V_s for all $s \in \mathcal{O}_n$ such that the following properties are fulfilled:

(C1) $V_\emptyset = X$, $U_\emptyset = M$ and $\varphi(V_s) = U_s$ for all $s \in \mathcal{O}_n$.
(C2) $V_s \in \mathcal{B}_k$ if $l(s) = k \in \{1, \dots, n\}$.
(C3) If $k < n$ and $s \in \omega^k$ then $\overline{V}_t \subset V_s$ for any $t \in \omega^{k+1}$ with $t|k = s$.
(C4) If $k < n$ and $s \in \omega^k$ then $U_s = \bigcup\{U_t : t \in \omega^{k+1}$ and $t|k = s\}$.
(C5) If $s \in \mathcal{O}_k$ and $k \in \{1, \dots, n\}$ then $\operatorname{diam}(U_s) < \frac{1}{k}$.

Take any $s \in \omega^n$; for any point $z \in U_s = \varphi(V_s)$ choose $y_z \in V_s$ with $\varphi(y_z) = z$. Since the mapping φ is open and continuous, we can find $W_z \in \mathcal{B}_{n+1}$ such that $y_z \in W_z \subset \overline{W}_z \subset V_s, \operatorname{diam}(\varphi(W_z)) < \frac{1}{n+1}$ and $\varphi(W_z) \subset U_s$. If $G_z = \varphi(W_z)$ for each $z \in U_s$ then $\{G_z : z \in U_s\}$ is an open cover of the second countable (and hence Lindelöf) space U_s. We can choose a subfamily $\mathcal{W}_s = \{W^s_n : n \in \omega\}$ of the family $\{W_z : z \in U_s\}$ such that $\mathcal{G}_s = \{G^s_n = \varphi(W^s_n) : n \in \omega\}$ covers U_s. After we have families $\mathcal{W}_s$ and $\mathcal{G}_s$ for all $s \in \omega^n$, let $U_t = W^{t|n}_{t(n)}$ for all $t \in \omega^{n+1}$.

Its is straightforward to verify that (C1)–(C5) are fulfilled for the families $\{U_s : s \in \mathcal{O}_{n+1}\}$ and $\{V_s : s \in \mathcal{O}_{n+1}\}$ so our inductive construction can go on giving us families $\{U_s : s \in \mathcal{O}\}$ and $\{V_s : s \in \mathcal{O}\}$ with (C1)–(C5) for all $n \in \omega$. Observe that (A1) and (A2) are fulfilled for the family $\mathcal{U} = \{U_s : s \in \mathcal{O}$ by (C1), (C4) and (C5). To see (A3) also holds, take any $f \in \omega^\omega$. If $s_n = f|n$ then $\operatorname{cl}_Z(V_{s_{n+1}}) \subset V_{s_n}$ by (C3) and $V_{s_n} \in \mathcal{B}_n$ for all $n \in \mathbb{N}$ by (C2). Since the sequence $\{\mathcal{B}_n : n \in \mathbb{N}\}$ is pseudocomplete, there is $y \in \bigcap\{V_{s_n} : n \in \omega\}$; if follows from $U_{f|n} = \varphi(V_{s_n})$ for all $n \in \omega$ that $\varphi(y) \in \bigcap\{U_{f|n} : n \in \omega\}$ so the property (A3) is proved for $\mathcal{U}$. As a consequence, $\mathcal{U}$ is an A-system in M and hence M is Čech-complete by Fact 3. Fact 4 is proved.

Given any space Z and any bases $\mathcal{B}$ and $\mathcal{C}$ in Z, let $\mathcal{B}[\mathcal{C}] = \{U \in \mathcal{B}$: there exists $V \in$ such that $\overline{U} \subset V\} \subset \mathcal{B}$. It is straightforward that $\mathcal{B}[\mathcal{C}]$ is a base in Z. We write $\mathcal{B} < \mathcal{C}$ if, for any $U \in \mathcal{B}$, there is $V \in \mathcal{C}$ such that $\overline{U} \subset V$. This relation is clearly transitive, i.e., for any bases $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in the space Z, if $\mathcal{A} < \mathcal{B}$ and $\mathcal{B} < \mathcal{C}$ then $\mathcal{A} < \mathcal{C}$. Another trivial observation is that $\mathcal{A}[\mathcal{B}] < \mathcal{B}$ for any bases $\mathcal{A}$ and $\mathcal{B}$ in the space Z.

Fact 5. Let Z be a strongly pseudocomplete space. Then there exists a pseudocomplete sequence $\{\mathcal{B}_n : n \in \omega\}$ of bases in Z such that

(P1) $\mathcal{B}_{n+1} < \mathcal{B}_n$ for each $n \in \omega$;
(P2) for each $m \in \omega$, if we have a family $\{U_i : i \geqslant m\}$ such that $U_i \in \mathcal{B}_i$ and $\overline{U}_{i+1} \subset U_i$ for all $i \geqslant m$ then $\bigcap\{U_i : i \geqslant m\} \neq \emptyset$.

Proof. Let $\{\mathcal{C}_n : n \in \omega\}$ be a pseudocomplete sequence of bases of Z. Note first that if we take any base $\mathcal{C}'_n \subset \mathcal{C}_n$ for all $n \in \omega$ then the sequence $\{\mathcal{C}'_n : n \in \omega\}$ is also pseudocomplete. If we let $\mathcal{B}_0 = \mathcal{C}_0$ and $\mathcal{B}_{n+1} = \mathcal{C}_{n+1}[\mathcal{B}_n]$ for each $n \in \omega$ then we get a pseudocomplete sequence $\{\mathcal{B}_n : n \in \omega\}$ of bases in Z such that $\mathcal{B}_{n+1} < \mathcal{B}_n$ for all $n \in \omega$ so (P1) is proved.

Suppose that $\{U_i : i \geqslant m\}$ is a family as in (P2). Since $\mathcal{B}_m < \mathcal{B}_{m-1}$, there exists $U_{m-1} \in \mathcal{B}_{m-1}$ such that $\overline{U}_m \subset U_{m-1}$. Going "backwards" in this manner, we obtain sets $U_{m-1},\ldots, U_1, U_0$ such that $U_i \in \mathcal{B}_i$ and $\overline{U}_{i+1} \subset U_i$ for all $i < m$. Since we have the same property for all $i \geqslant m$, we have obtained a sequence $\{U_i : i \in \omega\}$ such that $U_i \in \mathcal{B}_i$ and $\overline{U}_{i+1} \subset U_i$ for all $i \in \omega$. The sequence $\{\mathcal{B}_i : i \in \omega\}$ being pseudocomplete, we have $\bigcap\{U_i : i \geqslant m\} = \bigcap\{U_i : i \in m\} \neq \emptyset$ so Fact 5 is proved.

Fact 6. Any product of strongly pseudocomplete spaces is strongly pseudocomplete.

Proof. Let X_t be a strongly pseudocomplete non-empty space for each $t \in T$; fix a pseudocomplete sequence $\left\{\mathcal{B}^t_n : n \in \omega\right\}$ of bases in X_t for each $t \in T$. Fact 5 shows that we can assume, without loss of generality, that the sequence $\left\{\mathcal{B}^t_n : n \in \omega\right\}$ has the properties (P1) and (P2) for all $t \in T$. If $O_t \in \tau(X_t)$ for all $t \in T$, the set $\prod_{t\in T} O_t$ is called *standard* if $O_t \neq X_t$ for only finitely many t. The family of all standard sets is a base in the space $X = \prod\{X_t : t \in T\}$ (see Problem 101). If $O = \prod_{t\in T} O_t$ is a standard set then $\mathrm{supp}(O) = \{t \in T : O_t \neq X_t\}$. Let $pt : X \to X_t$ be the natural projection for all $t \in T$.

Let $\mathcal{B}_n = \left\{O = \prod_{t\in T} O_t \in \mathcal{S} : O_t \in \mathcal{B}^t_n \text{ for all } t \in \mathrm{supp}(O)\right\}$ for each $n \in \omega$. We claim that $\mathcal{B}_n$ is a base in X for each $n \in \omega$. Indeed, if $x \in U \in \tau^*(X)$ then there is $O = \prod_{t\in T} O_t \in \mathcal{S}$ with $x \in O \subset U$. Since $\mathcal{B}^t_n$ is a base in X_t for all $t \in \mathrm{supp}(O)$, there is $V_t \in \mathcal{B}^t_n$ with $x(t) \in V_t \subset O_t$ for each $t \in \mathrm{supp}(O)$. Let $V_t = X_t$ for all $t \in T\setminus \mathrm{supp}(O)$; then $V = \prod\{V_t : t \in T\} \in \mathcal{B}_n$ and $x \in V \subset O \subset U$ which proves that each $\mathcal{B}_n$ is a base in the space X.

We show next that the sequence $\{\mathcal{B}_n : n \in \omega\}$ is pseudocomplete. Take any family $\{U_n : n \in \omega\}$ such that $U_n \in \mathcal{B}_n$ and $\overline{U}_{n+1} \subset U_n$ for all $n \in \omega$. We have $U_n = \prod_{t\in T} U^n_t$ where $U^n_t \in \mathcal{B}^t_n$ for all $n \in \omega$ and $t \in \mathrm{supp}(U_n)$. Fix any $s \in T$; we claim that $\bigcap\left\{U^n_s : n \in \omega\right\} \neq \emptyset$. Indeed, if $s \notin \bigcup\{\mathrm{supp}(U_n):n \in \omega\}$ then $U^n_s = X_s$ for all $n \in \omega$ and hence $\bigcap\left\{U^n_s : n \in \omega\right\} = X_s \neq \emptyset$.

Observe that $\mathrm{supp}(U_{n+1}) \supset \mathrm{supp}(U_n)$ for all $n \in \omega$. Indeed, $U_{n+1} \subset U_n$ implies that $U^{n+1}_t = p_t(U_{n+1}) \subset p_t(U_n) = U^t_n$ for all $t \in T$. Consequently, if $U^n_t \neq X_t$ then $U^{n+1}_t \neq X_t$, i.e., $t \in \mathrm{supp}(U_n)$ implies $t \in \mathrm{supp}(U_{n+1})$.

Now assume that $s \in \mathrm{supp}(U_n)$ for some $n \in \omega$ and let $m \in \omega$ be the smallest of the numbers $n \in \omega$ such that $s \in \mathrm{supp}(U_n)$. Thus, $U^n_s = X_s$ for all $n < m$ and $s \in \mathrm{supp}(U_n)$ for all $n \geqslant m$ by our previous remark. Therefore $U^n_s \in \mathcal{B}^s_n$ for all $n \geqslant m$. Given any $n \geqslant m$, observe that $\overline{U}_{n+1} = \prod\left\{\overline{U^{n+1}_t} : t \in T\right\}$ and therefore $\overline{U^{n+1}_s} = p_s(\overline{U}_{n+1}) \subset p_s(U_n) = U^n_s$ for all $n \geqslant m$ (all closures are denoted by a bar; we hope there is no confusion as to in which space the closure is taken). By property (P2) for the sequence $\left\{\mathcal{B}^s_n : n \in \omega\right\}$, we have $\bigcap\left\{U^n_s : n \geqslant m\right\} \neq \emptyset$. However, $U^n_s = X_s$ for all $n < m$ so $\bigcap\left\{U^n_s : n \in \omega\right\} = \bigcap\left\{U^n_s : n \geqslant m\right\} \neq \emptyset$.

We proved that there is $x_t \in \bigcap\left\{U^n_t : n \in \omega\right\}$ for each $t \in T$. Letting $x(t) = x_t$ f or all $t \in T$ we obtain a point $x \in X$ such that $x \in \bigcap\{U_n : n \in \omega\}$ which shows that the sequence $\{U_n : n \in \omega\}$ of bases in X is pseudocomplete. Fact 6 is proved.

To finish our solution apply Fact 1 of S.465 to see that any Čech-complete space is strongly pseudocomplete. By Fact 6, any product of Čech-complete spaces is strongly pseudocomplete. Finally, it follows from Fact 4 that any product of Čech-complete spaces is projectively complete.

S.492. *Prove that a separable metrizable space is projectively complete if and only if it is Čech-complete. Give an example of a pseudocomplete space which is not projectively complete.*

Solution. If X is separable metrizable and projectively complete then the identity map of X onto itself is an open map of X onto a second countable space, so the image of X under this map, which is X, must be Čech-complete. This proves necessity.

Sufficiency follows from the fact that any Čech-complete space is projectively complete (see Problem 491; observe that second countability is not needed here).

Let $P = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and $Q = \mathbb{Q} \times \{0\}$. The space $X = P \cup Q$ with the topology inherited from $\mathbb{R}^2$ is pseudocomplete because P is a Čech-complete dense subspace of X (Problem 468). Observe that X is not Čech-complete because Q is a closed subspace of X which is not Čech-complete (Problem 260). Thus X is not projectively complete because every projectively complete second countable space must be Čech-complete.

S.493. *Give an example of a projectively complete space which is not pseudocomplete.*

Solution. Recall that a space X is called *a P-space* if every G_δ-subset of X is open in X. A space is *zero-dimensional* if it has a base which consists of clopen sets.

Fact 1. Every P-space is projectively complete.

Proof. Let X be a P-space; take any open map $f : X \to M$ onto a second countable space M. Since $\{y\}$ is a G_δ-set in M for any $y \in M$, the set $f^{-1}(y)$ is a G_δ-set in X so $f^{-1}(y)$ is open in X. The map f being open, the set $\{y\}$ is open in M for any $y \in M$. Hence all points of M are isolated, i.e., M is discrete. Since any discrete space is Čech-complete (Problem 204) the space M is Čech-complete so Fact 1 is proved.

Fact 2. Given any space Z and an infinite cardinal κ, let $(Z)_\kappa$ be the set Z with the topology generated by the family of all G_κ-subsets of Z. Then $(Z)_\kappa$ is a Tychonoff zero-dimensional space in which every G_κ-set is open. The space $(Z)_\kappa$ is called κ-*modification* of the space Z. In particular, the ω-modification of any space is a Tychonoff P-space.

Proof. If $\mathcal{G}(\kappa)$ is the family of all G_κ-subsets of Z then $\bigcup \mathcal{G}(\kappa) = Z$ and the intersection of any two elements from $\mathcal{G}(\kappa)$ belongs to $\mathcal{G}(\kappa)$. Thus $\mathcal{G}(\kappa)$ is a base for the topology of the space $Z' = (Z)_\kappa$ (Problem 006). For each $z \in Z$, the set $\{z\}$ is closed in Z and hence in Z' because every closed subspace of Z is closed in Z', Therefore Z' is a T_1-space. Besides, closed G_κ-subsets of Z form a base in Z' by Fact 2 of S.328. Recalling again that any closed subset of Z is closed in Z', we conclude that Z' has a base which consists of clopen subsets of Z', i.e., the space Z' is zero-dimensional; Fact 1 of S.232 implies that Z' is Tychonoff. To see that any G_κ-subset of the space Z' is open in Z', take any family $\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \subset \tau(Z')$. Since $\mathcal{G}(\kappa)$ is a base for Z', for every $z \in \bigcap\mathcal{U}$ and every $\alpha < \kappa$, there exists $V_\alpha \in \mathcal{G}(\kappa)$ such that $z \in V_\alpha \subset U_\alpha$. It is easy to see that $V = \bigcap\{V_\alpha : \alpha < \kappa\} \in \mathcal{G}(\kappa)$ and $z \in V \subset \bigcap\mathcal{U}$. Thus every point from $\bigcap\mathcal{U}$ has a neighbourhood contained in $\bigcap\mathcal{U}$ which shows that $\bigcap\mathcal{U}$ is open in Z' so Fact 2 is proved.

If X is the ω-modification of $\sigma(\omega_1) = \{x \in \mathbb{R}^{\omega_1} : |x^{-1}(\mathbb{R}\backslash\{0\})| < \omega\}$, then X is a P-space by Fact 2. Let $X_n = \{x \in X : |x^{-1}(\mathbb{R}\backslash\{0\})| \leqslant n\}$. It is clear that $X = \bigcup\{X_n : n \in \omega\}$; besides, each X_n is closed in X. Indeed, if $x \in \sigma(\omega_1)\backslash X_n$ then take $\alpha_1, \dots, \alpha_{n+1} < \omega_1$ such that $x(\alpha_i) \neq 0$ for all $i \leqslant n+1$. It is evident that the set $V = \{y \in \sigma(\omega_1) : y(\alpha_i) \neq 0$ for all $i = 1, \dots, n+1\}$ is open in $\sigma(\omega_1)$ and $x \in V \subset \sigma(\omega_1)\backslash X_n$. This shows that X_n is closed in $\sigma(\omega_1)$ and hence in X.

We show next that the set X_n is nowhere dense in X for all $n \in \omega$; it suffices to prove that every X_n has empty interior. Suppose that $U \in \tau^*(X)$ and $U \subset X_m$ for some $m \in \omega$. Take any $x \in U$; since G_δ-subsets of $\sigma(\omega_1)$ form a base in X, there is a G_δ-set H in the space $\sigma(\omega_1)$ such that $x \in H \subset U$. It is easy to find a G_δ-set H' in the space $\mathbb{R}^{\omega_1}$ such that $H' \cap \sigma(\omega_1) = H$. There exists a countable set $A \subset \omega_1$ such that $G(x,A) = \{y \in \mathbb{R}^{\omega_1} : y|A = x|A\} \subset H'$ (see Fact 1 of S.426 applied to the discrete space $D(\omega_1)$). Consequently, $G(x, A) \cap \sigma(\omega_1) \subset X_m$; since the set A is countable, we can take distinct $\beta 1, \dots, \beta_{m+1} \in \omega_1\backslash A$. Define a point $y \in X$ by $y|A = x|A$, $y(\beta_i) = 1$ for all $i \leqslant m + 1$ and $y(\alpha) = 0$ for all $\alpha \in \omega_1\backslash(A \cup \{\beta_1, \dots, \beta_{m+1}\})$. It is immediate that $y \in (X\backslash X_m) \cap G(x, A)$ which is a contradiction. This shows that each X_m is nowhere dense in X and therefore X is of first category in itself. As a consequence, X is a projectively complete space (Fact 1) that is not pseudocomplete because each pseudocomplete space has the Baire property by Problem 464.

S.494. *Suppose that $C_p(X)$ is projectively complete. Prove that any open subspace of $C_p(X)$ is also projectively complete.*

Solution. If X is finite then $C_p(X) = \mathbb{R}^X$ is Čech-complete as well as any of its open subspaces. As any Čech-complete space is projectively complete (Problem 491), there is nothing more to prove in this case. In what follows we assume that the space X is infinite. The expression $Y \simeq Z$ says that the spaces Y and Z are homeomorphic.

Given a space Z, distinct points $z_1, \dots, z_n \in Z$ and $O_1, \dots, O_n \in \tau^*(\mathbb{R})$, let $[z_1, \dots, z_n; O_1, \dots, O_n]_Z = \{f \in C_p(Z) : f(z_i) \in O_i$ for all $i \leqslant n\}$. The family $\mathcal{C}_Z = \{[z_1, \dots, z_n; O_1, \dots, O_n]_Z : n \in \mathbb{N}, z_1, \dots, z_n \in Z$ and $O_1, \dots, O_n$ are non-trivial rational intervals$\}$ is a base in the space $C_p(Z)$ (Problem 056).

Fact 1. Given an arbitrary space Z, for any distinct $\{z_1, \dots, z_m\} \subset Z$ and any rational non-empty intervals $O_1, \dots, O_m$, the space $W = [z_1, \dots, z_m; O_1, \dots, O_m]_Z$ is homeomorphic to the space $I_K^Z \times O_1 \times \cdots \times O_m$, where $K = \{z_1, \dots, z_m\}$ and $I_K^Z = \{f \in C_p(Z) : f|K \equiv 0\}$.

Proof. Fix $U_i \in \tau(z_i, Z)$, $i = 1, \dots, m$ such that the family $\{U_1, \dots, U_m\}$ is disjoint. There exists $f_i \in C(Z)$ such that $f_i(z_i) = 1$ and $f_i|(X \setminus U_i) \equiv 0$ for all $i \leqslant m$. Given an arbitrary function $f \in W$, let $r(f) = f(z_1) f_1 + \cdots + f(z_m) \cdot f_m$ and $\varphi(f) = (f - r(f),$

$f|K$). It is immediate that $r(f)(z_i) = f(z_i)$ for all $i \leqslant m$ so $f - r(f) \in I_K^Z$. It follows from $f(z_i) \in O_i$ for all $i \leqslant m$ that $f|K \in O = O_1 \times \cdots \times O_m$ so we have $\varphi(f) \in I_K^Z \times O$ for each $f \in W$, i.e., $\varphi : W \to I_K^Z \times O$. Given any $i \leqslant m$, the map $e_{z_i} : C_p(Z) \to \mathbb{R}$ defined by $e_{z_i}(f) = f(z_i)$, is continuous (Problem 166). Therefore the map $f \mapsto f(z_i) \cdot f_i$ is continuous for any $i \leqslant m$ (Problem 116). This shows that the map r is continuous and hence so is the map $f \mapsto f - r(f)$; an immediate consequence is that the map φ is continuous.

Now, if $g \in I_K^Z$ and $h \in O$ then letting $f = \delta(g, h) = g + \sum_{i=1}^m h(z_i) \cdot f_i$ we obtain $f \in W$ such that $\varphi(f) = (g, h)$, i.e., the map φ is onto. Now, suppose that $f, f' \in W$ and $f \neq f'$. If $f|K \neq f'|K$ then $\varphi(f) \neq \varphi(f')$ because the second coordinates of $\varphi(f)$ and $\varphi(f')$ are distinct. If $f|K = f'|K$ then $r(f) = r(f')$ and therefore $f - r(f) \neq f' - r(f')$ so again $\varphi(f) \neq \varphi(f')$. Thus the map φ is a bijection and $\delta : I_K^Z \times O \to W$ is the inverse of φ. To see that δ is a continuous map, observe that it maps $I_K^Z \times O$ into a product, namely $\mathbb{R}^Z$, so it suffices to verify that, for any $z \in Z$, the map $\delta_z = \pi\{z\} \circ \delta$ is continuous. Note first that the map $(g, h) \to g \to g(z)$ is continuous being a composition of two natural projections. Since $f_i(z)$ is a constant, the map $h \to h(z_i) \cdot f_i(z)$ is a natural projection multiplied by a constant; hence the map δ is continuous being a composition of arithmetical operations with natural projections. This shows that $\varphi : W \to I_K^Z \times O$ is a homeomorphism so Fact 1 is proved.

Take any $W = [x_1, \ldots, x_m; O_1, \ldots, O_m]_X \in {}_X$. We know that the space $C_p(X)$ is homeomorphic to $I_K^X \times \mathbb{R}^K$ where $K = \{x_1, \ldots, x_m\} \subset X$ (Fact 1 of S.409). Since $O_i \simeq \mathbb{R}$ for each $i \leqslant m$, we can apply Fact 1 to conclude that $W \simeq I_K^X \times O_1 \times \cdots \times O_n \simeq I_K^X \times \mathbb{R}^K \simeq C_p(X)$. This proves that

$(*)$ Every element of ${}_X$ is homeomorphic to $C_p(X)$.

Now take any open $U \subset C_p(X)$; suppose that M is second countable and $\varphi : U \to M$ is an open continuous onto map. For any $y \in M$, choose any $f \in U$ with $\varphi(f) = y$ and $V \in \mathcal{C}_X$ such that $f \in V \subset U$. The map $\varphi|V$ is open; since V is homeomorphic to $C_p(X)$ by $(*)$, it is projectively complete so $W_y = \varphi(V)$ is an open Čech-complete subspace of M with $y \in W_y$.

Consider the space $L = \oplus\{W_y : y \in M\}$. We will identify each W_y with the respective clopen subset of L. Since each W_y is Čech-complete, the space L is also Čech-complete (Problem 262). Any W_y is also a subset of M so if $x \in W_y \subset L$, we denote by x' is twin in M. This makes it possible to define a map $\xi : L \to M$ by $\xi(x) = x'$ for each $x \in L$. Given any $y \in M$, note that $y \in W_y$ so if we consider that $W_y \subset L$ then the point y has a twin z in W_y considered as a subspace of L. Of course, we have $\xi(z) = y$ so the map ξ is onto.

To see that ξ is continuous, take any $x \in L$ and $y \in Y$ with $x \in W_y \subset L$. It is clear that $\xi|W_y : W_y \to \xi(W_y)$ is a homeomorphism and hence the map $\xi|W_y : W_y \to M$ is continuous. Since W_y is open in L, the map ξ is continuous by Fact 1 of S.472.

To show that ξ is open, take any open $W \subset L$. Then $W \cap W_y$ is open in W_y for every $y \in M$. Since $\xi|W_y$ is a homeomorphism of W_y onto an open subset of M, the set $\xi(W \cap W_y)$ is open in M and therefore $\xi(W) = \bigcup\{\xi(W \cap W_y) : y \in M\}$ is open in X so the map ξ is open.

Finally, apply Fact 1 of S.491 to conclude that there is a closed set $F \subset L$ such that $\xi(f) = M$ and $\xi|F : F \to M$ is a perfect map. Since the space F is Čech-complete by Problem 260, the space M is also Čech-complete by Problem 261. This shows that U is projectively complete.

S.495. *Suppose that $C_p(X)$ is projectively complete. Prove that any countable closed $A \subset X$ is discrete and C-embedded in X.*

Solution. Recall that, since the set A is closed in the space X, the restriction map $\pi_A : C_p(X) \to C_p(A|X) = \pi_A(C_p(X)) \subset C_p(A)$ is open by Problem 152. As $C_p(X)$ is projectively complete, the space $C_p(A|X)$ has to be Čech-complete. The set $C_p(A|X)$ is dense in $C_p(A)$ and hence in $\mathbb{R}^A$ (Problem 152). Since $C_p(A)$ has a dense Čechcomplete subspace, the space A is discrete (Problem 265). Assume that $C_p(A|X) \neq \mathbb{R}^A$; note that $C_p(A|X)$ is an algebra and, in particular we have, $f - g \in C_p(A|X)$ for any $f, g \in C_p(A|X)$. Take any $h \in \mathbb{R}^A \setminus C_p(A|X)$; the map $T_h : \mathbb{R}^A \to \mathbb{R}^A$ defined by $T_h(g) = h + g$, is a homeomorphism (Problem 079) and $T_h(C_p(A|X)) \cap C_p(A|X) = \emptyset$. Indeed, if $g \in C_p(A|X)$ and $f = h+g \in C_p(A|X)$ then $h = f - g \in C_p(A|X)$ which is a contradiction. Therefore $\mathbb{R}^A$ has two disjoint dense Čech-complete subspaces $C_p(A|X)$ and $T_h(C_p(A|X))$ which contradicts Problem 264. Thus $C_p(A|X) = \mathbb{R}^A$ and this means that A is discrete and C-embedded in X.

S.496. *Prove that $C_p(\beta\omega)$ is not projectively complete.*

Solution. To avoid going into a general theory, let us define *a linear space* as a subspace L of some $\mathbb{R}^T$ such that $f, g \in L$ implies $af + bg \in L$ for any $a, b \in \mathbb{R}$. The set T will be always clear from the context; we will also consider L to be a topological space with the topology inherited from $\mathbb{R}^T$. For example, any $C_p(X)$ is a linear space with $T = X$. Observe that any linear space $L \subset \mathbb{R}^T$ contains the function $\mathbf{0}_L$ which is identically zero on T. For any linear spaces L and M, a map $u : L \to M$ is called *linear* if $u(af + bg) = au(f) + bu(g)$ for any $f, g \in L$ and $a, b \in \mathbb{R}$. It is easy to see that $u(\mathbf{0}_L) = \mathbf{0}_M$ for any linear map $u : L \to M$.

Fact 1. If L is a linear space and $z \in L$ then the map $Q^L_z : L \to L$ defined by $Q^L_z(x) = x + z$ for all $x \in L$, is a homeomorphism for any $z \in L$.

Proof. Indeed, if $L \subset \mathbb{R}^T$ then let $P_z(y) = y + z$ for any $y \in \mathbb{R}^T$. The map $P_z : \mathbb{R}^T \to \mathbb{R}^T$ is a homeomorphism (Problem 079). We have $P_z(L) = L$ because L is a linear space; besides, $P_z|L = Q^L_z$ so Fact 1 is proved.

Fact 2. Given linear spaces L and M, a linear map $u : L \to M$ is continuous if and only if it is continuous at $\mathbf{0}_L$.

Proof. It is clear that we only must prove sufficiency so assume that u is continuous at $\mathbf{0}_L$ and take any $x \in L$. Let $y = u(x)$; for any $W \in \tau(y, M)$, the set $W' = Q^M_{-y}(W)$ is an open neighbourhood of $\mathbf{0}_M$ by Fact 1. Since u is continuous at $\mathbf{0}_L$, there is $V' \in \tau(\mathbf{0}_L, L)$ such that $u(V') \subset W'$. Applying Fact 1 again, we convince ourselves that $V = Q^L_x(V')$ is an open neighbourhood of x.

For any $z \in V$ there is $z' \in V'$ such that $z' + x = z$. We have the equalities $u(z) = u(x + z') = u(x) + u(z') = y + u(z')$; in addition, $t' = u(z') \in W'$ because $u(V') \subset W'$. This shows that there exists $t \in W$ such that $t + (-y) = t'$. Therefore $u(z) = y + u(z') = y + t' = y + (t + (-y) = t \in W$. The point $z \in V$ has been chosen arbitrarily so $u(V) \subset W$ and hence we established continuity of u at an arbitrary point $x \in L$. Fact 2 is proved.

Fact 3. Given linear spaces L and M, a continuous linear onto map $u : L \to M$ is open if and only if it is open at $\mathbf{0}_L$, i.e., for any $W \in \tau(\mathbf{0}_L, L)$ there is $V \in \tau(\mathbf{0}_M, M)$ such that $u(W) \supset V$.

Proof. We must only prove that $u(G)$ is an open set for any $G \in \tau(L)$. It suffices to show that, for every $z \in u(G)$, there is $V_z \in \tau(z, M)$ such that $V_z \subset u(G)$.

Take any $y \in G$ with $u(y) = z$; the map $Q^L_{-y} : L \to L$ is a homeomorphism by Fact 1 so $W = Q^L_{-y}(G)$ is an open neighbourhood of $\mathbf{0}_L$. Since u is open at $\mathbf{0}_L$, there is $V \in \tau(\mathbf{0}_M, M)$ such that $u(W) \supset V$. The set $V_z = Q^M_z(V)$ is an open neighbourhood of z by Fact 1 so it suffices to show that $V_z \subset u(G)$.

Take any $z' \in V_z$; then $z' = z + t$ for some $t \in V$. Since $V, u(W)$, there exists $x' \in W$ with $u(x') = t'$. In addition, $x' = g - y$ for some $g \in G$ so we have $z' = z + t = z + u(x') = z + u(g - y) = z + u(g) - u(y) = z + u(g) - z = u(g) \in u(G)$, which proves that $V_z, u(G)$ and hence $u(G)$ is open in M. Fact 3 is proved.

Let $M = C^*(D(\omega)) = \{f \in \mathbb{R}^\omega : f \text{ is bounded}\} \subset \mathbb{R}^\omega$; clearly, M is a linear space which is dense in the space $\mathbb{R}^\omega$. We have $M = \bigcup\{[-n, n]^\omega : n \in \mathbb{N}\}$; observe that $M_n = [-n, n]^\omega$ is nowhere dense in M for each $n \in \mathbb{N}$. Indeed, if $U \in \tau^*(M)$ and $U \subset [-n, n]^\omega$ for some $n \in \mathbb{N}$ then there is $V \in \tau(R^\omega)$ with $V \cap M = U$; therefore $\overline{V} = \overline{U} \subset M_n$ (the closure is taken in $\mathbb{R}^\omega$) because M_n is compact. It turns out that $V \subset M_n$ and hence each point of V is a point of local compactness of $\mathbb{R}^\omega$. Take any $f \in V$; since there are homeomorphisms of $\mathbb{R}^\omega$ onto itself which send f onto any given element of $\mathbb{R}^\omega$ (Problem 079), the space $\mathbb{R}^\omega$ is locally compact which contradicts Problem 186. This proves that each M_n is nowhere dense so M is of first category in itself and, in particular, M is not Čech-complete.

It is easy to find a disjoint family $\{A_n : n \in \omega\} \subset \exp(\omega)$ such that $|A_n| = n + 1$ for all $n \in \omega$ and $\omega = \bigcup\{A_n : n \in \omega\}$. Given any function $f \in C_p(\beta\omega)$, let $\varphi(f)(n) = \sum\{\frac{1}{n+1} \cdot f(k) : k \in An\}$ for each $n \in \omega$. In other words, $\varphi(f)\ (n)$ is the arithmetic mean of the values of f on the set A_n. It is clear that $\varphi(f) \in \mathbb{R}^\omega$ for any $f \in C_p(\beta\omega)$; thus $\varphi : C_p(\beta\omega) \to \mathbb{R}^\omega$. In fact, φ maps $C_p(\beta\omega)$ to M; to see this, take any $f \in C_p(\beta\omega)$. There is $r \in \mathbb{R}$ such that $|f(k)| < r$ for any $k \in \omega$. Given any $n \in \omega$, we have $|\varphi(f)(n)| \leqslant \sum\{\frac{1}{n+1} \cdot |f(k)| : k \in An\} \leqslant \sum\{\frac{1}{n+1} \cdot r : k \in An\} = r$, so the function $\varphi(f)$ is bounded by r and hence $\varphi(f) \in M$.

We claim that the map $\varphi : C_p(\beta\omega) \to M$ is open, continuous and onto. Given any $g \in M$, there is $r \in \mathbb{R}$ such that $|g(n)| \leqslant r$ for all $n \in \omega$. For all $n \in \omega$, let $f(k) = g(n)$ for each $k \in A_n$. This determines a function $f : \omega \to \mathbb{R}$; for each $k \in \omega$ there is $n \in \omega$ such that $|f(k)| = |g(n)| \leqslant r$ so $|f(k)|\, r$ for each $k \in \omega$, i.e., f is bounded. Since the function f is continuous on $D(\omega)$, there exists a continuous $h : \beta\omega \to \mathbb{R}$ such that $h|\omega = f$. It is easy to see that $\varphi(h) = g$ so we established that φ is surjective.

It is straightforward that the map φ is linear so, by Fact 2, to prove continuity of φ it is sufficient to establish that φ is continuous at the point $\mathbf{0} \in C_p(\beta\omega)$ that is identically zero on $\beta\omega$. Given any $n \in \omega$ and $\varepsilon > 0$, let $H(n, \varepsilon) = \{y \in M : |y(i)| < \varepsilon$ for all $i \leqslant n\}$. It is evident that the family $\{H(n, \varepsilon) : n \in \omega$ and $\varepsilon > 0\}$ is a local base of M at $\mathbf{0}_M$. Given an arbitrary $n \in \omega$, $\varepsilon > 0$ and a finite set $C \subset \beta\omega \setminus \omega$, the set $G(n, C, \varepsilon) = \{f \in C_p(\beta\omega) : |f(x)| < \varepsilon$ for all $x \in A_0 \cup \dots \cup A_n \cup C\}$ is an open neighbourhood of $\mathbf{0}$. It is clear that the family $\{G(n, C, \varepsilon) : n \in \omega, \varepsilon > 0$ and $C \subset \beta\omega \setminus \omega$ is finite$\}$ is a local base of $C_p(\beta\omega)$ at $\mathbf{0}$. Let us show that
$(*)$ $\varphi(G(n, C, \varepsilon)) = H(n, \varepsilon)$ for any $n \in \omega$, $\varepsilon > 0$ and any finite set $C \subset \beta\omega \setminus \omega$.

Take an arbitrary function $f \in G(n, C, \varepsilon)$; then, for every $i \leqslant n$, we have $|\varphi(f)(i)| \leqslant \sum\{\frac{1}{i+1} \cdot |f(k)| : k \in A_i\} < \sum\{\frac{1}{i+1} \cdot \varepsilon : k \in A_i\} = \varepsilon$ which shows that $\varphi(f) \in H(n, \varepsilon)$. Therefore $\varphi(G(n, C, \varepsilon)) \subset H(n, \varepsilon)$.

Now, let $h \in H(n, \varepsilon)$; denote by m the number of elements of C and take $r \in \mathbb{R}$ for which $|h(k)| < r$ for all $k \in \omega$. For each $l \geqslant m + 1$, we have $l + 1 = k_l(m + 1) + r_l$ where $k_l, r_l \in \mathbb{N}$ and $0 \leqslant r_n \leqslant m$. For each $l \geqslant m + 1$, choose an arbitrary partition $\{A_l^i : i \in \{0, \dots, m\}\}$ of the set A_l such that $|A_l^0| = k_l + r_l$ and $|A_l^i| = k_l$ for $i \in \{1, \dots, m\}$. If $B_i = \bigcup\{A_l^i : l \geqslant m + 1\}$ for all $i \leqslant m$, then $B_0, \dots, B_m$ are infinite disjoint sets such that $t_l = \frac{|A_l|}{|A_l \cap B_i|} \leqslant 2m + 1$ for each $i \leqslant m$ and $l \geqslant m+1$. Indeed, $A_l \cap B_i = A_l^i$ and therefore $t_l \leqslant \frac{l+1}{k_l} = m + 1 + \frac{r_l}{k_l} \leqslant m + 1 + m = 2m + 1$. The sets $\overline{B}_0, \dots, \overline{B}_m$ are disjoint by Fact 2 of S.369; this implies $\overline{B}_k \cap C = \emptyset$ for some $k \leqslant m$ (the bar denotes the closure in $\beta\omega$). Define a function $g' \colon \omega \to \mathbb{R}$ as follows: $g'(x) = h(l)$ if $x \in A_l$ and $l \leqslant m$; if $x \in A_l$ for some $l \geqslant m + 1$ then $g'(x) = t_l h(l)$ if $x \in A_l \cap B_k$ and $g'(x) = 0$ for all $x \in A_l \setminus B_k$.

For every $x \in \omega$ we have $|g'(x)| = |h(l)|$ or $|g'(x)| = |t_l h(l)|$ for some $l \in \omega$. Therefore, $|g'(x)| \leqslant t_l |h(l)| < t_l r \leqslant (2m + 1)r$ which shows that g' is bounded on ω. As a consequence, there is $g \in C(\beta\omega)$ with $g|\omega = g'$. Observe that $g(x) = 0$ for all $x \in \omega \setminus B_k$; since $\overline{B}_k \cap C = \emptyset$, we have $C \subset \overline{\omega \setminus B_k}$ whence $g(C) = \{0\}$. It is easy to check that $\varphi(g) = h$; besides, $g(C) = \{0\}$ and, for all $x \in A_0 \cup \dots \cup A_n$, there is $i \leqslant n$ such that $|g(x)| = |h(i)| < \varepsilon$, i.e., $|g(x)| < \varepsilon$ for all $x \in A_0 \cup \dots \cup A_n$ which implies $g \in G(n, C, \varepsilon)$ and therefore $\varphi(G(n, C, \varepsilon)) \supset H(n, \varepsilon)$ so $(*)$ is proved.

Now, take an arbitrary $W \in \tau(\mathbf{0}_M, M)$. There exist $n \in \omega$ and $\varepsilon > 0$ such that $H(n,\varepsilon) \subset W$. As a consequence, $V = G(n, \emptyset, \varepsilon) \in \tau(\mathbf{0}, C_p(\beta\omega))$ and it follows from $(*)$ that $\varphi(V) = H(n, \varepsilon) \subset W$ which proves continuity at $\mathbf{0}$. Therefore φ is a continuous map by Fact 2.

Now, assume that $V \in \tau(\mathbf{0}, C_p(\beta\omega))$; there are $n \in \omega$, $\varepsilon > 0$ and a finite set $C \subset \beta\omega \setminus \omega$ such that $G(n, C, \varepsilon) \subset V$. It is an immediate consequence of $(*)$ that $\varphi(V) \supset \varphi(G(n, C, \varepsilon)) = H(n, \varepsilon) \in \tau(\mathbf{0}_M, M)$ which shows that φ is open at the point $\mathbf{0}$. Hence the map φ is open by Fact 3. Thus φ is an open map of $C_p(\beta\omega)$ onto a second countable space M that is not Čech-complete and hence $C_p(\beta\omega)$ is not projectively complete.

S.497. *Suppose that $\overline{A}$ has a countable network for each countable $A \subset X$. Prove that, if $C_p(X)$ is projectively complete, then it is pseudocomplete.*

Solution. Take any countable $A \subset X$; if the set $\overline{A}$ is countable then $\overline{A}$ is discrete and C-embedded in X by 495. This implies that A is also discrete and C-embedded in X.

Fact 1. Every uncountable space with a countable network has a non-trivial convergent sequence.

Proof. Take any uncountable Z with $nw(Z) \leqslant \omega$; let $\mathcal{N}$ be a countable network in the space Z. The set $B = \{z \in Z :$ the set $\{z\}$ is a finite intersection of some elements of $\mathcal{N}\}$ is countable so there exists $y \in Z \setminus B$. Since $\mathcal{N}' = \{N \in \mathcal{N} : y \in N\}$ is also countable, we can choose an enumeration $\{N_i : i \in \omega\}$ of the family $\mathcal{N}'$. Given any $k \in \omega$, the set $M_k = \bigcap \{N_i : i \leqslant k\}$ cannot be finite. Indeed, if M_k is finite, then $P = M_k \backslash \{y\}$ is also finite so there is $N \in \mathcal{N}$ such that $y \in N \subset Z \setminus P$; therefore $\{y\} = M_k \cap N$ is a finite intersection of elements of $\mathcal{N}$ which contradicts the choice of y. Thus M_k is infinite and hence it is possible to choose $y_k \in M_k \backslash \{y\}$ for each $k \in \omega$. Observe that the sequence $\{y_n : n \in \omega\}$ converges to y. Indeed, given any $U \in \tau(y, Z)$ there is $N \in \mathcal{N}$ such that $y \in N \subset U$. We have enumerated all elements of $\mathcal{N}$ that contain y and therefore $N = N_k$ for some $k \in \omega$. It is clear that $y_i \in M_i \subset M_k \subset N_k = N \subset U$ for all $i \geqslant k$ which shows that $y_n \to y$. The sequence $\{y_n : n \in \omega\}$ has to be non-trivial because $y_n \neq y$ for each $n \in \omega$ so Fact 1 is proved.

Now, assume that $\overline{A}$ is uncountable for some countable set $A \subset X$. Since $nw(\overline{A}) = \omega$, we can apply Fact 1 to observe that there is a non-trivial convergent sequence $C \subset \overline{A}$. If x is the limit of C then $C \cup \{x\}$ is a non-discrete countable closed subspace of X which contradicts Problem 495. This contradiction shows that the closure of every countable $A \subset X$ is countable; hence $\overline{A}$ is discrete and C-embedded in X by 495. This implies that A is also discrete and C-embedded in X. Therefore each countable subset of X is closed and C-embedded in X so we can apply Problem 485 to conclude that $C_p(X)$ is pseudocomplete.

S.498. *Let X be any space. Prove that, if $C_p(X)$ is pseudocomplete then it is projectively complete.*

Solution. Take any open continuous map $\varphi : C_p(X) \to M$ of $C_p(X)$ onto a second countable space M. Apply Problem 300 to find a countable $A \subset X$ and a continuous map $p : \pi_A(C_p(X)) \to M$ such that $p \circ \pi_A = \varphi$. Since $C_p(X)$ is pseudocomplete, we have $C_p(A|X) = \pi_A(C_p(X)) = \mathbb{R}^A$ by Problem 485 so $C_p(A|X)$ is Čech-complete (see Problems 205, 207 and 269).

Given any open $U \subset C_p(A|X)$, the set $p(U) = \varphi(\pi_A^{-1}(U))$ is open in M because the map φ is open. Thus we have an open map p of a Čech-complete space $C_p(A|X)$ onto a metrizable (and hence paracompact) space M. By Fact 1 of S.491 there is a closed $F \subset C_p(A|X)$ such that $p(f) = M$ and the map $p|F$ is perfect. Apply Problem 260 to conclude that F is Čech-complete. Any perfect image of a Čech-complete space is Čech-complete (Problem 261) so M is also Čech-complete. This proves that $C_p(X)$ is projectively complete.

S.499. *Prove that $C(X)$ and $C(Y)$ are isomorphic as algebraic rings if and only if υX is homeomorphic to υY.*

Solution. Given a space T, call a set $A \subset T$ *a zero-set in T* if there is $f \in C(T)$ such that $A = f^{-1}(0)$. It is evident that each zero-set is closed in T. Denote by $_T$ the family of all zero-sets of T. A non-empty family $\mathcal{F} \subset \mathcal{Z}_T$ is called *a z-filter on T* if it has the following properties:

(ZF1) $\emptyset \notin \mathcal{F}$.
(ZF2) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.
(ZF3) If $A \in \mathcal{F}$ and $A \subset B \in \mathcal{Z}_T$ then $B \in \mathcal{F}$.

A z-filter $\mathcal{U}$ on T is called *a z-ultrafilter on T* if it is a maximal z-filter, i.e., if is a z-filter on T and $\mathcal{U} \subset \mathcal{F}$ then $\mathcal{U} = \mathcal{F}$. A family $\subset \exp(T)$ is *countably centered* if $\bigcap \mathcal{F}' \neq \emptyset$ for any countable $\mathcal{F}' \subset \mathcal{F}$.

Given $r \in \mathbb{R}$, let $c_r(x) = r$ for any $x \in T$, i.e., c_r is a function on T which is identically r at all points of T. A non-empty set $I \subset C(T)$ is called an ideal in $C(T)$ if

(I1) $af + bg \in I$ for any $f, g \in I$ and $a, b \in \mathbb{R}$.
(I2) $I \neq C(T)$ and $fg \in I$ for any $f \in I$ and $g \in C(T)$.

An ideal I in $C(T)$ is *maximal* if it is maximal with respect to inclusion, i.e., for any ideal $I' \subset C(T)$, if $I \subset I'$ then $I' = I$. A maximal ideal $I \subset C(T)$ is called *a real ideal* if, for any $f \in C(T)$, there is $r \in \mathbb{R}$ such that $f + c_r \in I$.

Let us state some simple facts on zero-sets.

Fact 1. Given an arbitrary space T, we have

(1) Any finite union of zero-sets in T is a zero-set in T.
(2) If A is a zero-set in T then there is $g \in C(T, [0,1])$ such that $A = g^{-1}(0)$.
(3) Any countable intersection of zero-sets in T is a zero-set in T.
(4) If $f : T \to T'$ is a continuous map and P is a zero-set in T' then $f^{-1}(P)$ is a zero-set in T.
(5) If P is a closed subset of $\mathbb{R}$ and $f \in C(T)$ then $f^{-1}(P)$ is a zero-set in T.

Proof. (1) If $f_1, \ldots, f_n \in C(T)$ and $P_i = f_i^{-1}(0)$ for all $i \leqslant n$ then the function $f = f_1, \ldots, f_n$ is continuous on T and $P = P_1 \cup \cdots \cup P_n = f^{-1}(0)$ so $P_1 \cup \cdots \cup P_n$ is a zero-set in T.

(2) Fix a function $f \in C(T)$ such that $A = f^{-1}(0)$. The function $h = |f|$ defined by $h(x) = |f(x)|$ for any $x \in T$, is continuous, non-negative and $A = h^{-1}(0)$. The function $g = \min(h, c_1)$ is continuous on T (Problem 028); it is clear that we have $g \in C(T, [0, 1])$ and $A = g^{-1}(0)$.

(3) if A_i is a zero-set in T for all $i \in \omega$ then we can apply (2) to find a function $f_i \in C(T, [0, 1])$ with $A_i = f_i^{-1}(0)$ for all $i \in \omega$. If $g_n = \sum_{i=1}^{n} 2^{-i} \cdot f_i$ then the sequence $\{g_n : n \in \omega\}$ converges uniformly to a function $g \in C(T, [0, \infty))$ (see Problem 030) and it is immediate that $g^{-1}(0) = \cap \{A_n : n \in \omega\}$.

(4) Take any $h \in C(T')$ with $P = h^{-1}(0)$; then $g = h \circ f \in C(T)$ and we have $f^{-1}(P) = f^{-1}(h^{-1}(0)) = g^{-1}(0)$ which proves that $f^{-1}(P)$ is a zero-set in T.

(5) Any closed subset of $\mathbb{R}$ is a G_δ-set in $\mathbb{R}$; since $\mathbb{R}$ is normal, any closed subset of $\mathbb{R}$ is a zero-set in $\mathbb{R}$ by Fact 1 of S.358. Hence P is a zero-set in $\mathbb{R}$ so we can apply (4) to conclude that $f^{-1}(P)$ is a zero-set in T. Fact 1 is proved.

Fact 2. Given an arbitrary space T, we have

(1) Any z-filter is a centered family.
(2) For any centered family $\mathcal{C} \subset \mathcal{Z}_T$, there exists a z-ultrafilter $\mathcal{U} \supset \mathcal{C}$.

In particular, any z-filter on T is contained in a z-ultrafilter on T.

Proof. (1) if $\mathcal{F}$ is a z-filter and $F_1, \ldots, F_n \in \mathcal{F}$ then an evident consequence of (ZF2) is that $F = F_1 \cap \cdots \cap F_n \in \mathcal{F}$ so $F \neq \emptyset$ by (ZF1). This shows that $\mathcal{F}$ is a centered family.

(2) Let $\mathcal{P}$ be the family of all z-filters that contain $\mathcal{C}$. Observe first that $\mathcal{P} \neq \emptyset$; indeed, let $\mathcal{F} = \{F \in \mathcal{Z}_T :$ there exists a finite $\mathcal{C}' \subset \mathcal{C}$ such that $\bigcap \mathcal{C}' \subset F\}$: The axiom (ZF1) holds for $\mathcal{F}$ because each element of $\mathcal{F}$ contains a finite intersection of elements of $\mathcal{C}$ and no such intersection is empty due to the fact that $\mathcal{C}$ is centered.

Now, if $F_1, F_2 \in \mathcal{F}$ then there are finite $\mathcal{C}_1 \subset \mathcal{C}$ and $\mathcal{C}_2 \subset \mathcal{C}$ such that $F_1 \supset \bigcap \mathcal{C}_1$ and $F_1 \supset \bigcap \mathcal{C}_2$. It is clear that the family $\mathcal{C}' = \mathcal{C}_1 \cup \mathcal{C}_2 \subset \mathcal{C}$ is finite and $F_1 \cap F_2 \supset \bigcap \mathcal{C}'$. Since $F_1 \cap F_2$ is a z-set y Fact 1, we have $F_1 \cap F_2 \in \mathcal{F}$, i.e., (ZF2) is proved for $\mathcal{F}$. Finally, if $F \in \mathcal{F}$ and G is a z-set with $F \subset G$ then take any finite $\mathcal{C}' \subset \mathcal{C}$ such that $\bigcap \mathcal{C}' \subset F$; it is immediate that $\bigcap \mathcal{C}' \subset G$ so $G \in \mathcal{F}$ and we proved that $\mathcal{F}$ is a z-filter that contains $\mathcal{C}$. As a consequence, $\mathcal{P} \neq \emptyset$.

Now assume that $\mathcal{P}'$ is a chain of elements of $\mathcal{P}$; we claim that $\mathcal{F} = \bigcup \mathcal{P}'$ is a z-filter. Indeed, if $\emptyset \in \mathcal{F}$ then there is $\mathcal{G} \in \mathcal{P}'$ such that $\emptyset \in \mathcal{G}$ which contradicts the fact that $\mathcal{G}$ is a z-filter. Now, assume that $F, G \in \mathcal{F}$; there exist $\mathcal{G}, \mathcal{G}' \in \mathcal{P}$ such that $F \in \mathcal{G}$ and $G \in \mathcal{G}'$. Since $\mathcal{P}$ is a chain, we can assume, without loss of generality, that $\mathcal{G} \subset \mathcal{G}'$. Thus $F, G \in \mathcal{G}'$ and therefore $F \cap G \in \mathcal{G}'$ because $\mathcal{G}'$ is a z-filter. This proves that $F \cap G \in \mathcal{F}$ so the axiom (ZF2) is checked for $\mathcal{F}$. Finally, if G is a z-set in T and $G \supset F \in \mathcal{F}$ then there is $\mathcal{G} \in \mathcal{P}$ such that $F \in \mathcal{G}$. Since $\mathcal{G}$ is a z-filter, we have $G \in \mathcal{G}$ and therefore $G \in \mathcal{F}$ which proves that $\mathcal{F}$ is a z-filter.

This shows that we can apply the Zorn's lemma to conclude that there is a maximal z-filter $\mathcal{U} \in \mathcal{P}$. It is clear that $\mathcal{U}$ is a z-ultrafilter that contains $\mathcal{C}$ so Fact 2 is proved.

Fact 3. Let $\mathcal{F}$ be a z-filter on a space T. If $P \in \mathcal{Z}_T$ and $P \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ then the family $\mathcal{F} \cup \{P\}$ is centered. In particular, if $\mathcal{F}$ is a z-ultrafilter then $P \in \mathcal{F}$.

Proof. Take any $F_1, \ldots, F_n \in \mathcal{F} \cup \{P\}$. We can consider that $P = F_i$ for some $i \leqslant n$ for otherwise $\bigcap_{i=1}^n F_i \neq \emptyset$ because $F_1, \ldots, F_n \in \mathcal{F}$ and $\mathcal{F}$ is centered by Fact 2. Thus we do not lose generality supposing that $P = F_n$. An immediate consequence of (ZF2) is that $F = F_1 \cap \cdots \cap F_{n-1} \in \mathcal{F}$ so $F_1 \cap \cdots \cap F_n = F \cap P \neq \emptyset$ by our hypothesis. This shows that $\mathcal{F}' = \mathcal{F} \cup \{P\}$ is a centered family.

Now, if the family $\mathcal{F}$ is a z-ultrafilter then apply Fact 2 to observe that there is a z-ultrafilter $\mathcal{U} \supset \mathcal{F}' \supset \mathcal{F}$. Since every z-ultrafilter is a z-filter and $\mathcal{F}$ is maximal, we have $\mathcal{F} = \mathcal{F}' = \mathcal{U}$ so $P \in \mathcal{F}' = \mathcal{F}$ and Fact 3 is proved.

Fact 4. If T is a realcompact space then any countably centered z-ultrafilter on T has a non-empty intersection.

Proof. Let $\mathcal{F}$ be a countably centered z-ultrafilter on the space T. There exists a point $q \in \bigcap\{\text{cl}_{\beta T}(f) : F \in \mathcal{F}\}$. Assume first that $q \in \beta T \setminus T$. Since T is realcompact, there is a closed G_δ-set H in the space βT such that $q \in H \subset \beta T \setminus T$ (see Problem 401 and Fact 2 of S.328). Now apply Fact 1 of S.358 and Fact 1 of this Solution to conclude that there is $f \in C(\beta T, [0, 1])$ such that $f^{-1}(0) = H$. If $g = f|T$ then $g(x) > 0$ for all $x \in T$. Observe also that $q \in \text{cl}_{\beta T}(f)$ implies $f^{-1}([0, \frac{1}{n})) \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. Since $F \subset T$, we obtain $\emptyset \neq F \cap f^{-1}\left(\left[0, \frac{1}{n}\right]\right) \cap T = F \cap g^{-1}\left(\left[0, \frac{1}{n}\right]\right)$ for each n $\in \mathbb{N}$ and $F \in \mathcal{F}$.

The set $F_n = g^{-1}([0, \frac{1}{n}])$ is zero-set in T for all $n \in \mathbb{N}$ by Fact 1. In addition, $F_n \cap F \neq \emptyset$ or all $F \in \mathcal{F}$ and therefore $F_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ because the family $\mathcal{F}$ is a z-ultrafilter (see Fact 3). Since $g(x) > 0$ for all $x \in T$, we obtain $\bigcap\{F_n : n \in \mathbb{N}\} = \emptyset$ which is contradictory because $\mathcal{F}$ is countably centered. This contradiction shows that $q \in T$; thus $q \in \text{cl}_{\beta T}(f) \cap T = F$ for each $F \in \mathcal{F}$ because F is closed in T. This proves that $q \in \cap \mathcal{F}$ and hence $\bigcap \mathcal{F} \neq \emptyset$ so Fact 4 is proved.

Fact 5. Let T be an arbitrary space.

(1) If $I \subset C(T)$ is an ideal in $C(T)$ then the family $Z(I) = \{f^{-1}(0) : f \in I\}$ is a z-filter in T.
(2) If $\mathcal{F}$ is a z-filter on T then the set $I(\mathcal{F}) = \{f \in C(T) : f^{-1}(0) \in \mathcal{F}\}$ is an ideal in $C(T)$.
(3) If $I \subset C(T)$ is a maximal ideal then $Z(I)$ is a z-ultrafilter on T and $I(Z(I)) = I$. In particular, if $f \in C(T)$ and $f^{-1}(0) \in Z(I)$ then $f \in I$.
(4) If $\mathcal{F}$ is a z-ultrafilter on T then $I(\mathcal{F})$ is a maximal ideal in $C(T)$.

Proof. (1) If $\emptyset \in Z(I)$ then there is a function $f \in I$ such that $f(x) \neq 0$ for all $x \in T$. Then $g = \frac{1}{f} \in C(T)$ and therefore $c_1 = g \cdot f \in I$; this implies $f = f \cdot c_1 \in I$ for any function $f \in C(T)$, i.e., $I = C(T)$ which is a contradiction with (I2). This proves that $f^{-1}(0) \neq \emptyset$ for any $f \in I$ and hence $\emptyset \notin Z(I)$ so (ZF1) holds for $Z(I)$.

Given any $A, B \in Z(I)$ we have $A = f^{-1}(0), B = g^{-1}(0)$ for some $f, g \in I$. Then, for the function $h = f^2 + g^2$, we have $h \in I$ and $h^{-1}(0) = A \cap B$ so $A \cap B \in Z(I)$ and (ZF2) is also fulfilled for $Z(I)$.

To check (ZF3), take any $A \in Z(I)$ and any $B \in \mathcal{Z}_T$ with $A \subset B$. There is $f \in I$ and $g \in C(T)$ such that $A = f^{-1}(0)$ and B $= g^{-1}(0)$. We have $h = f \cdot g \in I$ by (I2) and $h^{-1}(0) = B$ so $B \in Z(I)$ and hence Z(I) is a z-filter in T.

(2) To prove (I1) for $I(\mathcal{F})$, observe that $f^{-1}(0) \in \mathcal{F}$ and $g^{-1}(0) \in \mathcal{F}$ imply that H $= f^{-1}(0) \cap g^{-1}(0) \in \mathcal{F}$. Now, if $h = af + bg$ then $H \subset h^{-1}(0)$; since $\mathcal{F}$ is a z-filter, we have $h^{-1}(0) \in \mathcal{F}$ and hence $h \in I(\mathcal{F})$ so (I1) is established for $I(\mathcal{F})$.

Since $C_1^{-1}(0) = \emptyset \notin \mathcal{F}$, we have $c_1 \notin I(\mathcal{F})$ and hence $I(\mathcal{F}) \neq C(T)$. Finally, take any $f \in I(1)$, any $g \in C(T)$ and let $h = f \cdot g$. It is clear that $f^{-1}(0) \subset h^{-1}(0)$; since $\mathcal{F}$ is a z-filter and $f^{-1}(0) \in \mathcal{F}$, we have $h^{-1}(0) \in \mathcal{F}$, i.e., $h \in I(\mathcal{F})$. This proves (I2) for $I(\mathcal{F})$ and shows that $I(\mathcal{F})$ is indeed an ideal in $C(T)$.

(3) The set $I(Z(I))$ is an ideal in $C(T)$ by (2); it is immediate that $I \subset I(Z(I))$. Since I is maximal, we have $I = I(Z(I))$. Suppose that $\mathcal{F}$ is a z-filter and $Z(I) \subset \mathcal{F}$. Then $I(\mathcal{F})$ is an ideal in $C(T)$ by (2) and it is immediate that $I \subset I(Z(I)) \subset I(\mathcal{F})$. The ideal I being maximal, we have $I = I(\mathcal{F})$ so, for any $F \in \mathcal{F}$ there is $f \in I$ such that $F = f^{-1}(0)$. But $f^{-1}(0) \in Z(I)$ by the definition of $Z(I)$ so $F \in Z(I)$. The set $F \in \mathcal{F}$ has been chosen arbitrarily, so we have $\mathcal{F} \subset Z(I)$ which proves that $\mathcal{F} = Z(I)$ and therefore $Z(I)$ is a z-ultrafilter.

(4) Suppose that $I \subset C(T)$ is an ideal and $I(\mathcal{F}) \subset I$. The family $Z(I)$ is z-filter on T by (1) and $\mathcal{F} = Z(I(\mathcal{F})) \subset Z(I)$. Since $\mathcal{F}$ is a z-ultrafilter, we have $\mathcal{F} = Z(I)$ and hence $f^{-1}(0) \in \mathcal{F}$ for any $f \in I$ so we have $f \in I(\mathcal{F})$ by the definition of $I(\mathcal{F})$. The function $f \in I$ has been chosen arbitrarily, so we proved that $I \subset I(\mathcal{F})$ whence $I = I(\mathcal{F})$ and hence $I(\mathcal{F})$ is a maximal ideal. Fact 5 is proved.

Fact 6. For any space T and any real ideal $I \subset C(T)$, the family $Z(I)$ is countably centered.

Proof. Any real ideal is maximal so $Z(I)$ is a z-ultrafilter in T by Fact 5. Take any sequence $\{F_i : i \in \mathbb{N}\} \subset Z(I)$ and fix a family of functions $\{f_i : i \in \mathbb{N}\} \subset I$ such that $F_i = f_i^{-1}(0)$ for all $i \in \mathbb{N}$. It is clear that $g_i = f_i^2 \in I$ and $g_i^{-1}(0) = F_i$ for all $i \in \mathbb{N}$. Furthermore, if $h_i = \min(g_i, 1)$ then $h_i \in C(T, [0, 1])$ and we have $h_i^{-1}(0) = g_i^{-1}(0) = F_i$ for all $i \in \mathbb{N}$. Thus $h_i^{-1}(0) \in Z(I)$ and hence $h_i \in I(Z(I)) = I$ for all $i \in \mathbb{N}$ by Fact 5. If $w_n = \sum_{i=1}^{n} 2^{-i} \cdot h_i$ for all $n \in \mathbb{N}$ then $w_n \in I$ for all $n \in \mathbb{N}$ and the sequence $\{w_n : n \in \mathbb{N}\}$ converges uniformly to a function $w \in C(T)$ (see Problem 030). It is immediate that $w^{-1}(0) = \bigcap\{f_i^{-1}(0) : i \in \mathbb{N}\}$; the ideal I is real so there is $r \in \mathbb{R}$ such that $u = w + c_r \in I$. If $w'_n = w_n - u$ then $w'_n \in I$ for all $n \in \mathbb{N}$ and the sequence $\{w'_n : n \in \mathbb{N}\}$ converges uniformly to $w - u = -c_r$ (see Problem 035). If $\varepsilon = |r| \neq 0$ then there is $n \in \mathbb{N}$ such that $|w'_n(x) + r| < \varepsilon$ for every $x \in T$ which shows that $w'_n(x) \neq 0$ for all $x \in T$. Consequently, $(w'_n)^{-1}(0) = \emptyset$ which is a contradiction with $(w'_n)^{-1}(0) \in Z(I)$ and the fact that no element of $Z(I)$ can be empty by (ZF1). This contradiction shows that $r = 0$ and hence $w = u \in I$. We established that $w^{-1}(0) = \bigcap\{F_i : i \in \mathbb{N}\} \in Z(I)$ so Fact 6 is proved.

Fact 7. For any space T and any $x \in T$, the set $I_x = I_x^T = \{f \in C(T) : f(x) = 0\}$ is a real ideal of $C(T)$. If T is realcompact then an ideal $I \subset C(T)$ is real if and only if there is $x \in T$ such that $I = Ix$.

Proof. It is easy to check that the set I_x is an ideal; to see that I_x is maximal, assume that J is an ideal in $C(T)$ with $I_x \subset J$. If there exists $f \in J \setminus Ix$ then $x \notin f^{-1}(0)$ and therefore we can find $g \in C(T)$ for which $g(x) = 0$ and $g(f^{-1}(0)) = \{1\}$. The function g belongs to I_x and hence to J. Therefore $h = f^2 + g^2 \in J$ and $h(z) > 0$ for any $z \in T$. It follows from $h \in J$ that $c_1 = h \cdot \frac{1}{h} \in J$. Therefore $g = g \cdot c_1 \in J$ for each $g \in C(T)$. It turns out that $J = C(T)$ which is a contradiction. This shows that I_x is a maximal ideal. Finally, the ideal I_x is real because, for any $f \in C(T)$ we have $f - c_{f(x)} \in I_x$.

Now assume that T is realcompact and I is a real ideal in $C(T)$. The family $Z(I)$ is a countably centered z-ultrafilter by Facts 5 and 6. Therefore $\bigcap Z(I) \neq \emptyset$ by Fact 4.

It is evident that $I \subset I_x$ for any $x \in \bigcap Z(I)$; thus $I = I_x$ by maximality of I. Fact 7 is proved.

Returning to our solution, assume that $C(X)$ is isomorphic to $C(Y)$. It is easy to check that the restriction map $\pi_X : C(\upsilon X) \to C(X)$ is an isomorphism. Analogously, $C(Y)$ is isomorphic to $C(\upsilon Y)$. This shows that there is an isomorphism $\varphi : \mathrm{C}(\upsilon X) \to C(\upsilon Y)$. Observe that the notion of a real ideal is defined in algebraic terms and hence $\varphi(I)$ is a real ideal in $C(\upsilon Y)$ for any real ideal $I \subset C(\upsilon X)$. For any $x \in \upsilon X$, the set $I_x^{\upsilon X}$ x is a real ideal of $C(\upsilon X)$ by Fact 7; hence $\varphi(I_x^{\upsilon X})$ is a real ideal of $C(\upsilon Y)$. Since υY is realcompact, we can apply Fact 7 again to conclude that there is $y \in \upsilon Y$ such that $\varphi(I_x^{\upsilon X}) = I_y^{\upsilon Y}$. Letting $y = f(x)$, we obtain a function $f : \upsilon X \to \upsilon Y$. Since φ is an isomorphism, the map $I_x^{\upsilon X} \mapsto \varphi(I_x)$ is a bijection between the families of all real ideals in $C(\upsilon X)$ and in $C(\upsilon Y)$. Applying Fact 7 we see that f is also a bijection. Finally, let $A \subset \upsilon X$. Given $x \in \overline{A}$, we have $I_x^{\upsilon X} \supset \bigcap\{I_z^{\upsilon X} : z \in (A)\}$ by Fact 3 of S.183. Since φ is a bijection, we have $I_{f(x)}^{\upsilon Y} \supset \bigcap\{I_z^{\upsilon Y} : z \in f(A)\}$ and hence $f(x) \in \overline{f(A)}$ by Fact 3 of S.183. This proves that $f(\overline{A}) \subset \overline{f(A)}$ and hence f is a continuous map. The same reasoning is applicable to f^{-1} so f^{-1} is also continuous which proves that f is a homeomorphism and settles necessity.

Finally, if $h : \upsilon X \to \upsilon Y$ is a homeomorphism then $h^* : C(\upsilon Y) \to C(\upsilon X)$ is an isomorphism: this is an easy exercise (see the first paragraph of S.183). We mentioned already that $C(X)$ is isomorphic to $C(\upsilon X)$ and $C(Y)$ is isomorphic to $C(\upsilon Y)$ so $C(X)$ is isomorphic to $C(Y)$ and hence our solution is complete.

S.500. *Prove that C^* (X) and C^* (Y) are isomorphic as algebraic rings if and only if βX is homeomorphic to βY.*

Solution. Observe that the restriction map $\pi_X : C(\beta X) \to C^*(X)$ is an isomorphism. Analogously, $\pi_Y : C(\beta Y) \to C^*(Y)$ is an isomorphism so if $C^*(X)$ is isomorphic to $C^*(Y)$ then $C(\beta X)$ is isomorphic to $C(\beta Y)$. As a consequence, βX is homeomorphic to βY by Problem 183.

On the other hand, if a map $h : \beta X \to \beta Y$ is a homeomorphism then the dual map $h^* : C(\beta Y) \to C(\beta X)$ is an isomorphism: this is an easy exercise (see the first paragraph of S.183). We mentioned already that $C^*X)$ is isomorphic to $C(\beta X)$ and $C^*Y)$ is isomorphic to $C(\beta Y)$ so $C^*(X)$ is isomorphic to $C^*(Y)$ and hence our solution is complete.

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