Chapter 2
Smooth Spaces

2.1 Introduction

In this chapter, we introduce the class of *smooth spaces*. We remark immediately that there is a *duality relationship* between uniform smoothness and uniform convexity. In the sequel, we shall examine this relationship. We begin with the following definition.

Definition 2.1. A normed space X is called smooth if for every $x \in X, ||x|| = 1$, there exists a *unique* x^* in X^* such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.

2.2 The Modulus of Smoothness

In this section, we shall define a function called the *modulus of smoothness* of a normed space X (denoted by $\rho_X : [0,\infty) \to [0,\infty)$) and prove three important properties of the function that will be used in the sequel, namely:

1. *(Proposition 2.3) For every normed space X, the modulus of smoothness, ρ_X, is a convex and continuous function.*
2. *(Theorem 2.5) A normed space X is uniformly smooth if and only if* $\lim\limits_{t\to 0^+} \dfrac{\rho_X(t)}{t} = 0.$
3. *(Corollary 2.8) For every normed space, $\frac{\rho_X(t)}{t}$ is a nondecreasing function on $[0,\infty)$ and $\rho_X(t) \le t \;\; \forall\, t \ge 0$.*

There exists a complete dual notion to uniformly smooth space which plays a central role in the structure of Banach spaces (see, Diestel [206]). In this section, we present this duality. Recall that a Banach space is called *smooth* if for every x in X with $\|x\| = 1$, there exists a *unique* x^* in X^* such that $\|x^*\| = \langle x, x^* \rangle = 1$. Assume now that X is not smooth and take x in X and u^*, v^* in X^* such that $\|x\| = \|u^*\| = \|v^*\| = \langle x, u^* \rangle = \langle x, v^* \rangle = 1$ and

$u^* \neq v^*$. Let y in X be such that $\|y\| = 1$, $\langle y, u^* \rangle > 0$ and $\langle y, v^* \rangle < 0$. Then for every $t > 0$ we have

$$1 + t\langle y, u^* \rangle = \langle x + ty, u^* \rangle \leq \|x + ty\|,$$

$$1 - t\langle y, v^* \rangle = \langle x - ty, v^* \rangle \leq \|x - ty\|$$

which imply $2 < 2 + t(\langle y, u^* \rangle - \langle y, v^* \rangle) \leq \|x + ty\| + \|x - ty\|$ or, equivalently

$$0 < t\Big(\frac{\langle y, u^* \rangle - \langle y, v^* \rangle}{2}\Big) \leq \frac{\|x + ty\| + \|x - ty\|}{2} - 1.$$

With this motivation we introduce the following definition.

Definition 2.2. Let X be a normed space with $\dim X \geq 2$. The *modulus of smoothness* of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\begin{aligned} \rho_X(\tau) &:= \sup\Big\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\Big\} \\ &= \sup\Big\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = 1 = \|y\|\Big\}. \end{aligned}$$

Note that evidently $\rho_X(0) = 0$. The following results explore some properties of the modulus of smoothness. We begin with the following important proposition.

Proposition 2.3. *For every normed space X, the modulus of smoothness, ρ_X, is a convex and continuous function.*

Proof. Fix x and y with $\|x\| = \|y\| = 1$ and consider

$$f_{x,y}(t) := \frac{\|x + ty\| + \|x - ty\|}{2} - 1.$$

Then for λ in $[0, 1]$

$$\begin{aligned} &f_{x,y}(\lambda t + (1 - \lambda)s) \\ &= \frac{\|x + (\lambda t + (1 - \lambda)s)y\| + \|x - (\lambda t + (1 - \lambda)s)y\|}{2} - 1 \\ &\leq \frac{\lambda\|x + ty\| + (1 - \lambda)\|x + sy\| + \lambda\|x - ty\| + (1 - \lambda)\|x - sy\|}{2} - 1 \\ &= \lambda f_{x,y}(t) + (1 - \lambda) f_{x,y}(s). \end{aligned}$$

Therefore $f_{x,y}$ is a convex function, for every choice of x and y.

Now for arbitrary $\varepsilon > 0$, there exist x, y with $\|x\| = \|y\| = 1$ such that

$$\begin{aligned}\rho_X(\lambda t+(1-\lambda)s)-\varepsilon &\le f_{x,y}(\lambda t+(1-\lambda)s)\\ &\le \lambda f_{x,y}(t)+(1-\lambda)f_{x,y}(s)\\ &\le \lambda\rho_X(t)+(1-\lambda)\rho_X(s).\end{aligned}$$

Therefore ρ_X is a convex function since ε is arbitrary. The continuity follows from lemma 1.9. □

Definition 2.4. A normed space X is said to be *uniformly smooth* whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = 1$ and $\|y\| \le \delta$, then

$$\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|.$$

As the modulus of convexity characterizes the uniformly convex spaces, the modulus of smoothness can be used to characterize the uniformly smooth spaces. This is the manner of the following theorem.

Theorem 2.5. *A normed space X is uniformly smooth if and only if*

$$\lim_{t\to 0^+}\frac{\rho_X(t)}{t} = 0.$$

Proof. If X uniformly smooth and $\varepsilon > 0$ then, there exists $\delta > 0$ such that

$$\frac{\|x+y\|+\|x-y\|}{2} - 1 < \frac{\varepsilon}{2}\|y\|,$$

for every $x, y \in X$ such that $\|x\| = 1$, $\|y\| = \delta$. This implies $\rho_X(t) < \frac{\varepsilon}{2}t$ for every $t < \delta$.

Conversely, let $\varepsilon > 0$, and suppose there exists $\delta > 0$ such that $\rho_X(t) < \frac{1}{2}\varepsilon t$, for every $t < \delta$. Let $\|x\| = 1$, $\|y\| = \delta$. Then with $t = \|y\|$ we have $\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|$ and the space is uniformly smooth. □

Proposition 2.6. *Every uniformly smooth normed space X is smooth.*

Proof. Suppose that X is not smooth, then there exist x_0 in X and x_1^*, x_2^* in X^* such that $x_1^* \ne x_2^*$, $\|x_1^*\| = \|x_2^*\| = 1$ and $\langle x_0, x_1^*\rangle = \|x_0\| = \langle x_0, x_2^*\rangle$. Evidently we can assume $\|x_0\| = 1$. Let y_0 in X be such that $\|y_0\| = 1$ and $\langle y_0, x_1^* - x_2^*\rangle > 0$. For every $t > 0$ we have

$$\begin{aligned}0 &< t\langle y_0, x_1^* - x_2^*\rangle\\ &= t(\langle y_0, x_1^*\rangle - \langle y_0, x_2^*\rangle)\\ &= \frac{\langle x_0 + ty_0, x_1^*\rangle + \langle x_0 - ty_0, x_2^*\rangle}{2} - 1\\ &\le \frac{\|x_0 + ty_0\| + \|x_0 - ty_0\|}{2} - 1,\end{aligned}$$

therefore, $0 < \langle y_0, x_1^* - x_2^*\rangle \le \frac{\rho_X(t)}{t}$ for any $t > 0$ and then X is not uniformly smooth. □

2.3 Duality Between Spaces

As usual in mathematics, the study of one concept is in close relation with another which reflects its characteristics. This kind of "duality" is present in our case. Now we state one of the fundamental links between the *Lindenstrauss duality formulas.*

Proposition 2.7. *Let X be a Banach space. For every $\tau > 0$, x in X, $\|x\| = 1$ and x^* in X^* with $\|x^*\| = 1$ we have*

$$\rho_{X^*}(\tau) = \sup\Big\{\frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) : 0 < \varepsilon \leq 2\Big\},$$

$$\rho_X(\tau) = \sup\Big\{\frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 < \varepsilon \leq 2\Big\}.$$

Proof. Let $\tau > 0$, $0 < \varepsilon \leq 2$, x, y in X with $\|x\| = \|y\| = 1$. By Hahn-Banach Theorem there exist x_0^*, y_0^* in X^* with $\|x_0^*\| = \|y_0^*\| = 1$ such that

$$\langle x + y, x_0^* \rangle = \|x + y\| \quad \text{and} \quad \langle x - y, y_0^* \rangle = \|x - y\|.$$

Then

$$\begin{aligned} \|x + y\| + \tau\|x - y\| - 2 &= \langle x + y, x_0^* \rangle + \tau\langle x - y, y_0^* \rangle - 2 \\ &= \langle x, x_0^* + \tau y_0^* \rangle + \langle y, x_0^* - \tau y_0^* \rangle - 2 \\ &\leq \|x_0^* + \tau y_0^*\| + \|x_0^* - \tau y_0^*\| - 2 \\ &\leq \sup\{\|x^* + \tau y^*\| + \|x^* - \tau y^*\| - 2 : \\ &\quad \|x^*\| = \|y^*\| = 1\} = 2\rho_{X^*}(\tau). \end{aligned}$$

If $\varepsilon \leq \|x - y\|$ then, from the last inequality,

$$\frac{\tau\varepsilon}{2} - \rho_{X^*}(\tau) \leq 1 - \frac{\|x + y\|}{2}.$$

Therefore $\frac{\tau\varepsilon}{2} - \rho_{X^*}(\tau) \leq \delta_X(\varepsilon)$ and since $0 < \varepsilon \leq 2$ is arbitrary, we get

$$\sup\Big\{\frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) : 0 < \varepsilon \leq 2\Big\} \leq \rho_{X^*}(\tau).$$

Now let x^*, y^* in X^* with $\|x^*\| = \|y^*\| = 1$ and let $\delta > 0$. For a given $\tau > 0$ there exist x_0, y_0 in X with $x_0 \neq y_0$ and $\|x_0\| = \|y_0\| = 1$ such that

$$\|x^* + \tau y^*\| \leq \langle x_0, x^* + \tau y^* \rangle + \delta \quad \text{and} \quad \|x^* - \tau y^*\| \leq \langle y_0, x^* - \tau y^* \rangle + \delta$$

(from the definition of $||.||$ in X^*). From these inequalities, we obtain

$$\begin{aligned}
&\|x^* + \tau y^*\| + \|x^* - \tau y^*\| - 2 \\
&\leq \langle x_0, x^* + \tau y^*\rangle + \langle y_0, x^* - \tau y^*\rangle - 2 + 2\delta \\
&= \langle x_0 + y_0, x^*\rangle + \tau\langle x_0 - y_0, y^*\rangle - 2 + 2\delta \\
&\leq \|x_0 + y_0\| - 2 + \tau|\langle x_0 - y_0, y^*\rangle| + 2\delta.
\end{aligned}$$

Hence, if we define $\varepsilon_0 := |\langle x_0 - y_0, y^*\rangle|$, then $0 < \varepsilon_0 \leq \|x_0 - y_0\| \leq 2$ and

$$\begin{aligned}
\frac{\|x^* + \tau y^*\| + \|x^* - \tau y^*\|}{2} - 1 &\leq \frac{\tau\varepsilon_0}{2} + \delta - \delta_X(\varepsilon_0) \\
&\leq \delta + \sup\left\{\frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) : 0 < \varepsilon \leq 2\right\}.
\end{aligned}$$

As $\delta > 0$ is arbitrary, we have

$$\rho_{X^*}(\tau) \leq \sup\left\{\frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) : 0 < \varepsilon \leq 2\right\},$$

and the first equation is proved.

In order to prove the second equation, let $\tau > 0$ and let x^*, y^* in X^* with $\|x^*\| = \|y^*\| = 1$. For any $\eta > 0$, from the definition of $||.||$ in X^* there exist x_0, y_0 in X with $\|x_0\| = \|y_0\| = 1$ such that

$$\|x^* + y^*\| - \eta \leq \langle x_0, x^* + y^*\rangle, \quad \|x^* - y^*\| - \eta \leq \langle y_0, x^* - y^*\rangle.$$

Then

$$\begin{aligned}
&\|x^* + y^*\| + \tau\|x^* - y^*\| - 2 \\
&\leq \langle x_0, x^* + y^*\rangle + \tau\langle y_0, x^* - y^*\rangle - 2 + \eta(1 + \tau) \\
&= \langle x_0 + \tau y_0, x^*\rangle + \langle x_0 - \tau y_0, y^*\rangle - 2 + \eta(1 + \tau).
\end{aligned}$$

Since in a Banach space, $\|x\| = \sup\{|\langle x, x^*\rangle| : \|x^*\| = 1\}$ we have

$$\begin{aligned}
&\|x^* + y^*\| + \tau\|x^* - y^*\| - 2 \\
&\leq \|x_0 + \tau y_0\| + \|x_0 - \tau y_0\| - 2 + \eta(1 + \tau) \\
&\leq \sup\{\|x + \tau y\| + \|x - \tau y\| - 2 : \|x\| = \|y\| = 1\} \\
&+\eta(1 + \tau) \\
&= 2\rho_X(\tau) + \eta(1 + \tau).
\end{aligned}$$

If $0 < \varepsilon \leq \|x^* - y^*\| \leq 2$ we have

$$\frac{\tau\varepsilon}{2} - \rho_X(\tau) - \eta(1 + \tau) \leq 1 - \left\|\frac{x^* + y^*}{2}\right\|$$

which implies that

$$\frac{\tau\varepsilon}{2} - \rho_X(\tau) - \eta(1+\tau) \le \delta_{X^*}(\varepsilon).$$

Now, since η is arbitrary we conclude that

$$\frac{\tau\varepsilon}{2} - \rho_X(\tau) \le \delta_{X^*}(\varepsilon)$$

for every ε in $(0,2]$ and therefore

$$\sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 < \varepsilon \le 2\right\} \le \rho_X(\tau).$$

Now let x, y be in X with $\|x\| = \|y\| = 1$ and let $\tau > 0$. By Hahn-Banach Theorem there exist x_0^*, y_0^* in X^* with $\|x_0^*\| = \|y_0^*\| = 1$ and such that

$$\langle x + \tau y, x_0^*\rangle = \|x + \tau y\|, \quad \langle x - \tau y, y_0^*\rangle = \|x - \tau y\|.$$

Then,

$$\begin{aligned} \|x + \tau y\| + \|x - \tau y\| - 2 &= \langle x + \tau y, x_0^*\rangle + \langle x - \tau y, y_0^*\rangle - 2 \\ &= \langle x, x_0^* + y_0^*\rangle + \tau\langle y, x_0^* - y_0^*\rangle - 2 \\ &\le \|x_0^* + y_0^*\| + \tau|\langle y, x_0^* - y_0^*\rangle| - 2. \end{aligned}$$

Hence, if we define $\varepsilon_0 = |\langle y, x_0^* - y_0^*\rangle|$, then $0 < \varepsilon_0 \le \|x_0 - y_0\| \le 2$ and

$$\begin{aligned} \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 &\le \frac{\|x_0^* + y_0^*\| + \tau|\langle y, x_0^* - y_0^*\rangle|}{2} - 1 \\ &= \frac{\tau\varepsilon_0}{2} - \left(1 - \frac{\|x_0^* + y_0^*\|}{2}\right) \\ &\le \frac{\tau\varepsilon_0}{2} - \delta_{X^*}(\varepsilon_0) \\ &\le \sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 < \varepsilon \le 2\right\}. \end{aligned}$$

Therefore

$$\rho_X(\tau) \le \sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 < \varepsilon \le 2\right\}$$

and the second equality holds. □

From the second formula of Proposition 2.7 we get the following corollaries.

Corollary 2.8 *For every Banach space X, the function $\frac{\rho_X(t)}{t}$ is non-decreasing and $\rho_X(t) \le t$.*

Corollary 2.9 *For every Banach space X and for every Hilbert space H we have*

$$\rho_H(\tau) = \sup\left\{\frac{\tau\varepsilon}{2} - 1 + \sqrt{1 - \frac{\varepsilon^2}{4}} : 0 < \varepsilon \leq 2\right\} = \sqrt{1+\tau^2} - 1 \leq \rho_X(\tau).$$

The following result gives the duality between uniformly convex and uniformly smooth spaces.

Theorem 2.10. *Let X be a Banach space.*
(a) X is uniformly smooth if and only if X^* is uniformly convex.
(b) X is uniformly convex if and only if X^* is uniformly smooth.

Proof. $(a) \rightarrow$. If X^* is not uniformly convex, there exists ε_0 in $(0, 2]$ with $\delta_{X^*}(\varepsilon_0) = 0$, and by the second formula in Proposition 2.7, we obtain for every $\tau > 0$,

$$0 < \frac{\varepsilon_0}{2} \leq \frac{\rho_X(\tau)}{\tau}$$

which means that X is not uniformly smooth.
$(a) \leftarrow$. Assume that X is not uniformly smooth. Then

$$\lim_{t\to 0^+} \frac{\rho_X(t)}{t} \neq 0,$$

that means, there exists $\varepsilon > 0$ such that for every $\delta > 0$ we can find t_δ with $0 < t_\delta < \delta$ and $t_\delta\varepsilon \leq \rho_X(t_\delta)$. Then there exists a sequence $(\tau_n)_n$ such that $0 < \tau_n < 1$, $\tau_n \to 0$ and $\rho_X(\tau_n) > \frac{\varepsilon}{2}\tau_n$. By the second formula in Proposition 2.7, for every n there exists ε_n in $(0, 2]$ such that

$$\frac{\varepsilon}{2}\tau_n \leq \frac{\tau_n\varepsilon_n}{2} - \delta_{X^*}(\varepsilon_n)$$

which implies

$$0 < \delta_{X^*}(\varepsilon_n) \leq \frac{\tau_n}{2}(\varepsilon_n - \varepsilon),$$

in particular $\varepsilon < \varepsilon_n$ and $\delta_{X^*}(\varepsilon_n) \to 0$. Given the fact that δ_{X^*} is a non-decreasing function we have $\delta_{X^*}(\varepsilon) \leq \delta_{X^*}(\varepsilon_n) \to 0$. Therefore X^* is not uniformly convex. Note that, interchanging the roles of X and X^* in this proof, and using the first formula in Proposition 2.7, we get the proof of part (b). □

The last part of the proof of Theorem 2.10 proves the following result.

Corollary 2.11 *Every uniformly smooth space is reflexive.*

SUMMARY

For the ease of reference, we summarize the key results obtained in this chapter. Here ρ_X denotes the modulus of smoothness of a Banach space X.

S1

(a) *Every uniformly smooth space is smooth.*
(b) *Every uniformly smooth space is reflexive.*
(c) X *is uniformly smooth if and only if* X^* *is uniformly convex.*

S2

(a) X *is uniformly smooth if and only if* $\lim_{t\to 0^+} \frac{\rho_X(t)}{t} = 0$.
(b) $\rho_X : [0.\infty) \to [0,\infty)$ *is a convex and continuous function.*
(c) $\frac{\rho_X(t)}{t}$ *is a nondecreasing function on* $[0,\infty)$.
(d) $\rho_X(t) \le t$ *for all* $t \ge 0$.

EXERCISES 2.1

1. Prove Corollaries 2.8 and 2.9.
2. Verify that L_p (or l_p) spaces, $1 < p < \infty$, are uniformly smooth (and are therefore smooth).
3. Prove Corollary 2.11.

   (Hint: A Banach space X is reflexive whenever the dual space X^* is).
4. Establish the following inequalities. In L_p (or l_p) spaces, $1 < p < \infty$,

$$\rho_{L_p}(\tau) = \begin{cases} (1+\tau^p)^{\frac{1}{p}} - 1 < \frac{1}{p}\tau^p, & 1 < p < 2, \\ \frac{p-1}{2}\tau^2 + o(\tau^2) < \frac{p-1}{2}\tau^2, & p \ge 2. \end{cases}$$

   (Hint: See Lindenstrauss and Tzafriri [312]).

2.4 Historical Remarks

Results in this chapter can be found in Beauzamy [26] or Diestel [206]. The Lindenstrauss duality formulas are proved in Lindenstrauss and Tzafriri [312].

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