CHAPTER 1

The language of symmetry

You boil it in sawdust, you salt it in glue
You condense it with locusts and tape
Still keeping one principal object in view–
To preserve its symmetrical shape.

The Hunting of the Snark, Lewis Carroll

Symmetry, to a mathematician, encompasses much more than it does in everyday usage. One of the pioneers of this grander view was the distinguished and influential German mathematician Felix Klein. In 1872, on the occasion of his appointment to a chair at the University of Erlangen at the remarkably early age of 23, Klein proposed to the mathematical world that it should radically extend its received view of symmetry, to encompass things which had never been thought of as symmetrical before. Our quotation from Lewis Carroll, alias Charles Dodgson, mathematician and Fellow of Christ Church College, Oxford, was written only four years later. Perhaps Dodgson had heard about Klein's ideas and had them in mind as he composed his nonsensical verse.

In his historic short article,[1] the young Klein synthesized over fifty years of mathematical development in a new and profoundly influential way. It is today difficult to fully appreciate the significance of what he said, because his lecture crystallized one of those paradigm shifts which, after they have happened, seem so obvious that it hard to imagine how anyone could ever have thought otherwise.

[1] With the erudite title *Vergleichende Betrachtungen über neuere geometrische Forschungen*, or 'Comparative considerations concerning recent researches in geometry', now universally known as the *Erlanger Programm*.

In a nutshell, Klein proposed viewing geometry as 'the study of the properties of a space which are invariant under a given group of transformations'. To study geometry, he said, one needed not only objects (triangles, circles, icosahedra, or much wilder things like the fractal pictures in this book), but also movements. In the classical Euclidean regime which had been around for over two millennia, these movements had always been rigid motions: pick up a figure and place an identical copy down in a new place. Klein's radical idea was

Cambridge University Press
0521352533 - Indra's Pearls: The Vision of Felix Klein
David Mumford, Caroline Series and David Wright
Excerpt
More information

Felix Christian Klein, 1849–1925

Felix Klein was born in Düsseldorf in what was then the Prussian empire in 1849. He studied mathematics and physics at the University of Bonn. He started out on his doctoral work aiming to be a physicist, but was drawn into geometry under the influence of his supervisor Plücker. Plücker died in 1868, the same year that Klein finished his doctorate, and Klein was the obvious person to complete his advisor's unfinished work. This brought him to the attention of Clebsch, one of the leading professors at Göttingen, who soon came to consider the young Klein likely to become the leading mathematician of his day.

Klein received his 'call' to the University of Erlangen in Bavaria in 1872 and did his most creative work in the next 10 years. In 1875 he moved to the Technische Hochschule at Munich where he found many excellent students and his great talent for teaching came into its own. In the same year he married Anne Hegel, granddaughter of the famous philosopher. In 1880 he moved to the highly stimulating mathematical environment at Leipzig. Here he developed the deep theory to which this book is devoted, but it was also here that his delicate health first collapsed under the strain of an intense rivalry in this work with the brilliant young French mathematician Poincaré. Klein spent the years 1883-1884 plagued by depression and never fully recovered his mathematical powers. His work relating to our subject was developed at great length in two treatises written jointly with Robert Fricke over the period 1890-1912.

In 1886 Klein moved to a chair in Göttingen where he remained until his retirement. Besides his mathematical ability, Klein had very considerable managerial and administrative talents, and it was his skill and energy which built up the famous mathematical school at Göttingen which flourished as the world's leading mathematical centre until it was dismantled by Hitler in the 1930's. Klein's influence spread far, partly via the many foreigners who studied with him, among them the Americans Frank Cole and William Osgood, the Italians Luigi Bianchi and Gregorio Ricci-Curbastro, and some pioneering women like Mary Newson and Grace Chisholm Young. Around the turn of the century, he began to take a lively interest in the teaching of mathematics, encouraging the introduction of calculus into the school curriculum. He retired due to ill health in 1913 and died in Göttingen at the age of 76.

that other movements, which might stretch or twist the objects quite drastically, could be thought of geometrical movements too. In this way geometry could be taken to encompass a much wider variety of set-ups than those previously conceived. Geometers should study those features

Photograph courtesy of the Archive of the Technical University of Braunschweig. Thanks also to Hans Opolka.

Robert Fricke, 1861–1930

Robert Fricke was born in Helmstedt, Germany. He studied and lectured in Göttingen before graduating in 1885 from Leipzig with a thesis written under Klein. His collaboration with Klein began with the publication of the two volumes of their first opus *Vorlesungen über die Theorie der elliptischen Modulfunktionen* in 1890 and 1892. During this period Fricke taught in two gymnasia in Braunschweig, and, more interestingly, was tutor to two sons of the Prussian Prince Regent Albrecht. Fricke did his 'Habilitation' in Kiel, following which in 1892 he lectured in Göttingen as a 'Privatdozent'. In 1894 he was appointed as Dedekind's successor in the Carolo-Wilhemina University in Braunschweig, and his always friendly relationship with Klein was cemented when he married Klein's niece Eleonora Flender later the same year.

Fricke was highly respected both as a mathematician and personally, working closely with Klein to develop much of the theory of what are now called Kleinian groups, the topic of our book. He played a leading role in the University administration, being Rektor from 1904–6 and again from 1921–3. His activity extended to state educational affairs where his experience as a school teacher was valued and he held several official posts. These many duties account for the long delay between the appearance of the first and second volumes (in 1897 and 1912 respectively) of the second opus *Vorlesungen über die Theorie der automorphen Funktionen*, of which Fricke was really the author, although with much input from Klein. In the final volume, Fricke took the opportunity to use new developments like Cantor's set theory and Brouwer's theory of dimension to solve a number of problems which had been unresolved in the past. He remained in post in Braunschweig until his death.

of the objects which the movements left unchanged, as so delightfully suggested in Carroll's verse.

There are two sides to the circle of ideas Klein was playing with: the idea of similar or symmetrical objects and the idea of 'transformation' or 'movement'. He brought these together using the idea of a **group**, a concept originally developed 50 years earlier by another very young mathematician, Évariste Galois. In his short career from 1829-1832, Galois saw that the solutions of a polynomial could be understood by defining their 'symmetries'; thus for example $+\sqrt{2}$ and $-\sqrt{2}$ can be considered as symmetrical solutions of the equation $x^2 = 2$. These ideas were way beyond the comprehension of any of his contemporaries and his work narrowly missed being completely lost.[1] Klein realised that, rather than trying to catalogue all possible kinds of symmetrical patterns and display them like the medieval Islamic builders of the Alhambra, the group concept gave a very simple and yet immensely powerful mathematical machinery for describing symmetries of all possible types.

[1]Galois' story is one of the great romances of mathematics, of which more on p. 20.

The group concept simply describes the rules which govern the *repetitive* aspect of symmetry. For example, if you are allowed to make a move once, then you can make the same move again, and again, and again. Such repetition may bring you back to exactly the position from which you started (as in reflecting in a mirror, when two reflections bring you back where you began) or it may lay out an ever expanding mosaic of objects, for example tiles, in a regular pattern over a larger and larger area, like a vast floor.

In short, the usual way to think of symmetry is in terms of design and proportion, a slightly elusive quality of being balanced and correct.[2] Mathematicians, since Klein, have had at their disposal a more precise version: symmetry is balance created by repetitions of many movements of the same kind, specifically, by all the movements in some particular group. These two ideas, introducing the group idea into geometry and widening the class of movements to be studied, formed the background to Klein's own work. These were the themes he brought together in his famous grand plan.

[2]In Chambers dictionary we find: 'exact correspondence of parts on either side of a straight line or plane, or about a centre of axis: balance or due proportion: beauty of form: disposition of parts'.

A large part of Klein's later work became bound up with exhibiting and studying one particular new kind of symmetry which we are going to be explaining in this book. Before we enter these new realms, though, let's spend some time getting acquainted with the familiar Euclidean symmetries from Klein's new point of view.

A taxonomy of symmetry

Our first picture, Figure 1.1, shows a satellite photograph of the agricultural state of Iowa. Beside it is an idealized version in which we have perfected the symmetry. The landscape stretches out as far as the eye can see, broken up into a regular pattern of mile-wide farms, each containing one farmhouse, one pond and one tree. This world is so symmetrical that travelling one mile either north, south, east or west, you will reach a new position from which your view is completely indistinguishable from that which you had before.

What interests the mathematician studying symmetry is not so much the details of the figure, whether each field contains one sheep or two cows, as the movements or **motions** you have to make to implement the repeats. A motion is an easy thing to perform on a computer: draw a

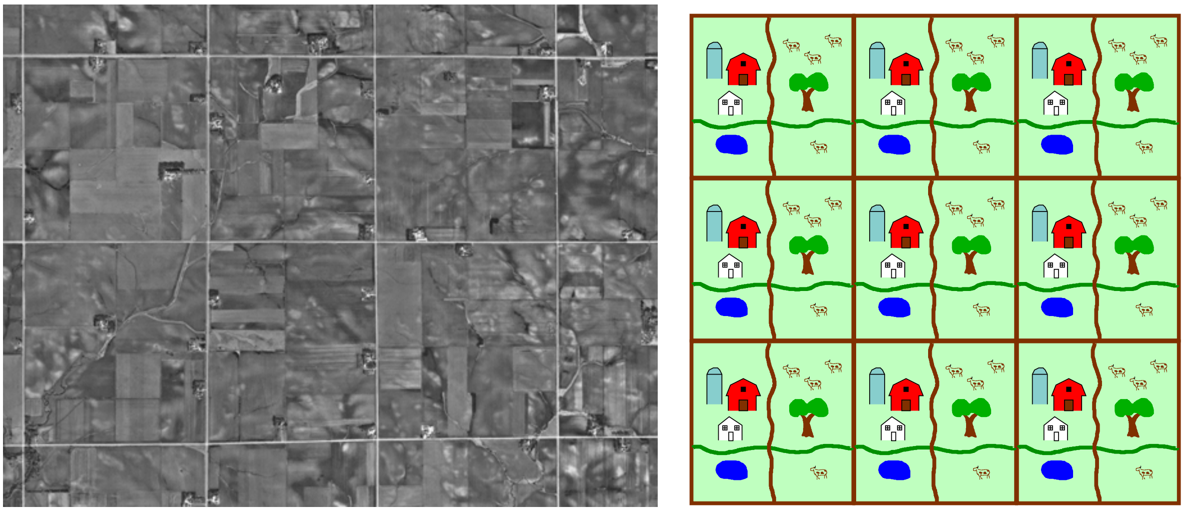

Figure 1.1. The real satellite photograph of Iowa on the left is not so very different from our idealized version on the right. Notice the brown 'north-south' and the green 'east-west' paths running across each farm.

flower, and then copy your picture over to another place on the page. You can either think that you moved the flower, or, and this is the point of view that the mathematician prefers, that you picked up the whole page and put it down again so that the flower appears in its new place. In this way, the movement of the flower can be implemented by a definite motion of the whole plane. Figure 1.2 shows two flowers which have been transported around by **translation**. The flowers are quite different, but to the mathematician the underlying **translational symmetry** of the two pictures is exactly the same.

Symmetry is created by repeating or **iterating** the same motion a number of times. A pattern or object is **symmetrical** with respect to the motion if its individual points change position, but the pattern or object as a whole remains unchanged. The simplest symmetry is that of a shape which repeats itself infinitely often, moving the same fixed distance in the same fixed direction each time. A good example is a straight railroad track[1] across a flat prairie, extending forwards and backwards to the horizon as far as the eye can see, as in Figure 1.3. In your mind's eye, slide the entire track forward along its length just enough to move each tie or sleeper from its original position on the prairie to the position of the next. This is the motion of **translation**: the translation distance is the distance between ties (alias sleepers) and the direction is the direction of the track. After the motion, the track looks exactly as it did before, but, in fact, each tie has moved ahead and taken the place of the next. Notice that once again two things are involved here: an abstract movement, translation, and a physical object, the track. To say that an object has translational symmetry means that

[1] Railway lines, found in England, aren't usually so long and straight.

when it is physically translated to a new position, then although its parts are shifted, the view of the object as a whole is unchanged.

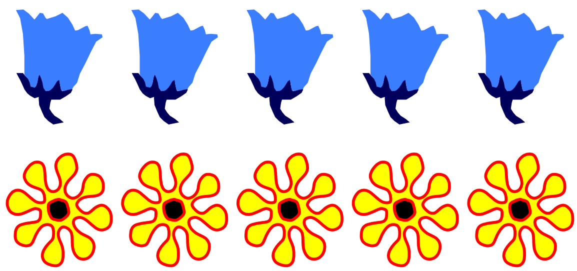

Figure 1.2. Two rows of flowers moved along by the same translations. The flowers are quite different but the symmetry of the two rows is the same.

Architects have made glorious use of translational symmetry. The visual effect of a repeating structure can be very graceful, as in the frieze also shown in Figure 1.3.

The starfish in Figure 1.4 is a good example of **rotational** symmetry. It has 12 arms, so if you rotate it by $360^\circ \div 12 = 30^\circ$ about its middle, each arm moves but the starfish looks just the same as it did before.

A third type of symmetry is **bilateral**, symmetry under **reflection**. This is the left-right symmetry of our bodies which you can see depicted in Leonardo's famous drawing in Figure 1.4. Imagine a vertical plane or mirror separating the left and right sides of a standing person and imagine moving every atom on the left side of the mirror to a point on the same horizontal level and at the same distance from the mirror on the right, and vice versa. Portraitists will object that the left side of the face expresses different facets of our personalities from the right and doctors will be confounded by the non-standard locations of colon and liver, but on the whole we can say that the body is unchanged. Most vehicles, like cars, boats, bicycles and planes, are nearly bilaterally symmetric, particularly on the exterior. Perhaps we subconsciously mould them on ourselves.

These three types of symmetries can often be seen in the same figure occurring in multiple ways. For instance, consider the wasps' nest shown in Figure 1.5. You should be able to spot each of the following types:

- translational symmetries in each of three different directions,
- rotational symmetry of 120° around points where 3 cells meet,
- rotational symmetry of 60° around the centre of each cell,
- reflectional symmetry in mirrors along edges where 2 cells meet,
- reflectional symmetry in mirrors through the midpoints of 2 opposite sides.

Cambridge University Press
0521352533 - Indra's Pearls: The Vision of Felix Klein
David Mumford, Caroline Series and David Wright
Excerpt
More information

Figure 1.3. Two manifestations of translational symmetry: train tracks across the prairies and a frieze from the ancient Mexican city of Oaxaca.

You can doubtless find other, more complicated symmetries, but all of them can be made from combinations of the ones above. Exactly how this is done mathematically we shall investigate below.

Before leaving the basic symmetry types, we would like to introduce Dr. Stickler, a long time associate of the authors, who will become a trusted guide in the pages which follow. You can see his photo in the top left frame of Figure 1.6, while the three other frames show him being moved around by translational, reflectional and rotational symmetries of the plane. As we extend our ideas about what we mean

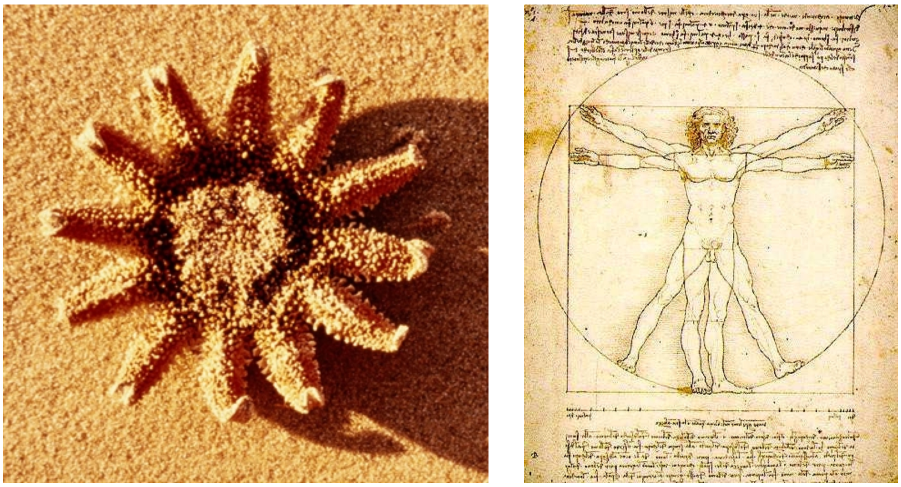

Figure 1.4. Left: A starfish displays rotational symmetry. This one has 12 arms, so it is symmetrical under rotation by $360° \div 12 = 30°$. Right: Leonardo's famous pen and ink study of the proportions of the male figure, showing nearly perfect bilateral symmetry.

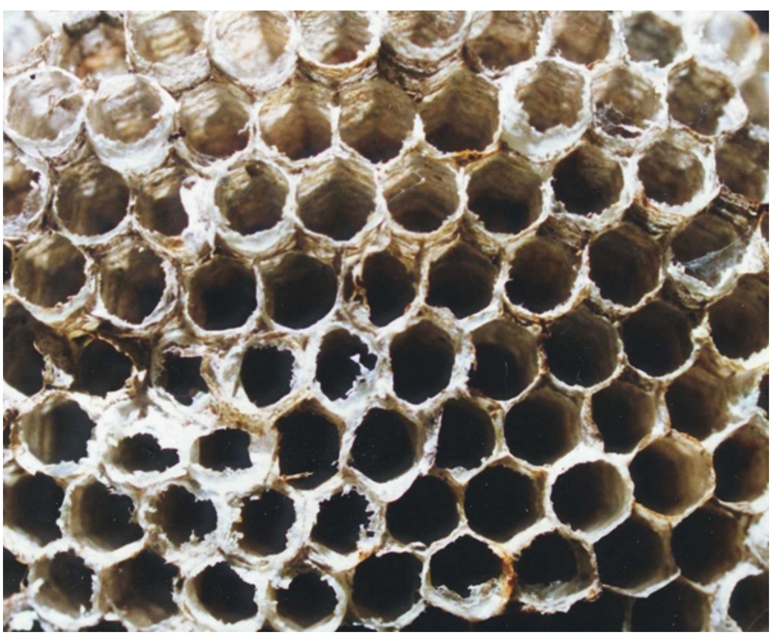

Figure 1.5. A wasps' nest is a wonderful example of an object exhibiting multiple symmetries of all the three kinds.

by symmetry, Dr. Stickler's rather staid progress here will be replaced by new kinds of motions which move him around in ever more exotic ways.

We have seen that there are two ways of thinking about symmetry. On the one hand, we can point to symmetrical objects, saying they

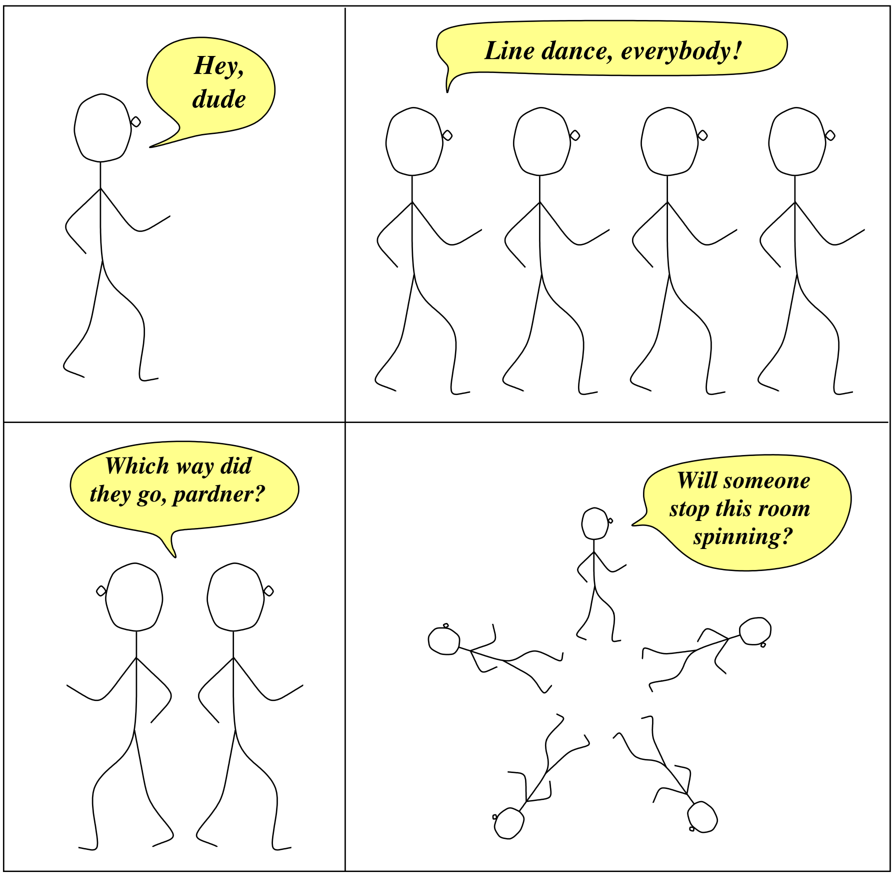

Figure 1.6. In the top left frame you see a still photo of Dr. Stickler, while the other three frames show him being moved around by translations, reflections and rotations respectively.

are examples of translation, rotational or reflectional symmetry as the case may be. However a second, deeper way of thinking about symmetry is to abstract from the picture and study the motions it embodies by themselves. The first way of thinking is more compelling because it gives you something tangible and visible, but the second is more fundamental because the abstraction contains no irrelevant detail, allowing us to focus on the underlying pattern itself.

Transformations of the plane

We have been talking in the last section about symmetries as 'movements' or 'motions' of the plane. Mathematicians, who even more than lawyers like to be extremely exact about their language, commonly speak in terms of a rather broader concept **transformation** or its common synonyms **mapping** or **map**. As in most of mathematics, these ordinary words are being used in specialised and very precise ways.

We had better explain carefully what is meant, because without using them it would be virtually impossible to write this book. In its widest sense, a **transformation** of the plane means simply a rule which assigns to each point P in the plane a new point Q. The rule might be: 'the new point is 3 inches to the left of the old one', or 'the new position is obtained by rotating 90° with respect to the centre point O'. The new point Q we get to is called the **image**[1] of the starting point P. Although a tremendous number of rules can be thought up, we shall only be thinking about rules whose effect can be undone. For example, the effect of the translation 'move 3 inches to the left', can be undone by the rule 'move 3 inches to the right', see Figure 1.7.

[1]Mathematicians have a predilection for giving special names to *everything* they talk about. They don't do this just to sound imposing: think of it like a surgeon laying out her instruments and checking everyone knows their correct names, to make sure she gets the right thing when she calls to the nurse.

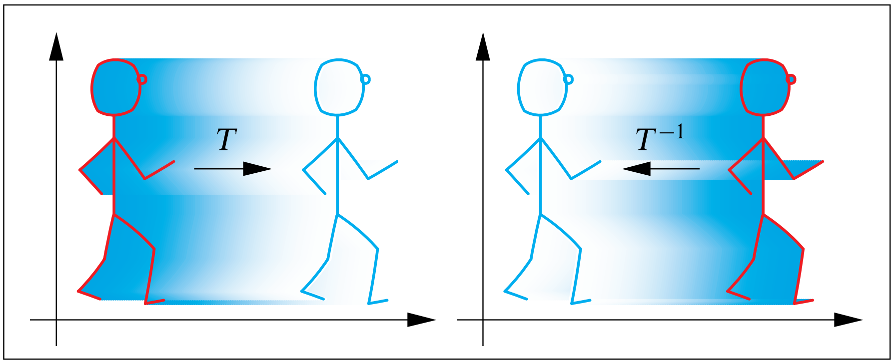

Figure 1.7. Dr. Stickler being transported by the transformation T which moves him 3 inches to the right. In the second frame you can see him transported by the inverse transformation T^{-1}, which moves him back 3 inches to the left.

A more concrete way to think of a transformation is to consider it as a procedure for physically moving the points of the plane to new locations. During the motion, the relative positions of points may get hugely distorted, but whatever figures or objects are in the plane get carried along and move to their new positions at the same time. You can see the effect of such a distortion on the classical bust of Paolina shown on the left in Figure 1.8. Even though her shape in the right hand picture has been radically altered, each of her features is still clearly identifiable. To create the right hand picture, we used a definite rule which told us exactly how to move each point in the left hand frame to a new point on the right. While leaving all her features intact, this transformation has fundamentally altered her shape.

To talk about this subject sensibly we need some notation to avoid the huge mouthfuls we found ourselves using above. We usually use letters like S and T to represent transformations and, following Euclid, use letters like P and Q to represent points in the plane. Like most mathematicians, we shall write $T(P)$ for the image of P under T; that is, $T(P)$ means 'the new point obtained by applying the rule T to the