1. FRÖHLICH'S DESCRIPTION OF CLASSGROUPS

§1. AN EXACT SEQUENCE FROM K-THEORY

Throughout all rings have a 1, and, for a given ring R, the multiplicative group of its invertible elements is denoted by R^*.

In general modules are taken to have their action on the right, and, unless stated otherwise, all modules are assumed to be finitely generated.

In this section we shall put the locally free classgroup in the context of K-theory. In particular, we describe its appearance in an exact sequence of K-theory. Our presentation is based on that given in chapter 1 of [F2].

Let K be a number field or a finite extension of the p-adic field Q_p. The ring of integers of K will be denoted by $\mathcal{O}_K$. Let A be a finite dimensional semi-simple K algebra and let $\mathcal{A}$ be an $\mathcal{O}_K$-order in A. The Grothendieck group of locally free $\mathcal{A}$ modules denoted $K_o(\mathcal{A})$, is the free abelian group on the isomorphism classes of locally free $\mathcal{A}$-modules, modulo the group generated by the relations [N] - [M] - [P] for each exact sequence of locally free $\mathcal{A}$-modules

$$0 \to M \to N \to P \to 0.$$

It is worth remarking that this Grothendieck group is more restrictive than the usual one where all projective $\mathcal{A}$ modules are considered. However, in the crucial case of a global group ring, Swan has shown

Theorem 1.1 [Sw2]

Let K be a number field, let Γ be a finite group and let M be an $\mathcal{O}_K\Gamma$ module. Then M is projective over $\mathcal{O}_K\Gamma$ if, and only if, it is a locally free $\mathcal{O}_K\Gamma$-module.

An $\mathcal{A}$-module M is said to be locally freely presented if there

Cambridge University Press
978-0-521-27870-6 - Classgroups of Group Rings
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exists an exact sequence of A-modules

$$0 \to P \to N \to M \to 0 \qquad (1.2)$$

with P and N locally free A-modules of the same rank. [Thus M is finite].

Remark If $A = \mathbb{Z}\Gamma$, then by Theorem 1.1, together with Rim's Theorem in [Ri], it follows that M is locally freely presented precisely when M is a finite $\mathbb{Z}\Gamma$ module which is cohomologically trivial.

Let $K_oT(A)$ denote the Grothendieck group of the category of such modules, taken with respect to exact sequences.

For a locally free A-module M (resp. locally freely presented A-module M) we shall write [M] for its class in $K_o(A)$ (resp. $K_oT(A)$).

There is a natural homomorphism

$$\delta: K_oT(A) \to K_o(A)$$

where for M, N, P as given in (1.2)

$$\delta([M]) = [P] - [N].$$

Recall that for a ring R,

$$K_1(R) = \frac{\varinjlim GL_n(R)}{\varinjlim E_n(R)}$$

where $E_n(R)$ denotes the group generated by the elementary matrices in $M_n(A)$.

From §I in [F2] we have a long exact sequence of K-theory

$$K_1(A) \xrightarrow{\kappa} K_1(A) \xrightarrow{\theta} K_oT(A) \xrightarrow{\delta} K_o(A) \xrightarrow{r} \mathbb{Z} \to 0 \qquad (1.3)$$

If one replaces locally free modules by projective modules in defining the above Grothendieck groups then the above becomes the well-known exact sequence of K-theory of Bass cf. [B] IX 6.3.

We now explain the homomorphisms used in (1.3): r is the homomorphism induced by associating to each locally free A-module its

A-rank, and κ is induced by the inclusion $A \hookrightarrow A$. The homomorphism θ is induced by the homomorphisms

$$\theta_n : GL_n(A) \to K_oT(A)$$

which are defined as follows. For $\alpha \in GL_n(A)$ we choose a locally free A module X which spans $\overset{n}{\underset{1}{\oplus}} A$ and some $c \in \mathcal{O}_K \setminus \{0\}$ such that $c\,X \subset X \cap x\alpha$; then

$$\theta_n(\alpha) = \left[\frac{X}{Xc}\right] - \left[\frac{X\alpha}{Xc}\right] .$$

The locally free classgroup of A, $Cl(A)$ is defined to be Ker(r). It is worth remarking that if K is a finite extension of Q_p, then of course $Cl(A) = \{1\}$.

In general there is a homomorphism $K_o(A) \to Cl(A)$, where for a locally free A-module M, [M] maps to $[M] - rk_A(M)\ [A]$. We shall write (M) for this class of M in $Cl(A)$.

$Cl(A)$, and therefore by necessity $K_oT(A)$, are the main subject of this study, in the case when A is a group ring.

Let M be a maximal order of A containing A, then $\otimes_A M$ induces a homomorphism (which is shown to be a surjection in §3)

$$Cl(A) \to Cl(M).$$

We denote the kernel of this homomorphism by $D(A)$. It is well-known that in fact $D(A)$ is independent of the particular choice of maximal order M (cf. (3.10)). In §3, it will be shown that $Cl(M)$ is a product of certain ideal classgroups - a result which is originally due to Eichler. A great deal is known about such classgroups, and so, for the most part, we restrict our attention to $D(A)$. Even when $D(A)$ and $Cl(M)$ are both known, there remains the delicate question of describing the extension

$$1 \to D(A) \to Cl(A) \to Cl(M) \to 1$$

The only results in this direction that I am aware of are [U5] and [Mc].

§2. NOTATION

For a field F, let F^c be an algebraic closure of F. Once and for all we fix algebraic closures Q^c, Q_p^c for all rational primes p. Let $\mathbb{R}$ (resp. $\mathbb{C}$) be the field of real numbers (resp. field of complex numbers). We shall frequently speak of the Archimedean valuations of a number field as the infinite primes. With this convention $Q_\infty = \mathbb{R}$, $Q_\infty^c = \mathbb{C}$.

If K is a number field (resp. a finite extension of Q_p), then we let $\Omega_K = \mathrm{Gal}(Q^c/K)$ (resp. $\Omega_K = \mathrm{Gal}(Q_p^c/K)$).

(2.1) Let $K \subset Q^c$ be a number field, suppose $p < \infty$ and let $h: K \hookrightarrow Q_p^c$ be a field embedding. Let P^c (resp. U_p) be the maximal ideal (resp. group of units) in the ring of integers Q_p^c.

Let p denote the maximal ideal $h^{-1}(P^c)$. $\mathrm{Gal}(Q_p^c/Q_p)$ acts on such embeddings by composition, and the orbits (or by abuse of language a representative of an orbit) is called a place of K. In this way a bijection between places with $p < \infty$ and the maximal ideals of $\mathcal{O}_K$ is obtained. Alternatively, relaxing the condition $p < \infty$, we obtain a bijection between the places of K and the equivalence classes of valuations of K.

For a place h of K associated to the (possibly infinite) prime p, let K_p be the closure of h(K) in Q_p^c (so that K_p is defined only up to Galois conjugacy). With the notation of the previous section, let A_p resp. $\mathcal{A}_p$ denote the closure of the image of A (resp. $\mathcal{A}$) under the K-linear extension of h to $A \hookrightarrow A \otimes_K Q_p^c$.

For a rational prime p we write A_p (resp. $\mathcal{A}_p$) for $A \otimes_Q Q_p$ (resp. $\mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p$). Then the isomorphism

$$K \otimes_Q Q_p \cong \prod_{p|p} K_p$$

induces isomorphisms $A_p \cong \prod A_p$, $\mathcal{A}_p \cong \prod \mathcal{A}_p$.

Let R_A denote the Grothendieck group of $A \otimes_K K^c$ modules. In the particular case when Γ is a finite group, $A = K\Gamma$, we write R_Γ (resp. $R_{\Gamma,p}$) for R_A when K is a number field (resp. a finite extension of Q_p). An embedding $Q^c \hookrightarrow Q_p^c$ determines an isomorphism $R_\Gamma \cong R_{\Gamma,p}$. Frequently such an embedding will be assumed to be given and $R_{\Gamma,p}$ and R_Γ will be identified.

Alternatively R_Γ (resp. $R_{\Gamma,p}$) will frequently be viewed as the group of virtual Q^c (resp. Q_p^c) characters of Γ.

Let V be a $Q^c\Gamma$ module with character χ. For $1 \leq i \leq n = \dim_{Q^c}(V)$, let $\lambda^i(\chi)$ be the character of i^{th} exterior product of V. By convention we put $\lambda^o(\chi) = \varepsilon_\Gamma$, the identity character of Γ, and of course $\lambda^i(\chi) = 0$ for $i > n$. Then R_Γ, together with the operations $\lambda^i: R_\Gamma \to R_\Gamma$, extended from characters to virtual characters by the relation

$$\lambda^i(\chi + \phi) = \sum_{0 \leq j \leq i} \lambda^j(\chi)\, \lambda^{i-j}(\phi)$$

is a λ-ring in the sense of Grothendieck. In particular if $\gamma \in \Gamma$ has eigenvalues $x_1, \ldots x_n$ on V, then for $1 \leq i \leq n$, γ has eigenvalues on the i^{th} exterior product of V,

$$x_{a_1} \ldots x_{a_i}$$

where $1 \leq a_1 < \ldots < a_i \leq n$. In other words $\lambda^i(\chi)(\gamma)$ is the i^{th} symmetric polynomial $S_i(x_1 \ldots x_n)$ in the eigenvalues. By convention $S_o(x_1 \ldots x_n) = 1$, and for $i > 0$ the r^{th} symmetric power sum is

$$P_r(x_1, \ldots, x_n) = \sum_{a=1}^{n} x_a^r .$$

By Newton's formulae

$$n\, S_n = \sum_{k=1}^{n} (-1)^{k-1} P_k \cdot S_{n-k}.$$

Thus, inductively, it can be deduced that the central function

$$\gamma \to \chi(\gamma^i) = P_i(x_1, \ldots, x_n)$$

lies in R_Γ. This virtual character will be written as $\psi_i(\chi)$. The operation $\psi_i: R_\Gamma \to R_\Gamma$ is called the i^{th} Adams operation (or sometimes i^{th} Frobenius operation).

Let p be a prime number and let Γ be a fixed finite group. For $\gamma \in \Gamma$ we may write $\gamma = \gamma' \gamma_p = \gamma_p \gamma'$ uniquely with γ' (resp. γ_p) having order prime to p (resp. p power order).

Suppose $|\Gamma| = p^n m$ with $(p,m) = 1$ and let r be a positive integer r such that

$$r \equiv 0 \bmod (p^n) \qquad r \equiv 1 \bmod (m) .$$

δ_p is defined to be the operation ψ_r. Then clearly for $\chi \in R_\Gamma$, $\gamma \in \Gamma$,

$$\delta_p(\chi)(\gamma) = \chi(\gamma'), \tag{2.2}$$

and $\delta_p^2 = \delta_p$ on R_Γ. These idempotent operations were first introduced by Philippe Cassou-Noguès [CN1].

Let χ be a character of Γ of degree $d = \chi(1)$. Then $\lambda^d(\chi)$ is an abelian character of Γ, and the map $\chi \to \lambda^d(\chi)$ induces a homomorphism

$$\det\colon R_\Gamma \to R_\Gamma^*,$$

which satisfies the relation

$$\det(\chi\phi) = \det(\chi)^{\phi(1)} \det(\phi)^{\chi(1)}$$

for $\chi, \phi \in R_\Gamma$.

If Δ is a subgroup of Γ, restriction from Γ to Δ yields a ring homomorphism

$$\mathrm{Res}_\Gamma^\Delta : R_\Gamma \to R_\Delta$$

and induction (i.e. $\otimes_{Q^c\Delta} Q^c\Gamma$) yields a group homomorphism

$$\mathrm{Ind}_\Delta^\Gamma\colon R_\Delta \to R_\Gamma .$$

If $q\colon \Gamma \to \Sigma$ is group epimorphism, then composition with q yields an injective ring homomorphism

$$\mathrm{Inf}_\Sigma^\Gamma\colon R_\Sigma \to R_\Gamma,$$

which will be called inflation.

Let K be a number field, let A be a finite dimensional semi-simple K-algebra, and let $\mathcal{A}$ be an $\mathcal{O}_K$ order in A. The group of ideles of A, J(A), is the subgroup of elements in ΠA_p^* almost all of whose entries lie in $\mathcal{A}_p^*$. [If p is infinite then $\mathcal{A}_p$ is defined to be A_p]. Note that J(A) does not depend on $\mathcal{A}$, since for any other order $\mathcal{B}$ in A, $\mathcal{B}_p = \mathcal{A}_p$ for almost all primes p. The group of unit ideles of $\mathcal{A}$, U($\mathcal{A}$), is the group $\prod_p \mathcal{A}_p^*$.

For a finite set S of prime numbers define $U_S(\mathcal{A}) = \prod_{p \in S} \mathcal{A}_p^*$, and for a number field F, $U_S(F) = \prod_q \mathcal{O}_{F_q}^*$, where q runs through the primes of F above those primes in S.

In the sequel we take F to be a number field containing K, which is Galois over Q and which is large enough in the sense that the absolutely irreducible classes of representations of A are to be realisable over F.

Given a representation T: A → $M_n(F)$, twisting by $\omega|_F$ for $\omega \in \Omega_K$, yields a further representation of A. Hence R_A is an Ω_K-module. Now J(F), the ideles of F, are also an Ω_K-module, so that the group $\mathrm{Hom}_{\Omega_K}(R_A, J(F))$ is now defined.

The principal goal of this first chapter is to describe Cl($\mathcal{A}$) as a quotient of this group of homomorphisms. This is Fröhlich's so-called Hom-description of the locally free classgroup. His description is simultaneously well-adapted both to questions of functoriality and to specific calculations. With this end in view, we now introduce various subgroups of homomorphisms.

$\mathrm{Hom}_{\Omega_K}(R_A, F^*)$ will be considered as a subgroup of $\mathrm{Hom}_{\Omega_K}(R_A, J(F))$ via the diagonal map $F^* \hookrightarrow J(F)$.

For $z \in GL_a(\mathcal{A}_p)$ we wish to define an Ω_K homomorphism

$$\mathrm{Det}(z): \; R_A \to F \otimes_K K_p \; .$$

To do this, it suffices by linearity to define Det(z) on classes of $A \otimes_K K^c$ modules, and then check that it is additive and commutes with Ω_K-action. Let T: A → $M_n(F)$ be a representation of A with class $\chi \in R_A$. Then we extend T to an algebra homomorphism

$$\tilde{T}: M_a(A_p) \to M_{na}(F \otimes_K K_p)$$

and define

$$\mathrm{Det}(z)(\chi) = \det(\tilde{T}(z)). \tag{2.3}$$

Additivity is immediate, and commutativity with Ω_K action follows from the equalities

$$\begin{aligned}(\mathrm{Det}(z)(\chi^\omega)) &= \det(\widetilde{\omega \circ T(z)}) \\ &= \det(\omega(\tilde{T}(z))) \qquad (2.4) \\ &= (\mathrm{Det}(z)(\chi))^\omega\end{aligned}$$

for $\omega \in \Omega_K$. We remark that Det is related to the map det: $R_\Gamma \to R_\Gamma^*$ by the rule that for $\gamma \in \Gamma$, $\chi \in R_\Gamma$,

$$\mathrm{Det}(\gamma)(\chi) = \det(\chi)(\gamma).$$

If K/L is normal then $\mathrm{Hom}_{\Omega_K}(R_A, J(F))$ may be viewed as an Ω_L-module with action

$$f^\omega(\chi) = f(\chi^{\omega^{-1}})^\omega$$

for $\omega \in \Omega_L$. Note that in the case $A = \mathcal{O}_K\Gamma$, as in (2.4), it follows that

$$\mathrm{Det}(z)^\omega = \mathrm{Det}(z^\omega) , \tag{2.5}$$

for $z \in \mathcal{O}_{K_p}\Gamma^*$, with Ω_L acting coefficient-wise on $\mathcal{O}_{K_p}\Gamma$.

We shall call the group of homomorphisms Det(z) for $z \in GL_a(A_p)$, $\mathrm{Det}(GL_a(A_p))$. In fact because A_p is semi-local, z can always be multiplied by elementary matrices in $GL_a(A_p)$ to change z into upper triangular form with all but one entry equal to 1 (cf. [B], V, (4.1)). Hence for all $a \geq 1$

$$\mathrm{Det}(GL_a(A_p)) = \mathrm{Det}(A_p^*). \tag{2.6}$$

In exactly the same way, we can define $\mathrm{Det}(GL_a(\mathcal{A}_p))$, $\mathrm{Det}(GL_a(A))$ etc. and obtain equalities corresponding to (2.6).

Next the corresponding local construction will be considered together with the relation between the global and local construction. Let $\mathfrak{q}$ be a prime of F over a prime $\mathfrak{p}$ of K. Then for $z \in A_{\mathfrak{p}}^*$ we have $\mathrm{Det}(z) \in \mathrm{Hom}_{\Omega_{K_{\mathfrak{p}}}}(R_{A_{\mathfrak{p}}}, F_{\mathfrak{q}}^*)$, and the natural isomorphisms

$$F \otimes_K K_{\mathfrak{p}} \cong \prod_{\mathfrak{q}|\mathfrak{p}} F_{\mathfrak{q}} \qquad A_p \cong \prod_{\mathfrak{p}|p} A_{\mathfrak{p}}$$

induce isomorphisms

$$\mathrm{Det}(A_p^*) \cong \prod_{\mathfrak{p}} \mathrm{Det}(A_{\mathfrak{p}}^*). \tag{2.7}$$

$$\mathrm{Det}(A_p^*) \cong \prod_{\mathfrak{p}} \mathrm{Det}(A_{\mathfrak{p}}^*).$$

For an irreducible class $\chi \in R_{A_{\mathfrak{p}}}$ and for $z \in A^*$, $\mathrm{Det}(z)(\chi)$ is, up to Galois conjugacy, the reduced norm of the image of z in the simple algebra corresponding to χ. We assert that

$$\mathrm{Det}(A_{\mathfrak{p}}^*) = \begin{cases} \mathrm{Hom}^+_{\Omega_{K_{\mathfrak{p}}}}(R_{A_{\mathfrak{p}}}, F_{\mathfrak{q}}^*) & \mathfrak{p} \text{ real}, \\ \mathrm{Hom}_{\Omega_{K_{\mathfrak{p}}}}(R_{A_{\mathfrak{p}}}, F_q^*) & \text{otherwise}. \end{cases} \tag{2.8}$$

where the superscript + on Hom, means that the homomorphisms are to be totally positive on all symplectic representations of $A_{\mathfrak{p}}$. This reduces immediately to the case where $A_{\mathfrak{p}}$ is simple and the result then follows from the fact that the reduced norm of a local division algebra maps onto the centre, unless $\mathfrak{p}$ is real and $A_{\mathfrak{p}}$ is a matrix algebra over the quaternions in which case the reduced norm maps onto the positive reals. In the same way, we see that if $\mathfrak{p}$ is finite and $\mathcal{A}_{\mathfrak{p}}$ is a maximal order of $A_{\mathfrak{p}}$, then

$$\mathrm{Det}(\mathcal{A}_{\mathfrak{p}}^*) = \mathrm{Hom}_{\Omega_{K_{\mathfrak{p}}}}(R_{A_{\mathfrak{p}}}, \mathcal{O}_{F_{\mathfrak{q}}}^*). \tag{2.9}$$

In the same vein, from the Hasse-Schilling norm theorem (cf. Theorem 7.6 in [SE]), we have

$$\mathrm{Det}(A^*) = \mathrm{Hom}^+_{\Omega_K}(R_A, F^*) \tag{2.10}$$

This section will be concluded by describing the relationship between K_1-groups and the above Det-groups, in certain cases.

Let $\alpha \in K_1(A)$ be represented by $x \in GL_a(A)$. We assert that the map $x \to \mathrm{Det}(x)$ induces an isomorphism

$$K_1(A) \cong \mathrm{Det}(A^*). \tag{2.11}$$

This follows from the remark after (2.7) together with the fact that the reduced norm yields an isomorphism of $K_1(A)$ into the centre of A, when A is a finitely generated, semi-simple algebra over either a number field or a local field. (Cf. Wang's Theorem in V Theorem 9.7 [B]).

§3. FRÖHLICH'S DESCRIPTION

Let K be a number field and let $\mathfrak{p}$ be a finite prime of K. A locally freely presented $\mathcal{A}_{\mathfrak{p}}$ module is automatically a locally freely presented $\mathcal{A}$ module. Moreover a torsion $\mathcal{O}_K$ module is a product of torsion $\mathcal{O}_{K_{\mathfrak{p}}}$ modules as $\mathfrak{p}$ ranges through the finite primes of K. Thus, by weak approximation,

$$K_oT(A) \cong \prod_{\mathfrak{p}} K_oT(A_{\mathfrak{p}}). \tag{3.1}$$

Theorem 3.2

Let K be a finite extension of Q_p with $p < \infty$; then there is a natural isomorphism

$$\sigma \colon K_oT(A) = \frac{\mathrm{Hom}_{\Omega_K}(R_A, Q_p^{c*})}{\mathrm{Det}(A^*)}$$

Moreover, if M is as in (1.2) with N,P spanning A^n, and $P = N\alpha$ for $\alpha \in GL_n(A)$, then $\sigma([M])$ is represented by the homomorphism $\mathrm{Det}(\alpha)$.

By (3.1) we will immediately be able to deduce the corresponding global result:

Theorem 3.3

Let K be a number field, then there is a natural isomorphism

$$\sigma \colon K_oT(A) \xrightarrow{\sim} \frac{\mathrm{Hom}^+_{\Omega_K}(R_A, J(F))}{\mathrm{Det}(U(A))} .$$