2

Mean Residual Life Orders

In this chapter we study two orders that are based on comparisons of functionals of mean residual lives. Like the orders in Chapter 1, the purpose of the orders here is to compare the "location" or the "magnitude" of random variables. Among other things, the relationship between the orders of Chapter 1 and the orders in this chapter will be analyzed.

2.A The Mean Residual Life Order

2.A.1 Definition

If X is a random variable with a survival function $\overline{F}$ and a finite mean μ, the mean residual life of X at t is defined as

$$m(t) = \begin{cases} E[X - t \big| X > t], & \text{for } t < t^*; \\ 0, & \text{otherwise}, \end{cases} \tag{2.A.1}$$

where $t^* = \sup\{t : \overline{F}(t) > 0\}$. Note that if X is an almost surely positive random variable, then $m(0) = \mu$. By the finiteness of μ we have that $m(t) < \infty$ for all $t < \infty$. However, it is possible that $m(\infty) \equiv \lim_{t\to\infty} m(t) = \infty$. A useful observation is that $m(t) = (\int_t^\infty \overline{F}(x)\mathrm{d}x)/\overline{F}(t)$ when $t^* = \infty$.

Although in (2.A.1) there is no restriction on the support of X, the mean residual life function is usually of interest when X is a nonnegative random variable. In that case X can be thought of as a lifetime of a device and $m(t)$ then expresses the conditional expected residual life of the device at time t given that the device is still alive at time t. Clearly, $m(t) \geq 0$, but not every nonnegative function is a *mean residual life* (mrl) function corresponding to some random variable. In fact, a function m is an mrl function of some nonnegative random variable with an absolutely continuous distribution function if, and only if, m satisfies the following properties:

(i) $0 \leq m(t) < \infty$ for all $t \geq 0$,

(ii) $m(0) > 0$,
(iii) m is continuous,
(iv) $m(t) + t$ is increasing on $[0, \infty]$, and
(v) when there exists a t_0 such that $m(t_0) = 0$, then $m(t) = 0$ for all $t \geq t_0$. Otherwise, when there does not exist such a t_0 with $m(t_0) = 0$, then

$$\int_0^\infty \frac{1}{m(t)} \mathrm{d}t = \infty.$$

Clearly, the smaller the mrl function is, the smaller X should be in some stochastic sense. This is the motivation for the order discussed in this section.

Let X and Y be two random variables with mrl functions m and l, respectively, such that

$$m(t) \leq l(t) \quad \text{for all } t. \tag{2.A.2}$$

Then X is said to be *smaller than* Y *in the mean residual life order* (denoted as $X \leq_{\mathrm{mrl}} Y$).

Analogously to (1.B.3), it can be shown that $X \leq_{\mathrm{mrl}} Y$ if, and only if,

$$\frac{\int_t^\infty \overline{G}(u)\mathrm{d}u}{\int_t^\infty \overline{F}(u)\mathrm{d}u} \quad \text{increases in } t \text{ over } \{t : \int_t^\infty \overline{F}(u)\mathrm{d}u > 0\}, \tag{2.A.3}$$

or equivalently, if, and only if,

$$\overline{G}(t) \int_t^\infty \overline{F}(u)\mathrm{d}u \leq \overline{F}(t) \int_t^\infty \overline{G}(u)\mathrm{d}u \quad \text{for all } t, \tag{2.A.4}$$

or equivalently, if, and only if,

$$\frac{E[(Y-t)_+]}{E[(X-t)_+]} \quad \text{increases in } t \text{ over } \{t : E[(X-t)_+] > 0\}, \tag{2.A.5}$$

where, for any real number a, we let a_+ denote the positive part of a; that is, $a_+ = a$ if $a \geq 0$ and $a_+ = 0$ if $a < 0$.

Analogously to (1.B.5), we also have that $X \leq_{\mathrm{mrl}} Y$ if, and only if,

$$\frac{\overline{F}(s)}{\int_t^\infty \overline{F}(u)\mathrm{d}u} \geq \frac{\overline{G}(s)}{\int_t^\infty \overline{G}(u)\mathrm{d}u} \quad \text{for all } s \leq t \tag{2.A.6}$$

such that the denominators are positive.

It is worthwhile to note that Condition (2.A.5) uses the expectations $E[(X-t)_+]$ and $E[(Y-t)_+]$ as (3.A.5) in Chapter 3 and (4.A.4) in Chapter 4 do.

For discrete random variables that take on values in $\mathbb{N}_+$ the definition of $\leq_{\mathrm{mrl}}$ should be modified. Let X be such a random variable with a finite mean μ. The mrl function of X at n is defined as

$$m(n) = \begin{cases} E[X - n \big| X \geq n], & \text{for } n \leq n^*; \\ 0, & \text{otherwise,} \end{cases}$$

where $n^* = \max\{n : P\{X \geq n\} > 0\}$. Note that for such a random variable $m(0) = \mu$. By the finiteness of μ we have that $m(n) < \infty$ for $n < \infty$. Let X and Y be two such random variables with mrl functions m and l, respectively. We denote $X \leq_{\rm mrl} Y$ if

$$m(n) \leq l(n) \quad \text{for all } n \geq 0. \tag{2.A.7}$$

The discrete analog of (2.A.3) is that (2.A.7) holds if, and only if,

$$\frac{\sum_{j=n}^{\infty} P\{Y \geq j\}}{\sum_{j=n}^{\infty} P\{X \geq j\}} \quad \text{increases in } n \text{ over } \mathbb{N}_+ \cap \{n : \sum_{j=n}^{\infty} P\{X \geq j\} > 0\}.$$

The discrete analog of (2.A.4) is that (2.A.7) holds if, and only if,

$$P\{Y \geq n\} \sum_{j=n+1}^{\infty} P\{X \geq j\} \leq P\{X \geq n\} \sum_{j=n+1}^{\infty} P\{Y \geq j\} \quad \text{for all } n \geq 0.$$

The discrete analog of (2.A.6) is that $X \leq_{\rm mrl} Y$ if, and only if,

$$\frac{P\{X \geq m\}}{\sum_{j=n+1}^{\infty} P\{X \geq j\}} \geq \frac{P\{Y \geq m\}}{\sum_{j=n+1}^{\infty} P\{Y \geq j\}} \quad \text{for all } m \leq n$$

such that the denominators are positive.

2.A.2 The relation between the mean residual life and some other stochastic orders

If X is a random variable with mrl function m and hazard rate function r, it is not hard to verify that

$$m(t) = \int_t^{t^*} \exp\Big\{ - \int_t^x r(u)\mathrm{d}u \Big\}\mathrm{d}x, \quad \text{for } t < t^*. \tag{2.A.8}$$

Therefore, if Y is another random variable with mrl function l and hazard rate function q and (1.B.2) is satisfied, that is, $X \leq_{\rm hr} Y$, then $X \leq_{\rm mrl} Y$. We thus have proved the following result.

Theorem 2.A.1. *If X and Y are two random variables such that $X \leq_{\rm hr} Y$, then $X \leq_{\rm mrl} Y$.*

Neither of the orders $\leq_{\rm st}$ and $\leq_{\rm mrl}$ implies the other; counterexamples can be found in the literature. The next result, however, gives a condition under which $X \leq_{\rm mrl} Y$ if, and only if, $X \leq_{\rm hr} Y$. Therefore, in particular, under that condition, $X \leq_{\rm mrl} Y \Longrightarrow X \leq_{\rm st} Y$.

Theorem 2.A.2. *Let X and Y be two random variables with mrl functions m and l, respectively. Suppose that $\frac{m(t)}{l(t)}$ increases in t. Then, if $X \leq_{\rm mrl} Y$, then $X \leq_{\rm hr} Y$.*

Proof. It is not hard to verify that m is differentiable over $\{t : P\{X > t\} > 0\}$ and that if X has the hazard rate function r, then

$$r(t) = \frac{m'(t) + 1}{m(t)},$$

where m' denotes the derivative of m. Similarly, if Y has the hazard rate function q, then

$$q(t) = \frac{l'(t) + 1}{l(t)}.$$

The monotonicity of $m(t)/l(t)$, together with (2.A.2), implies that

$$r(t) = \frac{m'(t)}{m(t)} + \frac{1}{m(t)} \geq \frac{l'(t)}{l(t)} + \frac{1}{l(t)} = q(t),$$

that is, $X \leq_{\mathrm{hr}} Y$. □

Under a condition that is weaker than the one in Theorem 2.A.2 one merely obtains that $X \leq_{\mathrm{mrl}} Y$ implies that $X \leq_{\mathrm{st}} Y$. This is shown in the next result.

Theorem 2.A.3. *Let X and Y be two nonnegative random variables with mrl functions m and l, respectively. Suppose that $\frac{m(t)}{l(t)} \geq \frac{m(0)}{l(0)}$ (that is, $\frac{m(t)}{l(t)} \geq \frac{EX}{EY}$ when X and Y are almost surely positive), $t \geq 0$. If $X \leq_{\mathrm{mrl}} Y$, then $X \leq_{\mathrm{st}} Y$.*

Proof. Let $\overline{F}$ be the survival function of X. It is not hard to verify that

$$\overline{F}(t) = \frac{EX}{m(t)} \exp\Big\{ -\int_0^t \frac{1}{m(x)} \mathrm{d}x \Big\} \quad \text{over } \{t : P\{X > t\} > 0\}.$$

Similarly, the survival function of Y can be expressed as

$$\overline{G}(t) = \frac{EY}{l(t)} \exp\Big\{ -\int_0^t \frac{1}{l(x)} \mathrm{d}x \Big\} \quad \text{over } \{t : P\{Y > t\} > 0\}.$$

Therefore, under the assumptions of the theorem, it is seen that $\frac{\overline{G}(t)}{\overline{F}(t)} \geq 1$. □

The mean residual life order can be characterized by means of the hazard rate order and the appropriate equilibrium age variables. Recall from (1.A.20) that for nonnegative random variables X and Y with finite means we denote by A_X and A_Y the corresponding asymptotic equilibrium ages. The following result follows at once from (1.B.3) and (2.A.3). It may be contrasted with Theorem 1.C.13.

Theorem 2.A.4. *For nonnegative random variables X and Y with finite means we have $X \leq_{\mathrm{mrl}} Y$ if, and only if, $A_X \leq_{\mathrm{hr}} A_Y$.*

In the next theorem the order $\le_{\text{mrl}}$ is characterized by ordering two related random variables in the sense of the hazard rate order. Let X and Y be two nonnegative random variables with finite means and suppose that $X \le_{\text{st}} Y$ and that $EX < EY$. Let F and G be the distribution functions of X and of Y, respectively. Define the random variable $Z_{X,Y}$ as the random variable that has the density function h given by (1.C.7), as in Theorem 1.C.14; see also Theorem 2.B.3.

Theorem 2.A.5. *Let X and Y be two nonnegative random variables with finite means such that $X \le_{\text{st}} Y$ and such that $EY > EX > 0$. Then*

$$X \le_{\text{mrl}} Y \Longleftrightarrow A_Y \le_{\text{hr}} Z_{X,Y} \Longleftrightarrow A_X \le_{\text{hr}} Z_{X,Y},$$

where $Z_{X,Y}$ has the density function given in (1.C.7).

Proof. Denote by $\overline{G}_\text{e}$ and $\overline{H}$ the survival functions of A_Y and $Z_{X,Y}$, respectively. Using (1.A.20) and (1.C.7) we compute

$$\frac{\overline{H}(x)}{\overline{G}_\text{e}(x)} = \frac{EY}{EY - EX}\left(1 - \frac{\int_x^\infty \overline{F}(u)\text{d}u}{\int_x^\infty \overline{G}(u)\text{d}u}\right), \quad x \ge 0,$$

and the first stated equivalence follows from (2.A.3) and (1.B.3). The second equivalence is proven similarly. □

Some characterizations of the hazard rate order by means of the order $\le_{\text{mrl}}$ are given below. We denote by $\text{Exp}(\mu)$ any exponential random variable with mean μ.

Theorem 2.A.6. *Let X and Y be two continuous nonnegative random variables. Then $X \le_{\text{hr}} Y$ if, and only if,*

$$\min\{X, \text{Exp}(\mu)\} \le_{\text{mrl}} \min\{Y, \text{Exp}(\mu)\} \quad \textit{for all } \mu > 0.$$

The proof of Theorem 2.A.6 uses the Laplace transform order which is discussed in Chapter 5, and it will be given in Remark 5.A.23.

Note that from Theorem 2.A.6 it follows, for continuous nonnegative random variables, that $X \le_{\text{hr}} Y$ if, and only if,

$$\min\{X, Z\} \le_{\text{mrl}} \min\{Y, Z\}$$

for any nonnegative random variable Z which is independent of X and of Y. This is so because $X \le_{\text{hr}} Y$ implies $\min\{X, Z\} \le_{\text{hr}} \min\{Y, Z\}$ by Theorem 1.B.33, and the latter implies the above inequality by Theorem 2.A.1.

The proof of the next result is not given here.

Theorem 2.A.7. *Let X and Y be two continuous nonnegative random variables. Then $X \le_{\text{hr}} Y$ if, and only if,*

$$1 - \text{e}^{-sX} \le_{\text{mrl}} 1 - \text{e}^{-sY} \quad \textit{for all } s > 0.$$

A characterization of the order $\le_{\text{mrl}}$, by means of the increasing convex order, is given in Theorem 4.A.24.

2.A.3 Some closure properties

In general, if $X_1 \le_{\rm mrl} Y_1$ and $X_2 \le_{\rm mrl} Y_2$, where X_1 and X_2 are independent random variables and Y_1 and Y_2 are also independent random variables, then it is not necessarily true that $X_1+X_2 \le_{\rm mrl} Y_1+Y_2$. However, if these random variables are IFR, then it is true. This is shown in Theorem 2.A.9, but first we state and prove the following lemma, which is of independent interest.

Lemma 2.A.8. *If the random variables X and Y are such that $X \le_{\rm mrl} Y$ and if Z is an* IFR *random variable which is independent of X and Y, then*

$$X + Z \le_{\rm mrl} Y + Z. \tag{2.A.9}$$

Proof. Denote by f_W and $\overline{F}_W$ the density function and the survival function of any random variable W. Note that

$$\int_{x=s}^{\infty} \overline{F}_{X+Z}(x)\mathrm{d}x = \int_{-\infty}^{\infty} \overline{F}_X(u)\overline{F}_Z(s-u)\mathrm{d}u \quad \text{for all } s.$$

Now, for $s \le t$, compute

$$\begin{aligned}
&\int_{x=s}^{\infty} \overline{F}_{X+Z}(x)\mathrm{d}x \int_{y=t}^{\infty} \overline{F}_{Y+Z}(y)\mathrm{d}y - \int_{x=t}^{\infty} \overline{F}_{X+Z}(x)\mathrm{d}x \int_{y=s}^{\infty} \overline{F}_{Y+Z}(y)\mathrm{d}y \\
&= \int_v \int_{u\ge v} \Big[\overline{F}_X(u)\overline{F}_Z(s-u)\overline{F}_Y(v)\overline{F}_Z(t-v) \\
&\qquad\qquad + \overline{F}_X(v)\overline{F}_Z(s-v)\overline{F}_Y(u)\overline{F}_Z(t-u)\Big]\mathrm{d}u\mathrm{d}v \\
&- \int_v \int_{u\ge v} \Big[\overline{F}_X(u)\overline{F}_Z(t-u)\overline{F}_Y(v)\overline{F}_Z(t-v) \\
&\qquad\qquad + \overline{F}_X(v)\overline{F}_Z(t-v)\overline{F}_Y(u)\overline{F}_Z(s-u)\Big]\mathrm{d}u\mathrm{d}v \\
&= \int_v \int_{u\ge v} \left[\int_{x=u}^{\infty} \overline{F}_X(x)\,dx \cdot \overline{F}_Y(v) - \int_{x=u}^{\infty} \overline{F}_Y(x)\,dx \cdot \overline{F}_X(v)\right] \\
&\qquad\qquad \times [f_Z(s-u)\overline{F}_Z(t-v) - f_Z(t-u)\overline{F}_Z(s-v)]\mathrm{d}u\mathrm{d}v,
\end{aligned}$$

where the second equality is obtained by integration of parts and by collection of terms. Since $X \le_{\rm mrl} Y$ it follows from (2.A.4) that the expression within the first set of brackets in the last integral is nonpositive. Since Z is IFR it can be verified that the quantity in the second pair of brackets in the last integral is also nonpositive. Therefore the integral is nonnegative. This proves (2.A.9). □

Theorem 2.A.9. *Let (X_i, Y_i), $i = 1, 2, \ldots, m$, be independent pairs of random variables such that $X_i \le_{\rm mrl} Y_i$, $i = 1, 2, \ldots, m$. If X_i, Y_i, $i = 1, 2, \ldots, m$, are all* IFR*, then*

$$\sum_{i=1}^m X_i \le_{\rm mrl} \sum_{i=1}^m Y_i.$$

Proof. Repeated application of (2.A.9), using the closure property of IFR under convolution, yields the desired result. □

Another interesting lemma is stated next. Recall that a random variable X is said to be (or to have) *decreasing mean residual life* (DMRL) if $m(t)$ is decreasing in t.

Lemma 2.A.10. *If the random variables X and Y are such that $X \le_{\rm hr} Y$ and if Z is a* DMRL *random variable independent of X and Y, then*

$$X + Z \le_{\rm mrl} Y + Z.$$

Proof. Integrating the identity in the proof of Lemma 1.B.3, we obtain that, for $s \le t$, one has

$$\begin{aligned}
&\int_{x=s}^{\infty} \overline{F}_{X+Z}(x)\mathrm{d}x \int_{y=t}^{\infty} \overline{F}_{Y+Z}(y)\mathrm{d}y - \int_{x=t}^{\infty} \overline{F}_{X+Z}(x)\mathrm{d}x \int_{y=s}^{\infty} \overline{F}_{Y+Z}(y)\mathrm{d}y \\
&= \int_{v}\int_{u\ge v} \Big[\overline{F}_X(u) f_Y(v) - f_X(v)\overline{F}_Y(u)\Big] \\
&\times \left[\int_{y=t}^{\infty} \overline{F}_Z(y-v)\mathrm{d}y \cdot \overline{F}_Z(s-u) - \int_{x=s}^{\infty} \overline{F}_Z(x-v)\mathrm{d}x \cdot \overline{F}_Z(t-u)\right]\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

The result now follows from the assumptions. □

It should be pointed out that a theorem such as Theorem 2.A.9 cannot be obtained from Lemma 2.A.10. The reason is that the inductive argument used to prove Theorem 2.A.9 does not have an analog based on Lemma 2.A.10.

Theorem 2.A.11. *Let X be a* DMRL *random variable, and let Z be a non-negative random variable independent of X. Then*

$$X \le_{\rm mrl} X + Z.$$

Proof. Let F_X, F_Z, and F_{X+Z} denote the distribution functions of the corresponding random variables, and let $\overline{F}_X$ and $\overline{F}_{X+Z}$ denote the corresponding survival functions. Then, for any $t \in \mathbb{R}$ we have

$$\begin{aligned}
\overline{F}_X(t)\int_t^{\infty} \overline{F}_{X+Z}(u)\mathrm{d}u &= \overline{F}_X(t)\int_t^{\infty}\int_0^{\infty} \overline{F}_X(u-z)\mathrm{d}F_Z(z)\mathrm{d}u \\
&= \overline{F}_X(t)\int_0^{\infty}\int_t^{\infty} \overline{F}_X(u-z)\mathrm{d}u\mathrm{d}F_Z(z) \\
&= \int_0^{\infty} \overline{F}_X(t)\int_{t-z}^{\infty} \overline{F}_X(u)\mathrm{d}u\mathrm{d}F_Z(z) \\
&\ge \int_0^{\infty} \overline{F}_X(t-z)\int_t^{\infty} \overline{F}_X(u)\mathrm{d}u\mathrm{d}F_Z(z) \\
&= \overline{F}_{X+Z}(t)\int_t^{\infty} \overline{F}_X(u)\mathrm{d}u,
\end{aligned}$$

where the inequality follows from the assumption that X is DMRL. The stated result now follows from (2.A.4). □

A mean residual life order comparison of random sums is given in the following result.

Theorem 2.A.12. *Let* $\{X_i,\ i = 1, 2, \dots\}$ *be a sequence of independent and identically distributed nonnegative* IFR *random variables. Let* M *and* N *be two discrete positive integer-valued random variables such that* $M \le_{\rm mrl} N$ *(in the sense of* (2.A.7)*), and assume that* M *and* N *are independent of the* X_i*'s. Then*

$$\sum_{i=1}^{M} X_i \le_{\rm mrl} \sum_{i=1}^{N} X_i.$$

The mean residual life order does not have the property of being simply closed under mixtures. However, under quite strong conditions the order $\le_{\rm mrl}$ is closed under mixtures. This is shown in the next theorem which may be compared with Theorem 1.B.8.

Theorem 2.A.13. *Let* X*,* Y*, and* Θ *be random variables such that* $[X \big| \Theta = \theta] \le_{\rm mrl} [Y \big| \Theta = \theta']$ *for all* θ *and* θ' *in the support of* Θ*. Then* $X \le_{\rm mrl} Y$*.*

Proof. The proof is similar to the proof of Theorem 1.B.8. Select a θ and a θ' in the support of Θ. Let $\overline{F}(\cdot\big|\theta)$, $\overline{G}(\cdot\big|\theta)$, $\overline{F}(\cdot\big|\theta')$, and $\overline{G}(\cdot\big|\theta')$ be the survival functions of $[X\big|\Theta = \theta]$, $[Y\big|\Theta = \theta]$, $[X\big|\Theta = \theta']$, and $[Y\big|\Theta = \theta']$, respectively. It is sufficient to show that for $\alpha \in (0, 1)$ we have

$$\frac{\alpha \int_t^\infty \overline{F}(u\big|\theta)\mathrm{d}u + (1-\alpha)\int_t^\infty \overline{F}(u\big|\theta')\mathrm{d}u}{\alpha\overline{F}(t\big|\theta) + (1-\alpha)\overline{F}(t\big|\theta')} \le \frac{\alpha \int_t^\infty \overline{G}(u\big|\theta)\mathrm{d}u + (1-\alpha)\int_t^\infty \overline{G}(u\big|\theta')\mathrm{d}u}{\alpha\overline{G}(t\big|\theta) + (1-\alpha)\overline{G}(t\big|\theta')} \quad \text{for all } t \ge 0.$$

The proof of this inequality is similar to the proof of (1.B.12). □

An analog of Theorem 1.B.12 exists for the order $\le_{\rm mrl}$. This is stated next.

Theorem 2.A.14. *Let* X *and* Y *be two nonnegative independent random variables. Then* $X \le_{\rm mrl} Y$ *if, and only if, for all functions* α *and* β *such that* β *is nonnegative and* α/β *and* β *are increasing, one has*

$$E[\alpha^*(X)]E[\beta^*(Y)] \le E[\alpha^*(Y)]E[\beta^*(X)],$$

provided the expectations exist, where

$$\alpha^*(x) = \int_0^x \alpha(u)\mathrm{d}u \quad \text{and} \quad \beta^*(x) = \int_0^x \beta(u)\mathrm{d}u.$$

In particular, if $X \le_{\rm mrl} Y$, then

$$\frac{E[Y^n]}{E[X^n]} \quad \text{is increasing in } n. \tag{2.A.10}$$

Consider now a family of distribution functions $\{G_\theta,\ \theta \in \mathcal{X}\}$ where $\mathcal{X}$ is a subset of the real line. As in Sections 1.A.3 and 1.C.3 let $X(\theta)$ denote a random variable with distribution function G_θ. For any random variable Θ with support in $\mathcal{X}$, and with distribution function F, let us denote by $X(\Theta)$ a random variable with distribution function H given by

$$H(y) = \int_{\mathcal{X}} G_\theta(y)\mathrm{d}F(\theta), \quad y \in \mathbb{R}.$$

The following result is comparable to Theorems 1.A.6, 1.B.14, 1.B.52, and 1.C.17.

Theorem 2.A.15. *Consider a family of distribution functions* $\{G_\theta,\ \theta \in \mathcal{X}\}$ *as above. Let* Θ_1 *and* Θ_2 *be two random variables with supports in* $\mathcal{X}$ *and distribution functions* F_1 *and* F_2*, respectively. Let* Y_1 *and* Y_2 *be two random variables such that* $Y_i =_{\mathrm{st}} X(\Theta_i)$*,* $i = 1, 2$*, that is, suppose that the distribution function of* Y_i *is given by*

$$H_i(y) = \int_{\mathcal{X}} G_\theta(y)\mathrm{d}F_i(\theta), \quad y \in \mathbb{R},\ i = 1, 2.$$

If

$$X(\theta) \le_{\mathrm{mrl}} X(\theta') \quad \textit{whenever } \theta \le \theta', \tag{2.A.11}$$

and if

$$\Theta_1 \le_{\mathrm{hr}} \Theta_2, \tag{2.A.12}$$

then

$$Y_1 \le_{\mathrm{mrl}} Y_2. \tag{2.A.13}$$

The proof of Theorem 2.A.15 uses the increasing convex order, and is therefore given in Remark 4.A.29 in Chapter 4.

A Laplace transform characterization of the order $\le_{\mathrm{mrl}}$ is given next; it may be compared to Theorems 1.A.13, 1.B.18, 1.B.53, and 1.C.25.

Theorem 2.A.16. *Let* X_1 *and* X_2 *be two nonnegative random variables, and let* $N_\lambda(X_1)$ *and* $N_\lambda(X_2)$ *be as described in Theorem* 1.A.13*. Then*

$$X_1 \le_{\mathrm{mrl}} X_2 \Longleftrightarrow N_\lambda(X_1) \le_{\mathrm{mrl}} N_\lambda(X_2) \quad \textit{for all } \lambda > 0,$$

where the notation $N_\lambda(X_1) \le_{\mathrm{mrl}} N_\lambda(X_2)$ *is in the sense of* (2.A.7)*.*

Proof. We use the notation of Theorem 1.A.13. Denote the distribution and survival functions of X_k by F_k and $\overline{F}_k$, $k = 1, 2$. For $k = 1, 2$, note that $\overline{\alpha}_\lambda^{X_k}(n)$ can be written as

$$\begin{aligned} \overline{\alpha}_\lambda^{X_k}(n) &= \int_0^\infty \sum_{i=n}^{\infty} \mathrm{e}^{-\lambda x}\frac{(\lambda x)^i}{i!}\mathrm{d}F_k(x) \\ &= \begin{cases} 1, & n = 0, \\ \int_0^\infty \lambda \mathrm{e}^{-\lambda x}\frac{(\lambda x)^{n-1}}{(n-1)!}\overline{F}_k(x)\mathrm{d}x, & n = 1, 2, \ldots. \end{cases} \end{aligned} \tag{2.A.14}$$

Therefore

$$P[X_k = n] = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^n}{n!} dF_k(x), \quad n = 0, 1, 2, \ldots. \tag{2.A.15}$$

From (2.A.15) it is seen that

$$E[N_\lambda(X_k)] = \lambda E[X_k], \quad k = 1, 2, \tag{2.A.16}$$

provided the expectations exist.

First assume that $X_1 \le_{\text{mrl}} X_2$. For the sake of this proof replace temporarily the notation $\overline{\alpha}_\lambda^{X_1}(n)$ and $\overline{\alpha}_\lambda^{X_2}(n)$, by $\overline{\alpha}_{\lambda,1}(n)$ and $\overline{\alpha}_{\lambda,2}(n)$, respectively. We also denote $E[X_1]$ and $E[X_2]$ by μ_1 and μ_2, respectively. The proof of $N_\lambda(X_1) \le_{\text{mrl}} N_\lambda(X_2)$ will consist of showing the following three inequalities:

$$\frac{\sum_{n=0}^\infty \overline{\alpha}_{\lambda,2}(n)}{\sum_{n=0}^\infty \overline{\alpha}_{\lambda,1}(n)} \le \frac{\sum_{n=1}^\infty \overline{\alpha}_{\lambda,2}(n)}{\sum_{n=1}^\infty \overline{\alpha}_{\lambda,1}(n)}, \tag{2.A.17}$$

$$\frac{\sum_{n=1}^\infty \overline{\alpha}_{\lambda,2}(n)}{\sum_{n=1}^\infty \overline{\alpha}_{\lambda,1}(n)} \le \frac{\sum_{n=2}^\infty \overline{\alpha}_{\lambda,2}(n)}{\sum_{n=2}^\infty \overline{\alpha}_{\lambda,1}(n)}, \tag{2.A.18}$$

and

$$\sum_{n=m}^\infty \overline{\alpha}_{\lambda,k}(n) \text{ is TP}_2 \text{ in } k \in \{1, 2\} \text{ and } m \ge 2. \tag{2.A.19}$$

In order to prove (2.A.17) note that from (2.A.16) it follows that

$$\sum_{n=0}^\infty \overline{\alpha}_{\lambda,k}(n) = 1 + \lambda\mu_k \quad k = 1, 2, \quad \text{and}$$
$$\sum_{n=1}^\infty \overline{\alpha}_{\lambda,k}(n) = \mu_k, \quad k = 1, 2. \tag{2.A.20}$$

But since $X_1 \le_{\text{mrl}} X_2$ implies that $\mu_1 \le \mu_2$ it follows that

$$\frac{1 + \lambda\mu_2}{1 + \lambda\mu_1} \le \frac{\lambda\mu_2}{\lambda\mu_1},$$

and (2.A.17) is obtained.

Next notice that (2.A.18) is equivalent to

$$\frac{\overline{\alpha}_{\lambda,2}(1)}{\overline{\alpha}_{\lambda,1}(1)} \le \frac{\sum_{n=1}^\infty \overline{\alpha}_{\lambda,2}(n)}{\sum_{n=1}^\infty \overline{\alpha}_{\lambda,1}(n)}. \tag{2.A.21}$$

Since $\sum_{n=1}^\infty \overline{\alpha}_{\lambda,k}(n) = \lambda\mu_k$, $k = 1, 2$, and

$$\overline{\alpha}_{\lambda,k}(1) = \int_0^\infty \lambda e^{-\lambda x} \overline{F}_k(x) dx = \lambda\Big[\mu_k - \int_0^\infty \lambda e^{-\lambda x} \int_x^\infty \overline{F}_k(u) du dx\Big],$$
$$k = 1, 2,$$

it follows that (2.A.21) is the same as

$$\mu_1 \int_0^\infty \lambda \mathrm{e}^{-\lambda x} \int_x^\infty \overline{F}_2(u)\mathrm{d}u\mathrm{d}x - \mu_2 \int_0^\infty \lambda \mathrm{e}^{-\lambda x} \int_x^\infty \overline{F}_1(u)\mathrm{d}u\mathrm{d}x \geq 0. \quad (2.A.22)$$

Rewriting the left-hand side of (2.A.22) we see that

$$\begin{aligned}
&\int_0^\infty \lambda \mathrm{e}^{-\lambda x} \Big[\mu_1 \int_x^\infty \overline{F}_2(u)\mathrm{d}u - \mu_2 \int_x^\infty \overline{F}_1(u)\mathrm{d}u\Big]\mathrm{d}x \\
&= \int_0^\infty \lambda \mathrm{e}^{-\lambda x} \left[\int_0^\infty \overline{F}_1(u)\mathrm{d}u \int_x^\infty \overline{F}_2(u)\mathrm{d}u - \int_0^\infty \overline{F}_2(u)\mathrm{d}u \int_x^\infty \overline{F}_1(u)\mathrm{d}u\right]\mathrm{d}x \\
&\geq 0,
\end{aligned}$$

where the inequality follows from the TP_2-ness of $\int_x^\infty \overline{F}_k(u)\mathrm{d}u$ in $k = 1, 2$, and $x \geq 0$ (see (2.A.3)). This proves (2.A.22), and hence (2.A.18).

Finally, in order to prove (2.A.19), notice, using a straightforward computation, that, for $m \geq 2$,

$$\sum_{n=m}^\infty \overline{\alpha}_{\lambda,k}(n) = \int_0^\infty \lambda^2 \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{m-2}}{(m-2)!} \int_x^\infty \overline{F}_k(u)\mathrm{d}u\mathrm{d}x. \quad (2.A.23)$$

By assumption, $\int_x^\infty \overline{F}_k(u)\mathrm{d}u$ is TP_2 in $k \in \{1, 2\}$ and $x \geq 0$. Furthermore, $\lambda^2 \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{m-2}}{(m-2)!}$ is TP_2 in $m \geq 2$ and $x \geq 0$. Thus, it follows that $\sum_{n=m}^\infty \overline{\alpha}_{\lambda,k}(n)$ is TP_2 in $k \in \{1, 2\}$ and $m \geq 2$, and this establishes (2.A.19).

Now suppose that $N_\lambda(X_1) \leq_{\mathrm{mrl}} N_\lambda(X_2)$ for all $\lambda > 0$. Then

$$\frac{\sum_{n=m}^\infty \overline{\alpha}_{\lambda,1}(n)}{\overline{\alpha}_{\lambda,1}(m)} \leq \frac{\sum_{n=m}^\infty \overline{\alpha}_{\lambda,2}(n)}{\overline{\alpha}_{\lambda,2}(m)}, \quad m = 0, 1, 2, \ldots.$$

For $m \geq 2$, by (2.A.23) and (2.A.14),

$$\frac{\int_0^\infty \lambda \mathrm{e}^{-\lambda u} \frac{(\lambda u)^{m-2}}{(m-2)!} \left[\int_u^\infty \overline{F}_1(x)\mathrm{d}x\right]\mathrm{d}u}{\int_0^\infty \lambda \mathrm{e}^{-\lambda u} \frac{(\lambda u)^{m-1}}{(m-1)!} \overline{F}_1(u)\mathrm{d}u} \leq \frac{\int_0^\infty \lambda \mathrm{e}^{-\lambda u} \frac{(\lambda u)^{m-2}}{(m-2)!} \left[\int_u^\infty \overline{F}_2(x)\mathrm{d}x\right]\mathrm{d}u}{\int_0^\infty \lambda \mathrm{e}^{-\lambda u} \frac{(\lambda u)^{m-1}}{(m-1)!} \overline{F}_2(u)\mathrm{d}u}. \quad (2.A.24)$$

For a fixed $y > 0$, define $\lambda = (m-1)/y$. Letting $m \to \infty$ ($\lambda \to \infty$), we have

$$\int_0^\infty \lambda \mathrm{e}^{-\lambda u} \frac{(\lambda u)^{m-2}}{(m-2)!} \left[\int_u^\infty \overline{F}_k(x)\mathrm{d}x\right]\mathrm{d}u \to \int_y^\infty \overline{F}_k(x)\mathrm{d}x,$$

and

$$\int_0^\infty \lambda \mathrm{e}^{-\lambda u} \frac{(\lambda u)^{m-1}}{(m-1)!} \overline{F}_k(u)\mathrm{d}u \to \overline{F}_k(y), \quad k = 1, 2,$$

as long as y is a continuity point of $\overline{F}_1(x)$ and $\overline{F}_2(x)$. For such y's, (2.A.24) gives us

$$\frac{\int_y^\infty \overline{F}_1(x)\mathrm{d}x}{\overline{F}_1(y)} \le \frac{\int_y^\infty \overline{F}_2(x)\mathrm{d}x}{\overline{F}_2(y)}.$$

It follows that $X_1 \le_{\mathrm{mrl}} X_2$ since the set of continuity points of $\overline{F}_1(x)$ and $\overline{F}_2(x)$ is dense in the set of positive real numbers. □

An analog of Theorem 1.B.21 is the following result.

Theorem 2.A.17. *Let X be a nonnegative* DMRL *random variable, and let $a \le 1$ be a positive constant. Then $aX \le_{\mathrm{mrl}} X$.*

Proof. It is easy to verify that the mean residual life function of aX is given by $am(\frac{t}{a})$, for all t, where m is the mean residual life function of X. Now

$$am(\frac{t}{a}) \le m(\frac{t}{a}) \le m(t) \quad \text{for all } t,$$

where the first inequality follows from $a \in [0,1]$ and the second inequality follows from the assumption that X is DMRL. The proof now follows from (2.A.2). □

In the next result it is shown that a random variable, whose distribution is the mixture of two distributions of mean residual life ordered random variables, is bounded from below and from above, in the mean residual life order sense, by these two random variables.

Theorem 2.A.18. *Let X and Y be two random variables with distribution functions F and G, respectively. Let W be a random variable with the distribution function $pF + (1-p)G$ for some $p \in (0,1)$. If $X \le_{\mathrm{mrl}} Y$, then $X \le_{\mathrm{mrl}} W \le_{\mathrm{mrl}} Y$.*

The proof of Theorem 2.A.18 is similar to the proof of Theorem 1.B.22, but it uses (2.A.3) instead of (1.B.3). We omit the details.

The following result is proven in Remark 4.A.25 of Section 4.A.3.

Theorem 2.A.19. *Let X and Y be two random variables. If $X \le_{\mathrm{mrl}} Y$, then $\phi(X) \le_{\mathrm{mrl}} \phi(Y)$ for every increasing convex function ϕ.*

Analogous to the result in Remark 1.A.18, it can be shown that the set of all distribution functions on $\mathbb{R}_+$ with finite means is a lattice with respect to the order $\le_{\mathrm{mrl}}$.

Let $X_1, X_2, \ldots, X_m$ be random variables, and let $X_{(k:m)}$ denote the corresponding kth order statistic, $k = 1, 2, \ldots, m$.

Theorem 2.A.20. *Let $X_1, X_2, \ldots, X_m$ be m independent random variables. If*

$$X_i \le_{\mathrm{mrl}} X_m, \quad i = 1, 2, \ldots, m-1,$$

then

$$X_{(m-1:m-1)} \le_{\mathrm{mrl}} X_{(m:m)}.$$

Let $X_1, X_2, \ldots, X_m$ be nonnegative random variables and let $U_{(i:m)} = X_{(i:m)} - X_{(i-1:m)}$ denote the corresponding spacings, $i = 1, 2, \ldots, m$ (where $U_{(1:m)} = X_{(1:m)}$). Similarly, let $Y_1, Y_2, \ldots, Y_n$ be nonnegative random variables and let $V_{(i:n)}$ denote the corresponding spacings, $i = 1, 2, \ldots, n$.

Theorem 2.A.21. *For positive integers m and n, let $X_1, X_2, \ldots, X_m$ be independent identically distributed nonnegative random variables, and let $Y_1, Y_2, \ldots, Y_n$ be other independent identically distributed nonnegative random variables. If $X_1 \le_{\text{mrl}} Y_1$, and if X_1 is* IMRL *and Y_1 is* DMRL*, then*

$$(m - j + 1)U_{(j:m)} \le_{\text{mrl}} (n - i + 1)V_{(i:n)} \quad \text{for } j \le m \text{ and } i \le n.$$

The following example may be compared to Examples 1.B.24, 1.C.48, 3.B.38, 4.B.14, 6.B.41, 6.D.8, 6.E.13, and 7.B.13.

Example 2.A.22. Let X and Y be two absolutely continuous nonnegative random variables with survival functions $\overline{F}$ and $\overline{G}$ and density functions f and g, respectively. Denote $\Lambda_1 = -\log \overline{F}$, $\Lambda_2 = -\log \overline{G}$, and $\lambda_i = \Lambda_i'$, $i = 1, 2$. Consider two nonhomogeneous Poisson processes $N_1 = \{N_1(t),\ t \ge 0\}$ and $N_2 = \{N_2(t),\ t \ge 0\}$ with mean functions Λ_1 and Λ_2 (see Example 1.B.13), respectively. Let $T_{i,1}, T_{i,2}, \ldots$ be the successive epoch times of process N_i, and let $X_{i,n} \equiv T_{i,n} - T_{i,n-1}$, $n \ge 1$ (where $T_{i,0} \equiv 0$), be the inter-epoch times of the process N_i, $i = 1, 2$. Note that $X =_{\text{st}} X_{1,1}$ and $Y =_{\text{st}} X_{2,1}$. It turns out that, under some conditions, the mean residual life ordering of the first two inter-epoch times implies the mean residual life ordering of all the corresponding later inter-epoch times. Explicitly, it will be shown below that if $X \le_{\text{mrl}} Y$, if X and Y are IMRL, and if (1.B.25) holds, then $X_{1,n} \le_{\text{mrl}} X_{2,n}$ for each $n \ge 1$.

For the purpose of this proof we denote $\overline{F}$ by $\overline{F}_1$ and $\overline{G}$ by $\overline{F}_2$. The stated result is obvious for $n = 1$. So let us fix $n \ge 2$. The survival function $\overline{G}_{i,n}$ of $X_{i,n}$, $i = 1, 2$, is given in (1.B.26). From (2.A.3) it is seen that the stated result is equivalent to

$$\int_t^\infty \overline{G}_{i,n}(x)\mathrm{d}x \quad \text{is TP}_2 \text{ in } (i, t);$$

that is, to

$$\int_{s=0}^\infty \lambda_i(s)\frac{\Lambda_i^{n-2}(s)}{(n-2)!}\int_{u=s+t}^\infty \overline{F}_i(u)\mathrm{d}u\mathrm{d}s \quad \text{is TP}_2 \text{ in } (i, t). \tag{2.A.25}$$

Now, from Example 1.B.24 we know that (1.B.25) implies that $\lambda_i(s)\frac{\Lambda_i^{n-2}(s)}{(n-2)!}$ is TP$_2$ in (i, s). The assumption $F_1 \le_{\text{mrl}} F_2$ means that

$$\int_{u=s+t}^\infty \overline{F}_i(u)\mathrm{d}u \quad \text{is TP}_2 \text{ in } (i, s) \text{ and in } (i, t).$$

Finally, the assumption that F_i is IMRL means that

$$\int_{u=s+t}^{\infty} \overline{F}_i(u)\mathrm{d}u \quad \text{is TP}_2 \text{ in } (s,t).$$

Thus (2.A.25) follows from Theorem 5.1 on page 123 of Karlin [275].

2.A.4 A property in reliability theory

The order $\leq_{\rm mrl}$ can be used to characterize DMRL random variables. As in Section 1.A.3, $[Z \big| A]$ denotes any random variable that has as its distribution the conditional distribution of Z given A.

Theorem 2.A.23. *The random variable X is* DMRL *if, and only if, any one of the following equivalent conditions holds:*

(i) $[X-t\big|X>t] \geq_{\rm mrl} [X-t'\big|X>t']$ *whenever* $t \leq t'$.
(ii) $X \geq_{\rm mrl} [X-t\big|X>t]$ *for all* $t \geq 0$ (*when X is a nonnegative random variable*).
(iii) $X+t \leq_{\rm mrl} X+t'$ *whenever* $t \leq t'$.

The proofs of all these statements are trivial and are thus omitted.

Other characterizations of DMRL and IMRL random variables, by means of other stochastic orders, can be found in Theorems 2.B.17, 3.A.56, 3.C.13, and 4.A.51.

A multivariate extension of parts (i) and (ii) of Theorem 2.A.23 is given in Section 6.F.3.

An interesting application of part (iii) of Theorem 2.A.23 is the following corollary. Its proof consists of a combination of Theorem 2.A.23(iii) with Lemma 2.A.8 (or, alternatively, a combination of Theorem 2.A.23(iii), Theorem 1.B.38(iii), and Lemma 2.A.10).

Corollary 2.A.24. *Let X be a* DMRL *random variable and let Y be an* IFR *random variable. If X and Y are independent, then $X+Y$ is* DMRL.

2.B The Harmonic Mean Residual Life Order

2.B.1 Definition

Let X and Y be two nonnegative random variables with mrl functions m and l, respectively, and suppose that the harmonic averages of m and l are comparable as follows:

$$\left[\frac{1}{x}\int_0^x \frac{1}{m(u)}\mathrm{d}u\right]^{-1} \leq \left[\frac{1}{x}\int_0^x \frac{1}{l(u)}\mathrm{d}u\right]^{-1} \quad \text{for all } x>0. \tag{2.B.1}$$

Then X is said to be *smaller than Y in the harmonic mean residual life order* (denoted as $X \leq_{\rm hmrl} Y$).

Notice that

$$\frac{1}{m(u)} = \frac{\overline{F}(u)}{\int_u^\infty \overline{F}(v)\mathrm{d}v} = -\frac{\mathrm{d}}{\mathrm{d}u}\log\Big(\int_u^\infty \overline{F}(v)\mathrm{d}v\Big).$$

Therefore

$$\int_0^x \frac{1}{m(u)}\mathrm{d}u = \log\left(\frac{EX}{\int_x^\infty \overline{F}(u)\mathrm{d}u}\right).$$

Similarly

$$\int_0^x \frac{1}{l(u)}\mathrm{d}u = \log\left(\frac{EY}{\int_x^\infty \overline{G}(u)\mathrm{d}u}\right).$$

Thus it is seen that (2.B.1) holds if, and only if,

$$\frac{\int_x^\infty \overline{F}(u)\mathrm{d}u}{EX} \le \frac{\int_x^\infty \overline{G}(u)\mathrm{d}u}{EY} \quad \text{for all } x \ge 0. \tag{2.B.2}$$

For discrete random variables that take on values in $\mathbb{N}_+$ the definition of $\le_{\mathrm{hmrl}}$ should be modified. Let X and Y be two such random variables. We denote $X \le_{\mathrm{hmrl}} Y$ if

$$\frac{\sum_{j=n}^\infty P\{X \ge j\}}{E[X]} \le \frac{\sum_{j=n}^\infty P\{Y \ge j\}}{E[Y]}, \quad n = 1, 2, \ldots. \tag{2.B.3}$$

2.B.2 The relation between the harmonic mean residual life and some other stochastic orders

Since the harmonic averages of m and l are increasing functionals of m and l, respectively, it follows that

$$X \le_{\mathrm{mrl}} Y \Longrightarrow X \le_{\mathrm{hmrl}} Y.$$

The order $\le_{\mathrm{hmrl}}$ is closely related to the order $\le_{\mathrm{icx}}$ which is studied in Section 4.A. The reader may find it helpful to browse over that section now, since some of the ideas that are explained there are used below.

Note that both (2.B.2) and (2.B.3) are equivalent to

$$\frac{E[(X-t)_+]}{E[X]} \le \frac{E[(Y-t)_+]}{E[Y]} \quad \text{for all } t \ge 0, \tag{2.B.4}$$

and from (2.B.4) it follows that $X \le_{\mathrm{hmrl}} Y$ if, and only if,

$$\frac{E[\phi(X)]}{E[X]} \le \frac{E[\phi(Y)]}{E[Y]} \quad \text{for all increasing convex functions } \phi : [0,\infty) \to \mathbb{R}, \tag{2.B.5}$$

such that the expectations exist. It is worthwhile to note that condition (2.B.4) uses the expectations $E[(X-t)_+]$ and $E[(Y-t)_+]$ as (2.A.5) and as (3.A.5) in Chapter 3 and (4.A.4) in Chapter 4 do. In Chapter 4, where the order $\leq_{\text{icx}}$ is studied, we will use (2.B.4) in order to derive a relationship between the orders $\leq_{\text{hmrl}}$ and $\leq_{\text{icx}}$ (see Theorem 4.A.28).

Neither of the orders $\leq_{\text{st}}$ and $\leq_{\text{hmrl}}$ implies the other; counterexamples can be found in the literature.

Letting $x \to 0$ in (2.B.1) we obtain $m(0) \leq l(0)$, that is,

$$X \leq_{\text{hmrl}} Y \Longrightarrow E[X \big| X > 0] \leq E[Y \big| Y > 0].$$

Thus, when X and Y are positive almost surely, then

$$X \leq_{\text{hmrl}} Y \Longrightarrow EX \leq EY. \tag{2.B.6}$$

If $EX = EY$, then

$$X \leq_{\text{hmrl}} Y \Longleftrightarrow X \leq_{\text{cx}} Y, \tag{2.B.7}$$

where the order $\leq_{\text{cx}}$ is studied in Section 3.A (see (3.A.7)). Thus, from (3.A.4) it follows that if $X \leq_{\text{hmrl}} Y$ and $EX = EY$, then $\text{Var}[X] \leq \text{Var}[Y]$. Under the proper condition, even if X and Y do not have the same mean, one can still get the variance inequality; this is shown in the next result.

Theorem 2.B.1. *Let X and Y be two almost surely positive random variables with finite second moments. If $X \leq_{\text{hmrl}} Y$, and if Y is* NWUE, *then* $\text{Var}[X] \leq \text{Var}[Y]$.

Proof. From (2.B.5) we get

$$\frac{E[X^2]}{E[X]} \leq \frac{E[Y^2]}{E[Y]}. \tag{2.B.8}$$

From Barlow and Proschan [36, page 187] it is seen that $\text{Var}[Y] \geq \{E[Y]\}^2$, since Y is NWUE. Thus, using (2.B.6), we see that $\text{Var}[Y] \geq E[Y]E[X]$. Therefore

$$\begin{aligned}
\text{Var}[Y] &\geq \frac{E[X]}{E[Y]}\text{Var}[Y] + \{E[Y] - E[X]\}E[X] \\
&= \frac{E[X]}{E[Y]} \cdot E[Y^2] - \{E[X]\}^2 \\
&\geq E[X^2] - \{E[X]\}^2 \\
&= \text{Var}[X],
\end{aligned}$$

where the last inequality follows from (2.B.8). □

The harmonic mean residual life order can be characterized by means of the usual stochastic order and the appropriate equilibrium age variables. Recall from (1.A.20) that for nonnegative random variables X and Y with finite means we denote by A_X and A_Y the corresponding asymptotic equilibrium ages. The following result follows at once from (1.A.1) and (2.B.2). It may be contrasted with Theorems 1.C.13 and 2.A.4.

Theorem 2.B.2. *For nonnegative random variables* X *and* Y *with finite means we have* $X \le_{\text{hmrl}} Y$ *if, and only if,* $A_X \le_{\text{st}} A_Y$.

In the next theorem the order $\le_{\text{hmrl}}$ is characterized by ordering two related random variables in the sense of the usual stochastic order. Let X and Y be two nonnegative random variables with finite means and suppose that $X \le_{\text{st}} Y$ and that $EX < EY$. Let F and G be the distribution functions of X and of Y, respectively. Define the random variable $Z_{X,Y}$ as the random variable that has the density function h given by (1.C.7), as in Theorem 1.C.14; see also Theorem 2.A.5.

Theorem 2.B.3. *Let* X *and* Y *be two nonnegative random variables with finite means such that* $X \le_{\text{st}} Y$ *and such that* $EY > EX > 0$. *Then*

$$X \le_{\text{hmrl}} Y \Longleftrightarrow A_Y \le_{\text{st}} Z_{X,Y} \Longleftrightarrow A_X \le_{\text{st}} Z_{X,Y},$$

where $Z_{X,Y}$ *has the density function given in* (1.C.7).

Proof. It is easy to see that (here $\overline{H}$ is the survival function of Z, $\overline{G}_{\text{e}}$ is the survival function of A_Y, and $\overline{F}_{\text{e}}$ is as in (1.A.20))

$$\overline{H}(x) - \overline{G}_{\text{e}}(x) = \frac{EX}{EY - EX}\Big[\overline{G}_{\text{e}}(x) - \overline{F}_{\text{e}}(x)\Big], \quad x \ge 0.$$

Thus the first stated equivalence follows from Theorem 2.B.2. The proof of the second equivalence is similar. □

The order $\le_{\text{hmrl}}$ can characterize the order $\le_{\text{mrl}}$ as follows.

Theorem 2.B.4. *Let* X *and* Y *be two nonnegative random variables with finite means. Then* $X \le_{\text{mrl}} Y$ *if, and only if,* $[X - t \big| X > t] \le_{\text{hmrl}} [Y - t \big| Y > t]$ *for all* $t \ge 0$.

The proof of Theorem 2.B.4 consists of applying (2.B.2) to $[X - t \big| X > t]$ and $[Y - t \big| Y > t]$, for each $t \ge 0$, and then showing that the resulting inequality is equivalent to (2.A.3). We omit the details.

2.B.3 Some closure properties

Under the proper conditions, the order $\le_{\text{hmrl}}$ is closed under the operation of convolution. First we prove the following lemma. Recall that a nonnegative random variable X with a finite mean is called NBUE (new better than used in expectation) if $E[X - t \big| X > t] \le E[X]$ for all $t > 0$. Note that a nonnegative NBUE random variable must be almost surely positive.

Lemma 2.B.5. *If the two almost surely positive random variables* X *and* Y *are such that* $X \le_{\text{hmrl}} Y$, *and if* Z *is an* NBUE *nonnegative random variable independent of* X *and* Y, *then*

$$X + Z \le_{\text{hmrl}} Y + Z.$$

Proof. Let F, G, and H [$\overline{F}$, $\overline{G}$, and $\overline{H}$] be the distribution [survival] functions corresponding to X, Y, and Z, respectively. The corresponding equilibrium age distribution [survival] functions will be denoted by F_{e}, G_{e}, and H_{e} [$\overline{F}_{\mathrm{e}}$, $\overline{G}_{\mathrm{e}}$, and $\overline{H}_{\mathrm{e}}$]. Let A_X, A_Y, A_Z, A_{X+Z}, and A_{Y+Z} denote the asymptotic equilibrium ages corresponding to X, Y, Z, $X+Z$, and $Y+Z$, respectively. Now compute

$$\begin{aligned}
P\{A_{X+Z}>t\} &= \frac{1}{E[X+Z]}\int_{v=t}^{\infty} P\{X+Z>v\}\mathrm{d}v \\
&= \frac{1}{EX+EZ}\int_{v=t}^{\infty}\int_{u=0}^{\infty}\overline{F}(v-u)\mathrm{d}H(u)\mathrm{d}v \\
&= \frac{1}{EX+EZ}\int_{u=0}^{\infty}\int_{v=t}^{\infty}\overline{F}(v-u)\mathrm{d}v\mathrm{d}H(u) \\
&= \frac{1}{EX+EZ}\int_{u=0}^{\infty}\int_{v=t-u}^{\infty}\overline{F}(v)\mathrm{d}v\mathrm{d}H(u) \\
&= \frac{1}{EX+EZ}\Bigg[\int_{u=0}^{t}\int_{v=t-u}^{\infty}\overline{F}(v)\mathrm{d}v\mathrm{d}H(u) \\
&\qquad + \int_{u=t}^{\infty}\int_{v=0}^{\infty}\overline{F}(v)\mathrm{d}v\mathrm{d}H(u) + \int_{u=t}^{\infty}\int_{v=t-u}^{0}\mathrm{d}v\mathrm{d}H(u)\Bigg] \\
&= \frac{1}{EX+EZ}\Bigg[EX\int_{0}^{t}\overline{F}_{\mathrm{e}}(t-u)\mathrm{d}H(u) \\
&\qquad\qquad + EX\cdot\overline{H}(t) + \int_{t}^{\infty}\overline{H}(u)\mathrm{d}u\Bigg] \\
&= \frac{1}{EX+EZ}\big[EX\cdot P\{A_X+Z>t\} + EZ\cdot\overline{H}_{\mathrm{e}}(t)\big],
\end{aligned}$$

where A_X and Z are taken to be independent in the above expression. Now, since Z is NBUE we have that $Z \geq_{\mathrm{st}} A_Z$. Therefore

$$P\{A_X+Z>t\} \geq P\{A_X+A_Z>t\} \geq P\{A_Z>t\} = \overline{H}_{\mathrm{e}}(t). \tag{2.B.9}$$

Now notice that

$$\begin{aligned}
P\{A_{X+Z}>t\} &= \frac{1}{EX+EZ}\big[EX\cdot P\{A_X+Z>t\} + EZ\cdot\overline{H}_{\mathrm{e}}(t)\big] \\
&\leq \frac{1}{EY+EZ}\big[EY\cdot P\{A_X+Z>t\} + EZ\cdot\overline{H}_{\mathrm{e}}(t)\big] \\
&\leq \frac{1}{EY+EZ}\big[EY\cdot P\{A_Y+Z>t\} + EZ\cdot\overline{H}_{\mathrm{e}}(t)\big] \\
&= P\{A_{Y+Z}>t\}
\end{aligned}$$

(A_Y and Z are taken to be independent in the above), where the first inequality follows from (2.B.6) and (2.B.9), and the second inequality follows from Theorem 2.B.2. The result now follows from Theorem 2.B.2. □

Repeated application of Lemma 2.B.5, using the closure property of NBUE under convolution, and noting that every NBUE random variable is almost surely positive, yields the following result.

Theorem 2.B.6. *Let* (X_i, Y_i), $i = 1, 2, \ldots, m$, *be independent pairs of nonnegative random variables such that* $X_i \le_{\text{hmrl}} Y_i$, $i = 1, 2, \ldots, m$. *If* X_i, Y_i, $i = 1, 2, \ldots, m$, *are all* NBUE, *then*

$$\sum_{i=1}^{m} X_i \le_{\text{hmrl}} \sum_{i=1}^{m} Y_i.$$

Using Theorem 2.B.6 we can prove the following result.

Theorem 2.B.7. *Let* $X_1, X_2, \ldots$ *and* $Y_1, Y_2, \ldots$ *each be a sequence of* NBUE *nonnegative independent and identically distributed random variables such that* $X_i \le_{\text{hmrl}} Y_i$, $i = 1, 2, \ldots$. *Let* M *and* N *be integer-valued positive random variables that are independent of the* $\{X_i\}$ *and the* $\{Y_i\}$ *sequences, respectively, such that* $M \le_{\text{hmrl}} N$. *Then*

$$\sum_{j=1}^{M} X_j \le_{\text{hmrl}} \sum_{j=1}^{N} Y_j.$$

Proof. The proof here is similar to the proof of Theorem 4.A.9. The reader may wish to look at that proof before continuing to read the present proof.

From Theorem 2.B.6 and (2.B.4) it is seen that $\frac{1}{mE[X_1]} E\big[\big(\sum_{i=1}^{m} X_i - u\big)_+\big] \le \frac{1}{mE[Y_1]} E\big[\big(\sum_{i=1}^{m} Y_i - u\big)_+\big]$ (all the X_i's have the same mean, and also all the Y_i's have the same mean). Therefore

$$\frac{E\big[\big(\sum_{i=1}^{m} X_i - u\big)_+\big]}{E[X_1]} \le \frac{E\big[\big(\sum_{i=1}^{m} Y_i - u\big)_+\big]}{E[Y_1]} \quad \text{for all } u \ge 0,\ m = 1, 2, \ldots.$$

Thus

$$\begin{aligned}
\frac{E\big[\big(\sum_{i=1}^{M} X_i - u\big)_+\big]}{E\big[\sum_{i=1}^{M} X_i\big]} &= \frac{\sum_{m=1}^{\infty} E\big[\big(\sum_{i=1}^{m} X_i - u\big)_+\big] P\{M = m\}}{E[M]E[X_1]} \\
&\le \frac{\sum_{m=1}^{\infty} E\big[\big(\sum_{i=1}^{m} Y_i - u\big)_+\big] P\{M = m\}}{E[M]E[Y_1]} \\
&= \frac{E\big[\big(\sum_{i=1}^{M} Y_i - u\big)_+\big]}{E\big[\sum_{i=1}^{M} Y_i\big]}.
\end{aligned}$$

Therefore (again by (2.B.4)) we have

$$\sum_{i=1}^{M} X_i \le_{\text{hmrl}} \sum_{i=1}^{M} Y_i. \tag{2.B.10}$$

Now let ϕ be an increasing convex function and denote $g(n) \equiv E[\phi(Y_1 + Y_2 + \cdots + Y_n)]$. In the proof of Theorem 4.A.9 it is shown that $g(n)$ is increasing and convex in n. Therefore, since $M \le_{\mathrm{hmrl}} N$, we have that $\frac{E\left[\phi\left(\sum_{i=1}^{M} Y_i\right)\right]}{E[M]} \le \frac{E\left[\phi\left(\sum_{i=1}^{N} Y_i\right)\right]}{E[N]}$, and since the Y_i's have the same mean we have that

$$\frac{E\left[\phi\left(\sum_{i=1}^{M} Y_i\right)\right]}{E\left[\sum_{i=1}^{M} Y_i\right]} = \frac{E\left[\phi\left(\sum_{i=1}^{M} Y_i\right)\right]}{E[M]E[Y_1]} \le \frac{E\left[\phi\left(\sum_{i=1}^{N} Y_i\right)\right]}{E[N]E[Y_1]} = \frac{E\left[\phi\left(\sum_{i=1}^{N} Y_i\right)\right]}{E\left[\sum_{i=1}^{N} Y_i\right]}.$$

Thus we have that

$$\sum_{i=1}^{M} Y_i \le_{\mathrm{hmrl}} \sum_{i=1}^{N} Y_i. \tag{2.B.11}$$

The inequalities (2.B.10) and (2.B.11) yield the stated result. □

A result that is related to Theorem 2.B.7 is given next. It is of interest to compare it to Theorem 1.A.5.

Theorem 2.B.8. *Let* $\{X_j,\ j = 1, 2, \ldots\}$ *be a sequence of nonnegative independent and identically distributed* NBUE *random variables, and let* M *be a positive integer-valued random variable which is independent of the* X_i*'s. Let* $\{Y_j,\ j = 1, 2, \ldots\}$ *be another sequence of nonnegative independent and identically distributed* NBUE *random variables, and let* N *be a positive integer-valued random variable which is independent of the* Y_i*'s. Suppose that for some positive integer* K *we have*

$$\sum_{i=1}^{K} X_i \le_{\mathrm{hmrl}} [\ge_{\mathrm{hmrl}}]\ Y_1,$$

and

$$M \le_{\mathrm{hmrl}} [\ge_{\mathrm{hmrl}}]\ KN.$$

Then

$$\sum_{j=1}^{M} X_j \le_{\mathrm{hmrl}} [\ge_{\mathrm{hmrl}}]\ \sum_{j=1}^{N} Y_j.$$

We do not give a detailed proof of Theorem 2.B.8 here since it is similar to the proof of Theorem 4.A.12 in Section 4.A.1. In order to construct a proof of Theorem 2.B.8 from the proof of Theorem 4.A.12 one just uses the equivalence (2.B.7) and one replaces the application of Theorem 4.A.9 by an application of Theorem 2.B.7.

Two other similar theorems are the following. Their proofs are similar to the proofs of Theorems 4.A.13 and 4.A.14 in Section 4.A.1.

Theorem 2.B.9. *Let* $\{X_j,\ j = 1, 2, \ldots\}$ *be a sequence of nonnegative independent and identically distributed* NBUE *random variables, and let* M

be a positive integer-valued random variable which is independent of the X_i's. Let $\{Y_j,\ j = 1, 2, \dots\}$ be another sequence of nonnegative independent and identically distributed NBUE *random variables, and let N be a positive integer-valued random variable which is independent of the Y_i's. Also, let $\{N_j,\ j = 1, 2, \dots\}$ be a sequence of independent random variables that are distributed as N. If for some positive integer K we have*

$$\sum_{i=1}^{K} X_i \le_{\text{hmrl}} Y_1 \quad \textit{and} \quad M \le_{\text{hmrl}} \sum_{i=1}^{K} N_i,$$

or if we have

$$KX_1 \le_{\text{hmrl}} Y_1 \quad \textit{and} \quad M \le_{\text{hmrl}} KN,$$

or if we have

$$KX_1 \le_{\text{hmrl}} Y_1 \quad \textit{and} \quad M \le_{\text{hmrl}} \sum_{i=1}^{K} N_i,$$

then

$$\sum_{j=1}^{M} X_j \le_{\text{hmrl}} \sum_{j=1}^{N} Y_j.$$

Theorem 2.B.10. *Let $\{X_j,\ j = 1, 2, \dots\}$ be a sequence of nonnegative independent and identically distributed* NBUE *random variables, and let M be a positive integer-valued random variable which is independent of the X_i's. Let $\{Y_j,\ j = 1, 2, \dots\}$ be another sequence of nonnegative independent and identically distributed* NBUE *random variables, and let N be a positive integer-valued random variable which is independent of the Y_i's. If for some positive integers K_1 and K_2, such that $K_1 \le K_2$, we have*

$$\sum_{i=1}^{K_1} X_i \le_{\text{hmrl}} \frac{K_1}{K_2} Y_1 \quad \textit{and} \quad M \le_{\text{hmrl}} K_2 N,$$

then

$$\sum_{j=1}^{M} X_j \le_{\text{hmrl}} \sum_{j=1}^{N} Y_j.$$

The harmonic mean residual life order does not have the property of being simply closed under mixtures. However, under quite strong conditions the order $\le_{\text{hmrl}}$ is closed under mixtures. This is shown in the next theorem which may be compared with Theorems 1.B.8 and 2.A.13.

Theorem 2.B.11. *Let X and Y be nonnegative random variables, and let Θ be another random variable, such that $[X \big| \Theta = \theta] \le_{\text{hmrl}} [Y \big| \Theta = \theta']$ for all θ and θ' in the support of Θ. Then $X \le_{\text{hmrl}} Y$.*

Proof. The proof is similar to the proof of Theorem 1.B.8. Select a θ and a θ' in the support of Θ. Let $\overline{F}(\cdot|\theta)$, $\overline{G}(\cdot|\theta)$, $\overline{F}(\cdot|\theta')$, and $\overline{G}(\cdot|\theta')$ be the survival functions of $[X|\Theta=\theta]$, $[Y|\Theta=\theta]$, $[X|\Theta=\theta']$, and $[Y|\Theta=\theta']$, respectively. Let $E[X|\theta]$, $E[Y|\theta]$, $E[X|\theta']$, and $E[Y|\theta']$ be the corresponding expectations. By (2.B.2) it is sufficient to show that for $\alpha \in (0,1)$ we have

$$\frac{\alpha\int_t^\infty \overline{F}(u|\theta)\mathrm{d}u + (1-\alpha)\int_t^\infty \overline{F}(u|\theta')\mathrm{d}u}{\alpha E[X|\theta] + (1-\alpha)E[X|\theta']} \le \frac{\alpha\int_t^\infty \overline{G}(u|\theta)\mathrm{d}u + (1-\alpha)\int_t^\infty \overline{G}(u|\theta')\mathrm{d}u}{\alpha E[Y|\theta] + (1-\alpha)E[Y|\theta']} \quad \text{for all } t \ge 0. \tag{2.B.12}$$

The proof of this inequality is similar to the proof of (1.B.12). □

Another condition under which the order $\le_{\text{hmrl}}$ is closed under mixtures is given in the following theorem.

Theorem 2.B.12. *Let X and Y be nonnegative random variables, and let Θ be another random variable, such that $[X|\Theta=\theta] \le_{\text{hmrl}} [Y|\Theta=\theta]$ for all θ in the support of Θ. Furthermore, assume that*

$$\frac{E[Y|\Theta=\theta]}{E[X|\Theta=\theta]} = k \quad (\textit{independent of } \theta). \tag{2.B.13}$$

Then $X \le_{\text{hmrl}} Y$.

Proof. As in the proof of Theorem 2.B.11, select a θ and a θ' in the support of Θ. Let $\overline{F}(\cdot|\theta)$, $\overline{G}(\cdot|\theta)$, $\overline{F}(\cdot|\theta')$, and $\overline{G}(\cdot|\theta')$ be the survival functions of $[X|\Theta=\theta]$, $[Y|\Theta=\theta]$, $[X|\Theta=\theta']$, and $[Y|\Theta=\theta']$, respectively. Let $E[X|\theta]$, $E[Y|\theta]$, $E[X|\theta']$, and $E[Y|\theta']$ be the corresponding expectations.

Let $\alpha \in (0,1)$. Note that from (2.B.13) we obtain

$$\frac{\alpha E[Y|\theta] + (1-\alpha)E[Y|\theta']}{\alpha E[X|\theta] + (1-\alpha)E[X|\theta']} = k. \tag{2.B.14}$$

Also, from $[X|\Theta=\theta] \le_{\text{hmrl}} [Y|\Theta=\theta]$, $[X|\Theta=\theta'] \le_{\text{hmrl}} [Y|\Theta=\theta']$, and (2.B.13), we get, for $t \ge 0$, that

$$k\int_t^\infty \overline{F}(u|\theta)\mathrm{d}u \le \int_t^\infty \overline{G}(u|\theta)\mathrm{d}u \quad \text{and} \quad k\int_t^\infty \overline{F}(u|\theta')\mathrm{d}u \le \int_t^\infty \overline{G}(u|\theta')\mathrm{d}u,$$

and hence

$$k\Big[\alpha\int_t^\infty \overline{F}(u|\theta)\mathrm{d}u + (1-\alpha)\int_t^\infty \overline{F}(u|\theta')\mathrm{d}u\Big] \le \alpha\int_t^\infty \overline{G}(u|\theta)\mathrm{d}u + (1-\alpha)\int_t^\infty \overline{G}(u|\theta')\mathrm{d}u.$$

From this inequality and (2.B.14) we obtain (2.B.12), and this completes the proof. □

Consider now a family of distribution functions $\{G_\theta,\ \theta \in \mathcal{X}\}$ where $\mathcal{X}$ is a subset of the real line. As in Sections 1.A.3 and 1.C.3 let $X(\theta)$ denote a random variable with distribution function G_θ. For any random variable Θ with support in $\mathcal{X}$, and with distribution function F, let us denote by $X(\Theta)$ a random variable with distribution function H given by

$$H(y) = \int_{\mathcal{X}} G_\theta(y)\mathrm{d}F(\theta), \quad y \in \mathbb{R}.$$

The following result is comparable to Theorems 1.A.6, 1.B.14, 1.B.52, 1.C.17 and 2.A.15.

Theorem 2.B.13. *Consider a family of distribution functions $\{G_\theta,\ \theta \in \mathcal{X}\}$ as above. Let Θ_1 and Θ_2 be two random variables with supports in $\mathcal{X}$ and distribution functions F_1 and F_2, respectively. Let Y_1 and Y_2 be two random variables such that $Y_i =_{\mathrm{st}} X(\Theta_i)$, $i = 1, 2$, that is, suppose that the distribution function of Y_i is given by*

$$H_i(y) = \int_{\mathcal{X}} G_\theta(y)\mathrm{d}F_i(\theta), \quad y \in \mathbb{R},\ i = 1, 2.$$

If

$$X(\theta) \le_{\mathrm{hmrl}} X(\theta') \quad \textit{whenever } \theta \le \theta', \tag{2.B.15}$$

and if

$$\Theta_1 \le_{\mathrm{hr}} \Theta_2, \tag{2.B.16}$$

then

$$Y_1 \le_{\mathrm{hmrl}} Y_2. \tag{2.B.17}$$

The proof of Theorem 2.B.13 uses the increasing convex order, and is therefore given in Remark 4.A.29 in Chapter 4.

A Laplace transform characterization of the order $\le_{\mathrm{hmrl}}$ is given next; it may be compared to Theorems 1.A.13, 1.B.18, 1.B.53, 1.C.25, and 2.A.16.

Theorem 2.B.14. *Let X_1 and X_2 be two nonnegative random variables, and let $N_\lambda(X_1)$ and $N_\lambda(X_2)$ be as described in Theorem* 1.A.13. *Then*

$$X_1 \le_{\mathrm{hmrl}} X_2 \Longleftrightarrow N_\lambda(X_1) \le_{\mathrm{hmrl}} N_\lambda(X_2) \quad \textit{for all } \lambda > 0,$$

where the notation $N_\lambda(X_1) \le_{\mathrm{hmrl}} N_\lambda(X_2)$ is in the sense of (2.B.3).

Proof. First assume that $X \le_{\mathrm{hmrl}} Y$. As in the proof of Theorem 2.A.16 we temporarily replace the notation $\overline{\alpha}_\lambda^{X_1}(n)$ and $\overline{\alpha}_\lambda^{X_2}(n)$, by $\overline{\alpha}_{\lambda,1}(n)$ and $\overline{\alpha}_{\lambda,2}(n)$, respectively. We also denote the survival function and the mean of X_k by $\overline{F}_k$ and μ_k, respectively, $k = 1, 2$. Let $m \ge 2$. Using (2.A.23) we have

$$\mu_1 \sum_{n=m}^{\infty} \overline{P}_2(n) - \mu_2 \sum_{n=m}^{\infty} \overline{P}_1(n)$$
$$= \int_0^\infty \lambda^2 \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{m-2}}{(m-2)!} \Big[\mu_1 \int_x^\infty \overline{F}_2(u)\mathrm{d}u - \mu_2 \int_x^\infty \overline{F}_1(u)\mathrm{d}u\Big]\mathrm{d}x.$$

The integrand is nonnegative by the assumption of the theorem, and one direction of the proof is complete.

The proof of the converse statement is similar to the proof of the converse of Theorem 2.A.16. □

The following result gives necessary and sufficient conditions for two random variables to be equal in the sense of the order $\leq_{\rm hmrl}$.

Theorem 2.B.15. *Let X and Y be two nonnegative random variables with positive expectations, such that $EX \leq EY$. Then $X =_{\rm hmrl} Y$ if, and only if, $X =_{\rm st} BY$ for some Bernoulli random variable B, independent of Y.*

Proof. First assume that $X =_{\rm st} BY$ for some Bernoulli random variable B, independent of Y. Then

$$\frac{E[(X-t)_+]}{E[X]} = \frac{E[(BY-t)_+]}{E[BY]} = \frac{E[(Y-t)_+]P\{B=1\}}{E[Y]P\{B=1\}} = \frac{E[(Y-t)_+]}{E[Y]} \quad \text{for all } t \geq 0,$$

and thus $X =_{\rm hmrl} Y$ follows from (2.B.4).

Conversely, suppose that $X =_{\rm hmrl} Y$. By (2.B.2) this means that

$$\frac{\int_t^\infty P\{X > u\}\mathrm{d}u}{EX} = \frac{\int_t^\infty P\{Y > u\}\mathrm{d}u}{EY} \quad \text{for all } t \geq 0,$$

which yields

$$P\{X > t\} = \frac{EX}{EY} \cdot P\{Y > t\}, \quad t \geq 0.$$

That is, $X =_{\rm st} BY$, where B is a Bernoulli random variable such that $P\{B = 1\} = EX/EY$. □

From the proof of Theorem 2.B.15 it is seen, in contrast to (2.B.6), that if $X \leq_{\rm hmrl} Y$, then it does not necessarily follow that $EX \leq EY$ (unless X and Y are positive almost surely).

In the next result it is shown that a random variable, whose distribution is the mixture of two distributions of harmonic mean residual life ordered random variables, is bounded from below and from above, in the harmonic mean residual life order sense, by these two random variables.

Theorem 2.B.16. *Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. Let W be a random variable with the distribution function $pF + (1-p)G$ for some $p \in (0,1)$. If $X \leq_{\rm hmrl} Y$, then $X \leq_{\rm hmrl} W \leq_{\rm hmrl} Y$.*

Proof. By assumption, (2.B.2) holds. Therefore

$$\frac{\int_x^\infty \overline{F}(u)\mathrm{d}u}{EX} \le \frac{p\int_x^\infty \overline{F}(u)\mathrm{d}u + (1-p)\int_x^\infty \overline{G}(u)\mathrm{d}u}{pEX+(1-p)EY} \le \frac{\int_x^\infty \overline{G}(u)\mathrm{d}u}{EY} \quad \text{for all } x \ge 0,$$

and the stated result follows from (2.B.2). □

2.B.4 Properties in reliability theory

The order $\le_{\text{hmrl}}$ can be used to characterize DMRL random variables. As in Section 1.A.3, $[Z\big|A]$ denotes any random variable that has as its distribution the conditional distribution of Z given A.

Theorem 2.B.17. *The nonnegative random variable X is* DMRL *if, and only if,* $[X-t\big|X>t] \ge_{\text{hmrl}} [X-t'\big|X>t']$ *whenever* $t' \ge t \ge 0$.

The proof is simple and thus omitted.

Other characterizations of DMRL and IMRL random variables, by means of other stochastic orders, can be found in Theorems 2.A.23, 3.A.56, 3.C.13, and 4.A.51.

The order $\le_{\text{hmrl}}$ can also be used to characterize NBUE random variables as follows.

Theorem 2.B.18. *Let X be a nonnegative random variable with a finite positive mean. Then the following assertions are equivalent:*

(i) $X \le_{\text{hmrl}} X+Y$ *for any nonnegative random variable Y with a finite positive mean, which is independent of X.*
(ii) X *is* NBUE.
(iii) $X+Y_1 \le_{\text{hmrl}} X+Y_2$ *whenever Y_1 and Y_2 are almost surely positive random variables with finite means, which are independent of X, such that* $Y_1 \le_{\text{hmrl}} Y_2$.

Proof. Suppose that (i) holds. Then, taking $Y =_{\text{a.s.}} y$ for some $y>0$, we get from (2.B.4) that

$$\frac{E[(X-t)_+]}{E[X]} \le \frac{E[(X+y-t)_+]}{E[X]+y}, \quad t \ge 0.$$

Upon rearrangement this gives

$$yE[(X-t)_+] \le E[X]\big\{E[(X+y-t)_+] - E[(X-t)_+]\big\}, \quad t \ge 0;$$

that is,

$$E[(X-t)_+] \le \frac{E[X]}{y}\int_{t-y}^{t} P\{X>u\}\mathrm{d}u, \quad t \ge 0.$$

Letting $y \to 0$ we obtain

$$E[(X-t)_+] \le E[X]P\{X>t\}, \quad t \ge 0,$$

that is, X is NBUE.

The statement (ii)$\Longrightarrow$(iii) is Lemma 2.B.5.

Now assume that (iii) holds. Let $Y_1 =_{\rm a.s} a$ and $Y_2 =_{\rm a.s} y$, where $0 < a < y$. It is easy to verify (for instance, using (2.B.4)) that $Y_1 \le_{\rm hmrl} Y_2$. That is,

$$(E[X] + y)E[(X + a - t)_+] \le (E[X] + a)E[(X + y - t)_+], \quad t \ge 0.$$

Letting $a \to 0$ we obtain

$$(E[X] + y)E[(X - t)_+] \le E[X]E[(X + y - t)_+], \quad t \ge 0,\ y \ge 0.$$

Integrating both sides of the above inequality with respect to the distribution of Y (Y is any random variable as described in (i)) we obtain

$$(E[X] + E[Y])E[(X - t)_+] \le E[X]E[(X + Y - t)_+], \quad t \ge 0,$$

that is, by (2.B.4), we have $X \le_{\rm hmrl} X + Y$. $\square$

Another characterization of NBUE random variables by means of the usual stochastic order is given in Theorem 1.A.31.

2.C Complements

Section 2.A: Basic properties of the mrl function (which is also called the *biometric function*) can be found in Yang [572] and references therein. Some properties of the mrl functions are summarized in Shaked and Shanthikumar [513], where further references can be found. The counterexamples mentioned after Theorem 2.A.1 can also be found in that paper and further counterexamples can be found in Gupta and Kirmani [216] and in Alzaid [12]. The conditions under which the $\le_{\rm mrl}$ order implies the $\le_{\rm hr}$ and the $\le_{\rm st}$ orders (Theorems 2.A.2 and 2.A.3) are taken from Gupta and Kirmani [216]. The equivalence of the order $\le_{\rm mrl}$ and (2.A.3) can be found, for example, in Singh [536]. The characterization of the order $\le_{\rm mrl}$ which is given in Theorem 2.A.5 is taken from Di Crescenzo [164]. The characterizations of the order $\le_{\rm hr}$ by means of the order $\le_{\rm mrl}$, given in Theorems 2.A.6 and 2.A.7, can be found in Belzunce, Gao, Hu, and Pellerey [67]. The closure under convolution results of the order $\le_{\rm mrl}$ in Section 2.A.3 were communicated to us by Pellerey [444]. A special case of Lemma 2.A.8 can be found in Mukherjee and Chatterjee [403]. Theorem 2.A.9 can be found in Pellerey [448] and Theorem 2.A.12 can be found in Fagiuoli and Pellerey [186]. The fact that a DMRL random variable increases in the order $\le_{\rm mrl}$ when a nonnegative random variable is added to it (Theorem 2.A.11) is a result that is slightly stronger than a result in Frostig [207]. The closure under mixtures result (Theorem 2.A.13) is taken from Nanda, Jain, and Singh [424]. The characterization of the mrl order that is given in Theorem 2.A.14 can be found in Joag-Dev, Kochar, and Proschan [259], whereas its special case given in (2.A.10) is taken from Fagiuoli and Pellerey [187]. Fagiuoli and Pellerey [187] have extended (2.A.10) to sums of mrl ordered random variables. The closure under mixtures property of the order $\leq_{\rm mrl}$ (Theorem 2.A.15) is a special case of a result of Hu, Kundu, and Nanda [236], and it can also be found in Hu, Nanda, Xie, and Zhu [237]; see also Theorem 3.4 in Ahmed [7]. The Laplace transform characterization of the order $\leq_{\rm mrl}$ (Theorem 2.A.16) is taken from Shaked and Wong [524]; see also Kan and Yi [274]. An extension of Theorem 2.A.16 to more general orders can be found in Nanda [422]. The mean residual life order comparisons of order statistics (Theorems 2.A.20 and 2.A.21) can be found in Hu, Zhu, and Wei [243] and in Hu and Wei [240]. The comparison of inter-epoch times of two nonhomogeneous Poisson processes in the sense of the mean residual life order (Example 2.A.22) is taken from Belzunce, Lillo, Ruiz, and Shaked [69]. The result that a convolution of an IFR and a DMRL random variables is DMRL (Corollary 2.A.24) can be found in Kopocinska and Kopocinski [320].

Nanda, Singh, Misra, and Paul [429] studied a notion of reversed residual lifetime, and introduced and studied a stochastic order based on it.

An order which is related to the mean residual life order is introduced in Ebrahimi and Zahedi [179]. If m and l are the mrl functions of X and Y, respectively, then the order is defined by requiring $\frac{\rm d}{{\rm d}t}(l(t)-m(t))$ to be monotone in t. Ebrahimi and Zahedi [179] show that this order implies the mean residual life order.

In Kirmani [297] it is claimed that the spacings, from a sample of independent and identically distributed IMRL random variables, are ordered in the mean residual life order. However, the proof of Kirmani is erroneous; see Kirmani [298].

Section 2.B: The order $\leq_{\rm hmrl}$ is studied, for example, in Deshpande, Singh, Bagai, and Jain [161] and in Heilmann and Schröter [219]. Baccelli and Makowski [28] call it the *forward recurrence times stochastic order* (see an additional comment on the paper of Baccelli and Makowski [28] in Section 4.C). The counterexamples mentioned after (2.B.5) can be found, for example, in Mi [394]. In fact, Gerchak and Golani [209] have noticed that the example given on page 489 of Wolff [567] shows that it is possible for both $X \leq_{\rm st} Y$ and $Y \leq_{\rm hmrl} X$ to hold simultaneously in the strict sense. The comparison of the expectations of $\leq_{\rm hmrl}$ ordered random variables, described in (2.B.6), is a special case of a result of Nanda, Jain, and Singh [425]. The variance inequality (Theorem 2.B.1) can be found in Kirmani [297]. The characterization of the order $\leq_{\rm hmrl}$ which is given in Theorem 2.B.3 is taken from Di Crescenzo [164]. The characterization of the order $\leq_{\rm mrl}$ by means of the order $\leq_{\rm hmrl}$ (Theorem 2.B.4) can be found in Hu, Kundu, and Nanda [236]. The preservation under convolution property of the order $\leq_{\text{hmrl}}$ (Theorem 2.B.6) is taken from Pellerey [448, 449] (the latter is a correction note), and the closure under random summations property of the order $\leq_{\text{hmrl}}$ (Theorem 2.B.7) is also taken from Pellerey [448, 449], though it is alluded to in Heilmann and Schröter [219]. These results (Theorems 2.B.6 and 2.B.7) can also be found in Baccelli and Makowski [28]. A slight extension of Theorem 2.B.6 is given in Lefèvre and Utev [340]. Theorems 2.B.8–2.B.10 have been communicated to us by Pellerey [447]. The closure under mixtures properties of the order $\leq_{\text{hmrl}}$ (Theorems 2.B.11 and 2.B.12) are taken from Nanda, Jain, and Singh [424] and from Lefèvre and Utev [340], respectively, whereas Theorem 2.B.13 is inspired by Ahmed, Soliman, and Khider [9]. The Laplace transform characterization of the order $\leq_{\text{hmrl}}$ (Theorem 2.B.14) is taken from Shaked and Wong [524]. An extension of Theorem 2.B.14 to more general orders can be found in Nanda [422]. The conditions under which $X =_{\text{hmrl}} Y$ (Theorem 2.B.15) can be found in Lefèvre and Utev [340]. The NBUE characterization, given in Theorem 2.B.18, is taken from Lefèvre and Utev [340].