Spectral Theory of Pseudo-Differential Operators on $\mathbb{S}^1$

Mohammad Pirhayati

Abstract. For a bounded pseudo-differential operator with the dense domain $C^\infty(\mathbb{S}^1)$ on $L^p(\mathbb{S}^1)$, the minimal and maximal operator are introduced. An analogue of Agmon-Douglis-Nirenberg [1] is proved and then is used to prove the uniqueness of the closed extension of an elliptic pseudo-differential operator of symbol of positive order. We show the Fredholmness of the minimal operator. The essential spectra of pseudo-differential operators on $\mathbb{S}^1$ are described.

Mathematics Subject Classification (2000). Primary 47G30.

Keywords. Pseudo-differential operators, Sobolev spaces, Fredholmness, ellipticity, essential spectra, indices.

1. Introduction

In this paper the focus is on pseudo-differential operators on the unit circle $\mathbb{S}^1$ centered at the origin. For $-\infty < m < \infty$, let $S^m(\mathbb{S}^1 \times \mathbb{Z})$ be the set all functions σ in $C^\infty(\mathbb{S}^1 \times \mathbb{Z})$ such that for all nonnegative integers α and β there exists a positive constant $C_{\alpha,\beta}$ for which

$$|(\partial_\theta^\alpha \partial_n^\beta \sigma)(\theta, n)| \leq C_{\alpha,\beta}(1 + |n|)^{m-\beta}, \quad \theta \in [-\pi, \pi],\ n \in \mathbb{Z}.$$

Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$. Then we define the pseudo-differential operator T_σ on $L^1(\mathbb{S}^1)$ by

$$(T_\sigma f)(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta} \sigma(\theta, n)(\mathcal{F}_{\mathbb{S}^1} f)(n), \quad \theta \in [-\pi, \pi],$$

where

$$(\mathcal{F}_{\mathbb{S}^1} f)(n) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta, \quad n \in \mathbb{Z}.$$

Basic properties of pseudo-differential operators with symbols in $S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$, can be found in [2, 3, 4, 6, 10, 9]. The basic calculi for the

L. Rodino et al. (eds.), *Pseudo-Differential Operators: Analysis, Applications and Computations*, Operator Theory: Advances and Applications 213, DOI 10.1007/978-3-0348-0049-5_2,

product and the formal adjoint of pseudo-differential operators with symbols in $S^m(\mathbb{S}^1 \times \mathbb{Z})$ can be found in [9].

A symbol σ in $S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$, is said to be elliptic if there exist positive constants C and R such that

$$|\sigma(\theta, n)| \geq C(1 + |n|)^m, \quad |n| \geq R, \quad \theta \in [-\pi, \pi].$$

The following theorem gives a parametrix for an elliptic pseudo-differential operator with symbol in $S^m(\mathbb{S}^1 \times \mathbb{Z})$, $\infty < m < -\infty$, see [9].

Theorem 1.1. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$ be elliptic. Then there exists a symbol $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$ such that*

$$T_\sigma T_\tau = I + K \quad \textit{and} \quad T_\tau T_\sigma = I + R,$$

where K and R are infinitely smoothing in the sense that they are pseudo-differential operators with symbols in $\cap_{m \in \mathbb{R}} S^m(\mathbb{S}^1 \times \mathbb{Z})$.

Similar results for the symbol class $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ of the pseudo-differential operators on $\mathbb{R}^n$ have been studied for example in [15].

In Section 2, we recall L^p-Sobolev spaces $H^{s,p}$, $-\infty < s < \infty$, $1 \leq p \leq \infty$, and we give some of the results in [7]. Then in Section 3, we consider bounded pseudo-differential operators T_σ on $L^p(\mathbb{S}^1)$, $1 < p < \infty$ with dense domain $C^\infty(\mathbb{S}^1)$. The smallest and largest closed extension of T_σ are provided. The analogue of Agmon-Douglis-Nirenberg [1], is given to prove that for an elliptic symbol σ of positive order m, the corresponding pseudo-differential operator has a unique closed extension with domain $H^{m,p}$ on $L^p(\mathbb{S}^1)$. In Section 4, we focus on Fredholmness of pseudo-differential operator and its essential spectrum. Results on the Fredholmness of pseudo-differential operators on $\mathbb{R}^n$ can be found in [16, 13]. By using Theorem 2.9 in [7], we see that the minimal operator of an elliptic pseudo-differential operator of positive order is Fredholm. The essential spectra of the pseudo-differential operator and the minimal (maximal) operator are then provided. Similar results for the SG Pseudo-differential operator on $\mathbb{R}^n$ are given in [5, 8].

2. L^p-Sobolev spaces

For $-\infty < s < \infty$, let J_s be the pseudo-differential operator with symbol σ_s given by

$$\sigma_s(n) = (1 + |n|^2)^{-s/2}, \quad n \in \mathbb{Z}.$$

J_s is called the Bessel potential of order s.

Now, for $-\infty < s < \infty$ and $1 \leq p \leq \infty$, we define the L^p-Sobolev space $H^{s,p}$ to be the set of all tempered distributions u for which $J_{-s}u$ is a function in $L^p(\mathbb{S}^1)$. Then $H^{s,p}$ is a Banach space in which the norm $\|\cdot\|_{s,p}$ is given by

$$\|u\|_{s,p} = \|J_{-s}u\|_{L^p(\mathbb{S}^1)}, \quad u \in H^{s,p}.$$

It is easy to show that for $-\infty < s, t < \infty$, J_t is an isometry of $H^{s,p}$ onto $H^{s+t,p}$.

The following theorem is known as Sobolev embedding theorem.

Theorem 2.1. *Let $1 < p < \infty$ and $s \leq t$. Then $H^{t,p} \subseteq H^{s,p}$ and*

$$\|u\|_{s,p} \leq \|u\|_{t,p}, \quad u \in H^{t,p}.$$

Proposition 2.2. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$. Then $T_\sigma : H^{s,p} \to H^{s-m,p}$ is a bounded linear operator for $1 < p < \infty$.*

Proposition 2.3. *Let $s < t$. Then the inclusion operator $i : H^{t,p} \hookrightarrow H^{s,p}$ is compact for $1 \leq p \leq \infty$.*

The results above can be found in [7].

3. Minimal and maximal operators

Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m \in \mathbb{R}$. Then the formal adjoint of T_σ, denoted T_σ^* is a linear operator on $C^\infty(\mathbb{S}^1)$ such that

$$(T_\sigma \varphi, \psi) = (\varphi, T_\sigma^* \psi), \quad \varphi, \psi \in C^\infty(\mathbb{S}^1).$$

It can be proved that the formal adjoint of T_σ is a pseudo-differential operator of symbol of order $-m$ (see [10]). The following proposition guarantee that the minimal operator of T_σ exists.

Proposition 3.1. *Let $S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$. Then $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is closable with dense domain $C^\infty(\mathbb{S}^1)$ for $1 < p < \infty$.*

Proof. Let $\{\varphi_k\}_{k=1}^\infty$ be a sequence in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to 0$ and $T_\sigma \varphi_k \to f$ for some f in $L^p(\mathbb{S}^1)$ as $k \to \infty$. We only need to show that $f = 0$. We have

$$(T_\sigma \varphi_k, \psi) = (\varphi_k, T_\sigma^* \psi), \quad \psi \in C^\infty(\mathbb{S}^1),\ k = 1, 2, \ldots.$$

Let $k \to \infty$, then $(f, \psi) = 0$ for all $\psi \in C^\infty(\mathbb{S}^1)$. By the density of $C^\infty(\mathbb{S}^1)$ in $L^p(\mathbb{S}^1)$, it follows that $f = 0$. □

Consider $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ with domain $C^\infty(\mathbb{S}^1)$. Then by Proposition 3.1, T_σ has a closed extension. Let $T_{\sigma,0}$ be the minimal operator of T_σ which is the smallest closed extension of T_σ. Then the domain $\mathcal{D}(T_{\sigma,0})$ of $T_{\sigma,0}$ consists of all functions $u \in L^p(\mathbb{S}^1)$ for which there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to u$ in $L^p(\mathbb{S}^1)$ and $T_\sigma \varphi_k \to f$ for some $f \in L^p(\mathbb{S}^1)$ in $L^p(\mathbb{S}^1)$ as $k \to \infty$. It can be shown that f does not depend on the choice of $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ and $T_{\sigma,0} u = f$.

We define the linear operator $T_{\sigma,1}$ on $L^p(\mathbb{S}^1)$ with domain $\mathcal{D}(T_{\sigma,1})$ by the following. Let f and u be in $L^p(\mathbb{S}^1)$. Then we say that $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$ if and only if

$$(u, T_\sigma^* \varphi) = (f, \varphi), \quad \varphi \in C^\infty(\mathbb{S}^1).$$

It can be proved that $T_{\sigma,1}$ is a closed linear operator from $L^p(\mathbb{S}^1)$ into $L^p(\mathbb{S}^1)$ with domain $\mathcal{D}(T_{\sigma,1})$ containing $C^\infty(\mathbb{S}^1)$. In fact, $C^\infty(\mathbb{S}^1)$ is contained in the domain $\mathcal{D}(T_{\sigma,1}^t)$ of the true adjoint $T_{\sigma,1}^t$ of $T_{\sigma,1}$. Furthermore, $T_{\sigma,1}(u) = T_\sigma(u)$ for all u in $\mathcal{D}(T_{\sigma,1})$.

It is easy to see that $T_{\sigma,1}$ is an extension of $T_{\sigma,0}$. In fact $T_{\sigma,1}$ is the largest closed extension of T_σ in the sense that if B is any closed extension of T_σ such that $C^\infty(\mathbb{S}^1) \subseteq \mathcal{D}(B^t)$, then $T_{\sigma,1}$ is an extension of B. $T_{\sigma,1}$ is called the maximal operator of T_σ. The following theorem is an analogue of Agmon-Douglis-Nirenberg in [1].

Proposition 3.2. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$ be elliptic. Then there exist positive constants C and $D > 0$ such that*

$$C\|u\|_{m,p} \leq \|T_\sigma u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)} \leq D\|u\|_{m,p}, \quad u \in H^{m,p}.$$

Proof. By the boundedness of T_σ in Proposition 2.2 and the boundedness of the inclusion operator in Theorem 2.1, there exists a positive constant D such that for all $u \in H^{m,p}$,

$$\|T_\sigma u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)} \leq D\|u\|_{m,p}, \quad u \in H^{m,p}.$$

Since $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ is elliptic, by Theorem 1.1, there exists a symbol $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$ such that

$$u = T_\tau T_\sigma u - Ru, \quad u \in H^{m,p},$$

where R is an infinitely smoothing operator in the sense that R is a pseudo-differential operator with symbol in $\cap_{m\in\mathbb{R}} S^m(\mathbb{S}^1 \times \mathbb{Z})$. By using Proposition 2.2 again, $T_\sigma u \in L^p(\mathbb{S}^1)$. Therefore, $T_\tau T_\sigma u \in H^{m,p}$, for all $u \in H^{m,p}$, Moreover there exists a positive constant C such that

$$\|u\|_{m,p} \leq C(\|T_\sigma u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)}), \quad u \in H^{m,p}.$$ □

We have the following result which we use in the next theorem.

Lemma 3.3. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then $C^\infty(\mathbb{S}^1)$ is dense in $H^{s,p}$.*

Proof. Let $u \in H^{s,p}$. Then $J_{-s}u \in L^p(\mathbb{S}^1)$. Since $C^\infty(\mathbb{S}^1)$ is dense in $L^p(\mathbb{S}^1)$, there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to J_{-s}u$ in $L^p(\mathbb{S}^1)$ as $k \to \infty$. Let $\psi_k = J_s\varphi_k$, $k = 1, 2, \ldots$. Then $\psi_k \in C^\infty(\mathbb{S}^1)$, $k = 1, 2, \ldots$, and

$$\begin{aligned} \|\psi_k - u\|_{s,p} &= \|J_{-s}\psi_k - J_{-s}u\|_{L^p(\mathbb{S}^1)} \\ &= \|\varphi_k - J_{-s}u\|_{L^p(\mathbb{S}^1)} \to 0, \end{aligned}$$

as $k \to \infty$, which completes the proof. □

The following theorem gives the domain of the minimal operator of an elliptic pseudo-differential operator with symbol of positive order.

Theorem 3.4. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$, be elliptic. Then $\mathcal{D}(T_{\sigma,0}) = H^{m,p}$.*

Proof. Let $u \in H^{m,p}$. Then by using the density of $C^\infty(\mathbb{S}^1)$ in $H^{m,p}$, there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to u$ in $H^{m,p}$ and therefore in $L^p(\mathbb{S}^1)$ as $k \to \infty$. By Proposition 3.2, φ_k and $T_\sigma\varphi_k$ are Cauchy sequences in $L^p(\mathbb{S}^1)$. Therefore $\varphi_k \to u$ and $T_\sigma\varphi_k \to f$ for some f in $L^p(\mathbb{S}^1)$ as $k \to \infty$. This implies that $u \in \mathcal{D}(T_{\sigma,0})$ and $T_{\sigma,0}u = f$. Now assume that $u \in \mathcal{D}(T_{\sigma,0})$. Then there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{S}^1)$ such that $\varphi_k \to u$ in $L^p(\mathbb{S}^1)$ and $T_\sigma\varphi_k \to f$, for some $f \in L^p(\mathbb{S}^1)$ as $k \to \infty$. So, by Proposition 3.2, $\{\varphi_k\}_{k=1}^\infty$ is a Cauchy sequence in $H^{m,p}$. Since $H^{m,p}$ is complete, there exists $v \in H^{m,p}$ such that $\varphi_k \to v$ in $H^{m,p}$ as $k \to \infty$. By Sobolev embedding theorem $\varphi_k \to v$ in $L^p(\mathbb{S}^1)$ which implies that $u = v \in H^{m,p}$. □

The following theorem shows that the closed extension of an elliptic pseudo-differential operator on $L^p(\mathbb{S}^1)$ with symbol $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$, is unique and moreover by Theorem 3.4, its domain is $H^{m,p}$.

Theorem 3.5. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$, be elliptic. Then $T_{\sigma,0} = T_{\sigma,1}$.*

Proof. Since $T_{\sigma,1}$ is a closed extension of $T_{\sigma,0}$, by Theorem 3.4, it is enough to show that $\mathcal{D}(T_{\sigma,1}) \subseteq H^{m,p}$. Let $u \in \mathcal{D}(T_{\sigma,1})$. By ellipticity of σ, there exists $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$ such that

$$u = T_\tau T_\sigma u - Ru,$$

where R is an infinitely smoothing operator. Since $T_\sigma u = T_{\sigma,1} u \in L^p(\mathbb{S}^1)$, by Proposition 2.2, it follows that $u \in H^{m,p}$, which completes the proof. □

4. Fredholm pseudo-differential operators

A closed linear operator A from a complex Banach space X into a complex Banach space Y with dense domain $\mathcal{D}(A)$ is said to be Fredholm if

- the range of A, $R(A)$ is closed subspace of Y and
- the null space of A, $N(A)$ and the null space of the true adjoint of A, $N(A^t)$ are finite dimensional.

The index of a Fredholm operator A is defined by

$$i(A) = \dim N(A) - \dim N(A^t)$$

By Atkinson's theorem, a closed linear operator $A : X \to Y$ with dense domain $\mathcal{D}(A)$ is Fredholm if and only if there exists a bounded linear operator $B : Y \to X$ such that $K_1 = AB - I : Y \to Y$ and $K_2 = BA - I : X \to X$ are compact operators.

Let $A : X \to X$ be a closed linear operator with dense domain $\mathcal{D}(A)$ in the complex Banach space X. Then the spectrum of A, $\Sigma(A)$ is defined by

$$\Sigma(A) = \mathbb{C} - \rho(A),$$

where $\rho(A)$ is the resolvent set of A given by

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is bijective}\}.$$

The essential spectrum $\Sigma_w(A)$ of A, which has been defined in [14] by Wolf given by

$$\Sigma_w(A) = \mathbb{C} - \Phi_w(A), \text{ where } \Phi_w(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm}\}.$$

Note that $i(A - \lambda I)$ is constant for all λ in a connected component of $\Phi_w(A)$.

The essential spectrum $\Sigma_s(A)$ of A in sense of Schechter [11] is defined by

$$\Sigma_s(A) = \mathbb{C} - \Phi_s(A), \text{ where } \Phi_s(A) = \{\lambda \in \Phi_w(A) : i(A - \lambda I) = 0\}.$$

For the properties of essential spectra see [12]. The following theorem gives a sufficient condition for $T_\sigma : H^{s,p} \to H^{s-m,p}$ to be a Fredholm operator. The proof can be found in [7].

Theorem 4.1. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $-\infty < m < \infty$ be elliptic. Then for all $-\infty < s < \infty$ and $1 < p < \infty$, $T_\sigma : H^{s,p} \to H^{s-m,p}$ is a Fredholm operator. In particular if $\sigma \in S^0(\mathbb{S}^1 \times \mathbb{Z})$, then the bounded linear operator $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is Fredholm.*

The following is an immediate corollary of Theorem 3.4 and Theorem 4.1.

Corollary 4.2. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$ be elliptic. Then for $1 < p < \infty$, $T_{\sigma,0}$ is a Fredholm operator on $L^p(\mathbb{S}^1)$ with the domain $H^{m,p}$.*

The following theorem gives the essential spectrum of an elliptic pseudo-differential operator of positive order.

Theorem 4.3. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m > 0$ be elliptic. Then*

$$\Sigma_w(T_{\sigma,0}) = \varnothing.$$

Proof. Let $\lambda \in \mathbb{C}$. By Corollary 4.2, we need only to show that $\sigma - \lambda$ is elliptic. The ellipticity of σ, implies that there exist constants $C, R > 0$ such that

$$|\sigma(\theta, n) - \lambda| \geq C(1 + |n|)^m - |\lambda| = (1 + |n|)^m (C - \frac{|\lambda|}{(1 + |n|)^m}), \ \theta \in [-\pi, \pi],$$

whenever $|n| \geq R$. Since $(1 + |n|)^m \to \infty$ as $|n| \to \infty$, there exists $M > 0$ such that

$$|\sigma(\theta, n) - \lambda| \geq \frac{C}{2}(1 + |n|)^m, \quad |n| \geq M, \ \theta \in [-\pi, \pi],$$

which implies that $\sigma - \lambda$ is elliptic. □

Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m \geq 0$. Then the following theorem is a result on the essential spectra of the bounded pseudo-differential operator T_σ with the domain $H^{m,p}$ on $L^p(\mathbb{S}^1)$.

Theorem 4.4. *Let $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$, $m \geq 0$. Then for T_σ on $L^p(\mathbb{S}^1)$ with the domain $H^{m,p}$, $1 < p < \infty$, we have*

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq L_i\},$$

where

$$L_i = \liminf_{|n| \to \infty} \{(\inf_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)|)(1 + |n|)^{-m}\}.$$

Proof. Let $\lambda \in \mathbb{C}$ be such that $|\lambda| < L_i$. Then there exists $\epsilon > 0$ such that

$$|\lambda| + \epsilon < L_i.$$

Since $m \geq 0$, it follows that $|\lambda| < (L_i - \epsilon)(1 + |n|)^m$. On the other hand, there exists a positive constant R such that

$$\inf_{|n| \geq R} \{(\inf_{\theta \in [-\pi, \pi]} |\sigma(n, \theta)|)(1 + |n|)^{-m}\} > L_i - \frac{\epsilon}{2}.$$

So, for $|n| \geq R$,

$$\begin{aligned} |\sigma(\theta, n) - \lambda| &\geq |\sigma(\theta, n)| - |\lambda| \\ &> (L_i - \frac{\epsilon}{2} - L_i + \epsilon)(1 + |n|)^m \\ &= \frac{\epsilon}{2}(1 + |n|)^m, \quad \theta \in [-\pi, \pi]. \end{aligned}$$

Therefore, $\sigma - \lambda$ is elliptic and hence $T_\sigma - \lambda I : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ with domain $H^{m,p}$ is Fredholm. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| < L_i\} \subseteq \Phi_w(T_\sigma),$$

which implies that

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq L_i\}. \qquad \square$$

We have the following theorem on the essential spectrum of a pseudo-differential operator of order 0 from $L^p(\mathbb{S}^1)$ into $L^p(\mathbb{S}^1)$.

Theorem 4.5. *Let $\sigma \in S^0(\mathbb{S}^1 \times \mathbb{Z})$. Then for $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$, $1 < p < \infty$, we have*

$$\Sigma_s(T_\sigma) \subseteq \{\lambda : |\lambda| \leq L_s\},$$

where

$$L_s = \limsup_{|n| \to \infty} \{ \sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| \}.$$

Proof. Let $\lambda \in \mathbb{C}$ such that $|\lambda| > L_s$. Then there exists $\epsilon > 0$ such that

$$|\lambda| - \epsilon > L_s,$$

and there exists a positive number R such that

$$\sup_{|n| \geq R} \{ \sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| \} < L_s + \frac{\epsilon}{2}.$$

For all $|n| \geq R$,

$$\begin{aligned} |\sigma(\theta, n) - \lambda| &\geq |\lambda| - |\sigma(\theta, n)| \\ &> L_s + \epsilon - L_s - \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2}, \quad \theta \in [-\pi, \pi]. \end{aligned}$$

Hence $\sigma - \lambda$ is elliptic and by Theorem 4.1, $T_\sigma - \lambda I : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is Fredholm. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| > L_s\} \subseteq \Phi_w(T_\sigma),$$

which is the same as

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq L_s\}.$$

Since $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$ is a connected component of $\Phi_w(T_\sigma)$, it follows that $i(T_\sigma - \lambda I)$ is a constant for all λ in $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$. On the other hand,

$$\rho(T_\sigma) \cap \{\lambda \in \mathbb{C} : |\lambda| > L_s\} \neq \varnothing.$$

Therefore, $i(T_\sigma - \lambda I) = 0$ for all $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$. This implies that

$$\Sigma_s(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq L_s\}. \qquad \square$$

We have the following spectral alternative for a pseudo-differential operator with symbol in $S^0(\mathbb{S}^1 \times \mathbb{Z})$.

Corollary 4.6. *Let $\sigma \in S^0(\mathbb{S}^1 \times \mathbb{Z})$ be such that*

$$\limsup_{|n|\to\infty}(\sup_{\theta\in[-\pi,\pi]} |\sigma(\theta, n)|) = \liminf_{|n|\to\infty}(\inf_{\theta\in[-\pi,\pi]} |\sigma(\theta, n)|) = L > 0.$$

Then

$$\Sigma_w(T_\sigma) = \{\lambda \in \mathbb{C} : |\lambda| = L\} \quad \textit{or} \quad \Sigma_s(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

Proof. By Theorem 4.4 and Theorem 4.5,

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

Suppose that

$$\Sigma_w(T_\sigma) \neq \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

Then there exists $\lambda_0 \in \mathbb{C}$ such that $|\lambda_0| = L$ and $\lambda_0 \in \Phi_w(T_\sigma)$. On the other hand, by Theorem 4.5,

$$\{\lambda \in \mathbb{C} : |\lambda| > L\} \subseteq \Phi_s(T_\sigma).$$

Hence using the fact that $\Phi_w(T_\sigma)$ is an open set and the index of $T_\sigma - \lambda I$ is constant on on every connected component of $\Phi_w(T_\sigma)$ we get $i(T_\sigma - \lambda I) = 0$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \neq L$, which is the same as

$$\Sigma_s(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = L\},$$

as asserted. $\square$

References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, *Comm. Pure Appl. Math.* **12** (1959), 623–727.

[2] M.S. Agranovich, Spectral properties of elliptic pseudodifferential operators on a closed curve, *Funktsional. Anal. i Prilozhen.* **13** (1979), 54–56 (in Russian).

[3] M.S. Agranovich, Elliptic pseudodifferential operators on a closed curve, *Trudy Moskov. Mat. Obshch.* **47** (1984), 22–67, 246 (in Russian); *Trans. Moscow Math. Soc.* (1985), 23–74.

[4] B.A. Amosov, On the theory of pseudodifferential operators on the circle, *Uspekhi Mat. Nauk.* **43** (1988), 169–170 (in Russian); *Russian Math. Surveys* **43** (1988), 197–198.

[5] A. Dasgupta and M.W. Wong, Spectral theory of SG pseudo-differential operators on $L^p(\mathbb{R}^n)$, *Studia Math.* **187** (2008), 185–197.

[6] S. Molahajloo and M.W. Wong, Pseudo-differential operators on $\mathbb{S}^1$, in *New Developments in Pseudo-Differential Operators*, Operator Theory: Advances and Applications **189**, Birkhäuser, 2008, 297–306.

[7] S. Molahajloo and M.W. Wong, Ellipticity, Fredholmness and spectral invariance of pseudo-differential operators on $\mathbb{S}^1$, J. Pseudo-Differ. Oper. Appl. **1** (2010), 183–205.

[8] F. Nicola and L. Rodino, SG pseudo-differential operators and weak hyperbolicity, *Pliska Stud. Math. Bulgar.* **15** (2002), 5–19.

[9] M. Ruzhansky and V. Turunen, On the Fourier analysis of operators on the torus, in *Modern Trends in Pseudo-Differential Operators,* Operator Theory: Advances and Applications **172**, Birkhäuser, 2007, 87–105.

[10] M. Ruzhansky and V. Turunen, *Pseudo-Differential Operators and Symmetries*, Birkhäuser, 2009.

[11] M. Schechter, On the essential spectrum of an arbitrary operator I, *J. Math. Anal. Appl.* **13** (1966), 205–215.

[12] M. Schechter, *Spectra of Partial Differential Operators,* Second Edition, North-Holland, 1986.

[13] E. Schrohe, Spectral invariance, ellipticity, and the Fredholm property for pseudodifferential operators on weighted Sobolev spaces, *Ann. Global Anal. Geom.* **10** (1992), 237–254.

[14] F. Wolf, On essential spectrum of partial differential boundary problems, *Comm. Pure Appl. Math.* **12** (1959), 211–228.

[15] M.W. Wong, *An Introduction to Pseudo-Differential Operators*, Second Edition, World Scientific, 1999.

[16] M.W. Wong, Fredholm pseudo-differential operators on weighted Sobolev spaces, *Ark. Mat.* **21** (1983), 271–282.

Mohammad Pirhayati
Department of Computer Science
Islamic Azad University
Malayer Branch
Seyfie Park
Malayer, Iran
e-mail: pirhayati.mohammad@gmail.com

http://www.springer.com/978-3-0348-0048-8

Pseudo-Differential Operators: Analysis, Applications and Computations
(Eds.)L. Rodino; M.W. Wong; H. Zhu
2011, VII, 309 p. 8 illus., 6 in color., Hardcover
ISBN: 978-3-0348-0048-8
A product of Birkhäuser Basel