Part I

Finite and affine reflection groups

Chapter 1

Finite reflection groups

In this chapter we begin the study of finite groups generated by reflections in (real) euclidean spaces. Our main tool will be a well-chosen set of vectors ('roots') orthogonal to reflecting hyperplanes (1.2). A set of 'simple roots' (1.3) yields an efficient generating set for the group (1.5), leading eventually to a very simple presentation by generators and relations as a 'Coxeter group' (1.9). The latter part of the chapter treats a number of geometric and group-theoretic topics, all of which involve the 'parabolic' subgroups generated by sets of simple reflections (1.10), e.g., Poincaré polynomials (1.11), fundamental domains (1.12), and the Coxeter complex (1.15).

1.1 Reflections

Recall what is meant by a **reflection** in a (real) euclidean space V endowed with a positive definite symmetric bilinear form (λ, μ). A reflection is a linear operator s on V which sends some nonzero vector α to its negative while fixing pointwise the hyperplane H_α orthogonal to α. We may write $s = s_\alpha$, bearing in mind however that $s_\alpha = s_{c\alpha}$ for any nonzero $c \in \mathbb{R}$. There is a simple formula:

$$s_\alpha \lambda = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha.$$

Indeed, this is correct when $\lambda = \alpha$ and when $\lambda \in H_\alpha$; so it is correct for all $\lambda \in V = \mathbb{R}\alpha \oplus H_\alpha$. A quick calculation (left to the reader) shows that s_α is an orthogonal transformation, i.e., $(s_\alpha\lambda, s_\alpha\mu) = (\lambda, \mu)$ for all $\lambda, \mu \in V$. It is clear that $s_\alpha^2 = 1$, so s_α has order 2 in the group $\mathrm{O}(V)$ of all orthogonal transformations of V.

A finite group generated by reflections (or **finite reflection group**, for short) is an especially interesting type of finite subgroup of $\mathrm{O}(V)$.

The purpose of this chapter and the next will be to classify and describe all such groups. In doing so, we shall explore alternately the internal structure of the group itself (e.g., the relations satisfied by the generating reflections) and the geometric aspects of the action of the group on V (e.g., fundamental domains).

Here are some basic examples, which should be kept in mind as the story unfolds. (They are labelled by 'types', in accordance with the classification to be carried out in Chapter 2.)

($I_2(m)$, $m \geq 3$) Take V to be the euclidean plane, and define $\mathcal{D}_m$ to be the **dihedral group** of order $2m$, consisting of the orthogonal transformations which preserve a regular m-sided polygon centered at the origin. $\mathcal{D}_m$ contains m rotations (through multiples of $2\pi/m$) and m reflections (about the 'diagonals' of the polygon). Here 'diagonal' means a line bisecting the polygon, joining two vertices or the midpoints of opposite sides if m is even, or joining a vertex to the midpoint of the opposite side if m is odd. Note that the rotations form a cyclic subgroup of index 2, generated by a rotation through $2\pi/m$. The group $\mathcal{D}_m$ is actually generated by reflections, because a rotation through $2\pi/m$ can be achieved as a product of two reflections relative to a pair of adjacent diagonals which meet at an angle of $\theta := \pi/m$ (see Figure 1). Let

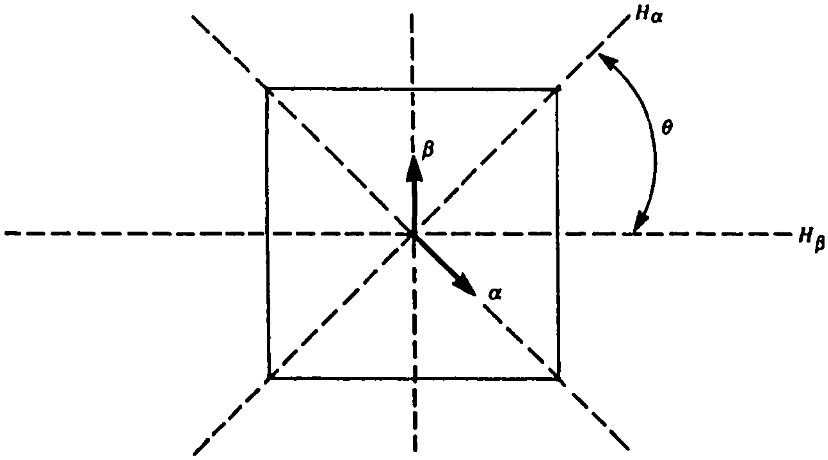

Figure 1: The case $m = 4$

the reflecting lines H_α and H_β contain these diagonals, and choose the orthogonal unit vectors $\alpha = (\sin\theta, -\cos\theta)$ and $\beta = (0, 1)$ which form an obtuse angle of $\pi - \theta$, so $(\alpha, \beta) = -\cos\theta$. To see that $s_\alpha s_\beta$ is a (counterclockwise) rotation through 2θ, take H_β to be the x-axis and compute with 2×2 matrices relative to the standard basis of $\mathbf{R}^2$:

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

Exercise 1. The reflections form a single conjugacy class in $\mathcal{D}_m$ when m is odd, but form two classes when m is even.

(A_{n-1}, $n \geq 2$) Consider the **symmetric group** $\mathcal{S}_n$. It can be thought of as a subgroup of the group $\mathrm{O}(n, \mathbf{R})$ of $n \times n$ orthogonal matrices in the following way. Make a permutation act on $\mathbf{R}^n$ by permuting the standard basis vectors $\varepsilon_1, \ldots, \varepsilon_n$ (permute the subscripts). Observe that the transposition (ij) acts as a reflection, sending $\varepsilon_i - \varepsilon_j$ to its negative and fixing pointwise the orthogonal complement, which consists of all vectors in $\mathbf{R}^n$ having equal ith and jth components. Since $\mathcal{S}_n$ is generated by transpositions, it is a reflection group. Indeed, it is already generated by the transpositions $(i, i+1), 1 \leq i \leq n-1$.

Exercise 2. Regarding $\mathcal{S}_n$ in this way as a subgroup of $\mathrm{O}(n, \mathbf{R})$, prove that the transpositions are the sole reflections belonging to $\mathcal{S}_n$.

When $\mathcal{S}_n$ acts on $\mathbf{R}^n$ in the way just described, it fixes pointwise the line spanned by $\varepsilon_1 + \ldots + \varepsilon_n$ (these are clearly the only fixed points) and leaves stable the orthogonal complement, the hyperplane consisting of vectors whose coordinates add up to 0. Thus $\mathcal{S}_n$ also acts on an $(n-1)$-dimensional euclidean space as a group generated by reflections, fixing no point except the origin. This accounts for the subscript $n-1$ in the label A_{n-1}. When a reflection group W acts on V with no nonzero fixed points, we say that W is **essential** relative to V. It is clear that any subgroup W of $\mathrm{O}(V)$ stabilizes the orthogonal complement V' of its space of fixed points and is essential relative to V'.

(B_n, $n \geq 2$) Again let $V = \mathbf{R}^n$, so $\mathcal{S}_n$ acts on V as above. Other reflections can be defined by sending an ε_i to its negative and fixing all other ε_j. These sign changes generate a group of order 2^n isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$, which intersects $\mathcal{S}_n$ trivially and is normalized by $\mathcal{S}_n$: conjugating the sign change $\varepsilon_i \mapsto -\varepsilon_i$ by a transposition yields another such sign change. Thus the semidirect product of $\mathcal{S}_n$ and the group of sign changes yields a reflection group W of order $2^n n!$. It is easy to check that W is essential.

(D_n, $n \geq 4$) We can get another reflection group acting on $\mathbf{R}^n$, a subgroup of index 2 in the group of type B_n just described: $\mathcal{S}_n$ clearly normalizes the subgroup consisting of sign changes which involve an *even* number of signs, generated by the reflections $\varepsilon_i + \varepsilon_j \mapsto -(\varepsilon_i + \varepsilon_j)$, $i \neq j$. So the semidirect product is also a reflection group (and is essential).

1.2 Roots

From now on we denote by W a finite reflection group, acting on the euclidean space V. The letter W is used because 'most' finite reflection groups turn out to be 'Weyl groups' (associated with semisimple Lie algebras or Lie groups). Much of the theory to be developed in this book is in fact motivated by the problems of Lie theory, cf. Bourbaki [1].

In order to understand the internal structure of W as an abstract group, we first explore the way in which W acts on V. Each reflection s_α in W determines a reflecting hyperplane H_α and a line $L_\alpha = \mathbf{R}\alpha$ orthogonal to it. The following result implies that W permutes the collection of all such lines.

Proposition *If $t \in \mathrm{O}(V)$ and α is any nonzero vector in V, then $ts_\alpha t^{-1} = s_{t\alpha}$. In particular, if $w \in W$, then $s_{w\alpha}$ belongs to W whenever s_α does.*

Proof. Obviously $ts_\alpha t^{-1}$ sends $t\alpha$ to its negative. So we need only show that $ts_\alpha t^{-1}$ fixes $H_{t\alpha}$ pointwise. Note that λ lies in H_α if and only if $t\lambda$ lies in $H_{t\alpha}$, since $(\lambda, \alpha) = (t\lambda, t\alpha)$. In turn, $(ts_\alpha t^{-1})(t\lambda) = ts_\alpha\lambda = t\lambda$ whenever λ lies in H_α. □

Thus W permutes the lines L_α, where s_α ranges over the set of reflections contained in W, via $w(L_\alpha) = L_{w\alpha}$. Only the lines L_α are determined by W, not the vectors α. However, if we select the pairs of unit vectors lying in all such lines, the collection of vectors so obtained will be stable under the action of W. It is this sort of geometric configuration which we shall emphasize below. Actually, we need not insist that the vectors be of equal length: only the stability under W is significant for our purposes. For example, the dihedral group $\mathcal{D}_4$ preserves the collection of eight vectors in $\mathbf{R}^2$:

$$\pm(1,0),\ \pm(1,1),\ \pm(0,1),\ \pm(-1,1)$$

For flexibility in some future arguments, it is most convenient to axiomatize the situation as follows. Take Φ to be a finite set of nonzero vectors in V satisfying the conditions:

(R1) $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$;
(R2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

Then define W to be the group generated by all reflections s_α, $\alpha \in \Phi$. Call Φ a **root system** with associated reflection group W. The elements of Φ are called **roots** because of the historical connection between Weyl groups and semisimple Lie algebras, where the notion of 'root' goes back ultimately to the characteristic roots of certain operators on the Lie algebra. However, our notion of 'root system' differs somewhat from that encountered in Lie theory; see 2.9 below.

As the previous discussion shows, any finite reflection group can be realized in this way, possibly for many different choices of Φ. Conversely, any group W arising from a root system is in fact finite. Indeed, each $s_\alpha (\alpha \in \Phi)$ and hence each element of W fixes pointwise the orthogonal complement of the subspace spanned by Φ. So only $w = 1$ can fix all elements of Φ. This means that the natural homomorphism of W into the symmetric group on Φ has trivial kernel, forcing W to be finite.

To recapitulate: our finite reflection group $W \subset \mathrm{O}(V)$ is henceforth to be studied in conjunction with a root system $\Phi \subset V$, subject only to (R1) and (R2) above. The choice of Φ is somewhat flexible. It might consist of unit vectors, or not. The reflections $s_\alpha (\alpha \in \Phi)$ might or might not be known to exhaust all reflections in W. The set Φ might span V, or not. All that really matters for later arguments is that (R1) and (R2) hold.

Remark. Given a root system Φ and corresponding reflection group W, define Φ' to be the set of unit vectors proportional to the vectors in Φ. Then Φ' is clearly a root system, with W as corresponding reflection group.

1.3 Positive and simple systems

Fix a root system Φ in the euclidean space V, so that W is the finite reflection group generated by all $s_\alpha (\alpha \in \Phi)$. While W is completely determined by the geometric configuration Φ, there is one serious drawback to using Φ as a tool in the classification of possible reflection groups: Φ may be extremely large compared with the dimension of V. For example, when W is a dihedral group, Φ may have just as many elements as W, even though $\dim V = 2$.

This leads us to look for a linearly independent subset of Φ (a 'simple system') from which Φ can somehow be reconstituted. More precisely, we ask that each root be an $\mathbf{R}$-linear combination of 'simple' roots with coefficients all of like sign. In this way a simple system will yield a partition of Φ into 'positive' and 'negative' roots, with precisely one of each pair $\{\alpha, -\alpha\}$ labelled as positive. Partitions of this sort are easy to find (by totally ordering V), so we take this as our starting point in the search for a simple system.

Recall that a **total ordering** of the real vector space V is a transitive relation on V (denoted $<$) satisfying the following axioms.

(1) For each pair $\lambda, \mu \in V$, exactly one of $\lambda < \mu, \lambda = \mu, \mu < \lambda$ holds.

(2) For all λ, μ, ν in V, if $\mu < \nu$, then $\lambda + \mu < \lambda + \nu$.

(3) If $\mu < \nu$ and c is a nonzero real number, then $c\mu < c\nu$ if $c > 0$, while $c\nu < c\mu$ if $c < 0$.

Given such an ordering, we say that $\lambda \in V$ is **positive** if $0 < \lambda$. The sum of positive vectors is positive, as is the scalar multiple of a positive vector by a positive real number.

To construct a total ordering of V is easy: choose an arbitrary ordered basis $\lambda_1, \ldots, \lambda_n$ of V and adopt the corresponding lexicographic order, where $\sum a_i\lambda_i < \sum b_i\lambda_i$ means that $a_k < b_k$ if k is the least index i for which $a_i \neq b_i$. The reader can quickly verify the axioms above. Note too that all λ_i are positive in this ordering.

Returning to the root system Φ, we call a subset Π a **positive system** if it consists of all those roots which are positive relative to some total ordering of V. It is clear that positive systems exist. Moreover, since roots come in pairs $\{\alpha, -\alpha\}$, it is clear that Φ must be the disjoint union of Π and $-\Pi$, the latter being called a **negative system**. When Π is fixed, we can write $\alpha > 0$ in place of $\alpha \in \Pi$.

Call a subset Δ of Φ a **simple system** (and call its elements **simple roots**) if Δ is a vector space basis for the $\mathbb{R}$-span of Φ in V and if moreover each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign (all nonnegative or all nonpositive). It is not at all evident that simple systems exist.

Theorem (a) *If Δ is a simple system in Φ, then there is a unique positive system containing Δ.*

(b) *Every positive system Π in Φ contains a unique simple system; in particular, simple systems exist.*

Proof. (a) Suppose the simple system Δ is contained in a positive system Π. Then all roots which are nonnegative linear combinations of Δ must also be in Π (and their negatives cannot be in Π). So Π is characterized uniquely as the set of all such roots. To see that such a positive system exists, extend the linearly independent set Δ to an ordered basis of V and take Π to be the set of positive elements of Φ in the corresponding lexicographic ordering. Evidently $\Delta \subset \Pi$.

(b) Suppose for a moment that the given positive system Π (coming from some total ordering of V) does contain a simple system Δ. Then Δ may be characterized as the set of all roots in $\alpha \in \Pi$ such that α is not expressible as a linear combination with strictly positive coefficients of two or more elements of Π. (This follows easily from the definitions.) So Δ is the unique simple system in Π.

How can we actually locate a simple system in Π? Choose as small a subset $\Delta \subset \Pi$ as possible subject to the requirement that each root in Π be a nonnegative linear combination of Δ. Obviously such a subset exists. We need only prove that Δ is linearly independent. This will follow from a key geometric condition, to be verified below:

$$(\alpha, \beta) \leq 0 \text{ for all pairs } \alpha \neq \beta \text{ in } \Delta. \qquad (1)$$

Assuming the truth of (1), consider what would happen if Δ failed to be linearly independent: $\sum_{\alpha\in\Delta} a_\alpha \alpha = 0$, with not all $a_\alpha = 0$. Rewrite this as $\sum b_\beta \beta = \sum c_\gamma \gamma$, where the sums are taken over disjoint subsets of Δ and the coefficients are strictly positive. If σ denotes the sum just written, we have $\sigma > 0$. But, thanks to (1),

$$0 \leq (\sigma, \sigma) = (\sum b_\beta \beta, \sum c_\gamma \gamma) \leq 0.$$

This forces $\sigma = 0$, which is absurd. Thus Δ must be linearly independent.

It remains to verify (1). Suppose it fails for some pair α, β. Then the formula for a reflection gives $s_\alpha \beta = \beta - c\alpha$, with $c = 2(\beta,\alpha)/(\alpha,\alpha) > 0$. Since $s_\alpha\beta \in \Phi$, either it or its negative must lie in Π. Say $s_\alpha\beta = \sum c_\gamma \gamma$ (sum over $\gamma \in \Delta, c_\gamma \geq 0$). In case $c_\beta < 1$, we get $s_\alpha \beta = \beta - c\alpha = c_\beta \beta + \sum_{\gamma\neq\beta} c_\gamma \gamma$, or $(1 - c_\beta)\beta =$ nonnegative linear combination of $\Delta \setminus \{\beta\}$. Since $1 - c_\beta > 0$, this allows us to discard β, contradicting the minimality of Δ. In case $c_\beta \geq 1$, we get instead $0 = (c_\beta - 1)\beta + c\alpha + \sum_{\gamma\neq\beta} c_\gamma\gamma$. But a nonnegative linear combination of Δ with at least one positive coefficient cannot equal 0, by definition of total ordering. So $s_\alpha\beta$ cannot be positive. A similar argument shows that $s_\alpha\beta$ cannot be negative either; here the cases to consider are $c + c_\alpha > 0$ and $c + c_\alpha \leq 0$. This contradiction implies that (1) must be true. □

Because of the uniqueness statements in the theorem, the proof actually shows that (1) must hold for any simple system. This is an important geometric constraint, which plays a role in the classification of possible reflection groups (Chapter 2):

Corollary (**of proof**) *If Δ is a simple system in Φ, then $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta$ in Δ.* □

The cardinality of any simple system is an invariant of Φ, since it measures the dimension of the span of Φ in V. We call it the **rank** of W. For example, $\mathcal{D}_m$ has rank 2, while $\mathcal{S}_n$ has rank $n - 1$.

Exercise 1. If Φ has rank 2, prove that W is a dihedral group. [This will be easier to do after Theorem 1.5.]

Exercise 2. Find simple systems for the various groups described in 1.1, taking for Φ in each case a convenient set of vectors (not necessarily unit vectors).

1.4 Conjugacy of positive and simple systems

We have shown that positive and simple systems in Φ determine each other uniquely. However, we have not ruled out the unpleasant possibility that differently chosen simple systems might differ drastically as geometric configurations. Here we examine the relationship between different systems.

It follows directly from the definition that, for any simple system Δ and for any $w \in W$, $w\Delta$ is again a simple system, with corresponding positive system $w\Pi$ (if Π is the positive system determined by Δ). To understand better the passage from Π to $w\Pi$, consider the special case $w = s_\alpha$ ($\alpha \in \Delta$). We find that Π and $s_\alpha\Pi$ differ only by one root:

Proposition *Let Δ be a simple system, contained in the positive system Π. If $\alpha \in \Delta$, then $s_\alpha(\Pi\backslash\{\alpha\}) = \Pi\backslash\{\alpha\}$.*

Proof. Let $\beta \in \Pi, \beta \neq \alpha$, and write $\beta = \sum_{\gamma\in\Delta} c_\gamma\gamma$ (with all $c_\gamma \geq 0$). Since the only multiples of α in Φ are $\pm\alpha$, some $c_\gamma > 0$ for $\gamma \neq \alpha$. Now apply s_α to both sides: $s_\alpha\beta = \beta - c\alpha$ is a linear combination of Δ involving γ with the same coefficient c_γ. Because all coefficients in such an expression have like sign, $s_\alpha\beta$ must be positive. It cannot be α, for then we reach the contradiction: $\beta = s_\alpha s_\alpha\beta = s_\alpha\alpha = -\alpha$ (which is not in Π). Thus s_α maps $\Pi\backslash\{\alpha\}$ into itself (injectively), hence onto itself. □

Besides being the key step in the proof of the theorem below, this result is often helpful in recognizing when a root is in fact equal to a given simple root α: it characterizes α as the sole positive root made negative by s_α.

Theorem *Any two positive (resp. simple) systems in Φ are conjugate under W.*

Proof. Let Π and Π' be positive systems, so each contains precisely half of the roots. Proceed by induction on $r = \text{Card}(\Pi \cap -\Pi')$. If $r = 0$, then $\Pi = \Pi'$ and we are done. If $r > 0$, then clearly the simple system Δ in Π cannot be wholly contained in Π'. Choose $\alpha \in \Delta$ with $\alpha \in -\Pi'$. The proposition above implies that $\text{Card}(s_\alpha\Pi\cap-\Pi') = r-1$. Induction, applied to the positive systems $s_\alpha\Pi$ and Π', furnishes an element $w \in W$ for which $w(s_\alpha\Pi) = \Pi'$. □

1.5 Generation by simple reflections

Fix a simple system Δ and corresponding positive system Π in Φ. (Theorem 1.4 shows that it makes no great difference which Δ we choose.) Our next goal is to show that W is generated by **simple reflections**,

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i.e., those s_α for which $\alpha \in \Delta$. First, a definition: if $\beta \in \Phi$, write uniquely $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$, and call $\sum c_\alpha$ the **height** of β (relative to Δ), abbreviated $\mathrm{ht}(\beta)$. For example, $\mathrm{ht}(\beta) = 1$ if $\beta \in \Delta$.

Theorem *For a fixed simple system Δ, W is generated by the reflections $s_\alpha (\alpha \in \Delta)$.*

Proof. Denote by W' the subgroup of W so generated. We proceed in several steps to show that $W' = W$.

(1) If $\beta \in \Pi$, consider $W'\beta \cap \Pi$. This is a nonempty set of positive roots (containing at least β), and we can choose from it an element γ of smallest possible height. We claim that $\gamma \in \Delta$. Write $\gamma = \sum_{\alpha\in\Delta} c_\alpha \alpha$, and note that $0 < (\gamma, \gamma) = \sum c_\alpha (\gamma, \alpha)$, forcing $(\gamma, \alpha) > 0$ for some $\alpha \in \Delta$. If $\gamma = \alpha$, we are satisfied. Otherwise consider the root $s_\alpha \gamma$, which is positive according to Proposition 1.4. Since $s_\alpha \gamma$ is obtained from γ by subtracting a positive multiple of α, we have $\mathrm{ht}(s_\alpha \gamma) < \mathrm{ht}(\gamma)$. But $s_\alpha \gamma \in W'\beta$ (since $s_\alpha \in W'$), contradicting the original choice of γ. So indeed $\gamma = \alpha$ must be simple.

(2) Now we can argue that $W'\Delta = \Phi$. We just showed that the W'-orbit of any positive root β meets Δ, so that $\Pi \subset W'\Delta$. On the other hand, if β is negative, then $-\beta \in \Pi$ is conjugate by some $w \in W'$ to some $\alpha \in \Delta$. Then $-\beta = w\alpha$ forces $\beta = (ws_\alpha)\alpha$, with $ws_\alpha \in W'$. Thus $-\Pi \subset W'\Delta$.

(3) Finally, take any generator s_β of W. Use step (2) to write $\beta = w\alpha$ for some $w \in W'$ and some $\alpha \in \Delta$. Then Proposition 1.2 shows that $s_\beta = ws_\alpha w^{-1} \in W'$. This proves that $W = W'$. □

A useful byproduct of the proof is the fact that every root can attain the status of a simple root (relative to some positive system):

Corollary (of proof) *Given Δ, for every $\beta \in \Phi$ there exists $w \in W$ such that $w\beta \in \Delta$.* □

Exercise 1. Let Φ be a root system of rank n consisting of unit vectors. If $\Psi \subset \Phi$ is a set of n roots whose mutual angles agree with those between the roots in some simple system, then Ψ must be a simple system.

Exercise 2. Given a simple system Δ, no proper subset of the simple reflections can generate W. [Otherwise find $\alpha \in \Delta$ for which s_α is not needed as a generator of W. Consider $w \in W$ for which $w(-\alpha) \in \Delta$.]

Exercise 3. If $\beta \in \Pi \backslash \Delta$, prove that $\mathrm{ht}(\beta) > 1$.

Having seen that W can be generated by relatively few reflections, we may go on to seek an efficient presentation of W as an abstract group, using these generators together with suitable relations. Certain relations