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A Modern Course in Aeroelasticity

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von
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Chapter 2
Static Aeroelasticity

Abstract The basics of static aeroelasticty, in contrast to dynamic aeroelasticity, are reviewed and some classic subjects such as divergence and control surface reversal are treated. The discussion starts with simple mathematical and physical models and progresses to more complex models and solution methods. Most of these models and methods prove to be useful in dynamic aeroelasticity as well.

2.1 Typical Section Model of an Airfoil

We shall find a simple, somewhat contrived, physical system useful for introducing several aeroelastic problems. This is the so-called 'typical section' which is a popular pedagogical device.[1] This simplified aeroelastic system consists of a rigid, flat, plate airfoil mounted on a torsional spring attached to a wind tunnel wall. See Fig. 2.1; the airflow over the airfoil is from left to right.

The principal interest in this model for the aeroelastician is the rotation of the plate (and consequent twisting of the spring), α, as a function of airspeed. If the spring were very stiff or airspeed were very slow, the rotation would be rather small; however, for flexible springs or high flow velocities the rotation may twist the spring beyond its ultimate strength and lead to structural failure. A typical plot of elastic twist, α_e, versus airspeed, U, is given in Fig. 2.2. The airspeed at which the elastic twist increases rapidly to the point of failure is called the 'divergence airspeed', U_D. A major aim of any theoretical model is to accurately predict U_D. It should be emphasized that the above curve is representative not only of our typical section model but also of real aircraft wings. Indeed the primary difference is not in the basic physical phenomenon of divergence, but rather in the elaborateness of the theoretical analysis required to predict accurately U_D for an aircraft wing versus that required for our simple typical section model.

To determine U_D theoretically we proceed as follows. The equation of static equilibrium simply states that the sum of aerodynamic plus elastic moments about any point on the airfoil is zero. By convention, we take the point about which moments

[1] See Chap. 6, BA, especially pp. 189–200.

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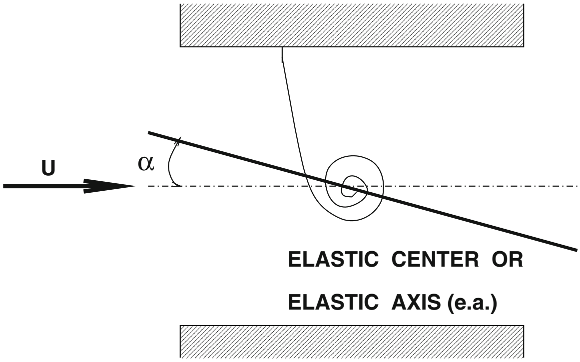

Fig. 2.1 Geometry of typical section airfoil

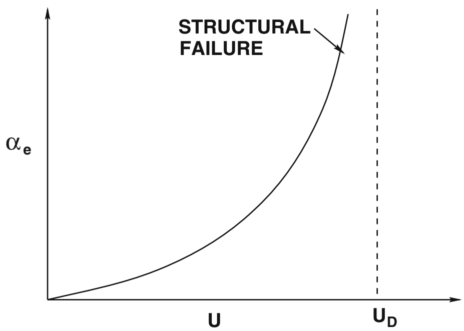

Fig. 2.2 Elastic twist versus airspeed

are summed as the point of spring attachment, the so-called 'elastic center' or 'elastic axis' of the airfoil.

The total aerodynamic angle of attack, α, is taken as the sum of some initial angle of attack, α_0 (with the spring untwisted), plus an additional increment due to elastic twist of the spring, α_e.

$$\alpha = \alpha_0 + \alpha_e \tag{2.1.1}$$

In addition, we define a point on the airfoil known as the 'aerodynamic center'.[2] This is the point on the airfoil about which the aerodynamic moment is independent of angle of attack, α. Thus, we may write the moment about the elastic axis as

$$M_y = M_{AC} + Le \tag{2.1.2}$$

where

[2] For two dimensional, incompressible flow this is at the airfoil quarter-chord; for supersonic flow it moves back to the half-chord. See Ashley and Landahl [1]. References are given at the end of each chapter.

M_y moment about elastic axis or center
M_{AC} moment about aerodynamic center, both moments are positive nose up
L lift, net vertical force positive up
e distance from aerodynamic center to elastic axis, positive aft.

From aerodynamic theory [1] (or experiment plus dimensional analysis) one has

$$L = C_L q S \tag{2.1.3a}$$

$$M_{AC} = C_{MAC} q S_c$$

where

$$C_L = C_{L_0} + \frac{\partial C_L}{\partial \alpha}\alpha, \quad \text{lift coefficient} \tag{2.1.3b}$$

$C_{MAC} = C_{MAC_0}$, a constant, aerodynamic center moment coefficient in which

$$q = \frac{\rho U^2}{2}, \text{ dynamic pressure and}$$

ρ air density
U air velocity
c airfoil chord
l airfoil span
S airfoil area, c × 1

(2.1.3a) defines C_L and C_{MAC} (2.1.3b) is a Taylor Series expansion of C_L for small α. C_{L_0} is the lift coefficient at $\alpha \equiv 0$. From (2.1.2), (2.1.3a) and (2.1.3b), we see the moment is also expanded in a Taylor series. The above forms are traditional in the aerodynamic literature. They are not necessarily those a nonaerodynamicist would choose.

Note that C_{L_0}, $\partial C_L/\partial\alpha$, C_{MAC_0} are nondimensional functions of airfoil shape, planform and Mach number. For a flat plate in two-dimensional incompressible flow [1]

$$\frac{\partial C_L}{\partial \alpha} = 2\pi, \qquad C_{MAC_0} = 0 = C_{L_0}$$

In what follows, we shall take $C_{L_0} \equiv 0$ for convenience and without any essential loss of information.

From (2.1.2), (2.1.3a) and (2.1.3b)

$$M_y = eqS\left[\frac{\partial C_L}{\partial \alpha}(\alpha_0 + \alpha_e)\right] + qScC_{MAC_0} \tag{2.1.4}$$

Now consider the elastic moment. If the spring has linear moment-twist characteristics then the elastic moment (positive nose up) is $-K_\alpha \alpha_e$ where K_α is the elastic spring constant and has units of moment (torque) per angle of twist. Hence, summing moments we have

$$eqS\left[\frac{\partial C_L}{\partial \alpha}(\alpha_0 + \alpha_e)\right] + qScC_{MAC_0} - K_\alpha \alpha_e = 0 \tag{2.1.5}$$

which is the equation of static equilibrium for our 'typical section' airfoil.

Solving for the elastic twist (assuming $C_{MAC_0} = 0$ for simplicity) one obtains

$$\alpha_e = \frac{qS}{K_\alpha} \frac{e\frac{\partial C_L}{\partial \alpha}\alpha_o}{1 - q\frac{Se}{K_\alpha}\frac{\partial C_L}{\partial \alpha}} \tag{2.1.6}$$

This solution has several interesting properties. Perhaps most important is the fact that at a particular dynamic pressure the elastic twist becomes infinitely large. This is, when the denominator of the right-hand side of (2.1.6) vanishes

$$1 - q\frac{Se}{K_\alpha}\frac{\partial C_L}{\partial \alpha} = 0 \tag{2.1.7}$$

at which point $\alpha_e \to \infty$.

Equation (2.1.7) represents what is termed the 'divergence condition' and the corresponding dynamic pressure which may be obtained by solving (2.1.7) is termed the 'divergence dynamic pressure',

$$q_D \equiv \frac{K_\alpha}{Se(\partial C_L/\partial \alpha)} \tag{2.1.8}$$

Since only the positive dynamic pressures are physically meaningful, note that only for $e > 0$ will divergence occur, i.e., when the aerodynamic center is ahead of the elastic axis. Using (2.1.6), (2.1.8) may be rewritten in a more concise form as

$$\alpha_e = \frac{(q/q_D)\alpha_0}{1 - q/q_D} \tag{2.1.9}$$

Of course, the elastic twist does not become infinitely large for any real airfoil; because this would require an infinitely large aerodynamic moment. Moreover, the linear relation between the elastic twist and the aerodynamic moment would be violated long before that. However, the elastic twist can become so large as to cause structural failure. For this reason, all aircraft are designed to fly below the divergence dynamic pressure of all airfoil or lifting surfaces, e.g., wings, fins, control surfaces. Now let us examine equations (2.1.5) and (2.1.9) for additional insight into our problem, again assuming $C_{MAC_0} = 0$ for simplicity. Two special cases will be informative. First, consider $\alpha_0 \equiv 0$. Then (2.1.5) may be written

$$\alpha_e \left[qS \frac{\partial C_L}{\partial \alpha} e - K_\alpha \right] = 0 \tag{2.1.5a}$$

Excluding the trivial case $\alpha_e = 0$ we conclude from (2.1.5a) that

$$qS \frac{\partial C_L}{\partial \alpha} e - K_\alpha = 0 \tag{2.1.7a}$$

which is the 'divergence condition'. This will be recognized as an eigenvalue problem, the vanishing of the coefficient of α_e in (2.1.5a) being the condition for nontrivial solutions of the unknown, α_e.[3] Hence, 'divergence' requires only a consideration of elastic deformations.

Secondly, let us consider another special case of a somewhat different type, $\alpha_0 \neq 0$, but $\alpha_e \ll \alpha_0$. Then (2.1.5) may be written approximately as

$$eqS \frac{\partial C_L}{\partial \alpha} \alpha_0 - K_\alpha \alpha_e = 0 \tag{2.1.10}$$

Solving

$$\alpha_e = \frac{qSe(\partial C_L / \partial \alpha) \alpha_0}{K_\alpha} \tag{2.1.11}$$

Note this solution agrees with (2.1.6) if the denominator of (2.1.6) can be approximated by

$$1 - q \frac{Se}{K_\alpha} \frac{\partial C_L}{\partial \alpha} = 1 - \frac{q}{q_D} \approx 1$$

Hence, this approximation is equivalent to assuming that the dynamic pressure is much smaller than its divergencevalue. Note that the term neglected in (2.1.5) is the aerodynamic moment due to the elastic twist. This term can be usefully thought of as the 'aeroelastic feedback'.[4] Without this term, solution (2.1.11) is valid only when $q/q_D \ll 1$; and it cannot predict divergence. A feedback diagram of Eq. (2.1.5) is given in Fig. 2.3. Thus, when the forward loop gain, G, exceeds unity, $G \equiv qeS(\partial C_L/\partial \alpha)/K_\alpha > 1$, the system is statically unstable, see Eq. (2.1.8). Hence, aeroelasticity can also be thought of as the study of aerodynamic + elastic

[3] Here in static aeroelasticity q plays the role of the eigenvalue; in dynamic aeroelasticity q will be a parameter and the (complex) frequency will be the eigenvalue. This is a source of confusion for some students when they first study the subject.

[4] For the reader with some knowledge of feedback theory as in, for example, Savant [2].

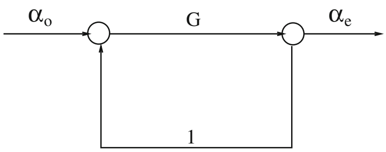

Fig. 2.3 Feedback representation of aeroelastic divergence

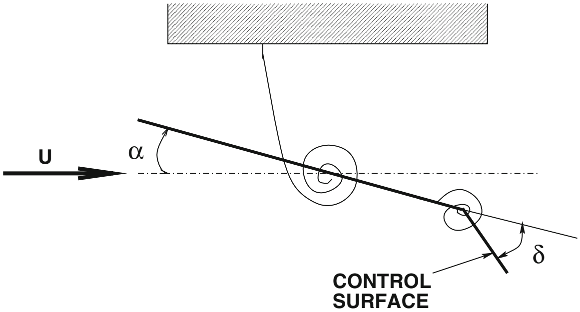

Fig. 2.4 Typical section with control surface

feedback systems. One might also note the similarity of this divergence problem to conventional 'buckling' of structures.[5] Having exhausted the interpretations of this problem, we will quickly pass on to some slightly more complicated problems, but whose physical content is similar.

2.1.1 Typical Section Model with Control Surface

We shall add a control surface to our typical section of Fig. 2.1, as indicated in Fig. 2.4. For simplicity, we take $\alpha_0 = C_{MAC_0} = 0$; hence, $\alpha = \alpha_e$. The aerodynamic lift is given by

$$L = qSC_L = qS\left(\frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial \delta}\delta\right) \quad \text{positive up} \tag{2.1.12}$$

and the moment by

$$M_{AC} = qScC_{MAC} = qSc\frac{\partial C_{MAC}}{\partial \delta}\delta \quad \text{positive nose up} \tag{2.1.13}$$

and the moment about the hinge line of the control surface by

---

[5] Timoshenko and Gere [3].

$$H = qS_H c_H C_H = qS_H c_H \left(\frac{\partial C_H}{\partial \alpha}\alpha + \frac{\partial C_H}{\partial \delta}\delta \right) \quad \text{positive tail down} \tag{2.1.14}$$

where S_H is the area of control surface, c_H the chord of the control surface and C_H the (nondimensional) aerodynamic hinge moment coefficient. As before, $\frac{\partial C_L}{\partial \alpha}$, $\frac{\partial C_L}{\partial \delta}$, $\frac{\partial C_{MAC}}{\partial \delta}$, $\frac{\partial C_H}{\partial \alpha}$, $\frac{\partial C_H}{\partial \delta}$ are aerodynamic constants which vary with Mach and airfoil geometry. Note $\frac{\partial C_H}{\partial \delta}$ is typically negative.

The basic purpose of a control surface is to change the lift (or moment) on the main lifting surface. It is interesting to examine aeroelastic effects on this lift.

To write the equations of equilibrium, we need the elastic moments about the elastic axis of the main lifting surface and about the hinge line of the control surface. These are $-K_\alpha \alpha$ (positive nose up), $-K_\delta(\delta - \delta_0)$ (positive tail down), and $\delta_e \equiv \delta - \delta_0$, where δ_e is the elastic twist of control surface in which δ_0 is the difference between the angle of zero aerodynamic control deflection and zero twist of the control surface spring.

The two equations of static moment equilibrium are

$$eqS \left(\frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial \delta}\delta \right) + qSc\frac{\partial C_{MAC}}{\partial \delta}\delta - K_\alpha \alpha = 0 \tag{2.1.15}$$

$$qS_H c_H \left(\frac{\partial C_H}{\partial \alpha}\alpha + \frac{\partial C_H}{\partial \alpha}\delta \right) - K_\delta(\delta - \delta_0) = 0 \tag{2.1.16}$$

The above are two algebraic equations in two unknowns, α and δ, which can be solved by standard methods. For example, Cramer's rule gives

$$\alpha = \frac{\begin{vmatrix} 0 & eqS\frac{\partial C_L}{\partial \delta} + qSc\frac{\partial C_{MAC}}{\partial \delta} \\ -K_\delta \delta_0 & qS_H c_H \frac{\partial C_H}{\partial \delta} - K_\delta \end{vmatrix}}{\begin{vmatrix} eqS\frac{\partial C_L}{\partial \delta} - K_\alpha & eqS\frac{\partial C_L}{\partial \delta} + qSc\frac{\partial C_{MAC}}{\partial \delta} \\ qS_H c_H \frac{\partial C_H}{\partial \alpha} & qs_h C_H \frac{\partial C_H}{\partial \delta} - K_\delta \end{vmatrix}} \tag{2.1.17}$$

and a similar equation for δ. To consider divergence we again set the denominator to zero. This gives a quadratic equation in the dynamic pressure q. Hence, there are two values of divergence dynamic pressure. Only the lower positive value of the two is physically significant.

In addition to the somewhat more complicated form of the divergence condition, there is a *new physical phenomenon* associated with the control surface called 'control surface reversal'. If the two springs were rigid, i.e., $K_\alpha \to \infty$ and $K_\delta \to \infty$, then $\alpha = 0$, $\delta = \delta_0$, and

$$L_r = qS\frac{\partial C_L}{\partial \delta}\delta_0 \tag{2.1.18}$$

With flexible springs, however,

$$L = qS\left(\frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial \delta}\delta\right) \tag{2.1.19}$$

where α, δ are determined by solving the equilibrium equations (2.1.15), and (2.1.16). In general, the latter value of the lift will be smaller than the rigid value of lift. Indeed, the lift may actually become zero or even negative due to aeroelastic effects. Such an occurrence is called 'control surface reversal'. To simplify matters and show the essential character of control surface reversal, we will assume $K_\delta \to \infty$ and hence, $\delta \to \delta_0$ from the equilibrium condition (2.1.16). Solving the equilibrium Eq. (2.1.15), we obtain

$$\alpha = \delta_0 \frac{\frac{\partial C_L}{\partial \delta} + \frac{c}{e}\frac{\partial C_{MAC}}{\partial \delta}}{\frac{K_\alpha}{qSe} - \frac{\partial C_L}{\partial \delta}} \tag{2.1.20}$$

But

$$\begin{aligned} L &= qS\left(\frac{\partial C_L}{\partial \delta}\delta_0 + \frac{\partial C_L}{\partial \alpha}\alpha\right) \\ &= qS\left(\frac{\partial C_L}{\partial \delta} + \frac{\partial C_L}{\partial \alpha}\frac{\alpha}{\delta_0}\right)\delta_0 \end{aligned} \tag{2.1.21}$$

so that, introducing (2.1.20) into (2.1.21) and normalizing by L_r, we obtain

$$\frac{L}{L_r} = \frac{1 + q\frac{Sc}{K_\alpha}\frac{\partial C_{MAC}}{\partial \delta}\left(\frac{\partial C_L}{\partial \alpha}/\frac{\partial C_L}{\partial \delta}\right)}{1 - q\frac{Se}{K_\alpha}\frac{\partial C_L}{\partial \alpha}} \tag{2.1.22}$$

Control surface reversal occurs when $L/L_r = 0$

$$1 + q_R\frac{Sc}{K_\alpha}\frac{\partial C_{MAC}}{\partial \delta}\left(\frac{\partial C_L}{\partial \alpha}/\frac{\partial C_L}{\partial \delta}\right) = 0 \tag{2.1.23}$$

where q_R is the dynamic pressure at reversal, or

$$q_R \equiv \frac{\frac{-K_\alpha}{Sc}\left(\frac{\partial C_L}{\partial \delta}/\frac{\partial C_L}{\partial \alpha}\right)}{\frac{\partial C_{MAC}}{\partial \delta}} \tag{2.1.24}$$

Typically, $\partial C_{MAC}/\partial \delta$ is negative, i.e., the aerodynamic moment for positive control surface rotation is nose down. Finally, (2.1.22) may be written

$$\frac{L}{L_r} = \frac{1 - q/q_r}{1 - q/q_D} \tag{2.1.25}$$

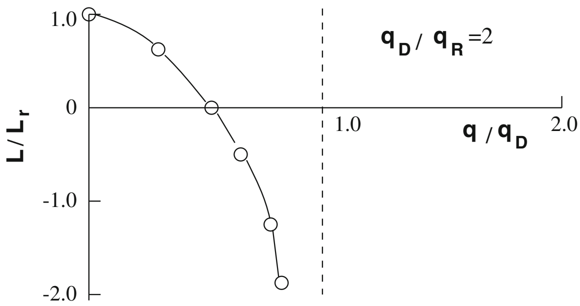

Fig. 2.5 Lift versus dynamic pressure

where q_R is given by (2.1.24) and q_D by (2.1.8). It is very interesting to note that when K_δ is finite, the reversal dynamic pressure is still given by (2.1.24). However, q_D is now the lowest root of the denominator of (2.1.17). Can you reason physically why this is so?[6]

A graphical depiction of (2.1.25) is given in the Fig. 2.5 where the two cases, $q_D > q_R$ and $q_D < q_R$, are distinguished. In the former case L/L_r, decreases with increasing q and in the latter the opposite is true. Although the graphs are shown for $q > q_D$, our analysis is no longer valid when the divergence condition is exceeded without taking into account nonlinear effects. It is interesting to note that the q_R given by (2.1.24) is still the correct answer even for finite K_δ. Consider (2.1.15). For reversal or zero lift, $L = 0$, (2.1.15) simplifies to

$$q_R S s \frac{\partial C_{MAC}}{\partial \delta} \delta - K_\alpha \alpha = 0 \tag{2.1.15\_R}$$

and (2.1.12) becomes

$$\frac{\partial C_L}{\partial \alpha} \alpha + \frac{\partial C_L}{\partial \alpha} \delta = 0 \tag{2.1.12\_R}$$

Eliminating α, δ from these two equations (or setting the determinant to zero for nontrivial solutions) gives

$$K_\alpha \frac{\partial C_L}{\partial \delta} + \frac{\partial C_L}{\partial \alpha} q_R S c \frac{\partial C_{MAC}}{\partial \delta} = 0 \tag{2.1.26}$$

Solving (2.1.26) for q_R gives (2.1.24). Note that by this approach an eigenvalue problem has been created. Also note the moment equilibrium about the control surface hinge line does not enter into this calculation. See Appendix B, Chap. 2 for a more conceptually straightforward, but algebraically more tedious approach.

At the generalized reversal condition, when $\alpha_0 \neq 0$, $C_{MAC_0} \neq 0$, the lift due to a *change* in δ is zero, by definition. In mathematical language,

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[6] See, [3], pp. 197–200.

$$\frac{dL}{d\delta} = 0 \text{ at } q = q_R \tag{2.1.27}$$

To see how this generalized definition relates to our earlier definition of the reversal condition, consider again the equation for lift and also the equation for overall moment equilibrium of the main wing plus control surface, viz.

$$L = qS\left[\frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial \delta}\delta\right] \tag{2.1.19}$$

and

$$qScC_{MAC_0} + qSc\frac{\partial C_{MAC}}{\partial \delta}\delta + eqS\left[\frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial \delta}\delta\right] - K_\alpha(\alpha - \alpha_0) = 0 \tag{2.1.28}$$

From (2.1.19)

$$\frac{dL}{d\delta} = qS\left[\frac{\partial C_L}{\partial \alpha}\frac{d\alpha}{d\delta} + \frac{\partial C_L}{\partial \delta}\right] \tag{2.1.29}$$

where $\frac{d\alpha}{d\delta}$ may be calculated from (2.1.29) as

$$\frac{d\alpha}{d\delta} = \frac{-\left[qSc\frac{\partial C_{MAC}}{\partial \delta} + qSe\frac{\partial C_L}{\partial \delta}\right]}{eqS\frac{\partial C_L}{\partial \delta} - K_\alpha} \tag{2.1.30}$$

Note that neither C_{MAC_0} nor α_0 appear in (2.1.30). Moreover when (2.1.30) is substituted into (2.1.29) and $dL/d\delta$ is set to zero, the same expression for q_R is obtained as before, (2.1.24), when reversal was defined as $L = 0$ (for $\alpha_0 = C_{MAC_0} = 0$).

This result may be given a further physical interpretation. Consider a Taylor series expansion for L in terms of δ about the reference condition, $\delta \equiv 0$. Note that $\delta \equiv 0$ corresponds to a wing without any control surface deflection relative to the main wing. Hence the condition $\delta \equiv 0$, may be thought of as a wing *without* any control surface.

The lift at any δ may then be expressed as

$$L(\delta) = L(\delta = 0) + \left.\frac{\partial L}{\partial \delta}\right|_{\delta=0}\delta + \ldots \tag{2.1.31}$$

Because a linear model is used, it is clear that higher order terms in this expansion vanish. Moreover, it is clear that $dL/d\delta$ is that same for any δ, cf. (2.1.29) and (2.1.30).

Now consider $L(\delta = 0)$. From (2.1.19)

$$L(\delta = 0) = qS\frac{\partial C_L}{\partial \alpha}\alpha(\delta = 0) \tag{2.1.32}$$

But from (2.1.28)

$$\alpha(\delta = 0) = \frac{K_\alpha \alpha_0 + qSC_{MAC_0}}{K_\alpha - eqS\frac{\partial C_L}{\partial \alpha}} \tag{2.1.33}$$

Note that $\alpha(\delta = 0) = 0$ for $\alpha_0 = C_{MAC_0} = 0$. Thus, in this special case, $L(\delta = 0) = 0$, and

$$L(\delta) = \left.\frac{dL}{d\delta}\right|_{\delta=0} \delta = \left.\frac{dL}{d\delta}\right|_{any\delta} \delta \tag{2.1.34}$$

and hence

$$L(\delta) = 0 \text{ or } \left.\frac{dL}{d\delta}\right|_{any\delta} \delta = 0 \tag{2.1.35}$$

are equivalent statements when $\alpha_0 = C_{MAC_0} = 0$.

For $\alpha_0 \neq 0$ and/or $C_{MAC_0} \neq 0$, however, the reversal condition is more meaningfully defined as the condition when the lift due to $\delta \neq 0$ is zero, i.e.,

$$\frac{dL}{d\delta} = 0 \text{ at } q = q_R \tag{2.1.27}$$

In this case, at the reversal condition from (2.1.32) and (2.1.33),

$$\begin{aligned} L(\delta)|_{\text{at reversal}} &= L(\delta = 0)|_{\text{at reversal}} \\ &= qS\frac{\partial C_L}{\partial \alpha}\left[\frac{\alpha_0 + \frac{qSc}{K_\alpha} C_{MAC_0}}{1 - \frac{eqS\frac{\partial C_L}{\partial \alpha}}{K_\alpha}}\right] \end{aligned} \tag{2.1.36}$$

and hence the lift at reversal per se is indeed not zero in general unless $\alpha_0 = C_{MAC_0} = 0$.

2.1.2 Typical Section Model—Nonlinear Effects

For sufficiently large twist angles, the assumption of elastic and/or aerodynamic moments proportional to twist angle becomes invalid. Typically the elastic spring becomes stiffer at larger twist angles; for example the elastic moment-twist relation might be

$$M_E = -K_\alpha \alpha_e - K_{\alpha_3} \alpha_e^3$$

where $K_\alpha > 0$, $K_{\alpha_3 > 0}$. The lift angle of attack relation might be

$$L = qS[(\partial C_L/\partial\alpha)\alpha - (\partial C_L/\partial\alpha)_3\alpha^3]$$

where $\partial C_L/\partial\alpha$ and $(\partial C_L/\partial\alpha)_3$ are positive quantities. Note the lift decreases for a large α due to flow separation from the airfoil. Combining the above in a moment equation of equilibrium and assuming for simplicity that $\alpha_0 = C_{MAC} = 0$, we obtain [recall (2.1.5)]

$$eqS[(\partial C_L/\partial\alpha)\alpha_e - (\partial C_L/\partial\alpha)_3\alpha_e^3] - [K_\alpha\alpha_e + K_{\alpha_3}\alpha_e^3] = 0$$

Rearranging,

$$\alpha_e[eq(S\partial C_L/\partial\alpha) - K_\alpha] - \alpha_e^3[eqS(\partial C_L/\partial\alpha)_3 + K_{\alpha_3}] = 0$$

Solving, we obtain the trivial solution $\alpha_e \equiv 0$, as well as

$$\alpha_e^2 = \frac{\left[eqS\frac{\partial C_L}{\partial\alpha} - K_\alpha\right]}{\left[eqS(\frac{\partial C_L}{\partial\alpha})_3 + K_{\alpha_3}\right]}$$

To be physically meaningful α_e must be a real number; hence the right hand side of the above equation must be a positive number for the nontrivial solution $\alpha_e \neq 0$ to be possible.

For simplicity let us first assume that $e > 0$. Then we see that only for $q > q_D$ (i.e., for $eqS(\partial C_L/\partial\alpha) > K_\alpha$) are nontrivial solutions possible. See Fig. 2.6. For $q < q_D$, $\alpha_e \equiv 0$ as a consequence of setting $\alpha_0 \equiv C_{MAC} \equiv 0$. Clearly for $e > 0$, $\alpha_e \neq 0$ when $q < q_D$ where

$$q_D \equiv \frac{K_\alpha}{eS\partial C_L/\partial\alpha}$$

Note that two (symmetrical) equilibrium solutions are possible for $q > q_D$. The actual choice of equilibrium position would depend upon how the airfoil is disturbed (by gusts for example) or possibly upon imperfections in the spring or airfoil geometry. α_0 may be thought of as an initial imperfection and its sign would determine which of the two equilibria positions occurs. Note that for the nonlinear model α_e remains finite for any finite q. For $e < 0$, the equilibrium configurations would be as shown in the Fig. 2.6 where

$$q_{D_3} = -K_{\alpha_3}/eS(\partial C_L/\partial\alpha)_3$$

and

$$\alpha_{e_\infty}^2 = \partial C_L/\partial\alpha(\partial C_L/\partial\alpha)_3$$

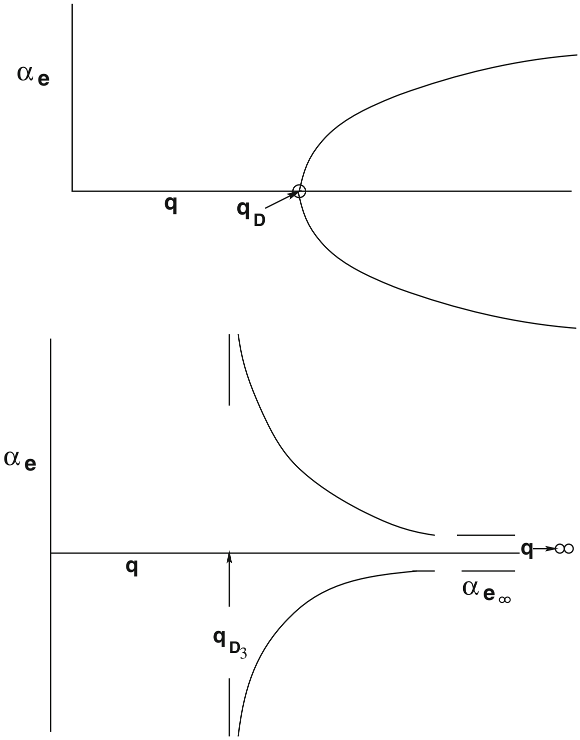


Fig. 2.6 (Nonlinear) equilibria for elastic twist: $e > 0$ (*top*). $e < 0$ (*bottom*)

As far as the author is aware, the behavior indicated in Fig. 2.6 has never been observed experimentally. Presumably structural failure would occur for $q > q_D$, even though α_{e_∞} is finite. It would be most interesting to try to achieve the above equilibrium diagram experimentally.

The above discussion does not exhaust the possible types of nonlinear behavior for the typical section model. Perhaps one of the most important nonlinearities in practice is that associated with the control surface spring and the elastic restraint of the control surface connection to the main lifting surface.[7]

2.2 One Dimensional Aeroelastic Model of Airfoils

2.2.1 Beam-Rod Representation of Large Aspect Ratio Wing

We shall now turn to a more sophisticated, but more realistic beam-rod model which contains the same basic physical ingredients as the typical section.[8] A beam-rod is

[7] Woodcock [4].

[8] See Chap. 7, BA, pp. 280–295, especially pp. 288–295.

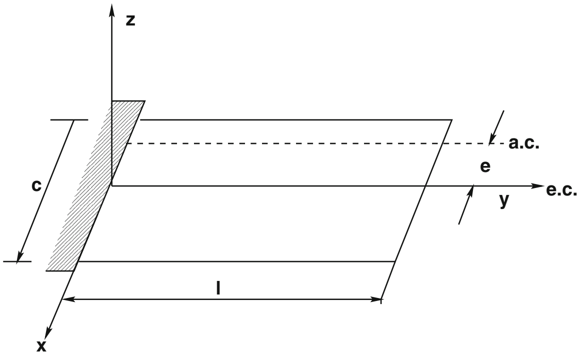


Fig. 2.7 Beam-rod representation of wing

here defined as a flat plate with rigid chordwise sections whose span, l, is substantially larger than its chord, c. See Fig. 2.7. The airflow is in the x direction. The equation of static moment equilibrium for a beam-rod is

$$\frac{d}{dy}\left(GJ\frac{d\alpha_e}{dy}\right) + M_y = 0 \tag{2.2.1}$$

$\alpha_e(y)$ nose up twist about the elastic axis, e.s., at station y
M_y nose up aerodynamic moment about e.a., per unit distance in the spanwise, y, direction
G shear modulus
J polar moment of inertia ($= ch^3/3$ for a rectangular cross-section of thickness, h, $h \ll c$)
GJ torsional stiffness

Equation (2.2.1) can be derived by considering a differential element dy (see Fig. 2.8) The internal elastic moment is GJ from the theory of elasticity.[9] Note for $d\alpha_e/dy > 0$, $GJ(d\alpha_e/dy)$ is positive nose down. Summing moments on the differential element, we have

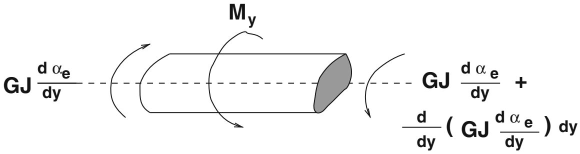


Fig. 2.8 Differential element of beam-rod

[9] Housner, and Vreeland [5].

$$-GJ\frac{d\alpha_e}{dy} + GJ\frac{d\alpha_e}{dy} + \frac{d}{dy}\left(GJ\frac{d\alpha_e}{dy}\right)dy + H.O.T. + M_y dy = 0$$

In the limit, as $dy \to 0$,[10]

$$\frac{d}{dy}\left(GJ\frac{d\alpha_e}{dy}\right) + M_y = 0 \tag{2.2.1}$$

Equation (2.2.1) is a second order differential equation in y. Associated with it are two boundary conditions. The airfoil is fixed at its root and free at its tip, so that the boundary conditions are

$$\alpha_e = 0 \text{ at } y = 0 \quad GJ\frac{d\alpha_e}{dy} = 0 \text{ at } y = l \tag{2.2.2}$$

Turning now to the aerodynamic theory, we shall use the 'strip theory' approximation. That is, *we shall assume that the aerodynamic lift and moment at station y depends only on the angle of attack at station y* (and is independent of the angle of attack at other spanwise locations).
Thus moments and lift per unit span are, as before,

$$M_y = M_{AC} + Le \tag{2.2.3a}$$

$$L \equiv qcC_L \tag{2.2.3b}$$

where now the lift and moment coefficients are given by

$$C_L(y) = \frac{\partial C_L}{\partial \alpha}[\alpha_0(y) + \alpha_e(y)] \tag{2.2.3c}$$

$$M_{AC} = qc^2 c_{MAC} \tag{2.2.3d}$$

(2.2.3b) and (2.2.3d) define C_L and C_{MAC} respectively.[11]

Using (2.2.3a) in (2.2.1) and nondimensionalizing (assuming for simplicity, constant wing properties)

---

[10] Higher Order Terms.

[11] A more complete aerodynamic model would allow for the effect of an angle of attack at one spanwise location, say η, on (nondimentional) lift at another, say y. This relation would then be replaced by $C_L(y) = \int A(y-\eta)[\alpha_0(\eta) + \alpha_e(\eta)]d\eta$ where A is an aerodynamic influence function which must be measured or calculated from an appropriate theory. More will be said about this later.

$$\tilde{y} \equiv \frac{y}{l}$$
$$\lambda^2 \equiv \frac{qcl^2}{GJ}\frac{\partial C_L}{\partial \alpha} e$$
$$K \equiv -\frac{qcl^2}{GJ}\left(e\frac{\partial C_L}{\partial \alpha}\alpha_0 + C_{MAC_0}c\right)$$

(2.2.1) becomes

$$\frac{d^2\alpha_e}{d\tilde{y}^2} + \lambda^2\alpha_e = K \tag{2.2.4}$$

which is subject to boundary conditions (2.2.2). These boundary conditions have the nondimensional form

$$\alpha = 0 \quad \text{at} \quad \tilde{y} = 0 \tag{2.2.5}$$
$$\frac{d\alpha_e}{d\tilde{y}} = 0 \quad \text{at} \quad \tilde{y} = 1$$

The general solution to (2.2.4) is

$$\alpha_e = A\sin\lambda\tilde{y} + B\cos\lambda\tilde{y} + \frac{K}{\lambda^2} \tag{2.2.6}$$

Applying boundary conditions (2.2.5), we obtain

$$B + \frac{K}{\lambda^2} = 0, \qquad \lambda[A\cos\lambda - B\sin\lambda] = 0 \tag{2.2.7}$$

Solving Eq. (2.2.7), $A = -(K/\lambda^2)\tan\lambda,\ B = -K/\lambda^2$, so that

$$\alpha_e = \frac{K}{\lambda^2}[1 - \tan\lambda\sin\lambda\tilde{y} - \cos\lambda\tilde{y}] \tag{2.2.8}$$

Divergence occurs when $\alpha_e \to \infty$, i.e., $\tan\lambda \to \infty$, or $\cos\lambda \to 0$.[12] Thus, for $\lambda = \lambda_m = (2m-1)\frac{\pi}{2}(m = 1, 2, 3, \ldots)$, $\alpha_e \to \infty$. The lowest of these, $\lambda_1 = \frac{\pi}{2}$ is physically significant. Using the definition of λ preceding Eq. (2.2.4), the divergence dynamic pressure is

$$q = (\pi/2)^2\frac{GJ}{l}/lce(\partial C_L/\partial\alpha) \tag{2.2.9}$$

[12] Note $\lambda \equiv 0$ is not a divergence condition!Expanding (2.2.8) for $\lambda \ll 1$, we obtain $\alpha_e = \frac{K}{\lambda^2}[1 - \lambda^2\tilde{y} - (1 - \frac{\lambda^2\tilde{y}^2}{2}) + \cdots] \to K[\frac{\tilde{y}^2}{2} - \tilde{y}]$ as $\lambda \to 0$.

Recognizing that $S = lc$, we see that (2.2.9) is equivalent to the typical section value, (2.1.8), with

$$K_\alpha = \left(\frac{\pi}{2}\right)^2 \frac{GJ}{l} \tag{2.2.10}$$

Consider again (2.2.8). A further physical interpretation of this result may be helpful. For simplicity, consider the case when $C_{MAC_0} = 0$ and thus $K = -\lambda^2\alpha_0$. Then the expression for α_e, (2.2.8), may be written as

$$\alpha_e = \alpha_0[-1 + \tan\lambda \sin\lambda\tilde{y} + \cos\lambda\tilde{y}] \tag{2.2.8a}$$

The tip of twist of $\tilde{y} = 1$ may be used to characterize the variation of α_e with λ, i.e.,

$$\alpha_e(\tilde{y} = 1) = \alpha_0\left[\frac{1}{\cos\lambda} - 1\right] \tag{2.2.8b}$$

and thus

$$\alpha = \alpha_0 + \alpha_e = \alpha_0/\cos\lambda \tag{2.2.8c}$$

From (2.2.8c), we see that for low flow speeds or dynamic pressure, $\lambda \to 0$, $\alpha = \alpha_0$. As $\lambda \to \pi/2$, α monotonically increases and $\alpha \to \infty$ as $\lambda \to \pi/2$. For a given wing design, a certain twist might be allowable. From (2.2.8c), or its counterpart for more complex physical and mathematical models, the corresponding allowable or design λ may be determined.

Another design allowable might be the allowable structural moment, $T \equiv GJd\alpha_e/dy$. Using (2.2.8) and the definition of T, for a given allowable T the corresponding allowable λ or q may be determined.

2.2.2 *Eigenvalue and Eigenfunction Approach*

One could have treated divergence from the point of view of an eigenvalue problem. Neglecting those terms which do *not* depend on the elastic twist, i.e., setting $\alpha_0 = C_{MAC_0} = 0$, we have $K = 0$ and hence

$$\frac{d^2\alpha}{d\tilde{y}^2} + \lambda^2\alpha = 0 \tag{2.2.11}$$

with

$$\alpha = 0 \text{ at } y = 0$$

$$\frac{d\alpha}{d\tilde{y}} = 0 \quad \text{at} \quad y = 1 \tag{2.2.12}$$

The general solution is

$$\alpha = A \sin \lambda \tilde{y} + B \cos \lambda \tilde{y} \tag{2.2.13}$$

Using (2.2.12) and (2.2.13)

$$B = 0$$

$$\lambda [A \cos \lambda - B \sin \lambda]$$

we conclude that

$$A = 0$$

or

$$\lambda \cos \lambda = 0 \text{ and } A \neq 0 \tag{2.2.14}$$

The latter condition, of course, is 'divergence'. Can you show that $\lambda = 0$, does not lead to divergence? What does (2.2.13) say? For each eigenvalue, $\lambda = \lambda_m = (2m - 1)\frac{\pi}{2}$ there is an eigenfunction,

$$\alpha_m \sim \sin \lambda_m \tilde{y} = \sin\,(2m - 1)\frac{\pi}{2}\tilde{y} \tag{2.2.15}$$

These eigenfunctions are of interest for a number of reasons:

1. They give us the twist distribution at the divergence dynamic pressure as seen above in (2.2.15).
2. They may be used to obtain a series expansion of the solution for any dynamic pressure.
3. They are useful for developing an approximate solution for variable property wings.

Let us consider further the second of these. Now we let $\alpha_0 \neq 0$, $C_{MAC_0} \neq 0$ and begin with (2.2.4)

$$\frac{d^2\alpha_e}{d\tilde{y}} + \lambda^2 \alpha_e = K \tag{2.2.4}$$

Assume a series solution of the form

$$\alpha_e = \sum_n a_n \alpha_n(\tilde{y}) \tag{2.2.16}$$

$$K = \sum_n A_n \alpha_n(\tilde{y}) \tag{2.2.17}$$

where a_n, A_n are to be determined. Now it can be shown that

$$\int_0^1 \alpha_n(\tilde{y})\alpha_m(\tilde{y})d\tilde{y} = \frac{1}{2} \quad \text{for} \quad m = n \tag{2.2.18}$$

$$= 0 \quad \text{for} \quad m \neg n$$

This is the so-called 'orthogonality condition'. We shall make use of it in what follows. First, let us determine A_n. Multiply (2.2.17) by α_m and $\int_0^1 \cdots d\tilde{y}$.

$$\int_0^1 K\alpha_m(\tilde{y})d\tilde{y} = \sum_n A_n \int_0^1 \alpha_n(\tilde{y})\alpha_m(\tilde{y})d\tilde{y} = A_m \frac{1}{2}$$

using (2.2.18). Solving for A_m,

$$A_m = 2\int_0^1 K\alpha_m(\tilde{y})d\tilde{y} \tag{2.2.19}$$

Now let us determine a_n. Substitute (2.2.16) and (2.2.17) into (2.2.4) to obtain

$$\sum_n \left[a_n \frac{d^2\alpha_n}{d\tilde{y}^2} + \lambda^2 a_n \alpha_n \right] = \sum_n A_n \alpha_n \tag{2.2.20}$$

Now each eigenfunction, α_n, satisfies (2.2.11)

$$\frac{d^2\alpha_n}{d\tilde{y}^2} + \lambda_n^2 \alpha_n = 0 \tag{2.2.11}$$

Therefore, (2.2.20) may be written

$$\sum_n a_n[-\lambda_n^2 + \lambda^2]\alpha_n = \sum A_n \alpha_n \tag{2.2.21}$$

Multiplying (2.2.21) by α_m and $\int_0^1 \cdots d\tilde{y}$,

$$[\lambda^2 - \lambda_m^2]a_m \frac{1}{2} = A_m \frac{1}{2} \tag{multiplication}$$

Solving for a_m,

$$a_m = \frac{A_m}{\left[\lambda^2 - \lambda_m^2\right]} \tag{2.2.22}$$

Thus,

$$\alpha_e = \sum a_n \alpha_n = \sum_n \frac{A_n}{\left[\lambda^2 - \lambda_m^2\right]} \alpha_n(\tilde{y}) \tag{2.2.23}$$

where A_n is given by (2.2.19).[13]

Similar calculations can be carried out for airfoils whose stiffness, chord, etc., are *not* constants but vary with spanwise location. One way to do this is to first determine the eigenfunction expansion *for the variable property wing* as done above for the constant property wing. The determination of such eigenfunctions may itself be fairly complicated, however. An alternative procedure can be employed which expands the solution for *the variable property* wing in terms of the eigenfunctions of the *constant property wing*. This is the last of the reasons previously cited for examining the eigenfunctions.

2.2.3 Galerkin's Method

The equation of equilibrium for a *variable* property wing may be obtained by substituting (2.2.3a) into (2.2.1). In dimensional terms

$$\frac{d}{dy}\left(GJ\frac{d}{dy}\alpha_e\right) + eqc\frac{\partial C_L}{\partial \alpha}\alpha_e = -eqc\frac{\partial C_L}{\partial \alpha}\alpha_0 - qc^2 C_{MAC_0} \tag{2.2.24}$$

In nondimensional terms

$$\frac{d}{dy}\left(\gamma\frac{d\alpha_e}{d\tilde{y}}\right) + \lambda^2 \alpha_e \beta = K \tag{2.2.25}$$

where

$$\gamma \equiv \frac{GJ}{(GJ)_{ref}} \qquad K = -\frac{qcl^2}{(GJ)_{ref}}\left[e\frac{\partial C_L}{\partial \alpha}\alpha_0 + C_{MAC_0}\right]$$

$$\lambda^2 \equiv \frac{ql^2 c_{ref}}{(GJ)_{ref}}\left(\frac{\partial C_L}{\partial \alpha}\right)_{ref} e_{ref} \qquad \beta = \frac{c}{c_{ref}}\frac{e}{e_{ref}}\frac{\left(\frac{\partial C_L}{\partial \alpha}\right)}{\left(\frac{\partial C_L}{\partial \alpha}\right)_{ref}}$$

Let

[13] For a more detailed mathematical discussion of the above , see Hildebrand [6], pp. 224–234. This problem is one of a type known as 'Sturm-Liouville Problems'.

$$\alpha_e = \sum_n a_n \alpha_n(\tilde{y})$$

$$K = \sum_n A_n \alpha_n(\tilde{y})$$

As before. Substituting the series expansions into (2.2.25), multiplying by α_m and $\int_0^1 \cdots d\tilde{y}$,

$$\sum a_n \left\{ \int_0^1 \frac{d}{d\tilde{y}} \left(\gamma \frac{d\alpha_n}{d\tilde{y}} \right) \alpha_m \, d\tilde{y} + \lambda^2 \int_0^1 \beta \alpha_n \alpha_m \, d\tilde{y} \right\} \tag{2.2.26}$$

$$= \sum_n A_n \int_0^1 \alpha_n \alpha_m \, d\tilde{y} = \frac{A_m}{2}$$

The first and second terms cannot be simplified further unless the eigenfunctions or 'modes' employed are eigenfunctions for the variable property wing. Hence, a_n is not as simply related to A_n as in the constant property wing example. Equation (2.2.26) represents a system of equations for the a_n. In matrix notation

$$[C_{mn}]\{a_n\} = \{A_m\}\frac{1}{2} \tag{2.2.27}$$

where

$$C_{mn} \equiv \int_0^1 \frac{d}{d\tilde{y}} \left(\gamma \frac{d\alpha_n}{d\tilde{y}} \right) \alpha_m d\tilde{y} + \lambda^2 \int_0^1 \beta \alpha_n \alpha_m \, d\tilde{y}$$

By truncating the series to a finite number of terms, we may formally solve for the a_n,

$$\{a_n\} = \frac{1}{2}[C_{mn}]^{-1}\{A_m\} \tag{2.2.28}$$

The divergence condition is simply that the determinant of C_{mn} vanish (and hence $a_n \to \infty$)

$$|C_{mn}| = 0 \tag{2.2.29}$$

which is a polynomial in λ^2. It should be emphasized that for an 'exact' solution, (2.2.27), (2.2.28) etc., are infinite systems of equations (in an infinite number of unknowns). In practice, some large but finite number of equations is used to obtain an accurate approximation. By systematically increasing the terms in the series, the convergence of the method can be assessed. This procedure is usually referred to

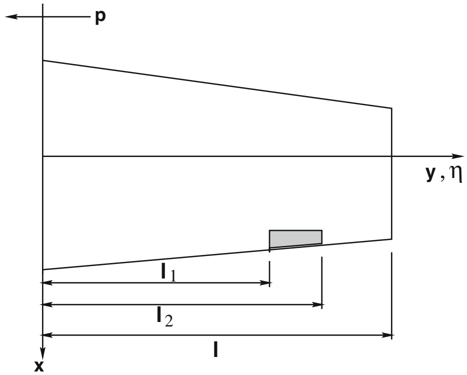


Fig. 2.9 Rolling of a straight wing

as Galerkin's method or as a 'modal' method.[14] The modes, α_n, used are called 'primitive modes' to distinguish them from eigenfunctions, i.e., they are 'primitive functions' for a variable property wing even though they are eigenfunctions for a constant property wing.

2.3 Rolling of a Straight Wing

We shall now consider a more complex physical and mathematical variation on our earlier static aeroelastic lifting surface (wing) studies. For variety, we treat a new physical situation, the rolling of a wing (rotation about the root axis). Nevertheless, we shall meet again our old friends, 'divergence' and 'control surface effectiveness' or 'reversal'.

The present analysis differs from the previous one as follows:

a. integral equation formulation versus differential equation formulation
b. aerodynamic induction effects versus 'strip' theory
c. 'lumped element' method of solution versus modal (or eigenfunction) solution.

The geometry of the problem is shown in Fig. 2.9.

2.3.1 Integral Equation of Equilibrium

The integral equation of equilibrium is

$$\alpha(y) = \int_0^1 C^{\alpha\alpha}(y, \eta) M_y(\eta) d\eta \tag{2.3.1}$$

[14] Duncan [7].

Before deriving the above equation,[15] let us first consider the physical interpretation of $C^{\alpha\alpha}$:

Apply a unit point moment at some point, say $y = \gamma$, i.e.,

$$M_y(\eta) = \delta(\eta - \gamma)$$

Then (2.3.1) becomes

$$\alpha(y) = \int_0^1 C^{\alpha\alpha}(y, \eta)\delta(\eta - \gamma)d\eta = C^{\alpha\alpha}(y,\ \gamma) \tag{2.3.2}$$

Thus $C^{\alpha\alpha}(y, \gamma)$ is the twist a y due to a unit moment at γ, or alternatively, $C^{\alpha\alpha}\ (y, \eta)$ is the twist at y due to a unit moment at η. $C^{\alpha\alpha}$ is called a *structural influence function*.

Also note that (2.3.1) states that to obtain the total twist, one multiplies the actual distributed torque, M_y, by $C^{\alpha\alpha}$ and sums (integrates) over the span. This is physically plausible.

$C^{\alpha\alpha}$ plays a central role in the integral equation formulation.[16] The physical interpretation of $C^{\alpha\alpha}$ suggests a convenient means of measuring $C^{\alpha\alpha}$ in a laboratory experiment. By successively placing unit couples at various locations along the wing and measuring the twists of all such stations for each loading position we can determine $C^{\alpha\alpha}$. This capability for measuring $C^{\alpha\alpha}$ gives the integral equation a preferred place in aeroelastic analysis where $C^{\alpha\alpha}$ and/or GJ are not always easily determinable from purely theoretical considerations.

2.3.2 Derivation of Equation of Equilibrium

Now consider a derivation of (2.3.1) taking as our starting point the differential equation of equilibrium. We have, you may recall,

$$\frac{d}{dy}\left(GJ\frac{d\alpha}{dy}\right) = -M_y \tag{2.3.3}$$

with

$$\alpha(0) = 0 \quad \text{and} \quad \frac{d\alpha}{dy}(l) = 0 \tag{2.3.4}$$

[15] For simplicity, $\alpha_0 \equiv 0$ in what follows.

[16] For additional discussion, see the following selected references: Hildebeand [6], pp. 388–394 and BAH, pp. 39–44.

as boundary conditions.

As a special case of (2.3.3) and (2.3.4) we have for a unit torque applied at $y = \eta$,

$$\frac{d}{dy} GJ \frac{dC^{\alpha\alpha}}{dy} = -\delta(y - \eta) \tag{2.3.5}$$

with

$$C^{\alpha\alpha}(0, \eta) = 0 \quad \text{and} \quad \frac{dC^{\alpha\alpha}}{dy}(l, \eta) = 0 \tag{2.3.6}$$

Multiply (2.3.5) by $\alpha(y)$ and integrate over the span,

$$\int_0^1 \alpha(y) \frac{d}{dy}\left(GJ \frac{dC^{\alpha,\alpha}}{dy}\right) dy = -\int_0^1 \delta(y - \eta)\alpha(y) dy = -\alpha(\eta) \tag{2.3.7}$$

Integrate LHS of (2.3.7) by parts,

$$\alpha GJ \frac{dC^{\alpha\alpha}}{dy}\bigg|_0^1 - GJ \frac{d\alpha}{dy} C^{\alpha\alpha}\bigg|_0^1 + \int_0^1 C^{\alpha\alpha} \frac{d}{dy}\left(GJ \frac{d\alpha}{dy}\right) dy = -\alpha(\eta) \tag{2.3.8}$$

Using boundary conditions (2.3.4) and (2.3.6), the first two terms of LHS of (2.3.8) vanish. Using (2.3.3) the integral term may be simplified and we obtain,

$$\alpha(\eta) = \int_0^1 C^{\alpha\alpha}(y, \eta) M_y(y) dy \tag{2.3.9}$$

Interchanging y and η,

$$\alpha(y) = \int_0^1 C^{\alpha\alpha}(\eta, y) M_y(\eta) d\eta \tag{2.3.10}$$

(2.3.10) is identical to (1), *if*

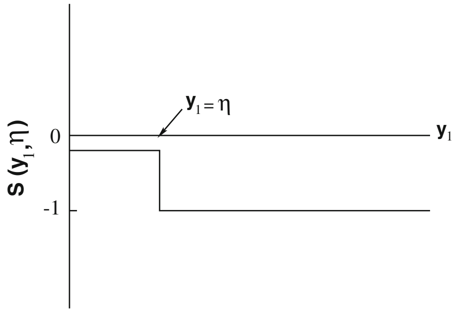
$$C^{\alpha\alpha}(\eta, y) = C^{\alpha\alpha}(y, \eta) \tag{2.3.11}$$

We shall prove (2.3.11) subsequently.

2.3.3 Calculation of $C^{\alpha\alpha}$

We shall calculate $C^{\alpha\alpha}$ from (2.3.5) using (2.3.6). Integrating (2.3.5) with respect to y from 0 to y_1,

$$GJ(y_1)\frac{dC^{\alpha\alpha}}{dy}(y_1,\eta) - GJ(0)\frac{dC^{\alpha\alpha}}{dy}(0,\eta) \tag{2.3.12}$$
$$= -1 \quad \text{if} \quad y_1 > \eta$$
$$= 0 \quad \text{if} \quad y_1 < \eta \equiv S(y_1,\eta)$$



2.3.4 Sketch of Function $S(y_1, \eta)$

Dividing (2.3.12) by $GJ\ (y_1)$ and integrating with respect to y_1 from 0 to y_2,

$$C^{\alpha\alpha}(y_2,\ \eta) - C^{\alpha\alpha}(0,\eta) - GJ(0)\frac{dC^{\alpha\alpha}}{dy}(0,\eta)\int_0^{y_2}\frac{1}{GJ}dy_1 \tag{2.3.13}$$
$$= \int_0^{y_2}\frac{S(y_1,\eta)}{GJ(y_1)}dy_1 = -\int_\eta^{y_2}\frac{1}{GJ(y_1)}dy_1 \quad \text{for} \quad y_2 > \eta$$
$$= 0 \quad \text{for} \quad y_2 < \eta$$

From boundary conditions, (2.3.6),

$$\text{(a)} \quad C^{\alpha\alpha}(0,\eta) = 0$$

$$\text{(b)} \quad \frac{dC^{\alpha\alpha}}{dy}(l,\eta) = 0$$

These may be used to evaluate the unknown terms in (2.3.12) and (2.3.13). Evaluating (2.3.12) at $y_1 = l$

$$\text{(c)}\quad GJ(l)\underbrace{\frac{dC^{\alpha\alpha}}{dy}(l,\eta)}_{\longrightarrow 0} - GJ\frac{dC^{\alpha\alpha}}{dy}(0,\eta) = -1$$

Using (a) and (c), (2.3.13) may be written,

$$\begin{aligned} C^{\alpha\alpha}(y_2,\eta) &= \int_0^{y_2} \frac{1}{GJ}dy_1 - \int_\eta^{y_2} \frac{1}{GJ}dy_1 \\ &= \int_0^{\eta} \frac{1}{GJ}dy_1 \quad \text{for} \quad y_2 > \eta \\ &= \int_0^{y_2} \frac{1}{GJ}dy_1 \quad \text{for} \quad y_2 < \eta \end{aligned}$$

One may drop the dummy subscript on y_2, of course. Thus

$$\begin{aligned} C^{\alpha\alpha}\ (y,\eta) &= \int_0^{y} \frac{1}{GJ}dy_1 \quad \text{for} \quad y < \eta \\ &= \int_0^{\eta} \frac{1}{GJ}dy_1 \quad \text{for} \quad y > \eta \end{aligned} \tag{2.3.14}$$

Note from the above result we may conclude by interchanging y and η that

$$C^{\alpha\alpha}(y,\eta) = C^{\alpha\alpha}(\eta,y)$$

This is a particular example of a more general principle known as Maxwell's Reciprocity Theorem[17] which says that all structural influence functions for linear elastic bodies are symmetric in their arguments. In the case of $C^{\alpha\alpha}$ these are y and η, of course.

2.3.5 Aerodynamic Forces (Including Spanwise Induction)

First, let us identify the aerodynamic angle of attack; i.e., the angle between the airfoil chord and relative airflow. See Fig. 2.10. Hence, the total angle of attack due

[17] Bisplinghoff, Mar, and Pian [8], p. 247.

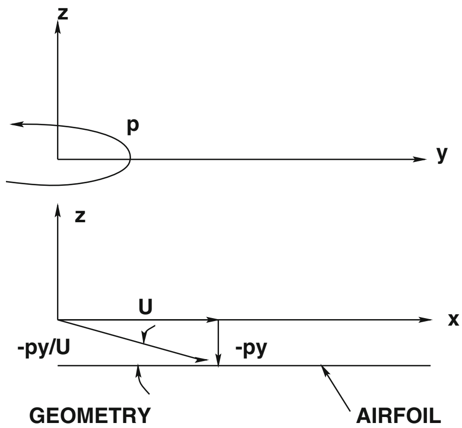


Fig. 2.10 Coordinate system and velocity diagram

to twisting and rolling is

$$\alpha_{Total} = \alpha(y) - \frac{py}{U}$$

The control surface will be assumed rigid and its rotation is given by

$$\delta(y) = \delta_R \quad \text{for} \quad l_1 < y < l_2 = 0 \quad \text{otherwise}$$

From aerodynamic theory or experiment

$$C_L \equiv \frac{L}{qc} = \int_0^1 A^{L\alpha}(y,\eta)\alpha_T(\eta)\frac{d\eta}{l} + \int_0^1 A^{L\delta}(y,\eta)\delta(\eta)\frac{d\eta}{l} \tag{2.3.15}$$

Here $A^{L\alpha}$, $A^{L\delta}$ are aerodynamic influence functions; as written, they are nondimensional. Thus, $A^{L\delta}$ is nondimensional lift at y due to unit angle of attack at η. Substituting for α_T and δ, (2.3.15) becomes,

$$C_L = \int_0^1 A^{L\alpha}\alpha\frac{d\eta}{l} - \frac{pl}{U}\int_0^1 A^{L\alpha}\frac{\eta}{l}\frac{d\eta}{l} + \delta_R\int_{l_1}^{l_2} A^{L\delta}\frac{d\eta}{l}$$

$$C_L = \int_0^1 A^{L\alpha}\alpha\frac{d\eta}{l} + \frac{pl}{U}\frac{\partial C_L}{\partial\left(\frac{pl}{U}\right)} + \delta_R\frac{\partial C_L}{\partial\delta_R} \tag{2.3.16}$$

where

$$\frac{\partial C_L}{\partial\left(\frac{pl}{U}\right)}(y) \equiv -\int_0^1 A^{L\alpha}\frac{\eta}{l}\frac{d\eta}{l}$$

and

$$\frac{\partial C_L}{\partial \delta_R}(y) \equiv \int_{l_1}^{l_2} A^{L\delta}\frac{d\eta}{l}$$

Physical Interpretation of $A^{L\alpha}$ and $A^{L\delta}$:$A^{L\alpha}$ is the lift coefficient at y due to unit angle of attack at n. $A^{L\delta}$ is the lift coefficient at y due to unit rotation of control surface at η.

Physical Interpretation of $\partial C_L/\partial\,(pl/U)$ and $\partial C_L/\partial\delta_R$:$\partial C_L/\partial\,(pl/U)$ is the lift coefficient at y due to unit rolling velocity, pl/U. $\partial C_L/\partial\delta_R$ is the lift coefficient at y due to unit control surface rotation, δ_R.

As usual

$$C_{MAC} \equiv \frac{M_{AC}}{qc^2} = \frac{\partial C_{MAC}}{\partial \delta_R}\delta_R \tag{2.3.17}$$

is the aerodynamic coefficient moment (about a.c.) at y due to control surface rotation. Note

$$\partial C_{MAC}/\partial\alpha_T \equiv 0$$

by definition of the aerodynamic center. Finally the total moment loading about the elastic axis is

$$M_y = M_{AC} + Le = qc[C_{MAC}c + C_L e] \tag{2.3.18}$$

Using (2.3.16) and (2.3.17), the above becomes

$$M_y = qc\left[c\frac{\partial C_{MAC}}{\partial \delta_R}\delta_R + e\left\{\int_0^1 A^{L\delta}\alpha\frac{d\eta}{l} + \frac{\partial C_L}{\partial\left(\frac{pl}{l}\right)}\left(\frac{pl}{U}\right) + \frac{\partial C_L}{\partial \delta_R}\delta_R\right\}\right] \tag{2.3.19}$$

Note that $A^{L\alpha}$, $A^{L\delta}$ are more difficult to measure than their structural counterpart, $C^{\alpha\alpha}$. One requires an experimental model to which one can apply unit angles of attack at various discrete points along the span of the wing. This requires a rather sophisticated model and also introduces experimental difficulties in establishing and maintaining a smooth flow over the airfoil. Conversely

$$\frac{\partial C_L}{\partial \frac{pl}{U}},\ \frac{\partial C_L}{\partial \delta_R} \quad \text{and} \quad \frac{\partial C_{MAC}}{\partial \delta_R}$$

are relatively easy to measure since they only require a rolling or control surface rotation of a *rigid wing with the same geometry* as the flexible airfoil of interest.

2.3.6 Aeroelastic Equations of Equilibrium and Lumped Element Solution Method

The key relations are (2.3.1) and (2.3.19). The former describes the twist due to an aerodynamic moment load, the latter the aerodynamic moment due to twist as well as rolling and control surface rotation.

By substituting (2.3.19) into (2.3.1), one could obtain a single equation for α. However, this equation is not easily solved analytically except for some simple cases, which are more readily handled by the differential equation approach. Hence, we seek an approximate solution technique. Perhaps the most obvious and convenient method is to approximate the integrals in (2.3.1) and (2.3.19) by sums, i.e., the wing is broken into various spanwise segments or 'lumped elements'. For example, (2.3.1) would be approximated as:

$$\alpha\ (y_i) \cong \sum_{i=1}^{N} C^{\alpha\alpha}\ (y_i, \eta_j) M_y\ (\eta_j) \Delta\eta \quad i = 1, \ldots, N \tag{2.3.20}$$

where $\Delta\eta$ is the segment width and N the total number of segments. Similarly, (2.3.19) may be written

$$M_y(y_i) \cong qc\{\, [\, c\frac{\partial C_{MAC}}{\partial \delta_R} + e\frac{\partial C_L}{\partial \frac{pl}{U}}\frac{pl}{U} + e\frac{\partial C_L}{\partial \delta_R}\delta_R\,] + e\sum_{j=1}^{N} A^{L\alpha}(y_i, \eta_j)\alpha(\eta_j)\frac{\Delta\eta}{l}\,\} \quad i = 1, \ldots, N \tag{2.3.21}$$

To further manipulate (2.3.20) and (2.3.21), it is convenient to use matrix notation. That is,

$$\{\alpha\} = \Delta\eta[C^{\alpha\alpha}]\{M_y\} \tag{2.3.20}$$

and

$$\{M_y\} = q\begin{bmatrix} \diagdown & & \\ & c^2 & \\ & & \diagdown \end{bmatrix}\left\{\frac{\partial C_MAC}{\partial \delta_R}\right\}\delta_R + q\begin{bmatrix} \diagdown & & \\ & ce & \\ & & \diagdown \end{bmatrix}\left\{\frac{\partial C_L}{\partial \frac{pl}{U}}\right\}\frac{pl}{U} + q\begin{bmatrix} \diagdown & & \\ & ce & \\ & & \diagdown \end{bmatrix}\left\{\frac{\partial C_L}{\partial \delta_R}\right\}\delta_R + q\begin{bmatrix} \diagdown & & \\ & ce & \\ & & \diagdown \end{bmatrix}[A^{L\alpha}]\{\alpha\}\frac{\Delta\eta}{l} \tag{2.3.22}$$

All full matrices are of order $N \times N$ and row or column matrices of order N. Substituting (2.3.21) into (2.3.20), and rearranging terms gives,

$$\left[\begin{bmatrix} \diagdown & & \\ & 1 & \\ & & \diagdown \end{bmatrix} - q\frac{(\Delta\eta)^2}{l}[E][A^{L\alpha}]\right]\{\alpha\} = \{f\} \tag{2.3.23}$$

where the following definitions apply

$$\{f\} \equiv q[E]\left[\left\{\frac{\partial C_L}{\partial \delta_R}\right\}\delta_R + \left\{\frac{\partial C_L}{\partial\left(\frac{pl}{U}\right)}\right\}\frac{pl}{U}\right]\Delta\eta$$

$$+ q[F]\left\{\frac{\partial C_{MAC}}{\partial \delta_R}\right\}\delta_R\ \Delta\eta$$

$$[E] \equiv [C^{\alpha\alpha}]\begin{bmatrix} \diagdown & & \\ & ce & \\ & & \diagdown \end{bmatrix}$$

$$[F] \equiv [C^{\alpha\alpha}]\begin{bmatrix} \diagdown & & \\ & c^2 & \\ & & \diagdown \end{bmatrix}$$

Further defining

$$[D] \equiv \begin{bmatrix} \diagdown & & \\ & 1 & \\ & & \diagdown \end{bmatrix} - q\frac{(\Delta\eta)^2}{l}[E][A^{L\alpha}]$$

we may formally solve (2.3.23) as

$$\{\alpha\} = [D]^{-1}\{f\} \tag{2.3.24}$$

Now let us interpret this solution.

2.3.7 Divergence

Recall that the inverse does not exist if

$$|D| = 0 \tag{2.3.25}$$

and hence,

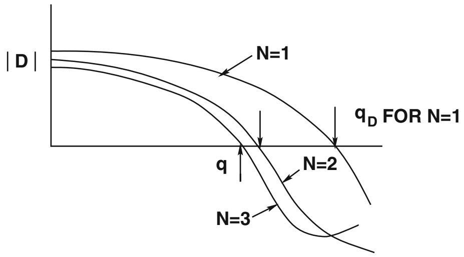


Fig. 2.11 Characteristic determinant versus dynamic pressure

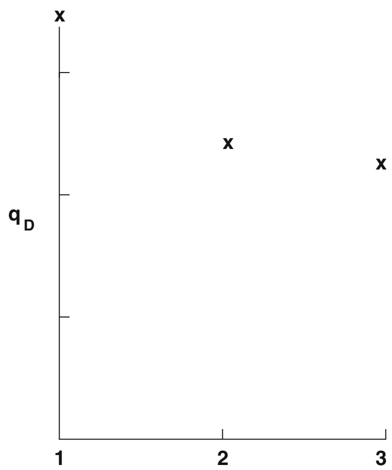


Fig. 2.12 Convergence of divergence dynamic pressure with modal number

$$\{\alpha\} \to \{\infty\}$$

(2.3.25) gives rise to an eigenvalue problem for the divergence dynamic pressure, q_D. Note (2.3.25) is a polynomial in q.

The lowest possible root (eigenvalue) of (2.3.25) gives the q of physical interest, i.e., $q_{\text{Divergence}}$. Rather than seeking the roots of the polynomial we might more simply plot $|D|$ versus q to determine the values of dynamic pressure for which the determinant is zero. A schematic of such results for various choices of N is shown below in Fig. 2.11. From the above results we may plot q_D (the lowest positive q for which $|D| = 0$) versus N as shown below in Fig. 2.12. The 'exact' value of q_D is obtained at $N \to \infty$. Usually reasonably accurate results can be obtained for small values of N, say 10 or so. The divergence speed calculated above does not depend upon the rolling of the wing, i.e., p is considered prescribed, e.g., $p = 0$.

2.3.8 Reversal and Rolling Effectiveness

In the above we have taken pl/U as known; however,in reality it is a function of δ_R and the problem parameters through the requirement that the wing be in static rolling equilibrium, i.e., it is an additional degree of freedom. For rolling equilibrium at a steady roll rate, p, the rolling moment about the x-axis is zero.

$$M_{\text{Rolling}} \equiv 2\int_0^1 Lydy = 0 \tag{2.3.26}$$

Approximating (2.3.26),

$$\sum_i L_i y_i \Delta y = 0 \tag{2.3.27}$$

or, in matrix notation,

$$2\lfloor y \rfloor \{L\} \Delta y = 0 \tag{2.3.28}$$

or

$$2q\lfloor cy \rfloor \{C_L\} \Delta y = 0$$

From (2.3.16), using the 'lumped element' approximation and matrix notation,

$$\{C_L\} = \frac{\Delta\eta}{l}[A^{L\alpha}]\{\alpha\} + \left\{\frac{\partial C_L}{\partial \delta_R}\right\}\delta_R + \left\{\frac{\partial C_L}{\partial\left(\frac{pl}{U}\right)}\right\}\frac{pl}{U} \tag{2.3.16}$$

Substitution of (2.3.16) into (2.3.28) gives

$$\lfloor cy \rfloor \left\{\frac{\Delta\eta}{l}[A^{L\alpha}]\{\alpha\} + \left\{\frac{\partial C_L}{\partial \delta_R}\right\}\delta_R + \left\{\frac{\partial C_L}{\partial\left(\frac{pl}{U}\right)}\right\}\frac{pl}{U}\right\} = 0 \tag{2.3.29}$$

Note that (2.3.29) is a single algebraic equation. Equation (2.3.29) plus (2.3.20) and (2.3.21) are $2N + 1$ linear algebraic equations in the $N(\alpha)$ plus $N(M_y)$ plus $1(p)$ unknowns. As before $\{M_y\}$ is normally eliminated using (2.3.21) in (2.3.20) to obtain N, Eq. (2.3.23), plus 1, Eq. (2.3.29), equations in $N(\alpha)$ plus $1(p)$ unknowns. In either case the divergence condition my be determined by setting the determinant of coefficients to zero and determining the smallest positive eigenvalue, $q = q_D$.

For $q < q_D$, pl/U (and α) may be determined from (2.3.23) and (2.3.29). Since our mathematical model is linear

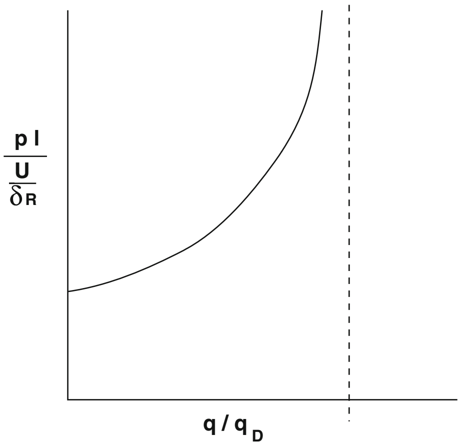


Fig. 2.13 Roll rate versus dynamic pressure

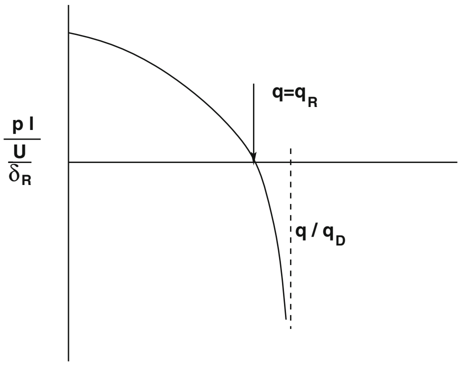


Fig. 2.14 Roll rate versus dynamic pressure

$$pl/U \sim \delta_R$$

and hence a convenient plot of the results is as shown in Fig. 2.13. As

$$q \to q_D, \frac{pl}{U}(\text{and}\{\alpha\}) \to \infty$$

Another qualitatively different type of result may sometimes occur. See Fig. 2.14. If

$$\frac{pl}{U/\delta_R} \to 0 \quad \text{for} \quad q \to q_R < q_D$$

then 'rolling reversal' is said to have occurred and the corresponding $q = q_R$ is called the 'reversal dynamic pressure'. The basic phenomenon is the same as that encountered previously as 'control surface reversal'. Figures 2.13 and 2.14 should be compared to Fig. 2.5a, b.

It is worth emphasizing that the divergence condition obtained above by permitting p to be determined by (static) rolling equilibrium will be different from that obtained previously by assuming $p = 0$. The latter physically corresponds to an aircraft constrained not to roll, as might be the case for some wind tunnel models. The former corresponds to a model or aircraft completely free to roll.[18]

The above analysis has introduced the simple yet powerful idea of structural and aerodynamic influence functions. While the utility of the concept has been illustrated for a one-dimensional aeroelastic model, not the least advantage of such an approach is the conceptual ease with which the basic notion can be extended to two-dimensional models, e.g., plate-like structures, or even three-dimensional ones (though the latter is rarely needed for aeroelastic problems).

In a subsequent section we briefly outline the generalization to two-dimensional models. Later this subject will be considered in more depth in the context of dynamic aeroelasticity.

2.3.9 Integral Equation Eigenvalue Problem and the Experimental Determination of Influence Functions

For the special case of a constant section wing with 'strip theory' aerodynamics one may formulate a standard integral equation eigenvalue problem for the determination of divergence. In itself this problem is of little interest. However, it does lead to some interesting results with respect to the determination of the structural and aerodynamic influence functions by experimental means.

For such a wing,

$$M_y = Le + M_{AC} = eqc\frac{\partial C_L}{\partial \alpha}\alpha + \cdots$$

where the omitted terms are independent of twist and may therefore be ignored for the divergence (eigenvalue) problem. Also the coefficients of α may be taken as constants for a constant section wing. Substituting the above expression into the integral equation of structural equilibrium we have

$$\alpha(y) = eqc\frac{\partial C_L}{\partial \alpha}\int_0^1 C^{\alpha\alpha}(y,\eta)\alpha(\eta)d\eta$$

This is an eigenvalue problem in integral form where the eigenvalue is

[18] This distinction between the two ways in which the aircraft may be restrained received renewed emphasis in the context of the oblique wing concept. Weisshaar and Ashley [9].

$$\lambda \equiv eqc\frac{\partial C_L}{\partial \alpha}$$

One may solve this problem for the corresponding eigenvalues and eigenfunctions which satisfy the equation

$$\alpha_n(y) = \lambda_n \int_0^1 C^{\alpha\alpha}(y, \eta)\alpha_n(\eta)d\eta$$

Incidentally, the restriction to a constant section wing was unnecessary and with a moderate amount of effort one could even use a more sophisticated aerodynamic model. Such complications are not warranted here.

These eigenfunctions or similar functions may be usefully employed to determine by experimental means the structural, $C^{\alpha\alpha}$, and aerodynamic, $A^{l\alpha}$, influence functions. The former is not as attractive as the use of point unit structural loads as we shall see; however, the procedure outlined below for the determination of $A^{L\alpha}$ probably deserves more attention than it has previously received.

Assume the structural influence function can be expanded in terms of the eigenfunctions

$$C^{\alpha\alpha}(y, \eta) = \sum_n C_n(y)\alpha_n(\eta) \tag{2.3.30}$$

where the C_n are to be determined. Also recall that

$$\alpha_n(y) = \lambda_n \int_0^1 C^{\alpha\alpha}(y, \eta)\alpha_n(\eta)d\eta \tag{2.3.31}$$

and the α_n are the eigenfunctions and λ_n the eigenvalues of $C^{\alpha\alpha}$ satisfying (2.3.31) and an orthogonality condition

$$\int \alpha_n \alpha_m dy = 0 \quad \text{for} \quad m \neq \eta$$

Then multiply (2.3.30) by $\alpha_m(\eta)$ and integrate over the span of the wing; the result is

$$C_m(y) = \frac{\int_0^1 C^{\alpha\alpha}(y, \eta)\alpha_m(\eta)d\eta}{\int_0^1 \alpha_m^2(\eta)d\eta}$$

and from (2.3.31)

$$C_m cyl = \frac{\alpha_m(y)}{\lambda_m \int_0^1 \alpha_m^2(\eta)d\eta} \tag{2.3.32}$$

Hence (2.3.32) in (2.3.30) gives

$$C^{\alpha\alpha}(y,\eta) = \sum_n \frac{\alpha_n(y)\alpha_n(\eta)}{\lambda_n \int_0^1 \alpha_m^2(\eta)d\eta} \tag{2.3.33}$$

Thus if the eigenfunctions are known then the Green's function is readily determined from (2.3.33). Normally this holds no special advantage since the determination of the α_n, theoretically or experimentally, is at least as difficult as determining the Green's function, $C^{\alpha\alpha}$, directly. Indeed as discussed previously if we apply unit moments at various points along the span the resulting twist distribution is a direct measure of $C^{\alpha\alpha}$. A somewhat less direct way of measuring $C^{\alpha\alpha}$ is also possible which makes use of the expansion of the Green's (influence) function. Again using (2.3.30)

$$C^{\alpha\alpha}(y,\eta) = \sum_n C_n\alpha_n(\eta) \tag{2.3.29}$$

and assuming the α_n are orthogonal (although not necessarily eigenfunctions of the problem at hand) we have

$$C_n(y) = \frac{\int_0^1 C^{\alpha\alpha}(y,\eta)\alpha_n(\eta)d\eta}{\int_0^1 \alpha_n^2(\eta)d\eta} \tag{2.3.34}$$

Now we have the relation between twist and moment

$$\alpha(y) = \int_0^1 C^{\alpha\alpha}(y,\eta)M_y(\eta)d\eta \tag{2.3.35}$$

Clearly if we use a moment distribution

$$M_y(\eta) = \alpha_n(\eta)$$

the resulting twist distribution will be [from (2.3.34)]

$$\alpha(y) = C_n(y)\int_0^1 \alpha_n^2(\eta)d\eta \tag{2.3.36}$$

Hence we may determine the expansion of the Green's function by successively applying moment distribution in the form of the expansion functions and measuring the resultant twist distribution. For the structural influence function this offers no advantage in practice since it is easier to apply point moments rather than moment distributions.

However, for the aerodynamic Green's functions the situation is different. In the latter case we are applying a certain twist to the wing and measuring the resulting aerodynamic moment distribution. It is generally desirable to maintain a smooth (if twisted) aerodynamic surface to avoid complications of flow separation and roughness and hence the application of a point twist distribution is less desirable than a distributed one. We quickly summarize the key relations for determining the aerodynamic influence function. Assume

$$A^{L\alpha}(y,\eta) = \sum_n A_n^{L\alpha}(y)\alpha_n(\eta) \tag{2.3.37}$$

We know that

$$C_L(y) = \int_0^1 A^{L\alpha}(y,\eta)\alpha(\eta)d\eta \tag{2.3.38}$$

For orthogonal functions, α_n we determine from (2.3.37) that

$$A_n^{L\alpha}(y) = \frac{\int_0^1 A^{L\alpha}(y,\eta)\alpha_n d\eta}{\int_0^1 \alpha_n^2(\eta)d\eta} \tag{2.3.39}$$

Applying the twist distribution $\alpha = \alpha_n(\eta)$ to the wing, we see from (2.3.38) and (2.3.39) that the resulting lift distribution is

$$C_L(y) = A_n^{L\alpha}(y)\int_0^1 \alpha_n^2(\eta)d\eta \tag{2.3.40}$$

Hence by measuring the lift distributions on 'warped wings' with twist distributions $\alpha_n(\eta)$ we may completely determine the aerodynamic influence function in terms of its expansion (2.3.37). This technique or a similar one has been used occasionally,[19] but not as frequently as one might expect, possibly because of the cost and expense of testing the number of wings sufficient to establish the convergence of the series. In this regard, if one uses the α_n for a Galerkin or modal expansion solution for the complete aeroelastic problem one can show that the number of C_n, $A_n^{L\alpha}$ required is equal to the number of modes, α_n, employed in the twist expansion.

2.4 Two Dimensional Aeroelastic Model of Lifting Surfaces

We consider in turn, structural modeling, aerodynamic modeling, the combining of the two into an aeroelastic model, and its solution.

[19] Covert [10].

2.4.1 Two Dimensional Structures—Integral Representation

The two dimensional or plate analog to the one-dimensional or beam-rod model is

$$w(x, y) = \iint C^{wp}(x, y; \xi, \eta) p(\xi, \eta) d\xi \, d\eta \tag{2.4.1}$$

where

w	vertical deflection at a point, x, y, on plate
p	force/area (pressure) at point ξ, η on plate
C^{wp}	deflection at x, y due to unit pressure at ξ, η

Note that w and p are taken as positive in the same direction. For the special case where

$$w(x, y) = h(y) + x\alpha(y) \tag{2.4.2}$$

and

$$C^{wp}(x, y; \xi, \eta) = C^{hF}(y, \eta) + xC^{\alpha F}(y, \eta) + \xi C^{hM}(y, \eta) + x\xi C^{\alpha M}(y, \eta) \tag{2.4.3}$$

with the definitions

C^{hF} is the deflection of y axis at y due to unit force F
$C^{\alpha F}$ is the twist about the y axis at y due to unit force F, etc.,

we may retrieve our beam-rod result. Note that (2.4.2) and (2.4.3) may be thought of as polynomial (Taylor Series) expansions of deflections.

Substituting (2.4.2), (2.4.3) into (2.4.1), we have

$$\begin{aligned} h(y) + x\alpha(y) = [&\int C^{hF} \left(\int p(\xi, \eta) d\xi \right) d\eta \\ &+ \int C^{hM} \left(\int \xi p(\xi, \eta) d\xi \right) d\eta \;] \\ &+ x \, [\int C^{\alpha F} \left(\int p(\xi, \eta) d\xi \right) d\eta \\ &+ \int C^{\alpha M} \left(\int \xi p(\xi, \eta) d\xi \right) d\eta \;] \end{aligned} \tag{2.4.4}$$

If y, η lie along an elastic axis, then $C^{hM} = C^{\alpha F} = 0$. Equating coefficients of like powers of x, we obtain

$$h(y) = \int C^{hF}(y,\eta)F(\eta)\,d\eta \tag{2.4.5}$$

$$\alpha(y) = \int C^{\alpha M}(y,\eta)M(\eta)\,d\eta \tag{2.4.6}$$

where

$$F \equiv \int p\,d\xi, \quad M \equiv \int p\xi\,d\xi$$

(2.4.6) is our previous result. Since for static aeroelastic problems, M is only a function of α (and not of h), (2.4.6) may be solved independently of (2.4.5). Subsequently (2.4.6) may be solved to determine h if desired. (2.4.5) has no effect on divergence or control surface reversal, of course, and hence we were justified in neglecting it in our previous discussion.

2.4.2 Two Dimensional Aerodynamic Surfaces—Integral Representation

In a similar manner (for simplicity we only include deformation dependent aerodynamic forces to illustrate the method),

$$\frac{p(x,y)}{q} = \iint A^{pw_x}(x,y;\xi,\eta)\frac{\partial w}{\partial \xi}(\xi,\eta)\frac{d\xi}{c_r}\frac{d\eta}{l} \tag{2.4.7}$$

where

A^{pw_x} nondimensional aerodynamic pressure at x, y due to unit $\partial w/\partial \xi$ at point ξ, n

c_r reference chord, l reference span

For the special case

$$w = h + x\alpha$$

and, hence,

$$\frac{\partial w}{\partial x} = \alpha$$

we may retrieve our beam-rod aerodynamic result.

For example, we may compute the lift as

$$L \equiv \int p dx = q c_r \int_0^1 A^{L\alpha}(y, \eta)\alpha(\eta)\frac{d\eta}{l} \tag{2.4.8}$$

where

$$A^{L\alpha} \equiv \iint A^{pw_x}(x, y; \xi, \eta)\frac{d\xi}{c_R}\frac{dx}{c_r}$$

2.4.3 Solution by Matrix-Lumped Element Approach

Approximating the integrals by sums and using matrix notation, (2.4.1) becomes

$$\{w\} = \Delta\xi\,\Delta\eta[C^{wp}]\{p\} \tag{2.4.9}$$

and (2.4.7) becomes

$$\{p\} = q\frac{\Delta\xi}{c_r}\frac{\Delta\eta}{l}[A^{pw_x}]\left(\frac{\partial w}{\partial \xi}\right) \tag{2.4.10}$$

Now

$$\left(\frac{\partial w}{\partial \xi}\right) \cong \frac{w_{i-1} - w_{i-1}}{2\Delta\xi}$$

is a difference representation of the surface slope. Hence

$$\left(\frac{\partial w}{\partial \xi}\right) = \frac{1}{2\Delta\xi}[W]\{w\} = \frac{1}{2\Delta\xi}\begin{bmatrix} [W] & [0] & [0] & [0] \\ & [W] & [0] & [0] \\ & & [W] & [0] \\ & & & [W] \end{bmatrix}\{w\} \tag{2.4.11}$$

is the result shown for *four* spanwise locations,[20] where

[20] For definiteness consider a rectangular wing divided up into small (rectangular) finite difference boxes. The weighting matrix [(W)] is for a given spanwise location and various chordwise boxes. The elements in the matrices, $\{\partial w/\partial \xi\}$ and $\{w\}$, are ordered according to fixed spanwise location and then over all chordwise locations. This numerical scheme is only illustrative and not necessarily that which one might choose to use in practice.

$$[W] = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdot \\ -1 & 0 & 1 & 0 & \cdot \\ 0 & -1 & 0 & 1 & \cdot \\ & & \cdots & & \\ \cdots & 0 & 0 & -1 & 0 \end{bmatrix}}_{number\ of\ chordwise\ location} \tag{2.4.12}$$

is a numerical weighting matrix. From (2.4.9)–(2.4.11), we obtain an equation for w,

$$[D]\{w\} \equiv \left[\begin{bmatrix} \diagdown & & \\ & 1 & \\ & & \diagdown \end{bmatrix} - q \frac{(\Delta\xi)^2}{c_r} \frac{(\Delta\eta)^2}{l} \frac{l}{2\Delta\xi} [C^{wp}][A^{pw_x}][W] \right] \{w\} = \{0\} \tag{2.4.13}$$

For divergence

$$|D| = 0$$

which permits the determination of q_D.

2.5 Other Physical Phenomena

2.5.1 Fluid Flow Through a Flexible Pipe

Another static aeroelastic configuration exhibiting divergence is a long slender pipe with a flowing fluid.[21] See Fig. 2.15. We shall assume the fluid is incompressible and has no significant variation across the cross-section of the pipe. Thus, the aerodynamic loading per unit length along the pipe is (invoking the concept of an equivalent fluid added mass moving with the pipe and including the effect of convection velocity[22] U),

$$-L = \rho A \left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right]^2 w = \rho A \left[\frac{\partial^2 w}{\partial t^2} + 2U \frac{\partial^2 w}{\partial x \partial t} + U^2 \frac{\partial^2 w}{\partial x^2} \right] \tag{2.5.1}$$

where

$A \equiv \pi R^2$	open area for circular pipe
ρ, U	fluid density, axial velocity
w	transverse deflection of the pipe
x	axial coordinate

[21] Housner [11].

[22] See Sect. 2.3.4.

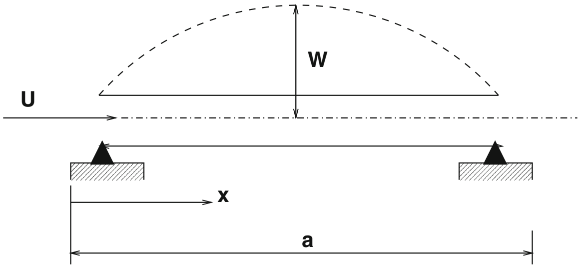


Fig. 2.15 Fluid flow through a flexible pipe

t time

The equation of motion for the beam-like slender pipe is

$$EI\,\frac{\partial^4 w}{\partial x^4} + m_p \frac{\partial^2 w}{\partial t^2} = L \tag{2.5.2}$$

where
$m_p \equiv \rho_p 2\pi Rh$ for a thin hollow circular pipe of thickness h, mass per unit length
EI beam bending stiffness
Both static and dynamic aeroelastic phenomena are possible for this physical model but for the moment we shall only consider the former. Further we shall consider for simplicity simply supported or pinned boundary conditions, i.e.,

$$w = 0$$

and

$$M \equiv EI\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, a \tag{2.5.3}$$

where M is the elastic bending moment and a, the pipe length.

Substituting (2.5.1) into (2.5.2) and dropping the time derivatives consistent with limiting our concern to static phenomena, we have

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A U^2 \frac{\partial^2 w}{\partial x^2} = 0 \tag{2.5.4}$$

subject to boundary conditions

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, a \tag{2.5.5}$$

The above equations can be recognized as the same as those governing the buckling of a beam under a compressive load of magnitude,[23] P. The equivalence is

$$P = \rho U^2 A$$

Formally we may compute the buckling or divergence dynamic pressure by assuming[24]

$$w = \sum_{i=1}^{4} A_i e^{p_i x}$$

where the p_i are the four roots of the characteristic equation associated with (2.5.4),

$$EIp^4 + \rho U^2 A p^2 = 0$$

Thus

$$p_{1,2} = 0$$
$$p_3, p_4 = \pm i \left(\frac{\rho U^2 A}{EI} \right)^{\frac{1}{2}}$$

and

$$w = A_1 + A_2 x + A_3 \sin \frac{\lambda x}{a} + A_4 \cos \frac{\lambda x}{a} \tag{2.5.6}$$

where

$$\lambda^2 \equiv \left(\frac{\rho U^2 A}{EI} \right) a^2$$

Using the boundary conditions (2.5.5) with (2.5.6) we may determine that

$$A_1 = A_2 = A_4 = 0$$

and either $A_3 = 0$ or $\sin \lambda = 0$

For nontrivial solutions

$$A_3 \neq 0$$

and

---

[23] Timoshenko and Gere [3].

[24] Alternatively one could use Galerkin's method for (2.5.4) and (2.5.5) or convert them into an integral equation to be solved by the 'lumped element' method.

$$\sin \lambda = 0$$

or

$$\lambda = \pi, 2\pi, 3\pi, etc. \tag{2.5.7}$$

Note that $\lambda = 0$ is a trivial solution, e.g., $w \equiv 0$.

Of the several eigenvalue solutions the smallest nontrivial one is of the greatest physical interest, i.e.,

$$\lambda = \pi$$

The corresponding divergence or buckling dynamic pressure is

$$\rho U^2 = \frac{EI}{Aa^2}\pi^2 \tag{2.5.8}$$

Note that λ^2 is a nondimensional ratio of aerodynamic to elastic stiffness; we shall call it and similar numbers we shall encounter an 'aeroelastic stiffness number'. It is as basic to aeroelasticity as Mach number and Reynolds number are to fluid mechanics. Recall that in our typical section study we also encountered an 'aeroelastic stiffness number', namely,

$$\frac{qs\frac{\partial C_L}{\partial \alpha}e}{K_\alpha}$$

as well as in the (uniform) beam-rod wing model,

$$\frac{q(lc)e\frac{\partial C_L}{\partial}\alpha}{\frac{GJ}{l}}$$

2.5.2 (Low Speed) Fluid Flow Over a Flexible Wall

A mathematically similar problem arises when a flexible plate is embedded in an otherwise rigid surface. See Fig. 2.16. This is a simplified model of a physical situation which arises in nuclear reactor heat exchangers, for example. Aeronautical applications may be found in the local skin deformations on aircraft and missiles. Early airships may have encountered aeroelastic skin buckling.[25]

For a one dimensional (beam) structural representation of the wall, the equation of equilibrium is, as in our previous example,

$$EI\frac{\partial^4 w}{\partial x^4} = L$$

---

[25] Shute [12], p. 95.

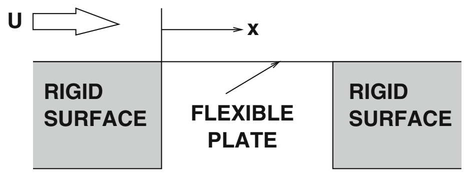


Fig. 2.16 Fluid flow over a flexible wall

Also, as a rough approximation, it has been shown that the aerodynamic loading may be written as[26]

$$L \sim \rho U^2 \frac{\partial^2 w}{\partial x^2}$$

Hence using this aerodynamic model, there is a formal mathematical analogy to the previous example and the aeroelastic calculation is the same. For more details and a more accurate aerodynamic model, the cited references should be consulted.

2.6 Sweptwing Divergence

A swept wing, one whose elastic axis is at an oblique angle to an oncoming fluid stream, offers an interesting variation on the divergence phenomenon. Consider Fig. 2.17. The angle of sweep is that between the axis perpendicular to the oncoming stream (y axis) and the elastic axis ($\overline{y}$ axis). It is assumed that the wing can be modeled by the bending-torsion deformation of a beam-rod. Thus the two structural equations of equilibrium are

Bending equilibrium of a beam-rod

$$\frac{d^2}{d\bar{y}^2}\left(EL\frac{d^2h}{d\bar{y}^2}\right) = -\bar{L} \tag{2.6.1}$$

Torsional equilibrium of a beam-rod

$$\frac{d^2}{d\bar{y}^2}\left(GJ\frac{d\alpha_e}{d\bar{y}}\right) + \bar{M}_y = 0 \tag{2.6.2}$$

Here h is the bending displacement of the elastic axis and is assumed positive downward. α_e, the elastic twist about the $\overline{y}$ axis, is positive nose up.

Now consider the aerodynamic model. Consider the velocity diagram, Fig. 2.18. A strip theory aerodynamic model will be invoked with respect to chords perpendicular to the $\overline{y}$ axis. Thus the lift and aerodynamic moment per unit span are given by

[26] Dowell [13], p. 19, Kornecki [14], Kornecki, Dowell and O'Brien [15].

$$\bar{L} = \bar{C}_L \bar{c} \bar{q} \tag{2.6.3}$$

and

$$\begin{aligned}\bar{M}_y &= \bar{L}\bar{e} + \bar{M}_{AC} \\ &= \bar{C}_L c \bar{q} e + \bar{C}_{MAC} \bar{c}^2 \bar{q}\end{aligned} \tag{2.6.4}$$

where $\bar{q} = \frac{1}{2}\rho(U \cos \Lambda)^2 = q \cos^2 \Lambda$. Also $\bar{C}_L$ is related to the (total) angle of attack, α_T, by

$$\bar{C}_L(\bar{y}) = \frac{\partial \bar{C}_L}{\partial \alpha} \alpha_T(\bar{y}) \tag{2.6.5}$$

where

$$\alpha_T = \alpha_e + \frac{dh}{d\bar{y}} \tan \Lambda \tag{2.6.6}$$

To understand the basis of the second term in (2.6.6), consider the velocity diagram of Fig. 2.19. From this figure we see the fluid velocity normal to the wing is $U \sin \Lambda dh/d\bar{y}$ and thus the effective angle of attack due to bending of a swept wing is

$$U \sin \Lambda \frac{dh}{d\bar{y}} / U \cos \Lambda = U \frac{dh}{d\bar{y}} \tan \Lambda \tag{2.6.7}$$

From (2.6.1)–(2.6.6), the following form of the equations of equilibrium is obtained.

$$\frac{d^2}{d\bar{y}^2}\left(EI \frac{d^2 h}{d\bar{y}^2}\right) = -\frac{\partial \bar{C}_L}{\partial \alpha}\left[\alpha_e + \frac{dh}{d\bar{y}} \tan \Lambda\right] \bar{c} q \cos^2 \Lambda \tag{2.6.7}$$

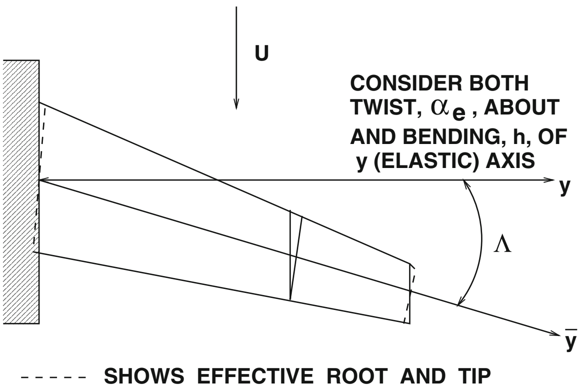


Fig. 2.17 Sweptwing geometry

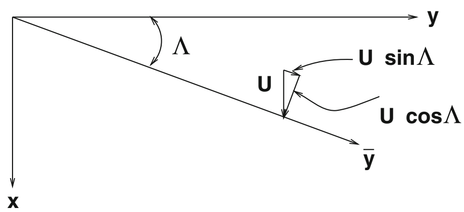


Fig. 2.18 Velocity diagram in the $x, y(\bar{x}, \bar{y})$ plane

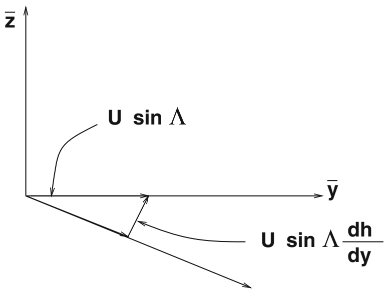


Fig. 2.19 Velocity diagram in $\bar{y}, \bar{z}$ plane

$$\frac{d}{d\bar{y}}\left(GJ\frac{d\alpha_e}{d\bar{y}}\right) + \frac{\partial \bar{C}_L}{\partial \alpha}\left[\alpha_e + \frac{dh}{d\bar{y}}\tan\Lambda\right]\bar{c}q\cos^2\Lambda\bar{e} + \bar{C}_{MAC}\bar{c}^2 q\cos^2\Lambda = 0 \tag{2.6.8}$$

Special cases;

- If the beam is very stiff in bending, $EI \to \infty$, then from (2.6.7), $h \to 0$. (2.6.8) then is very similar to the torsional equation for an unswept wing with slightly modified coefficients.
- If the beam-rod is very stiff in torsion, $GJ \to \infty$, then from (2.6.8), $\alpha \to 0$. (2.6.7) then reduces to

$$\frac{d^2}{d\bar{y}^2}\left(EI\frac{d^2h}{d\bar{y}^2}\right) + \frac{\partial \bar{C}_L}{\partial \alpha}\sin\Lambda\cos\Lambda\bar{c}q\frac{dh}{d\bar{y}} = 0 \tag{2.6.9}$$

As we shall see, divergence in bending alone is possible even for a swept wing which is very stiff in torsion. This is not possible for an unswept wing.

To illustrate this, consider a further special case, namely a beam with constant spanwise properties. Introducing appropriate non-dimensionalization then (2.6.9) becomes

$$\frac{d^4h}{d\tilde{y}^4} + \lambda\frac{dh}{d\tilde{y}} = 0 \tag{2.6.10}$$

where $\tilde{y} \equiv \bar{y}/l$ and

$$\lambda = \frac{\frac{\partial \bar{C}_L}{\partial \alpha} q \bar{c} \bar{l}^3}{EI} \sin \Lambda \cos \Lambda$$

The boundary conditions associated with this differential equation are zero deflection and slope at the root:

$$h = \frac{dh}{d\tilde{y}} = 0 \quad @ \quad \tilde{y} = 0 \tag{2.6.11}$$

and zero bending moment and shear force at the tip

$$EI\frac{d^2h}{d\tilde{y}^2} = EI\frac{d^3h}{d\tilde{y}^3} = 0 \quad @ \quad \tilde{y} = 1 \tag{2.6.12}$$

(2.6.10)–(2.6.12) constitute an eigenvalue problem. The eigenvalues of λ are all negative and the lowest of these provides the divergence condition.

$$\lambda_D = -6.33 = \frac{\partial \bar{C}_L}{\partial \alpha} \frac{\sin \Lambda \cos \Lambda \bar{c} \bar{l}^3 q}{EI} \tag{2.6.13}$$

The only way the right hand side of (2.6.13) can be less than zero is if $\sin \Lambda < 0$ or $\Lambda < 0$.

Thus only swept forward wings can diverge in bending without torsional deformation. This suggests that swept forward wings are more susceptible to divergence than swept back wings. This proves to be the case when both bending and torsion are present as well.

For many years, the divergence tendency of swept forward wings precluded their use. In recent years composite materials provide a mechanism for favorable bending-torsion coupling which alleviates this divergence. For a modern treatment of these issues including the effects of composite structures two reports by Weisshaar [16, 17] are recommended reading.

A final word on how the eigenvalues are calculated. For (2.6.10)–(2.6.12), classical techniques for constant coefficient differential equations may be employed. See BAH, pp. 479–489. Even when both bending and torsion are included (2.6.7, 2.6.8), if the wing properties are independent of spanwise location, then classical techniques may be applied. Although the calculation does become more tedious. Finally, for a variable spanwise properties Galerkin's method may be invoked, in a similar though more elaborate manner to that used for unswept wing divergence.

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