Introduction

In this book you will study the processes that lead to the formation of stars, the energy sources that fuel them, what they do during their lifetimes, and what happens when their fuel runs out. It is assumed that you already have a general, qualitative idea of some events in the life cycles of stars. This book is intended to take you beyond a *description* of events to an understanding of the physical processes that bring them about. That is, the book will not only tell you *what* happens, but will provide you with the physical tools to understand *why*, and to use the physics to work out what will happen as a star ages.

A key point is that stellar evolution is a cyclic process: stars are born, live their lives, and then die, but in doing so they seed the interstellar medium with nuclear-processed material that forms the building blocks of a subsequent generation of stars. We can, therefore, chose to begin an exploration of this cycle at any point, and follow the process until we get back to where we started.

This book is divided into eight chapters, each of which provides a fairly self-contained discussion of a particular phase in the life cycle of a star, or a particular physical process. We begin in Chapter 1 by looking at the general properties of stars on the main sequence of the Hertzsprung–Russell diagram, then in Chapter 2 explore the key driver of stellar evolution, namely the process of gravitational collapse. In Chapter 3 we explore the physics of nuclear fusion in more detail, concentrating on the fusion reactions that power a star on the main sequence. Chapter 4 examines how stars evolve from the main sequence to the giant branch whilst the phase of helium burning is described in Chapter 5. The next two Chapters, 6 and 7, consider advanced stages of stellar evolution and the fate of stars including white dwarfs, neutron stars and black holes. The last part of the book, in Chapter 8, considers how stars form out of the interstellar material which has been enriched by previous generations of stars. This brings us full circle back to the main sequence again, where we started.

The book is designed to be worked through in sequence; some aspects of later chapters build on the knowledge gained in earlier chapters. So, whilst you could dip in at any point, you will find if you do so that you are often referred back to concepts developed elsewhere in the book. Our intention is, that if the book is studied sequentially, it provides a self-contained, self-study course in stellar astrophysics.

A special comment should be made about the exercises in this book. You may be tempted to regard them as optional extras that are only there to help you refresh your memory about certain concepts when you re-read the text in preparation for an exam. *Do not fall into this trap!* The exercises are part of the *learning*. Several of the important concepts are developed through the exercises and nowhere else. Therefore, you should attempt each of them when you come to it. You will find full solutions for every exercise at the end of this book, but do try to complete the exercise yourself first before looking at the answer. A table of physical constants is also given at the end of the book; use these values as appropriate in your calculations.

For most calculations presented here, use of a scientific calculator is essential. In some cases, you will be able to work out order of magnitude estimates without the use of a calculator, and such estimates are invariably useful to check whether an expression is correct. In some calculations you may find that use of a computer spreadsheet, or graphing calculator, provides a convenient means of visualizing a particular function. If you have access to such tools, please feel free to use them.

Finally, a note about the genesis of this book. It was originally written for an astrophysics course which used, in addition, a book called *The Physics of Stars* by A. C. Phillips. We commend that book to readers, and acknowledge the influence that it had on the development of our approach to teaching this subject.

Chapter 1 Main-sequence stars

Introduction

In this chapter we present a set of topics concerning the physics of stars on the main sequence. These topics serve to illustrate some of the general principles of stellar astrophysics and allow you to begin a quantitative exploration of the structure and behaviour of stars. We review the Hertzsprung–Russell diagram, introduce some parameters of the Sun as a typical star, and consider the general equations of stellar structure. We then present a summary of the nuclear fusion reactions that occur in main-sequence stars and consider the amount of energy released by the fusion of hydrogen. At this stage, we will not go into the detailed quantum physics which underlies the nuclear fusion reactions of hydrogen burning; that is left for Chapter 3. We begin by considering the question of just what we mean by ‘a star’.

The starting point for the formation of a star is a cloud of cold gas, composed primarily of hydrogen and helium, with traces of heavier elements (usually referred to as *metals*). The cloud collapses due to its own gravity, and as gravitational potential energy is released and converted into heat, the pressure, density and temperature of the material increase. This causes the cloud to begin glowing, initially at infrared and later at optical wavelengths. Ultimately, if the cloud core reaches sufficiently high temperatures – in excess of several million kelvin – nuclear reactions begin. These thermonuclear reactions provide a non-gravitational source of energy release, whose heating provides pressure to support the gas against further collapse. This changes the behaviour of the object for which, previously, gravitation was the dominant factor, and marks the transition of the object to a star. By definition, a *star* is an object in which nuclear reactions *are* (or *have been*) sufficient to balance surface radiation losses.

1.1 The Hertzsprung–Russell diagram

When we observe stars, the two most obvious characteristics are their brightness and colour. Using a star’s distance to calculate its luminosity (L) from its brightness, and determining its temperature (T) from its colour, reveals that distinct relationships are found between luminosity and temperature as illustrated in the **Hertzsprung–Russell** (H–R) diagram (e.g. Figure 1.1 overleaf). Stars are not distributed randomly in this diagram. The location of a star in the H–R diagram reflects its mass, radius, age, evolutionary state and chemical composition. The most densely populated regions of the figure are where stars spend most of their lives, whereas the probability of finding a star in a short-lived phase is much lower, so fewer stars are seen in such states. Note that the H–R diagram is a snapshot – a view at a single instant – of the evolutionary states of *many* stars. It is not the evolutionary track that an *individual* star traces out over its lifetime.

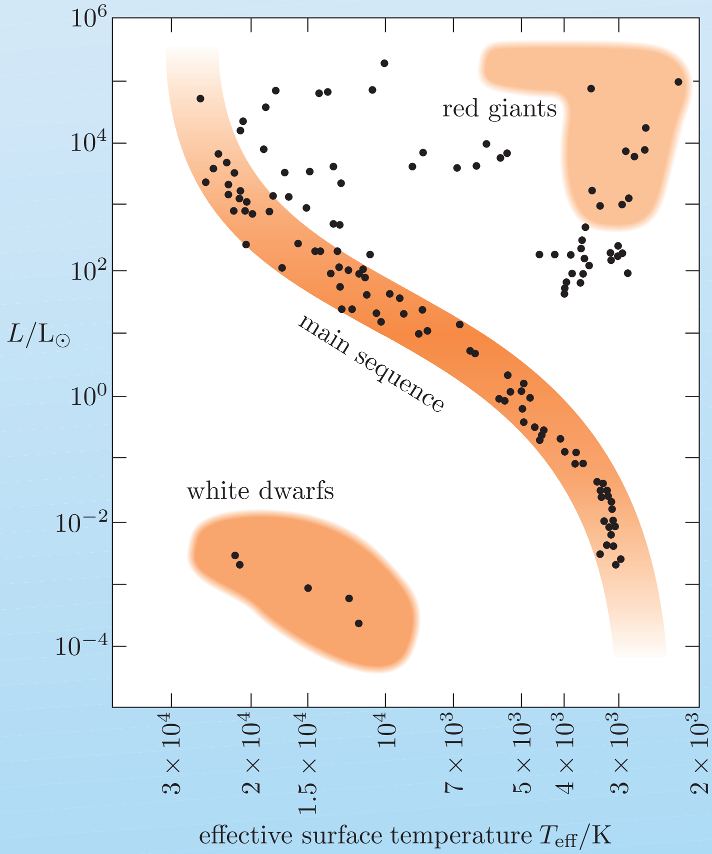

Figure 1.1 A schematic Hertzsprung–Russell diagram. This provides a snapshot of the luminosity L and effective surface temperature T_{eff} of many stars at different stages of their evolution. Note that the temperature increases from right to left. Most of the stars lie along a diagonal band, from upper-left (hot and luminous) to lower-right (cooler and fainter), called the main sequence. The second most populated region of the diagram, top right, is called the red-giant branch. The stars toward the lower left lie on the white-dwarf branch.

The H–R diagram can be presented in different ways. The original, observational diagram showed star brightness (measured as the **absolute visual magnitude**, M_V) on the vertical axis and stellar **spectral type** on the horizontal axis. Spectral type is very closely related to temperature and colour. Temperatures cannot be measured *directly* for many stars, but colours can be, so the *observational* H–R diagram is often a plot of M_V against **colour index** (such as $B - V$). The *theoretical* H–R diagram uses quantities more closely related to computations of stellar models, and plots luminosity, L, (energy radiated per unit time) against effective surface temperature, T_{eff}.

The **Stefan–Boltzmann law** states that the radiant flux F (radiant energy per square metre of surface area per second) from a **black body** of temperature T is given by $F = \sigma T^4$, where $\sigma = 5.671 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}$ and is called the Stefan–Boltzmann constant.

From now on radiant flux will be referred to simply as *flux*.

The radiant flux F passing through the surface of a spherical object of luminosity L and radius R (whose surface area is $4\pi R^2$) is simply

$$F = \frac{L}{4\pi R^2}.$$

If the object is a black body, then its luminosity L, radius R and temperature T are related by the equation

$$\frac{L}{4\pi R^2} = \sigma T^4.$$

How does this equation for a black body relate to a star? Moreover, if the Sun is a ball of gas, what do we *mean* by its *surface*? Light is continually emitted and reabsorbed by the hot gas that is the Sun. Near its outer layers, the fog-like gas eventually becomes transparent enough that some of the light escapes without being reabsorbed. However, the transition from being opaque to being transparent happens over a considerable distance. This zone, which is almost 500 km thick, is called the **photosphere**, and is the best definition we have for a surface for the Sun. The same is true in other stars, though the thickness of the photosphere is even greater in giants.

Because a star does not have an opaque, solid surface, light reaching an observer comes from a *range* of depths in its gaseous outer layers, each layer having a different temperature. What temperature should be used to characterize the surface? A useful *convention* is to refer to the temperature which a black body of the same luminosity and radius would have. This is called the **effective surface temperature** T_{eff}, and is defined as follows:

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4. \quad (1.1)$$

From the definition, we see that the effective surface temperature is related to the flux through the surface of the star, but how does it compare to the temperatures of the gas in its outer layers? Fortunately, stars are very good approximations to black bodies, since they absorb any light falling on them. Consequently, *the effective surface temperature coincides with the temperature at an intermediate depth in the photosphere* of the star. Deeper layers have temperatures higher than T_{eff}, while shallower layers in the photosphere have temperatures lower than T_{eff}. We will use the effective surface temperature to characterize the outer layers or surface of the star, even though a star does not have a solid surface in the conventional sense.

Note that Equation 1.1 involves three variables, L, R and T_{eff}, so if two are specified then the third can be calculated. Moreover, two of these are the axes of the theoretical H–R diagram, T_{eff} and L, so the third variable, R, can also be calculated for the H–R diagram.

● Equation 1.1 gives the relationship between the luminosity, effective surface temperature and radius of stars. How should it be changed for white dwarfs, which are fading away, or young protostars, which have yet to begin thermonuclear burning?

○ Don't mess with that equation! It works just fine as it is, for any black body. It therefore applies to *all* stages of stellar evolution.

Exercise 1.1 Draw curves on Figure 1.1 to show where stars of radii $100\,R_\odot$, $10\,R_\odot$, $1\,R_\odot$ and $0.1\,R_\odot$ lie. To do this, calculate the luminosities of stars having radii of $100\,R_\odot$, $10\,R_\odot$, $1\,R_\odot$ and $0.1\,R_\odot$, for six values of effective surface temperature: 2000 K, 4000 K, 6000 K, 10 000 K, 20 000 K and 40 000 K. Use the values for the solar luminosity and radius, and the Stefan–Boltzmann constant σ given at the end of this book. You could use a calculator to do this, but it will be less tedious if you use a spreadsheet. ■

Figure references beginning with 'S' are to be found in the Solutions to exercises, at the end of the book.

Make sure you understand the spacing and shape of the curves in Figure S1.1, which shows the result of the exercise above. The curves are straight lines, separated by equal amounts, and the one corresponding to $1\,R_\odot$ passes through the datum point for the Sun, although this is not shown. A key thing to note is that both Figure 1.1 and Figure S1.1 are plotted using log–log scales, rather than simple linear scales. The spacing and shapes of the curves can be understood by taking logarithms of the equation $L = 4\pi R^2 \sigma T_{\text{eff}}^4$, giving

$$\log_{10} L = \log_{10}(4\pi\sigma) + 2\log_{10} R + 4\log_{10} T_{\text{eff}}.$$

This is the equation for a straight line where $\log_{10} L$ is on the y-axis and $\log_{10} T_{\text{eff}}$ is on the x-axis. The coefficient of the x-axis term, i.e. the slope, is 4. The intercept is $\log_{10}(4\pi\sigma) + 2\log_{10} R$, which clearly depends on the value of R. Increasing or decreasing R by a factor of 10 changes $2\log_{10} R$ (and hence $\log_{10} L$) by $+2$ or -2 respectively, thus offsetting the curves of the $0.1\,R_\odot$ and $10\,R_\odot$ stars by equal amounts, but in opposite directions, from the $1\,R_\odot$ line.

1.2 The Sun as a typical star

Although it would be wrong to get the idea that all main-sequence stars are like the Sun, it is the star that we know best, and so it is a useful reference point. We will therefore consider the structure of the Sun, to help illustrate the properties of main-sequence stars in general.

Only the surface properties of the Sun are directly observable, but these measurements may be combined with theoretical models of the Sun's interior to predict its physical characteristics. The principal physical properties of the Sun are listed in Table 1.1

Table 1.1 The physical properties of the Sun.

Measured property	Value
Mass	$M_\odot = 1.99 \times 10^{30}$ kg
Radius	$R_\odot = 6.96 \times 10^{8}$ m
Luminosity	$L_\odot = 3.83 \times 10^{26}$ W
Effective surface temperature	$T_{\odot,\text{eff}} = 5780$ K
Calculated property	**Value**
Age	$t_\odot = 4.55 \times 10^{9}$ years
Core density	$\rho_{\odot,\text{c}} = 1.48 \times 10^{5}$ kg m^{-3}
Core temperature	$T_{\odot,\text{c}} = 15.6 \times 10^{6}$ K
Core pressure	$P_{\odot,\text{c}} = 2.29 \times 10^{16}$ Pa

In order to deduce further physical parameters of the Sun (and of other stars) it is necessary to know a little more about how the gravitational potential energy of a ball of gas is related to its other parameters, such as its mass, radius and internal pressure. We explore the details of such relationships in Chapter 2, but for now simply note the results in the box below.

The gravitational potential energy of a star

The astrophysics of stars is intimately concerned with self-gravitating balls of plasma. Gravity, which acts to make a body collapse, can be opposed by internal pressure if the pressure is greater in the core, i.e. if a pressure gradient exists, with pressure decreasing outwards. **Hydrostatic equilibrium** exists when gravity is just balanced by the pressure gradient, so the body neither contracts nor expands.

A very useful result describing the properties of such a system is encapsulated in the **virial theorem**. This states that the volume-averaged pressure needed to support a self-gravitating body in hydrostatic equilibrium is minus one-third of the gravitational potential energy density (i.e. the gravitational energy per unit volume). In symbols

$$\langle P\rangle = -\frac{1}{3}\frac{E_{\rm GR}}{V}. \tag{1.2}$$

The negative sign is required because the gravitational potential energy is defined as negative, while the pressure must be positive.

The gravitational potential energy of a sphere of *uniform density* may be calculated by integrating over the mass contained within it. The procedure is straightforward, but an unnecessary distraction, so we simply state the result:

$$E_{\rm GR} = -\frac{3GM^2}{5R}, \tag{1.3}$$

where M and R are the mass and radius of the star respectively. Real stars will, of course, not have a uniform density, but the relationship above provides a useful approximation in many cases.

The mean pressure inside the Sun (and indeed within any star in hydrostatic equilibrium) is given by the virial theorem (see the box above). Assuming the Sun to have a uniform density, combining Equations 1.2 and 1.3, and using solar values for the mass and radius, gives

$$\langle P_\odot\rangle = \left(-\frac{1}{3V_\odot}\right)\times\left(-\frac{3G\mathrm{M}_\odot^2}{5\mathrm{R}_\odot}\right) = \frac{G\mathrm{M}_\odot^2}{5V_\odot\mathrm{R}_\odot}.$$

Since the Sun is a sphere, its volume is

$$V_\odot = \frac{4}{3}\pi\mathrm{R}_\odot^3 \tag{1.4}$$

so the virial theorem for the average pressure inside the Sun may be rewritten as

$$\langle P_\odot\rangle = \frac{3G\mathrm{M}_\odot^2}{20\pi\mathrm{R}_\odot^4}. \tag{1.5}$$

- Using the values from Table 1.1, calculate the mean pressure inside the Sun.
- The mean pressure inside the Sun $\langle P_\odot\rangle$ is

$$\langle P_\odot\rangle = \frac{3\times 6.673\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}\times(1.99\times 10^{30}\,\mathrm{kg})^2}{20\pi\times(6.96\times 10^{8}\,\mathrm{m})^4}$$
$$= 5.38\times 10^{13}\,\mathrm{Pa}.$$

This is about 400 times smaller than the Sun's core pressure.

The mean density of the Sun is clearly just its mass divided by its volume. Since the Sun is a sphere this is simply

$$\langle\rho_\odot\rangle = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3} = \frac{3M_\odot}{4\pi R_\odot^3}. \tag{1.6}$$

- Using the values from Table 1.1, calculate the mean density of the Sun.

- The mean density inside the Sun $\langle\rho_\odot\rangle$ is

$$\langle\rho_\odot\rangle = \frac{3 \times 1.99 \times 10^{30}\ \text{kg}}{4\pi \times (6.96 \times 10^8\ \text{m})^3}$$
$$= 1.41 \times 10^3\ \text{kg m}^{-3}.$$

This is about 100 times smaller than its core density.

In order to calculate the mean temperature inside the Sun, we need to consider the particles of which it is composed, since the mass of the particles determines how the temperature and pressure are linked, via the ideal gas law. To do this, we need to find a way of describing the average composition in terms of the masses of the component particles. This is described in the following box.

Mean molecular mass

The composition of a star can be described in several ways, such as by stating the mass fractions X_Z of each of its constituent elements. However, often a more convenient measure is the mean molecular mass of the material. The **mean molecular mass**, μ, is the mean mass in amu (u) of the particles making up a gas. The amu scale itself is defined such that the mass of a carbon-12 atom, $m(^{12}\text{C})$ = 12 amu exactly, so

$$1\ \text{amu} = u = 1.661 \times 10^{-27}\ \text{kg}.$$

Hence, the mean molecular mass is given by the sum of the mass of the particles in amu divided by the total number of particles. In symbols, this is

$$\mu = \frac{\sum_i n_i \frac{m_i}{u}}{\sum_i n_i}, \tag{1.7}$$

where n_i is the number of particles of a particular type, and m_i is the mass of each of those particles of that type. The word *molecular* is a little misleading, because the interiors of stars are too hot for molecules to exist, and even most atoms are completely ionized, but the expression has survived from its use in cooler environments. The mean molecular mass is sensitive to both the chemical composition of the gas *and* its degree of dissociation and ionization.

A related concept is the mean molecular mass in units of kilograms. This is usually represented by the symbol $\overline{m}$ ('m-bar') and is simply

$$\overline{m} = \mu u. \tag{1.8}$$

In most cases in stellar astrophysics it is sufficient to assume that the mass of a hydrogen atom or ion is $m_{\rm H} \approx m_{\rm p} \approx 1u$ and the mass of a helium atom or ion is $m_{\rm He} \approx m_{\rm He^{2+}} \approx 4u$. Since the mass of an electron is almost 2000 times smaller than the mass of a proton it is usually sufficient to assume $m_{\rm e}/u \approx 0$.

For example, the mean molecular mass of a neutral gas of pure molecular hydrogen (H_2) is $\mu_{\rm H_2} \approx 2$, and the mean molecular mass of a neutral gas of pure atomic hydrogen is $\mu_{\rm H} \approx 1$. Finally, if we consider a neutral gas of completely ionized hydrogen, there are two different types of particles present: protons *and* electrons in a one-to-one ratio. The mean molecular mass is therefore

$$\mu_{\rm H^+} = \frac{N_{\rm p}(m_{\rm p}/u) + N_{\rm e}(m_{\rm e}/u)}{N_{\rm p} + N_{\rm e}}.$$

But since $N_{\rm p} = N_{\rm e}$ for a neutral gas and $m_{\rm p}/u \approx 1$ and $m_{\rm e}/u \approx 0$, we can write

$$\begin{aligned}\mu_{\rm H^+} &\approx \frac{N_{\rm p}}{N_{\rm p} + N_{\rm p}}\\ &\approx 0.5.\end{aligned}$$

Note that the mean molecular mass of ionized hydrogen is half that of neutral hydrogen, which is half that of molecular hydrogen, even though exactly the same numbers of protons and electrons are involved in each sample!

- What is the mean molecular mass of a neutral plasma containing fully ionized atoms whose atomic number is of order ~ 20 to 30?

- Atoms with atomic numbers ~ 20 to 30 will typically contain as many neutrons as protons. So a fully ionized atom of a given species with atomic number Z will have a mass of $m_Z \sim 2Zu$. Each ion will also produce Z electrons in the plasma, each of mass $m_{\rm e}$. The mean molecular mass of a plasma containing N_Z ions is therefore

$$\begin{aligned}\mu_Z &\sim \frac{(N_Z(2Zu)/u) + (ZN_Z(m_{\rm e})/u)}{N_Z + ZN_Z}\\ &\sim \frac{(2Z+0)\,N_Z}{(Z+1)N_Z} \sim \frac{2Z}{Z+1}.\end{aligned}$$

  Since $Z \sim 20$ to 30 then an approximate answer is $\mu_Z \sim 2$. Hence any neutral plasma of fully ionized heavy atoms will have a mean molecular mass of about 2.

Exercise 1.2 Assume the Sun is fully ionized (and neutral) and comprises 92.7% hydrogen ions and 7.3% helium ions, plus the appropriate number of electrons. Calculate the mean molecular mass of the particles in the Sun. ■

Now, if we assume that the gas of which the Sun is composed obeys the ideal gas law, then we may write its average pressure as

$$\langle P_\odot \rangle = \frac{\langle \rho_\odot \rangle \, kT_\mathrm{I}}{\overline{m}}, \tag{1.9}$$

where T_I is a typical temperature inside the Sun and $\overline{m}$ is the mean mass of the particles. Combining this with Equations 1.6 and 1.5, we have

$$T_\mathrm{I} = \frac{\overline{m} G \mathrm{M}_\odot}{5k\mathrm{R}_\odot}. \tag{1.10}$$

- Assuming that the average mass of the particles (nuclei and electrons) in the Sun is $\overline{m} \approx 0.6u$, use the values from Table 1.1 to calculate the typical temperature T_I inside the Sun.

- The typical temperature inside the Sun T_I is

$$T_\mathrm{I} = \frac{0.6 \times 1.661 \times 10^{-27}\ \mathrm{kg} \times 6.673 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}} \times 1.99 \times 10^{30}\ \mathrm{kg}}{5 \times 1.381 \times 10^{-23}\ \mathrm{J\ K^{-1}} \times 6.96 \times 10^{8}\ \mathrm{m}}$$
$$= 2.75 \times 10^{6}\ \mathrm{K}.$$

This is about 6 times smaller than the Sun's core temperature.

1.3 The equations of stellar structure

There is a set of equations which can be used to describe the conditions inside the Sun, and indeed within any main-sequence star. These are known as the *equations of stellar structure*, and comprise four first-order differential equations which describe the way in which a star's mass distribution, pressure, luminosity and temperature each varies as a function of distance from the star's core. They are supplemented by a set of four linking equations, which relate a star's pressure, luminosity, opacity and energy generation rate to its temperature and density. You have already met the first of these linking equations above: Equation 1.1 links a star's luminosity to its effective surface temperature using the Stefan–Boltzmann law.

A second of the linking equations is provided by a gas law. Clearly, the temperature, density and pressure within a star are not constant, and in fact each of these quantities will increase towards the core. In main-sequence stars, we can assume that the pressure, density and temperature inside the star are linked by the ideal gas law, which can be written as

$$P(r) = \frac{\rho(r) kT(r)}{\overline{m}}, \tag{1.11}$$

where $\overline{m}$ is the mean mass per particle inside the star. Recall that the notation $\rho(r)$ means that the density ρ is a function of the radial coordinate r in the star. It is also worth noting that $\overline{m}$ may also vary as a function of radius, and such abundance gradients are important in the post-main sequence stages of stellar evolution. For now, however, we assume this quantity may be treated as a constant.

We now proceed with the four differential equations of stellar structure, at this stage simply presenting them with a little justification. Clearly the Sun does not appear to change in size, and moreover there is no evidence that it has changed much in size over its recent lifetime. Hence, it is safe to assume that the Sun and other stars on the main sequence, are in a state of hydrostatic equilibrium. The pressure gradient at any point in the Sun must therefore be balanced by the gravitational force per unit volume, and we can write:

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{G\,m(r)\,\rho(r)}{r^2}, \qquad (1.12)$$

where $m(r)$ is the mass interior to a sphere of radius r (sometimes referred to as the **enclosed mass** at radius r) and $\rho(r)$ is the density at radius r. The minus sign reminds us that the pressure gradient increases *inwards* (i.e. the pressure is greater at smaller radii) whilst the radial coordinate (r) increases *outwards*.

The mass and density are related by the equation of **mass continuity**. This is the second of the four differential equations, and in a spherical geometry may be written as:

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r). \qquad (1.13)$$

For stars in which the majority of the energy transport is provided by radiation (rather than convection), another equation linking various properties is that of **radiative diffusion**. The temperature gradient $\mathrm{d}T(r)/\mathrm{d}r$ at some radius r within a star depends on the temperature $T(r)$, luminosity $L(r)$, density $\rho(r)$ and opacity $\kappa(r)$ (see the box below about opacity) at that radius according to:

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{3\kappa(r)\,\rho(r)\,L(r)}{(4\pi r^2)(16\sigma)\,T^3(r)} \qquad (1.14)$$

where σ is the Stefan–Boltzmann constant. The minus sign again indicates that the temperature increases *inwards* (i.e. the temperature is greater at smaller r). In the case of the Sun, and other stars of similar mass, radiative diffusion will dominate in the region known (not surprisingly) as the radiative zone. In the Sun's case this lies between about 0.2 and 0.7 solar radii from the centre.

Finally, we note that the luminosity increases outwards from the core of a star, as new sources of energy are encountered. The energy generation equation describes the increase in luminosity L as a function of radius r:

$$\frac{\mathrm{d}L(r)}{\mathrm{d}r} = 4\pi r^2\,\epsilon(r) \qquad (1.15)$$

where $\epsilon(r)$ is the energy generation rate per unit volume.

Equations 1.12 to 1.15 comprise the fundamental differential equations of stellar structure. They are based on simple assumptions of hydrostatic equilibrium and spherical symmetry, and assume that energy is transported within the star only by radiative diffusion. Modifications have to be made to these equations in cases where convective energy transport is important (such as in the outer layers of the Sun and other low-mass main-sequence stars), but in general they provide a means of calculating the structures of typical stars.

In order to solve these differential equations, other information is needed which relates the luminosity $L(r)$, pressure $P(r)$, opacity $\kappa(r)$ and energy generation rate $\epsilon(r)$ to the stellar density and temperature. The Stefan–Boltzmann law via

Equation 1.1 and the ideal gas law via Equation 1.11 provide this information in the first and second cases, whilst the Kramers opacity approximation (Equation 1.16 below) can provide the link in the third case. You will meet the final linking equation, namely an equivalent expression for the energy generation rate, in Chapter 3. In order to then solve this complete set of equations, a set of boundary conditions must also be selected. These quantify the behaviour of the variable quantities at the extremes where $r = 0$ and $r = R$.

Opacity

The opacity, κ (the Greek lower case letter kappa), of some material is its ability to block radiation. It is expressed as an absorption cross-section per unit mass, so has the unit $m^2\ kg^{-1}$. Perfectly transparent matter would have an opacity of zero (see Figure 1.2).

The opacity of material is caused by several physical processes, the four most common being: **bound–bound** atomic transitions, **bound–free** transitions (i.e. ionization), **free–free** (thermal **bremsstrahlung**) interactions, and **electron scattering** (especially in fully ionized [hot and/or low-density] material where there are a lot of free electrons). To calculate the opacity of some material, the contribution of each process must be computed, and all contributions added.

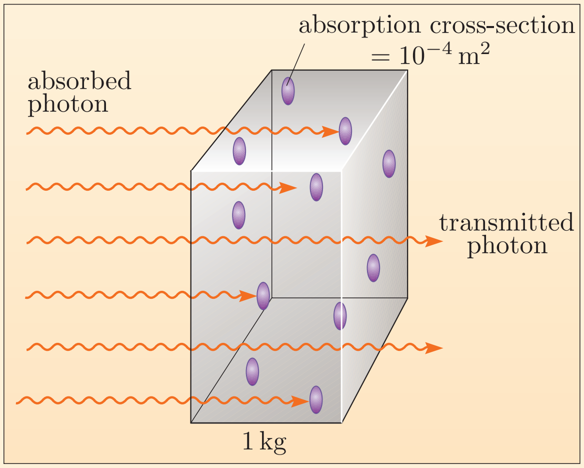

Figure 1.2 Schematic of opacity: if 1 kg of material contains 11 particles, each presenting an absorption cross-section of $10^{-4}\ m^2$, the opacity of the material is $11 \times 10^{-4}\ m^2\ kg^{-1}$. In practice, the number of particles will be $\gg 11$, and the absorption cross-section of each will be $\ll 10^{-4}\ m^2$.

Fortunately, these calculations have been done for us, and two useful generalizations can be made:

1. For stellar-composition material at temperatures $T \geq 30\,000$ K, the opacity κ is dominated by free–free and bound–free absorption, for which an approximate form is $\kappa \propto \rho T^{-3.5}$ (see Figure 1.3 on page 19). Any opacity of this form,

$$\kappa(r) = \kappa_0 \rho(r) T^{-3.5}(r), \tag{1.16}$$

is called a **Kramers opacity**, after the Dutch physicist Hendrik (Hans) Kramers who first found this solution in 1923. The Kramers opacity is a very useful approximation, but you should note that it is just that, and not a fundamental physical law.

2. In low-density environments and at very high temperature, scattering by free electrons dominates. In electron scattering, the absorption per free electron is independent of temperature or density, so this opacity source has an almost constant absorption per electron, and therefore per unit mass of ionized material. It is responsible for the flat tail at high temperature in Figure 1.3, which sets a lower limit on the opacity at high temperature and low density.

- If the absorbing particles in Figure 1.2 exhibit a Kramers opacity, and their temperature is doubled, what happens to the opacity and their absorption cross-sections? If the particles are electrons, so that the opacity is due entirely to electron scattering, what happens to the opacity if the temperature is doubled?

- The opacity of particles exhibiting a Kramers opacity decreases if the temperature is increased. The overall opacity would decrease by a factor $2^{3.5} \approx 11$, to $\approx 10^{-4}$ m^2 kg^{-1}. Since there are still 11 particles, we infer that their individual absorption cross-sections must also decrease by a factor of ≈ 11, to $\approx 9 \times 10^{-5}$ m^2. Electron scattering, on the other hand, depends only on the *number* of free electrons, *not* their *temperature*, so in the second case the opacity would be unchanged.

- Table 1.1 gives the central temperature and central density of the Sun. Use these values to mark a cross on Figure 1.3 indicating the conditions in the solar core. Based on where it lies, do you expect the opacity of the material in the core of the Sun to be dominated by a Kramers opacity or by electron scattering?

- The point for the solar core lies just above the curve for $\rho = 10^5$ kg m^{-3}, at $\log_{10}(T/\mathrm{K}) = 7.19$, where the opacity is seen to be $\approx 2 \times 10^{-1}$ m^2 kg^{-1}. This is still on the sloping part of the opacity–temperature curves, indicating that Kramers opacity still dominates over electron scattering.

The stellar structure equations describe the radial gradients d$P(r)$/dr, d$m(r)$/dr, d$T(r)$/dr and d$L(r)$/dr in terms of a number of variables including $m(r)$, $P(r)$, $\rho(r)$, $\kappa(r)$, $L(r)$, $T(r)$, $\epsilon(r)$ and r. Ideally one would like to be able to write down expressions for all of these quantities as a function of the radial coordinate r, but alas it is rather difficult to do so. Over the years there have been several attempts to develop simplistic models that use approximations to achieve this, and they can in many cases provide valuable insights into the physics of stars.

In fact, it is possible to infer some important relationships between the stellar structure variables by making just a few simple approximations. The mathematical treatment that we will follow to illustrate this is straightforward enough, but rather tedious and would serve as a distraction at this stage of the book, so we provide just a few insights rather than mathematical rigour.

The first step is to divide each stellar structure variable by its maximum value or its average value in the star. For example, whenever the enclosed mass $m(r)$ occurs in a stellar structure equation, we divide it by the total mass of the star M. Of course, we must perform the same division on both sides of the stellar structure equation. This step recasts the stellar structure equations in terms of normalized variables $m(r)/M$, $P(r)/P_\mathrm{c}$, $T(r)/T_\mathrm{c}$ etc., where P_c and T_c are the pressure and temperature at the core of the star.

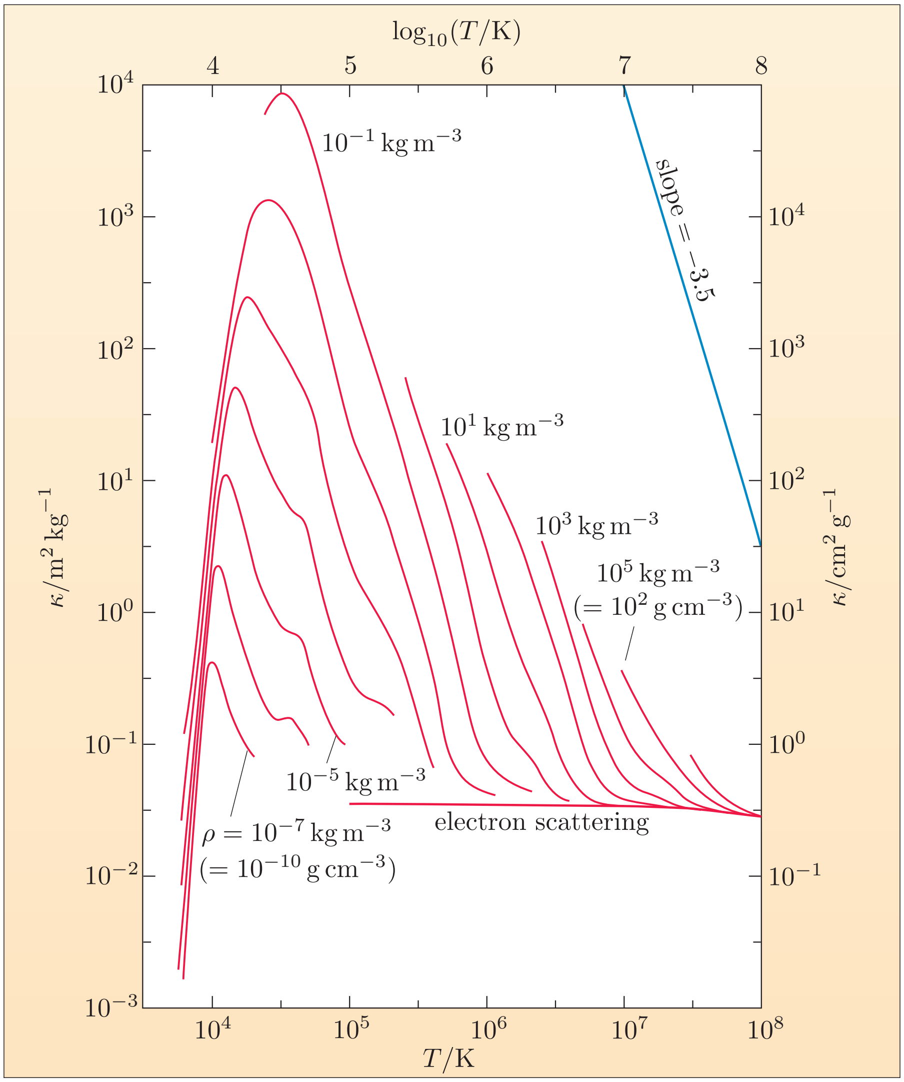

Figure 1.3 Opacity for solar composition material, as a function of temperature (on the horizontal axis) and density, in a log–log plane. Each curve is labeled by the value of the density ρ in kg m^{-3}. In the regime $T \geq 30\,000$ K, i.e. $T \geq 10^{4.5}$ K, the curves (in the log–log plane) are roughly linear with slope -3.5, and more or less uniformly spaced for equal increments of log density, confirming that $\log_{10}\kappa = \text{const} + \log_{10}\rho - 3.5\log_{10}T$, or equivalently $\kappa \propto \rho T^{-3.5}$. Electron scattering has an almost constant cross-section per unit mass, and is the dominant opacity at high temperatures and low densities, where it sets a lower bound on the opacity values.

The second step is to assume that all stars have the same structure when compared using normalized variables. (This assumption is reasonable — and hence useful — for some stars, but poor — and hence unhelpful — for others.) According to this assumption, although the total masses of two stars, M_1 and M_2, may differ, and their central temperatures $T_{c,1}$ and $T_{c,2}$ also may differ, their structures are similar enough that the variation of normalized temperature $T(r)/T_c$ with normalized enclosed mass $m(r)/M$ is the same for both objects.

The outcome of taking the two steps described above is that we can write a *companion equation* for each of the stellar structure equations given above. The companion equation differs from the original in four respects:

1. each occurrence of a derivative is replaced by a simple ratio of the corresponding variables,
2. each occurrence of a stellar structure variable is replaced by either its maximum or average value,

3. the equals sign = is replaced by the symbol $\propto$ meaning 'is proportional to', and
4. any physical constants can be dropped.

A worked example is given below to make this clearer.

Worked Example 1.1

As an illustration of the use of the stellar structure equations, we use them to derive an approximate mass–luminosity relationship for high-mass stars on the main sequence. In such stars, the main source of opacity is electron scattering, so for them the opacity is roughly constant (i.e. independent of temperature or density).

Solution

In the following, we use M and R to represent the star's total mass and radius, L for its surface luminosity, $P_{\rm c}$ for its central pressure, $T_{\rm c}$ for its central temperature, $\overline{\rho}$ for its mean density and $\overline{\kappa}$ for its mean opacity.

We begin by writing the companion equation to Equation 1.13 for mass continuity. In this case we can replace d$m(r)$/dr, by the simple ratio, M/R. We can also replace r by the radius R and $\rho(r)$ by the mean density $\overline{\rho}$. So, the companion equation to the mass continuity equation is simply $M/R \propto R^2\rho$ which may be re-written as (i) $\overline{\rho} \propto M/R^3$. This of course makes sense, since the average density is just $\overline{\rho} = M/\frac{4}{3}\pi R^3$.

The equation of hydrostatic equilibrium (Equation 1.12) gives rise to the companion equation $P_{\rm c}/R \propto M\overline{\rho}/R^2$ which using (i) from above may be re-written as (ii) $P_{\rm c} \propto M^2/R^4$.

For an unchanging chemical composition, the ideal gas law leads to the expression $P_{\rm c} \propto \overline{\rho}T_{\rm c}$. Substituting for $\overline{\rho}$ and $P_{\rm c}$ using (i) and (ii) from above, this becomes (iii) $T_{\rm c} \propto M/R$.

Finally, the temperature gradient equation leads to $T_{\rm c}/R \propto \overline{\kappa\rho}L/R^2T_{\rm c}^3$, which can immediately be re-arranged as $T_{\rm c}^4 \propto \overline{\kappa\rho}L/R$. Now substituting for $\overline{\rho}$ using (i) this becomes $T_{\rm c}^4 \propto \overline{\kappa}ML/R^4$ and then substituting for $T_{\rm c}$ using (iii) we have (iv) $L \propto M^3/\overline{\kappa}$.

Now, as noted at the beginning of this example, in a high-mass main-sequence star, the opacity is constant, so the mass–luminosity relationship for such stars is approximately $L \propto M^3$.

Exercise 1.3 Following a similar process to that in the above worked example, what would be an approximate relationship between the mass, luminosity and radius of a low-mass main-sequence star, in which the opacity may be represented by a Kramers opacity? ■

The two mass–luminosity relationships derived in the previous worked example and exercise may be summarized as:

$$L \propto M^3 \quad \text{for high-mass main-sequence stars} \tag{1.17}$$

$$L \propto M^{5.5} R^{-0.5} \quad \text{for low-mass main-sequence stars.} \tag{1.18}$$

The latter equation is known as the Eddington mass–luminosity–radius relationship. Note, these equations are only approximations to the real situation, for the reasons mentioned, and we have ignored factors such as differing chemical composition, for instance. Nonetheless, they are good approximations for real stars and demonstrate the power of this simple analysis in tackling complex situations.

1.4 The proton–proton (p–p) chain

With hydrogen being the most abundant element in the Universe it is understandable that hydrogen is both the starting point for stellar nuclear burning and, as it turns out, the longest-lasting nuclear fuel. In this section we examine the first of two hydrogen-burning processes, beginning with that which dominates in the Sun and other low-mass main-sequence stars.

As a contracting pre-main-sequence star heats up, it reaches a temperature of about 10^6 K at which protons can undergo fusion with any pre-existing light nuclei, such as deuterium (D), lithium (Li), beryllium (Be) and boron (B). However, these reactions are extremely rapid and release only a limited amount of energy because the light nuclei are present in such small quantities. In order to begin life properly as star, a different reaction must be initiated, one in which protons are combined with each other to make heavier nuclei. If one could simply combine two protons to make a nucleus of ${}^2_2$He, then hydrogen burning would be very rapid. However, a nucleus of helium containing no neutrons is not stable, so this is not how hydrogen fusion proceeds.

Instead, at a temperature of about 10^7 K, the first step in the **proton–proton chain** occurs when two protons react to form a nucleus of deuterium, which is called a **deuteron**:

$$\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{d} + \mathrm{e}^+ + \nu_\mathrm{e},$$

where the deuteron (d) comprises a proton (p) and a neutron (n). This first step is therefore a weak interaction in which a proton has been converted into a neutron, with the release of a positron (e^+) and an electron neutrino (ν_e). It is important to note that this first step is incredibly slow. On average, an individual proton in the core of the Sun will have to wait for about 5 billion years before it undergoes such a reaction. (Although, since the core of the Sun contains a huge number of protons, some of them will react as soon as the conditions are right for them to do so.) This reaction therefore effectively governs the length of time for which a star will undergo hydrogen fusion – its main sequence lifetime.

Once a deuterium nucleus has formed it will rapidly capture another proton to form a nucleus of helium-3, in the reaction:

$$\mathrm{d} + \mathrm{p} \longrightarrow {}^3_2\mathrm{He} + \gamma.$$

However, from here, the helium-3 nucleus can undergo one of two reactions. In the Sun, 85% of the time, the helium-3 nucleus will react with another helium-3 nucleus to form a helium-4 nucleus directly, in the reaction:

$$ {}^{3}_{2}\text{He} + {}^{3}_{2}\text{He} \longrightarrow {}^{4}_{2}\text{He} + \text{p} + \text{p}. $$

This set of reactions constitutes the first branch of the p–p chain, (ppI).

In the rest of the cases, the helium-3 nucleus will react with an existing helium-4 nucleus to form a nucleus of beryllium-7 as follows:

$$ {}^{3}_{2}\text{He} + {}^{4}_{2}\text{He} \longrightarrow {}^{7}_{4}\text{Be} + \gamma. $$

Virtually every beryllium-7 nucleus will subsequently capture an electron to form lithium-7, which then reacts with a further proton to produce two helium-4 nuclei:

$$ \text{e}^{-} + {}^{7}_{4}\text{Be} \longrightarrow {}^{7}_{3}\text{Li} + \nu_{\text{e}} $$
$$ \text{p} + {}^{7}_{3}\text{Li} \longrightarrow {}^{4}_{2}\text{He} + {}^{4}_{2}\text{He}. $$

This set of reactions constitutes the second branch of the p–p chain, (ppII).

In a very tiny proportion of cases, the beryllium-7 nucleus will instead react directly with a proton forming a nucleus of boron-8 which subsequently undergoes beta-plus decay to beryllium-8, which in turn splits into two helium-4 nuclei:

$$ \text{p} + {}^{7}_{4}\text{Be} \longrightarrow {}^{8}_{5}\text{B} $$
$$ {}^{8}_{5}\text{B} \longrightarrow {}^{8}_{4}\text{Be} + \text{e}^{+} + \nu_{\text{e}} $$
$$ {}^{8}_{4}\text{Be} \longrightarrow {}^{4}_{2}\text{He} + {}^{4}_{2}\text{He}. $$

This set of reactions constitutes the third branch of the p–p chain, (ppIII).

All three branches are summarized in Figure 1.4 (overleaf). In stars with higher core temperatures, the proportions of nuclei undergoing fusion via the ppII and ppIII branches are higher than in the Sun.

- The proton–proton chain comprises nine reactions in three branches, but just one reaction determines the overall rate at which hydrogen burning proceeds. Which reaction is the bottleneck and why?

- The slowest reaction in the series is the first one, $\text{p} + \text{p} \longrightarrow \text{d} + \text{e}^{+} + \nu_{\text{e}}$. This is slowest because during the collision of the protons, one of them must undergo a β^{+}-decay, $\text{p} \longrightarrow \text{n} + \text{e}^{+} + \nu_{\text{e}}$. The β^{+}-decay is mediated by the weak nuclear force, so has a very low probability of occurring.

1.5 The mass defect

The proton–proton chain converts four hydrogen nuclei (protons) into a ${}^{4}_{2}\text{He}$ nucleus, two positrons that quickly collide with electrons and are annihilated, and two neutrinos. Hence, branch I of the p–p chain may be summarized as:

$$ 2\text{e}^{-} + 4\text{p} \longrightarrow {}^{4}_{2}\text{He} + 2\nu_{\text{e}} + 2\gamma_{\text{pd}} + 4\gamma_{\text{e}}. $$

(The *unconventional* subscripts used here on the γ-rays are to distinguish the γ-rays from the p + d reaction from those from the electron–positron annihilation.)

The energy released in these reactions can be assessed from the differences in the masses of each particle, and the masses of the nuclei can be obtained from the atomic masses of their isotopes. We use the atomic mass unit (amu) scale where $m(^{12}\mathrm{C})$ = 12 amu exactly, so 1 amu = $1.660\,540 \times 10^{-27}$ kg. Isotope tables give the atomic masses of hydrogen and helium-4 as:

$$m(^1_1\mathrm{H}) = 1.007\,825\ \mathrm{amu} = 1.673\,534 \times 10^{-27}\ \mathrm{kg}$$

$$m(^4_2\mathrm{He}) = 4.002\,60\ \mathrm{amu} = 6.646\,478 \times 10^{-27}\ \mathrm{kg}.$$

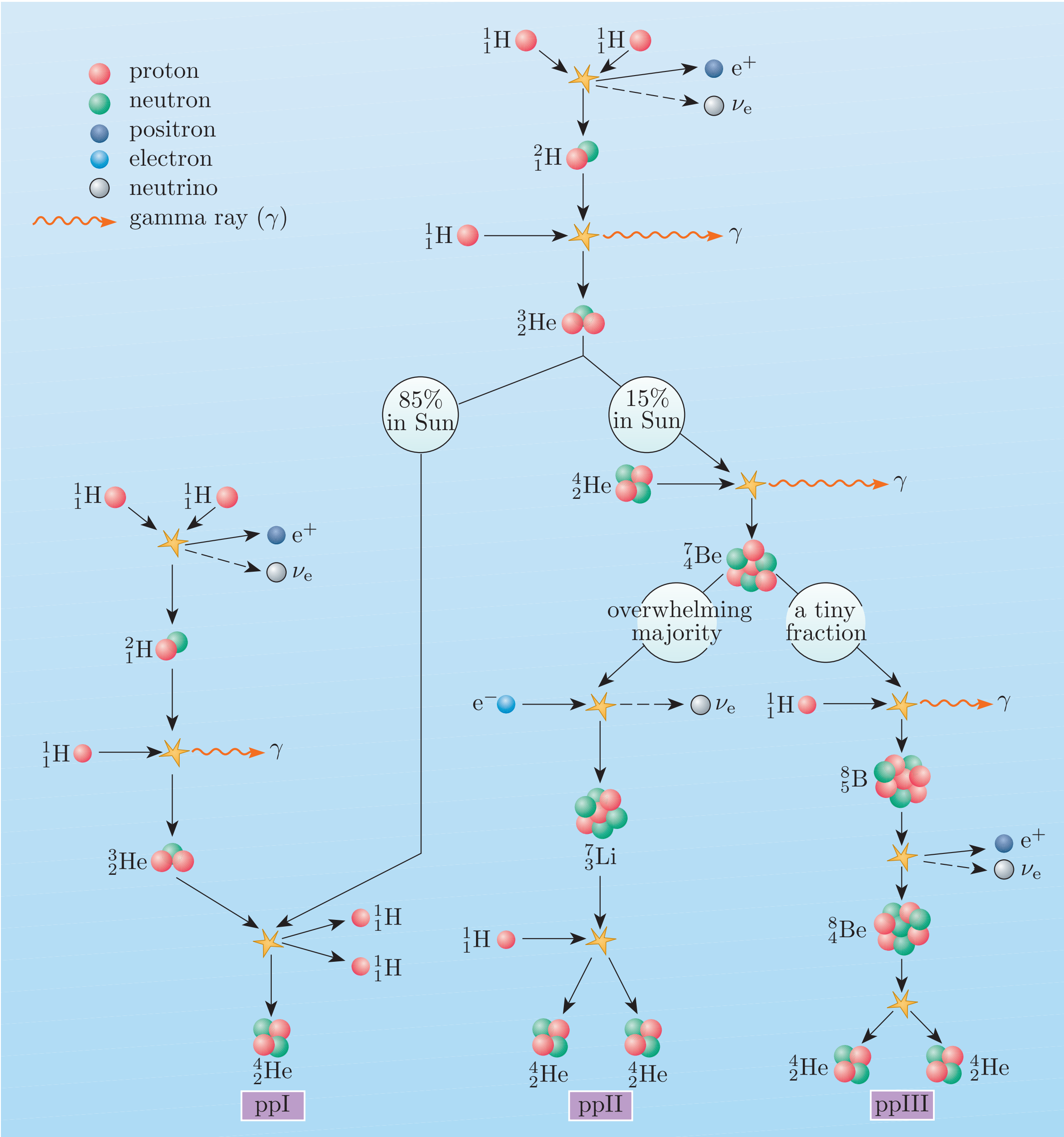

Figure 1.4 The p–p chain consists of three branches (ppI, ppII and ppIII). Each of them results in the creation of nuclei of helium-4 from four protons. The relative proportions of each branch occurring in the Sun are shown.

These include one electron for hydrogen and two electrons for helium, where $m(\mathrm{e}^-) = 9.109 \times 10^{-31}$ kg, so subtracting the electronic component gives nuclear masses $m_\text{n}(^1_1\text{H}) = 1.672\,623 \times 10^{-27}$ kg for the $^1_1$H *nucleus* and $m_\text{n}(^4_2\text{He}) = 6.644\,656 \times 10^{-27}$ kg for the $^4_2$He *nucleus*.

The reaction sequence has the following mass difference:

$$\begin{aligned}\Delta m &= \text{initial mass} - \text{final mass}\\ &= 2m(\text{e}^-) + m(4\text{p}) - m_\text{n}(^4_2\text{He}) - m(2\nu_\text{e}) - m(2\gamma_\text{pd}) - m(4\gamma_\text{e})\\ &= (2 \times 9.109 \times 10^{-31}\ \text{kg}) + (4 \times 1.672623 \times 10^{-27}\ \text{kg})\\ &\quad - 6.644656 \times 10^{-27}\ \text{kg} - 0 - 0 - 0\\ &= 4.7658 \times 10^{-29}\ \text{kg}.\end{aligned}$$

This mass difference is called the **mass defect**. Note for future reference that this corresponds to the fraction ≈ 0.0066 of the original mass of four protons.

Einstein's famous equation $E = mc^2$ expresses the mass–energy equivalence and gives the energy released from this change in mass. The energy associated with the mass defect is given the symbol ΔQ, and in our example is $\Delta Q = (\Delta m)c^2 = 4.2833 \times 10^{-12}$ J $= 26.74$ MeV. Some of this is in the form of the kinetic energy of reaction products, and some is in the form of the two γ_pd-rays and the four γ_e-rays. Most of this energy is quickly absorbed by the surrounding particles and thus appears as the increased kinetic energy of the gas. However, the two neutrinos do not interact with the local gas, and escape from the star unimpeded. They carry off a small amount of energy, on average 0.26 MeV each for the neutrinos in the first branch of the p–p chain, reducing the effective energy contribution to the star to $26.74\,\text{MeV} - (2 \times 0.26\,\text{MeV}) = 26.22$ MeV.

- In most of the exercises in this book only four significant figures are given for the physical constants. However, in the calculation above we have used seven for the atomic and nuclear masses. Why?

- Whenever you have to subtract nearly equal quantities, the number of significant figures decreases. Consider

$$\begin{aligned}4m(\text{p}) - m_\text{n}(^4_2\text{He}) &= 6.690\,492 \times 10^{-27}\ \text{kg} - 6.644\,656 \times 10^{-27}\ \text{kg}\\ &= 0.045\,836 \times 10^{-27}\ \text{kg}.\end{aligned}$$

Although the nuclear masses are quoted to 7 significant figures, the numbers are so similar that only 5 significant figures remain after the subtraction. If we had begun with only 4 figures, we would have been left with only 2, and numerical accuracy would have been lost.

Exercise 1.4 Following a similar procedure to that outlined above for the first branch of the proton–proton chain, what is the energy released by the reactions comprising the second branch of the proton–proton chain? (Note: Of the neutrinos released in the reaction where a beryllium-7 nucleus captures an electron, 90% of them carry away an energy of 0.86 MeV, whilst the remaining 10% carry away an energy of 0.38 MeV, depending on whether or not the beryllium-7 is created in an excited state.) ■

You have seen that each instance of the first branch of the p–p chain contributes 26.22 MeV of energy to the star and each instance of the second branch of the p–p chain contributes 25.67 MeV of energy. Since the first branch occurs 85% of the time and involves two proton–proton reactions, and the second branch occurs 15% of the time but involves only one proton–proton reaction, the average energy released for each reaction between two protons is $(0.85 \times 26.22\,\text{MeV})/2 + (0.15 \times 25.67\,\text{MeV}) = 15.0\,\text{MeV}$.

1.6 The carbon–nitrogen–oxygen (CNO) cycle

The proton–proton chain is the main fusion reaction in the Sun, but it is *not* the main one in many *more-massive* main-sequence stars. Here there are two closed cycles of reactions which not only convert hydrogen into helium, but also convert carbon-12 nuclei (formed in an earlier generation of stars and incorporated when the star formed) into carbon-13 and various isotopes of nitrogen and oxygen.

The **carbon–nitrogen (CN) cycle** uses $^{12}_{6}\text{C}$ as a catalyst. That is, *if* the CN cycle completes, any $^{12}_{6}\text{C}$ involved in the reaction sequence is returned at the end without being consumed. An illustrative view, emphasizing the cyclical nature, and also showing the associated **oxygen–nitrogen (ON) cycle** which requires higher temperatures, is shown in Figure 1.5. Collectively these two cycles are known as the carbon–nitrogen–oxygen cycle or **CNO cycle**.

The CN-cycle can be considered as starting with a carbon-12 nucleus. This captures a proton to form nitrogen-13, which then undergoes beta-plus decay to carbon-13. This captures two further protons to form nitrogen-14 and then oxygen-15. A beta-plus decay to nitrogen-15 is then followed by the capture of a proton, emission of an alpha-particle (i.e. helium-4 nucleus) and the recovery of the original carbon-12 nucleus.

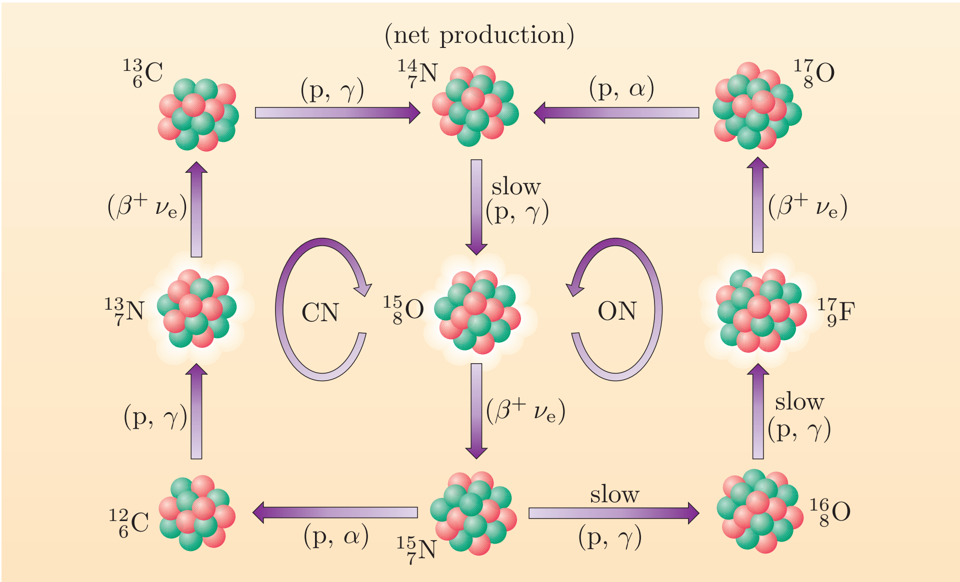

Figure 1.5 The CNO cycle is made up of two parts: the CN cycle which operates at lower temperatures, and the ON cycle which becomes more important at higher temperatures. The cycles contain essentially only three types of reactions: The notation p,γ indicates the capture of a proton with the emission of a γ-ray, which increase the atomic number and atomic mass number by one, the notation $\beta^+\nu_e$ indicates a beta-plus decay accompanied by neutrino emission, and the notation p,α indicates a reaction that closes each cycle with the capture of a fourth proton, followed by the emission of a helium nucleus (α-particle) and the recovery of the starting nucleus.

Similarly, the ON-cycle can be considered as starting with a nitrogen-14 nucleus. This captures a proton to form oxygen-15, which then undergoes beta-plus decay to nitrogen-15. This captures two further protons to form oxygen-16 and then fluorine-17. A beta-plus decay to oxygen-17 is then followed by the capture of a proton, emission of an alpha-particle (i.e. helium-4 nucleus) and the recovery of the original nitrogen-14 nucleus.

We shall return to the physics of the p–p chain and the CNO-cycle in Chapter 3.

Summary of Chapter 1

1. The Hertzsprung–Russell diagram is of fundamental importance for describing the properties of stars and for tracking their evolution. It has two forms: the observational H–R diagram plots absolute visual magnitude M_V versus colour index (often $B - V$) or spectral type; the theoretical H–R diagram plots luminosity L versus effective surface temperature T_{eff} on a log–log scale.
2. $L = 4\pi R^2 \sigma T_{\text{eff}}^4$ relates stellar radius, luminosity and effective surface temperature. Loci of constant radius in the H–R diagram are diagonal lines from upper left to lower right.
3. The virial theorem, which is derived from the condition for hydrostatic equilibrium, concludes that the average pressure needed to support a self-gravitating system is minus one-third of the gravitational potential energy density $\langle P \rangle = -E_{\text{GR}}/3V$.
4. The gravitational potential energy of a spherical cloud of gas of uniform density is $E_{\text{GR}} = -3GM^2/5R$. This is a useful approximation for most stars.
5. The mean molecular mass of a sample of gas is given by the sum of the mass of the particles in amu divided by the total number of particles. The mean molecular masses for three forms of hydrogen are: $\mu_{\text{H}_2} \approx 2$; $\mu_{\text{H}} \approx 1$ and $\mu_{\text{H}^+} \approx 0.5$. The mean molecular mass of the Sun is $\mu_\odot \approx 0.6$.
6. We can express some average properties of a star using the following equations:
   - the mean density: $\langle \rho \rangle = 3M/4\pi R^3$;
   - the mean (volume-averaged) pressure: $\langle P \rangle = -E_{\text{GR}}/3V \approx 3GM^2/20\pi R^4$
   - and the typical internal temperature: $T_{\text{I}} \approx GM\overline{m}/5kR$
   - where, by the ideal gas law: $\langle P \rangle = \langle \rho \rangle \, kT_{\text{I}}/\overline{m}$.
7. There are four differential equations which characterize stellar structure in terms of the star's pressure gradient, mass distribution, temperature gradient and luminosity gradient, each as a function of radius:

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{G\,m(r)\,\rho(r)}{r^2} \qquad \text{(Eqn 1.12)}$$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2\,\rho(r) \qquad \text{(Eqn 1.13)}$$

$$\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -\frac{3\kappa(r)\,\rho(r)\,L(r)}{(4\pi r^2)(16\sigma)\,T^3(r)} \qquad \text{(Eqn 1.14)}$$

$$\frac{\mathrm{d}L(r)}{\mathrm{d}r} = 4\pi r^2\,\epsilon(r). \qquad \text{(Eqn 1.15)}$$

   They can be solved by combining them with four linking equations which relate a star's pressure, luminosity, opacity and energy generation rate to its temperature and density.

8. Useful approximations to the structure of stars may be obtained by rewriting the stellar structure equations and the linking equations as simple proportionalities between the maximum or average values of the various stellar parameters.
9. For stellar-composition material at temperatures $T \geq 30\,000$ K, the opacity κ is dominated by free–free and bound–free absorption, for which $\kappa(r) \propto \rho(r) T^{-3.5}(r)$. Any opacity of this form is called a Kramers opacity. In low-density environments and at very high temperature, scattering by free electrons dominates. In this case the absorption per free electron is independent of temperature or density, so this opacity source has an almost constant absorption per electron.
10. Based on a simple analysis, the stellar structure equations and the two different opacity relations indicate the following mass–luminosity relationships:
$$L \propto M^3 \quad \text{for high-mass main-sequence stars}$$
$$L \propto M^{5.5} R^{-0.5} \quad \text{for low-mass main-sequence stars.}$$
11. Thermonuclear burning begins at $T_c \approx 10^6$ K to 10^7 K with the most abundant and least complex nucleus, the proton (hydrogen nucleus). The main branch (ppI) of the proton–proton chain can be written:
$$\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{d} + \mathrm{e}^+ + \nu_\mathrm{e}$$
$$\mathrm{p} + \mathrm{d} \longrightarrow {}^3_2\mathrm{He} + \gamma$$
$${}^3_2\mathrm{He} + {}^3_2\mathrm{He} \longrightarrow {}^4_2\mathrm{He} + 2\mathrm{p}.$$
A crucial step in the $\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{d} + \mathrm{e}^+ + \nu_\mathrm{e}$ reaction is a β^+-decay, $\mathrm{p} \longrightarrow \mathrm{n} + \mathrm{e}^+ + \nu_\mathrm{e}$. This is the bottleneck in the reaction chain.
12. The mass defect – the difference in the masses of the reactants and products of the nuclear reactions – quantifies the energy liberated into the gas, via $E = mc^2$. The main branch of the p–p chain liberates ≈ 26.7 MeV, of which ≈ 0.5 MeV is carried away from the star by the neutrinos from the two $\mathrm{p} \longrightarrow \mathrm{n} + \mathrm{e}^+ + \nu_\mathrm{e}$ reactions.
13. There are three branches to the p–p chain, each delivering slightly different energy per event and occurring with different frequency. In the Sun, ppI occurs 85% of the time, ppII accounts for almost all of the rest, and ppIII occurs in a very tiny proportion of cases. The average energy released per proton–proton fusion event is ≈ 15 MeV.
14. The second main hydrogen-burning process is the CNO cycle, in which carbon, nitrogen and oxygen nuclei are used as catalysts while synthesizing helium from hydrogen. The CN cycle operates first, and the ON cycle comes into play at higher temperatures.

Acknowledgements

Grateful acknowledgement is made to the following sources:

Figures

Cover image courtesy of NASA and The Hubble Heritage Team (AURA/STScl); Figures 2.2 and 3.5: Adapted from Phillips, A. C. (1999) *The Physics of Stars*, 2nd edition, John Wiley and Sons Limited; Figure 3.2: Wikipedia.com; Figure 3.6: Adapted from Schwarzschild, M. (1958) *Structure and Evolution of the Stars*, Dover Publications Inc; Figure 4.1: Adapted from Arp, H. C. (1958) *Handbook of Physics*, Vol. 51. Springer Verlag; Figure 4.2: Adapted from Pont, F. et al. (1998) 'Hipparcos subdwarfs and globular cluster ages: M92', *Astronomy & Astrophysics*, Vol. 329, pp. 87–100, Kluwer Academic Publishers; Figure 4.3a: Royal Observatory Edinburgh/Anglo–Australian Telescope Board/David Malin; Figure 4.3b: Nigel Sharp, Mark Hanna/AURA/NOAO/NSF; Figure 4.3c: Hubble Heritage Team (AURA/ST ScI/NASA); Figure 4.6: Böhm-Vitense, E. (1992) *Introduction to Stellar Astrophysics*, Cambridge University Press; Figure 5.1: Adapted from Mateo, M., (1987) Ph.D. Thesis, University of Washington; Figure 5.2: Adapted from Stetson, P. B. et al. (1999) 'Ages for globular clusters in the outer galactic halo', *The Astronomical Journal*, Vol. 117, January 1999, University of Chicago Press, ©Stetson et al.; Figure 5.3: Strohmeier, W. (1972) *Variable Stars*, Pergamon Press. ©1972 W. Strohmeier; Figure 5.6: Hesser, Harris and Vandenberg (1987) *Publications of the Astronomical Society of the Pacific*, Vol. 99, 1987, The Astronomical Society of the Pacific; Figure 6.1: S. Kwok (University of Calgary)/R. Rubin (NASA Ames Research Center)/ H. Bond (ST ScI) and NASA; Figure 6.2a: H. Bond (ST ScI) and NASA; Figure 6.2b: NASA, A. Fruchter and the ERO team (ST ScI); Figure 6.4: Kitt Peak National Observatory 0.9-meter telescope, National Optical Astronomy Observatories, courtesy of M. Bolte, University of California, Santa Cruz; Figures 7.1a and b: Anglo–Australian Observatory; Figure 7.1c: Hubble Heritage Team (AURA/ST ScI/NASA); Figure 7.2a: MDM Observatory; Figures 7.2b, 7.2c and 7.3: NASA/CXC/SAO; Figure 7.2d: NASA/CXC/SAO/Rutgers/J Hughes; Figure 7.4: Adapted from Lorimer, D. R. and Kramer, M. (2004) *Handbook of Pulsar Astronomy*, Cambridge University Press; Figure 8.1: Jeff Hester and Paul Scowen (AZ State Univ)/NASA; Figure 8.8: N. Walborn (ST ScI), R. Barbva (La Plata Observatory) and NASA; Figure 8.9: Yohko Tsuboi, Pennsylvania State University; Figure 8.10: C. Burrows (ST ScI)/J. Hester (AZ State University)/ J. Morse (ST ScI)/NASA; Figure 8.11: Adapted from Strahler, S. (1983), Astrophysical Journal, vol. 274, University of Chicago Press; Figure 8.12: A. Schultz (Computer Sciences Corp), S. Head (NASA Goddard Space Flight Center and NASA); Figure 8.13: M. J. McCaughrean (MPIA)/ C. R. O'Dell (Rice University)/NASA.

Every effort has been made to contact copyright holders. If any have been inadvertently overlooked the publishers will be pleased to make the necessary arrangements at the first opportunity.