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Japanese Mathematics in the Edo Period (1600-1868)

A study of the works of Seki Takakazu (?-1708) and Takebe Katahiro (1664-1739)

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CHAPTER 2

The *Jugairoku* (1639) by Imamura Tomoaki

Though the *Jinkōki* played an essential role in the rapid spread of arithmetic knowledge during the first half of the 17th century, it is far from reflecting all the richness of the mathematical activity of this period. The considerable gap which separates it from Seki's works is partly filled in by the *Jugairoku*.

2.1. Presentation of the Author and of the Textbook

The only information about the author available to us comes from the preface which is translated below. Born in a village in Kawachi province, close to Nara, Imamura studied mathematics with Mōri Shigeyoshi in Kyoto. Knowing that Mōri was the master of both Yoshida and Imamura, one might be tempted to see in him the founder of the Japanese mathematical tradition. But such a vision would certainly be deceptive, insofar as these two pupils with very different background each in their own way surpassed their master, and as neither one of them credited him with extensive knowledge. It is more appropriate to keep in mind the particular place which united these three mathematicians, Kyoto, which at that time was at the crossroads of commerce and cultural activities as well as the main hub for the publication and trade of books.

Imamura's main period of activity is situated in the years 1639–1660.[1] It is not known to what milieu he belonged or whether he had any other activity besides teaching mathematics (which seems probable). The *Jugairoku* was printed with movable type in one hundred copies, of which only one has survived.[2]

It is a work which in numerous points contrasts with the *Jinkōki*. Little known to the general reader, it circulated only in the inner circle of Imamura's disciples.[3] It was published in Chinese, the learned language of this era, and was composed solely of rules which were called *shiki*, formulated in full generality without reference

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to specific numerical examples. A large space is devoted to questions of metrical geometry.

The author seems to have been aware of the fact that, as such, his book was not accessible to everyone. The following year he published the *Inki Sanka* (Mathematical Poems for Multiplication and Division) which looks like the antithesis of the *Jugairoku*. The same rules are this time versified and accompanied by numerical examples. In 1662, one of his disciples took the initiative for printing the *Jugairoku Kanashō* (The *Jugairoku* in Kana) which reproduced the original text of the *Jugairoku* with commentaries in the vernacular language.[4] It was only after its appearance that the *Jugairoku* became a book well-known to mathematicians.

2.2. Preface of the *Jugairoku*

> "Having decided from my early youth to devote myself to the art of calculation (*sanjutsu*), I consulted various books and applied myself to it to the best of my ability without, however, succeeding in solving all the problems. One day I heard it said of Mōri Shigeyoshi of Kyoto that he was a discerning lover of this art. I went to see him and told him about my difficulties.
>
> Shigeyoshi then said: 'When one wants to apply oneself to the art of calculation, one must write down the operand (*jitsu*) and the operator (*hō*), build up the procedures of multiplication and division, or the procedures of increase and reduction (*zōgenjutsu*) or the procedure of the side-and-diagonal (*hōgenjutsu*). By modifying [the inscribed numbers], one obtains the result (*shō*). What one designates by side–diagonal is nothing else but the base-perpendicular-hypotenuse (*kōkogen*). This is the compass (*ki*)[5].'"

Let us pause for a moment at this paragraph, which introduces some of the key notions of the Japanese calculus. The *jitsu*, the *hō* and the *shō*, translated here as operand, operator and result, are generic terms by means of which, since the *Nine Chapters*, the algorithms of division and, by extension, of the extraction of roots, are described and carried out. They refer to positions on the calculation table or on the division, the *jitsu* is in the place where the dividend is put initially and where the abacus and indicate at the same time the numbers which are placed there. For a successive remainders of the division are then placed. The *hō* is the position that receives the divisor, which remains fixed through the whole procedure. Finally, the *shō* is the place where the digits of the quotient are noted down progressively. In the *Jugairoku*, this vocabulary is used for all operations, including the multiplication.

In the paragraph quoted, Imamura emphasizes the importance of three types of procedures. While the procedures of multiplication and division are explained in the book, it is not known what he meant by the procedures of increase and reduction. Finally, the "side-and-diagonal" refers to the calculation by which one obtains the diagonal of a square, knowing its side. It is obviously by Pythagoras's theorem that this result is acquired. The general calculation in the case of a right triangle is named *kōkogen*[6], which decomposes into *kō*, the short side of the triangle, *ko*, the long side, and *gen*, the hypotenuse. For convenience, we translate the first two terms by base and perpendicular, but it is a good idea to keep in mind that the corresponding Japanese words do not make any reference to the triangle's orientation.

We return to the continuation of the preface:

> "Respecting these words [of Shigeyoshi], I set out to produce numbers and procedures. But though I could in this way treat the square, it did not go the same way with the circle. And yet, were lengths not already measured at the beginning of the world? This being the case, was it not indispensable to clarify this procedure from the beginning of the studies? And thus I lamented in my heart of hearts, and months and years passed by. I asked a[nother] master who answered me: 'As to the origin of mathematics, it begins at the moment where Fu Xi draws the Eight Trigrams (*hakke*), where Huang Di fixes the three types of notation of numbers (*sansū*) and establishes the ten stages (*jittō*). It is this way that Li Shou carried out the *Nine Chapters*.[7] That is the reason why, whether regarding signs of divination or the *Nine Chapters*, the origin of ten million things is One. But alas, how to know this unique origin?' These words convinced me and I pushed [my] research further in spite of my mediocre abilities. To the procedures related to the chords of the circle I gave the names of diameter – sagitta* – chord and arc-sagitta-chord. This is the set square (*ku*).
>
> If there were no compass and no set square, what would we use as a model? This is why the compass and the set square are at the origin of mathematics and also at the heart of phenomena and things.
>
> There are quite a few books like the *Keimō* by Jonan, but this is an overloaded work, with careless mistakes and difficult to understand. That is the reason why I have, despite my mediocrity, worked out the nine rules [relating to the] round and rectangular [forms], planar and solid, which constitute the essence of mathematics, and I have made of it a work to which I have given the name of *Jugairoku* (Register of Jugai).
>
> On reflection, it is divination and land-measuring which are described in the steps of Jugai. Now, with this book in one's hand, one will be able to determine lengths, to know the height without climbing and the distance without moving. How can these not be prescient words? I therefore wish to give them a long life and to pass them on to my disciples. The mockery of the people will not touch me. It will even make my happiness[8]."

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* The word "sagitta," little used in English, means "arrow" (which is the word used in both the Japanese and French texts – *shi* and *flèche*, respectively) and refers to the distance from the midpoint of a chord of a circle to the nearest point on the circle. The reason for the name is obvious from the picture – the arc, chord and sagitta of a circle segment resemble a bow, bowstring and arrow. [Translator's note.]

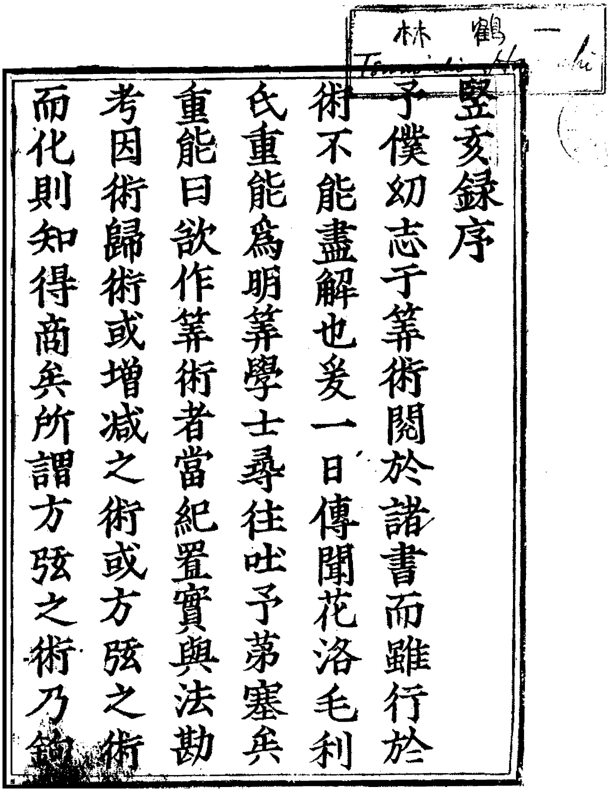

竪亥録序
予僕幼志于筭術閲於諸書而雖行於
術不能盡解也爰一日傳聞花洛毛利
氏重能爲明筭學士尋往吐予茅塞矣
重能曰欲作筭術者當紀置實與法勘
考因術歸術或増減之術或方弦之術
而化則知得商矣所謂方弦之術乃鉤

Figure 2.1: Imamura Tomoaki's *Jugairoku*, 1639, Tōhoku University, Hayashi Collection. Preface by the author himself.

In spite of several obscure points, the foreword of the *Jugairoku* remains valuable for the light it sheds on the preoccupations of the author. It is also revealing of his attempt to revive the Chinese mathematical tradition.

The art of calculation which captured Imamura's attention was predominantly geometric. The title *Jugairoku* signifies Register of Jugai (Shuhai in Chinese), Jugai being a character in the Chinese classic *Shanhaijing* (Classic of Mountains and Rivers) who finds himself entrusted with the mission of measuring the empire from one end to the other.[9] Indeed, more than half of Imamura's work is dedicated to the measurement of distances, areas and volumes. This interest appears also in his references to the compass and the set square as the models and origin of mathematics.

The reference to these two symbolic instruments (the set square was the insignia of Fu Xi, the first Sovereign of China, traditionally presented as the founder of mathematics, and the compass that of his wife) could have been inspired by a Chinese work.[10] Whatever the origin of this reference, it is important to note that Imamura on this occasion gives them a precise mathematical content: on the one hand, the Pythagorean relation linking the three sides of a right triangle, and on the other, the relation linking the three fundamental magnitudes of a circle segment, the chord, the sagitta and the arc. Both terms were to be adopted by his disciples.[11]

In the preceding chapter, we saw that Yoshida drew a part of his problems from the *Suanfa tongzong*. Imamura for his part cites a different title, the *Keimō* (Introduction) by Jonan, which he criticizes as being overloaded and not very clear. It has not been possible to identify any work or any author of this name. If we admit the possibility that an error may have occurred, *Keimō* suggests the *Suanxue qimeng* (1299) (Introduction to Mathematical Science) (*Sangaku Keimō in Japanese*), a work of the Yuan period which greatly inspired the Japanese mathematicians of the second half of the 17th century. On the other hand, the name of Jonan is closer to Cheng Dawei's literary name, Joshi, the first character being identical. Furthermore, if the word "overloaded" applies well to Cheng Dawei's book, whose circulation at the time is attested, the same cannot be said of the former, whose conciseness and richness had strongly impressed the mathematicians immediately after Imamura. Thus the balance inclines slightly in favor of the latter, and in the following we will take the *Suanfa tongzong* to have been the object of this remark. We thus perceive that, ten years after the publication of the *Jinkōki*, the Chinese text no longer benefited from the same prestige and that there were mathematicians who deliberately considered moving away from the model it represented.

2.3. Analysis of the Textbook

Though historians unanimously acknowledge the originality of Imamura's work and its influence on his successors, there have been very few studies devoted to him. That he accomplished an essential work in shaping and organizing mathematical knowledge is, however, an incontestable fact. We will recall here its most outstanding features.

The content of the *Jugairoku* is presented in the form of "rules" of calculation (*shiki*),[12] classified in nine categories:

1) rules relating to numbers
2) fixed rules
3) rules relating to procedures
4) rules for opening the square
5) rules for opening the cube
6) rules for the plane square
7) rules for the plane circle
8) rules for the solid square
9) rules for the solid circle

Postponing for the moment the explanation of this terminology, we would like to examine more closely the rules relating to the extraction of roots, which we consider to be representative of Imamura's arithmetic, and then those that concern geometrical forms.

2.3.1. The Extraction of Square and Cube Roots in the *Jugairoku*

For the operations of extracting square or cube roots, the Japanese mathematicians, who had adopted the Chinese terminology, appealed to the notion of "opening" (or "development") of squares or cubes. In our translations we will normally keep this terminology, which not only calls to mind the original geometric character,[13] but also emphasizes the fact that the extraction of roots is perceived as an operation applied to a number identified with a square, a cube or a product. This is an important point to which we will come back.

One can find two reasons for the fact that these operations are explained separately. The first is that they are considered as difficult and hardly known to the mathematicians of this time. The second is the essential role of these operations in the geometric rules expounded in the four last chapters. Indeed, they intervene in a crucial manner in the calculation of hypotenuses or the calculation of the dimensions of figures knowing their area or their volume. It is also significant that these operations are defined by Imamura in a geometrical framework:

Before quoting them, let us recall the principal units in use, which are:

- the *sun*, or inch, equivalent approximately to 3 cm,
- the *shaku*, equal to 10 *sun*, and
- the *bu*, a unit of area corresponding to one *sun*^{2}.

Imamura considers three types of opening of the square.[14]

1) The opening of the square (*kaihei*), defined by:

   > "The rule by which, having made of the area [literally, of the *bu*] a square, one knows the number of *sun* of the four equal sides."

   The rule defined is the one by which the number x such that $x^2 = a$ can be determined.

2) The proportional opening of the square (*sōō kaihei*), defined by:

   > "The rule by which, having made from the area a [rectangle] whose proportions (*sōō*) are, for example, as 8 *shaku* long and 2 *sun* wide, one knows the number of *shaku* of the length and of the width."

   The rule defined is the one that permits to calculate x and y such that $xy = a$ and $x/y = b/c$ (the example given is 80/2).

3) The opening of the square with excess length[15] (*taijū kaihei*), defined by:

   > "The rule by which, having made from the area [a rectangle] whose length surpasses its width by, for example, 1 *shaku* and 5 *sun*, one knows the number of *shaku* of the length and the width."[16]

   The rule defined is the one which permits to calculate x and y such that $xy = a$ and $x - y = m$ (the example considered is $m = 15$).

Both as regards the classification just expounded and the actual presentation of the procedures, a significant rationalization has taken place in comparison with the *Suanfa tongzong* of Cheng Dawei. The numerous examples considered by the latter (who himself drew his inspiration from older authors[17]) have been sorted into three categories of extraction, and a precise terminology has been assigned to them. One can also note that, in contrast to Yoshida and Cheng Dawei, the rules are formulated here without any geometric underpinning.

Inasmuch as Imamura's procedures reveal a certain ease in the domain of arithmetic, we reproduce here an example which will will serve as a point of comparison when we study the new procedures that appeared in the second half of the century.

The problem of extraction that we choose here is that of finding the square root of 152.2756 *bu*.[18] The description of this operation makes use of the terminology already mentioned. The numbers are called by the names of the positions where they are operated on in the chosen instrument of calculation, here the abacus.

In the case of the square root, the four positions destined to play a role are the *jitsu* (operand, dividend), the *hō* (operator, divisor), the *shō* (quotient), and the *gū* (corner).

To facilitate comprehension, we accompany the text with a table where the numbers calculated in the course of the procedure are entered in modern notation:

> "Take for example 152 *bu* 2756 [19] as dividend.
> 1 [times] 1, one subtracts 100 *bu*;
> 1 *shaku* as the quotient. There remain 52 *bu* 2756;
> one doubles 1 *shaku*, one obtains 2 *shaku*, which gives the divisor:
> one carries out the division up to the first digit (*hitoketa ni kijo su*)
> 2 *sun* as the quotient.
> Then in the corner, 2 [times] 2, one subtracts 4 *bu*;
> there remain 8 *bu* 2756;
> one doubles 1 *shaku* 2 *sun*, one obtains 2 *shaku* 4 *sun*, which gives the divisor;
> one carries out the division to the first digit;
> 3 *bu* as the quotient.
> Then in the corner, 3 [times] 3, one subtracts 9 *ri*.
> There remains 0.9856;
> one doubles 1 *shaku* 2 *sun* 3 *bu* and obtains 2 *shaku* 4 *sun* 6 *bu*, which gives the divisor;
> one carries out the division to the first digit;
> 4 *ri* as the quotient.
> Then in the corner, 4 [times] 4, one subtracts 1 *go* 6 *kei*;
> Giving a quotient of 1 *shaku* 2 *sun* 3 *bu* 4 *ri*, which is the value in *sun* of the side."[20]

The extraction of the root is thus carried out by iterating the following two operations:

- the number designated as dividend is divided by the number called divisor and the first digit of the quotient is entered in the position "quotient."
- the remainder of this division is substituted for the dividend and the square of the quotient is subtracted from it.[21]

Table 2.1: Extraction of the square root in the *Jugairoku*.

	dividend	**divisor**	**corner**	**quotient**
N	152.2756			
				$a = 10$
$M = N - a^2$	52.2756			
		$2a = 20$		
$N' = M - 2aa'$	12.2756			$a' = 2$
			$a'^2 = 4$	
$M' = N' - a'^2$	8.2756			
		$2(a + a') = 24$		
$N'' = M' - 2(a + a')a''$	1.0756			$a'' = 0.3$
			$a''^2 = 0.009$	
$M'' = N'' - a''^2$	0.9856			
		$2(a + a' + a'') = 24.6$		
$N''' = M'' - 2(a + a' + a'')a'''$	0.0016		$a'''^2 = 0.0016$	
$M''' = N''' - a'''^2$	0			

For the case of the opening of the square with "excess length," the calculations carried out are the following (preserving the preceding notations and denoting by m the excess length):

$$\begin{aligned} M &= N - a^2 - am \\ N' &= M - (2a + m)\, a' \\ M' &= N' - a'^2 \\ N'' &= M' - \left[2(a + a') + m\right] a'' \\ M'' &= N'' - a''^2 \qquad \text{etc.} \end{aligned}$$

In the same manner as for the simple openings, the successive quotients entered a, a', etc. are obtained by interrupting the divisions after obtaining the first digit.

2.3.2. The Study of Areas and Volumes in the *Jugairoku*

One of the features which distinguishes the *Jugairoku* from the other textbooks produced at this time is the space that it devotes to geometric calculations and the manner in which these questions are approached.

A first distinction is established between plane figures (*hei*) and solid figures (*choku*) and then, within these categories, between curved and square figures.

In doing so, the *Jugairoku* breaks with a tradition going back to the *Nine Chapters*, still very much alive in the *Suanfa tongzong*, which consists of classifying problems according to plane and solid figures in different chapters according as the goal is to measure the surfaces of fields (chapter *fangtian*), to determine the length of a rectangular field of known area (chapter *shaoguang*), to evaluate volumes to be excavated or built (dykes, canals, ramparts) (chapter *shanggong*), or to evaluate distances using Pythagoras's theorem (chapter *gougu*).

In Imamura's text, references to the contexts of use are eliminated. The only object of discussion is the geometric figure, considered here as an abstract form without materiality.

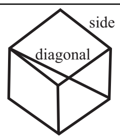
To each plane or solid form are associated a name and a visual representation which highlight their geometrical characteristics. A caption accompanies these representations, defining the names of the segments destined to play a role in the subsequent calculations.

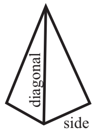
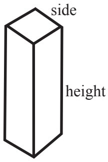
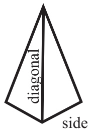
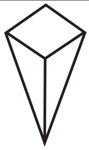
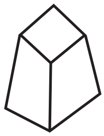
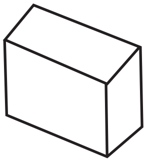
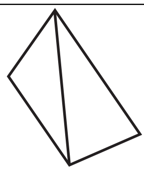
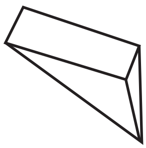
For each of the plane or solid figures, Imamura's rules indicate the method of calculation of the area or volume starting from the figure's dimensions or, inversely, the method of calculation of one particular dimension starting from the area (or volume) and the other dimensions. The author of the *Jugairoku* thus gathers in one chapter all rules relating to the calculation of lengths, areas and volumes. An interesting feature of these rules is that they are formulated in full generality, without reference to numerical examples.

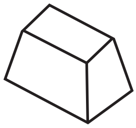
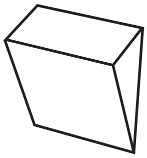
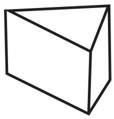
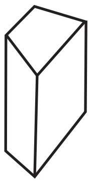
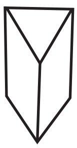
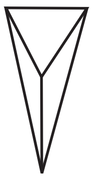
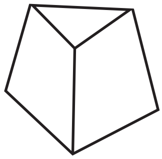
We observe that Imamura, as well as introducing a different classification principle, follows in the exposition of each chapter the order of the stages of geometric reasoning that had permitted finding each of the rules, particularly those giving areas and volumes. An example will enable us to illustrate this remark.

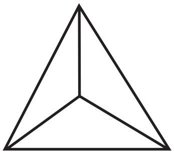
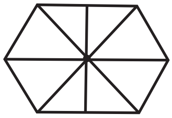
2.3.3. Examples of Square and Solid Figures

The majority of figures studied in the eighth chapter of the *Jugairoku* are classical in the sense that they had been treated regularly in Chinese textbooks ever since the *Nine Chapters*, the *Suanfa tongzong* not excluded. The figures worked out most carefully are those which we designate as pyramids with square or rectangular base, truncated pyramids, prisms with triangular or trapezoidal base, etc. Certain forms appeared for the first time in the *Jugairoku*, such as the *sobagata* (regular tetrahedron) and the *kirikogata* (cubic octahedron) obtained by splitting up a classic figure. Below we give the list of figures singled out by Imamura with the original names and their translation[22] in the first column, the representation given by Imamura in the second, and finally the present-day name or, if this is lacking, an explanation in modern language of the figure studied in the third.

	Original name	**Picture**	**Explanation**
1	cube *(rippō)*	 	cube

	Original name	Picture	Explanation
2	side-diagonal *(hōsha)*		pyramid inscribed in a cube, having a face of the cube as base and one of the opposite vertices as apex
3	side-height *(hōju)*		parallelepiped with two square faces
4	base-double perpendicular-diagonal *kōsōkosha)*		pyramid inscribed in the "side-height" having a square face as base and a vertex of the opposite face as apex
5	pointed square *(hōsui)*		pyramid with square base
6	platform square *(hōdai)*		truncated pyramid with square base
7	thickness-depth-height *(kōfukuju)*		parallelepiped
8	base-perpendicular-fold-diagonal *(kōko kyokusha)*		pyramid inscribed in the "thickness-depth-height" with one of its faces as its base and a vertex of the opposite face as its apex
9	pointed thickness-depth *(kōfukusui)*		pyramid with rectangular base

	Original name	**Picture**	**Explanation**
10	platform thickness-depth *(kōfukudai)*		truncated pyramid with rectangular base
11	wedge shape *(kusabigata)*		wedge
12	mountain shape height *(sangyōju)*		prism with triangular base
13	narrow edge-height *(henkyōju)*		prism with trapezoidal base
14	three sides-height *(sanhōju)*		prism whose base is an equilateral triangle
15	pointed three sides *(sanhōsui)*		pyramid whose base is an equilateral triangle
16	platform three sides *(sanhōdai)*		truncated pyramid whose base is an equilateral triangle

	Original name	Picture	Explanation
17	*(sobagata)*		regular tetrahedron
18	*(kirikogata)*		cubic octahedron

Some of the names of the figures are of Chinese origin and figure in the *Suanfa tongzong*. (This is the case for 1, 5 and 6.) The intermediate figures corresponding to special cases of figures considered later in the table (e.g. 2 and 8) are designated by juxtaposition of the names of the segments which compose them. Finally, we observe that the representations of the figures do not respect the laws of perspective and do not always clearly express the expected properties. This holds particularly for the last figure.

This chapter of the *Jugairoku* reveals the systematic character of the method employed by Imamura for constructing his rules. It is neither formulated nor visualized explicitly, but the order followed, the introduction of intermediate figures and the presence of references to earlier rules suggest that his study was based on the cutting up of figures on the one hand and on the use of proportionality on the other.

For example, after studying the cube, Imamura considers the "side–diagonal," the pyramid having as its base one of the cube's faces and as its vertex one of the vertices of the opposite face. Its volume can easily be determined by cutting the cube into three similar pyramids having the same vertex (Figure 2.3). From this decomposition, the coefficient 3, called "the operator of the side-diagonal," can be determined easily.[23]

The three pyramids in this case have the vertex A and the bases $EFGH$, $BCGF$ and $CDHG$, respectively.

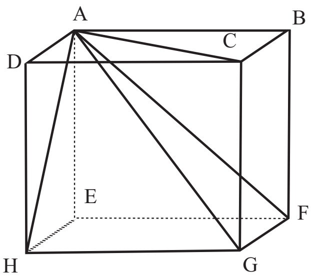


Figure 2.3

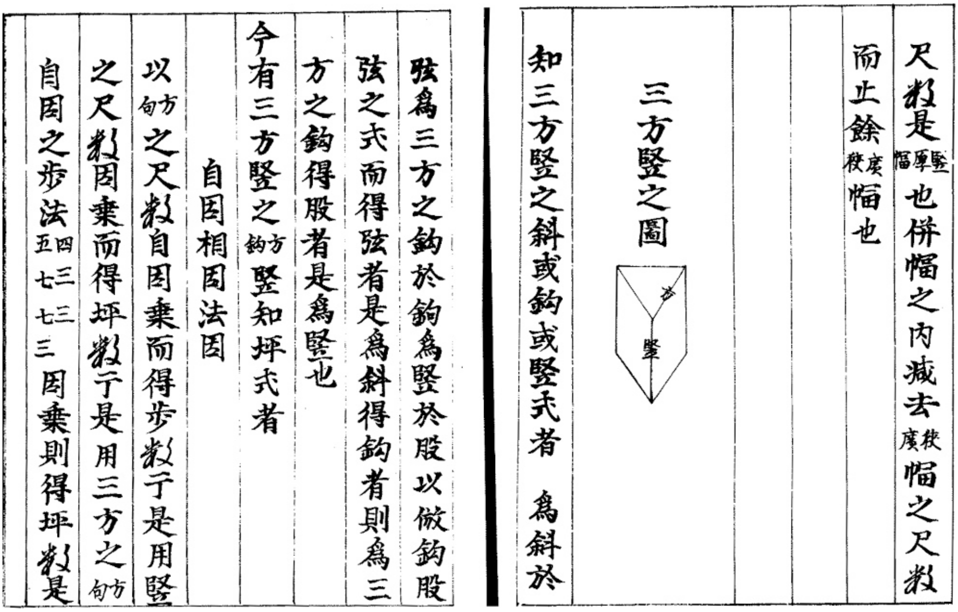
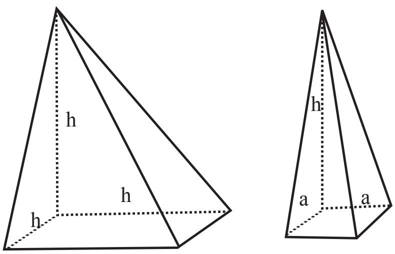

尺数是竪厚幅也併幅之内減去狭廣幅之尺数
而止餘廣狭幅也
三方竪之圖
知三方竪之斜或鈎或竪式者　爲斜於
弦爲三方之鈎於鉤爲竪於股以倣鈎股
弦之式而得弦者是爲斜得鈎者則爲三
方之鈎得股者是爲竪也
今有三方竪之方鈎竪知坪式者
自因相因法因
以方句之尺数自因乗而得歩数于是用竪
之尺数因乗而得坪数于是用三方之方句
自因之歩法四三三五七七三因乗則得坪数是

Figure 2.2: Imamura Tomoaki's *Jugairoku*, 1639, Tōhoku University, Hayashi Collection. Algorithms concerning the "platform triangle", that is, a truncated pyramid whose base is an equilateral triangle.

The next volume considered is a parallelepiped with two square faces. The presence of this intermediate stage, before studying the parallelepiped with three different dimensions, is explained when one considers that Imamura deduces this new volume from that of the cube by identifying the ratio of the volumes with the ratio of the corresponding heights.

In the same way, the volume of Figure 4 in the table (corresponding to the "side–diagonal" for a parallelepiped with square base) may have been obtained from that of the "side–diagonal" by identifying the ratio of the volumes of two pyramids of the same height with the ratio of their respective bases (Figure 2.4).



$V_1 = \frac{1}{3} h^3$ $\qquad V_2 = \frac{a^2 V_1}{h^2} = \frac{1}{3} a^2 h$

Figure 2.4

The passage from the "side–diagonal" to the classical pyramid, where the vertex is situated vertically above the center of the base, may have been obtained by decomposing the latter into four pyramids of the first type (Figure 2.5).

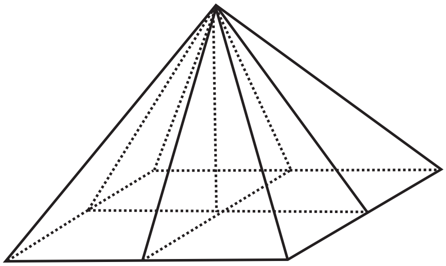


Figure 2.5

Our interpretation, which is inspired by the works of the immediate successors of Imamura,[24] is of course only a hypothesis, but we can point out that it has the advantage of being based solely on attested knowledge of the epoch, including the use of proportionality to evaluate volumes.[25]

2.3.4. The New Figures

Apart from inaugurating a methodical approach to the evaluation of areas and volumes, the *Jugairoku* also opened up new fields of research whose continuation one finds in Seki's work. The two most significant examples are the study of regular polygons and that of the length of an arc of a circle.

The objectives are the same as for the other figures: the main question is to determine a way of calculating the area from the primary dimensions – the side and the "bases"[26] for polygons and the sagitta, the chord and the diameter for the arc of a circle.

The two studies are in fact linked. Indeed, it seems that Imamura relied on the rules concerning the arc of the circle to evaluate the coefficients related to polygons.[27] This is one of the reasons why we propose to examine here only the rules concerning segments of a circle. Another reason is that it is this which the author sees as the climax of his work.[28]

Imamura's rules concerning the circle segment are of three types. A first series of three rules, grouped under the name of "diameter–sagitta–chord" (*keishigen*), permits the calculation of one of these magnitudes knowing the two others. In modern notation these rules are written as follows:

$$d = s + \frac{c^2}{4s} \tag{1}$$

$$c = \sqrt{4s(d-s)} \tag{2}$$

$$s = \frac{d - \sqrt{d^2 - c^2}}{2}. \tag{3}$$

Here and in the following we denote the diameter, the sagitta and the chord by their initials d, s and c, respectively (Figure 2.6).

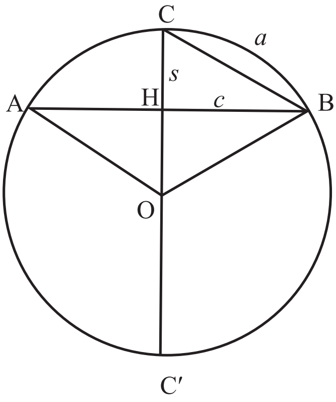


Figure 2.6

The second series of rules is grouped under the name of "arc–sagitta–chord" (*koshigen*). They indicate the way of calculating the length of the arc starting from two parameters chosen among the diameter, the chord and the sagitta. Here, as an example, we give the calculation rule starting with the sagitta and the diameter:

$$a^2 = 4s\left(d + \frac{s}{2}\right). \tag{4}$$

The third series of rules concerns the calculation of the area of a segment of a circle (*enketsu*) knowing the sagitta, the chord, the diameter and the length of the arc:

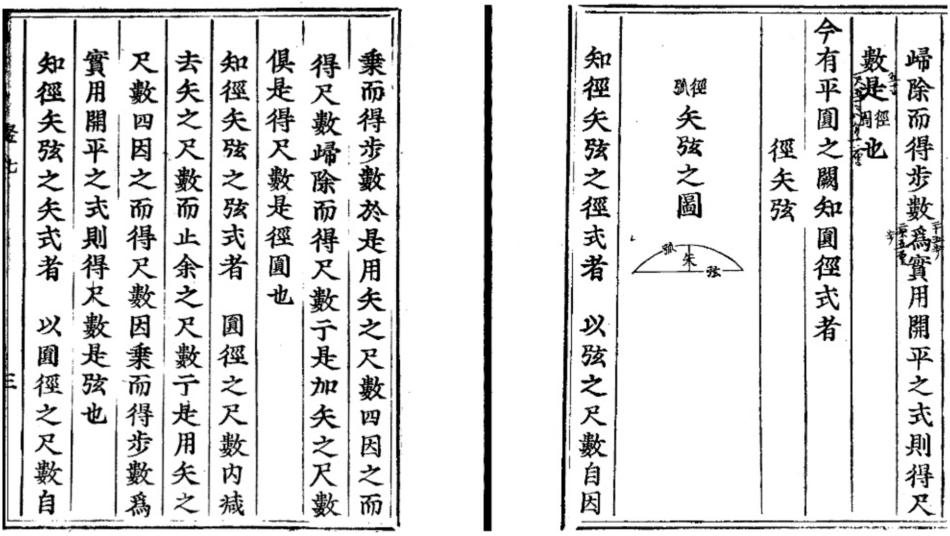
$$A = \frac{da}{4} - \frac{c}{2}\left(\frac{d}{2} - s\right). \tag{5}$$

While Imamura does not give any indication of the way these rules were obtained, one can again formulate hypotheses from what one knows about his geometric tools. The rules (1), (2) and (3) were probably deduced from the fact that the triangle OHB has a right angle in H (Figure 2.6).

The procedure given for the surface of the circle segment suggests that the surface was obtained as the difference of the areas of the angular segment $OACB$ and the triangle OAB. Indeed, the former is equal to $da/4$, which is an exact value.

Only the origin of rule (4) is problematic. One can observe that for the values 0 and $d/2$ of the sagitta, the corresponding values of the arc are 0 and $\sqrt{10}\,d/2$. Since Imamura used $\sqrt{10}$ for the value of π,[29] one can suppose that the expression for the arc was set up to yield the half-circumference when the sagitta equals the radius of the circle.[*]

[*] More precisely: when the sagitta is very small, then the lengths a and c of the arc and the chord are nearly the same, so by formula (2) one has $a^2 \approx c^2 \approx 4sd$. On the other hand, when $s = d/2$, so that the arc



帰除而得歩數爲實用開平之式則得尺
數是也
今有平圓之闕知圓徑式者
徑矢弦
矢弦之圖
知徑矢弦之徑式者 以弦之尺數自因

乗而得歩數於是用矢之尺數四因之而
得尺數帰除而得尺數于是加矢之尺數
倶是得尺數是徑圓也
知徑矢弦之弦式者 圓徑之尺數内減
去矢之尺數而止余之尺數于是用矢之
尺數四因之而得尺數因乗而得歩數爲
實用開平之式則得尺數是弦也
知徑矢弦之矢式者 以圓徑之尺數自

Figure 2.7: Imamura Tomoaki's *Jugairoku*, 1639, Tōhoku University, Hayashi Collection. Algorithms concerning the arc of the circle.

The *Sanpō Ketsugishō* (1659) by Isomura Yoshinori, which provides explanations for most of the rules of the *Jugairoku*, furnishes us with an additional piece of information. After having formulated the expression for the arc in a different – but equivalent – manner ($a^2 = c^2 + 6s^2$), Isomura writes:

> "The preceding rules for the 'diameter-sagitta-chord' and the 'arc-sagitta-chord' are ancient precepts. Their *kiku* (compass and ruler) is not clear from them. One knows only that the rule of augmentation (*zōjutsu*) is used."[30]

According to this, the key to the method should be found in the "rule of augmentation" cited, but the text does not give any further information about the content of this rule. The type of reasoning suggested by the rule of arc–sagitta–chord is not foreign to the Chinese tradition. Since the Tang, the Chinese calendarists had been in possession of a method of interpolation [31] of second degree [32] (*sōgensōjō no hō*, method of mutual subtraction and multiplication) which, when applied to the present problem with the value $\sqrt{10}\ \frac{d}{2}$ of the half-circumference cited before, leads exactly to the result given by Imamura.

Can Imamura have had recourse to the methods of the calendarists? This question has not yet been elucidated.

---

is a semicircle, then the value of a is $\pi d/2$ or, since Imamura used $\sqrt{10}$ for the value of π, $a^2 \approx 5d^2/2$. Formula (4) gives both of these extreme cases correctly, and in fact the value of a that it yields never deviates from the correct value by more than about 1 part in 75. [Translator's note.]

2.4. Imamura and the Chinese Works

Until now we have chosen to focus our attention on the knowledge that one can consider as familiar to the mathematicians of Japan, without lingering too much over the origin of that knowledge. One should, however, keep in sight the fact that Chinese textbooks were a constant source of stimulation for Imamura.

Besides the references to Chinese tradition that we were able to pick out in the foreword, one must also note that the new mathematical ideas developed in the *Jugairoku* usually have their origin in the *Suanfa tongzong*. Thus, to cite only one example, the study of the circle segment is a subject treated at length by Cheng Dawei.[33] The various relations that one can establish among the diameter, the sagitta and the chord are justified there by referring to the rule of the base–perpendicular (equivalent to Pythagoras's theorem).

For the area of a segment, Cheng reproduces the expression given in the *Nine Chapters*, which is written in modern language as

$$A = \frac{s(s+c)}{2}\,.$$

For the arc, on the other hand, no expression is given in any of the problems studied in the Chinese manual. Nevertheless some interesting pieces of information are provided in a commentary with the title "Remarks on the Square and the Circle":

> "[...] The rule for the division of the circle (*geyuan*)," writes Cheng, "consists in determining the sagitta and the chord and obtaining by this way the length of the arc. One must fear that the arc remains inaccessible. How might one get to know it? If one takes a circle of diameter 10 *cun* and splits it in its center, the width of the sagitta of the part thus cut off is 5 *cun*. Raised to the square this gives 25 *cun*. Dividing by the diameter, one obtains 2 *cun* and 5 *fen*. Which gives one-half of the difference between the arc and the chord. By doubling one obtains 5 *cun*; by adding the chord, 15 *cun*. This coincides with [the value of the semi-circumference] on the hypothesis that, for a diameter equal to 1, one has a circumference equal to 3. But actually, the circumference is equal to a bit more than three. So that its value cannot be exhausted in this way. Thus one knows that the question of the measure of the arc is not yet resolved [...]."[34]

Here an approximate expression for the length of the arc is implicitly used which possesses the property of giving "exactly" the value of the half-circumference (assuming that this is calculated taking a relation of one to three between the diameter and the circumference) in the specific case where the sagitta equals the radius, namely:

$$a = c + 2\,\frac{s^2}{d}\,.$$

There is a manifest analogy between the work of Cheng Dawei and that of Imamura. The latter merely delves more deeply into the study of questions that had already been formulated by his Chinese predecessor. His main contribution lies in the fact that he clarified the subject by proposing for the area of a circle segment an exact expression as a function of the length of the arc, thus reducing the study to the calculation of this single length. A first step in this direction was made with the formula (4) cited above.

2.5. Conclusion

The studies of the *Jugairoku* are still too fragmentary to attempt here a satisfactory clarification of its style and content. One can say more about the impact which this work had on its period. We have already mentioned the fact that the *Jugairoku* was very little read in its original form. It was to reach the public only twenty years later, when a disciple named Andō Arimasu published a vernacular version.

> "It was [he writes] in the autumn of the third year of Manji [1660] that I had the occasion to see it for the first time. I consulted it furtively. Its phrases were clear and their meaning profound. Somebody remarked: 'This work is difficult to read because there are no marks for the Japanese readings (*kun*), and is difficult to understand because one doesn't find any numbers there. Would it not be desirable to add the Japanese readings and to give numerical values so that it could be appreciated by a greater public?'"

Thus the conciseness of Imamura's book is perceived at this period as an impediment to its understanding. His immediate successors kept above all his terminology and endeavored to make his rules for calculating areas and volumes more accessible. Later they tried to improve the special numerical coefficients occurring in these rules. Among the works belonging to this development, one can cite the *Sanpō Ketsugishō* (1661) (Book Removing Doubts in Mathematics)[35] by Isomura Yoshinori. This very popular work contributed a lot to clarifying the rules of Imamura and making them better known. We will not spend more time with this mathematician, whose concerns remains close to those of Imamura, but will turn instead to an author who is just as squarely situated in the sphere of influence of the latter but whose achievements bring us closer to the work of Seki.

Notes

1. See *MNS*, vol. 1, p. 229.
2. See *MNS*, vol. 1, p. 50.
3. According to the preface of the *Jugairoku*, translated below.
4. See *MNS*, vol. 1, p. 229.
5. Observe that Imamura here confuses the *ki*, the compass, and the *ku*, the set square, mentioned further below. Isomura Yoshinori makes the same error in the *Sanpō Ketsugishō* (1661), and we have to wait for the *Sanso* (see the next chapter) to see it corrected.

6. In the *Nine Chapters*, the right-angled triangle and the numerical relation between its three sides are globally designated by *gougu*, where *gou* and *gu* correspond to the small and the large side of the right angle, respectively.
7. This reference to Li Shou as having introduced the enumeration system is frequent in the prefaces of ancient Chinese manuals. Fujiwara observed a striking similarity between this passage and the preface of the *Xiahouyang suanjing*, a classic of Chinese mathematics. See *MNS*, vol 1, p. 217 (footnote). For a general discussion of the prefaces of Chinese mathematical works, see Martzloff (5'), pp. 46–47.
8. Imamura (1), p. 35; Imamura (2), pp. 26–27; *MNS*, vol. 1, pp. 216–217.
9. We find here the following reference: "The emperor ordered Shuhai to travel across his lands from the extreme East to the extreme West, over a distance of five million one hundred nine thousand eight hundred paces. Shuhai held in his right hand a measuring instrument, indicating the north of the Qiungqiu with his left hand." Rémi Mathieu (1'), p. 434.
10. This symbolism was standard in ancient Chinese works. It can be found for example in the *Liji*, the writings of Mengzi, or the *Zhoubi suanjing*. See Qian (1), p. 13 and *SCC*, vol. 3, pp. 22–23 and p. 570.
11. The notion of *kiku* will be taken up by various mathematicians, among them Isomura Yoshinori and Seki Takakazu. See Isomura (2), p. 112 and Seki (1), pp. 3–4.
12. The use of the term *shiki* to designate the methods of solution is, as far as we know, without precedent either in Japan or in China. In the *Suanfa tongzong* this term, which by the way is used only very little, seems to be reserved for numbers arranged in columns or tables, reproducing on paper configurations belonging to a different medium (counting-rods or abacus), and it is the term *fa* (method) that generally introduces each solution. Under Imamura's pen, this meaning seems to be excluded, and it seems that we should take *shiki* as a synonym for *kakushiki*, that is, a rule in the sense of unanimously accepted convention. In any case, this is what is suggested by the preface of the *Jugairoku Kanashō* (*Kana* Edition of the *Jugairoku*), written by a disciple. Imamura (2), p. 36.
13. Lam Lay Yong (1').
14. More precisely, Imamura distinguishes the procedures according to the order of magnitude of the numbers to which the extraction is being applied.
15. The translation of the word *taijū* (*daicong* in Chinese) that we propose here is literal. The term, borrowed from the *Suanfa tongzong*, is not commented on anywhere.
16. Imamura (2), pp. 77–85. Similar categories are considered in the case of the opening of the cube. The *taijū kairitsu*, or the opening of the cube with excess length, is defined as "the rule by which, having made of the volume [literally, of the *tsubo*] a parallelepiped whose height is for example bigger by 2 *shaku* than the side [of the square base], one knows the value of the side in *shaku*." Imamura (2), p. 98.
17. *MNS*, pp. 419–429.

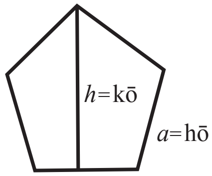
18. Imamura (2), p. 84.
19. The units used are the following: the *shaku*, the *sun*, the *fu*, the *ri*, the *gō* and the *kei* (each being a tenth of the preceding one). The *bu* is the area of a square of side one *sun*. To lighten the translation, we abbreviate 152 *ho* 2 *bu* 7 *ri* 5 *gō* 6 *kei* by 152 *ho* 2756.
20. Imamura (2), p. 77.
21. The procedure for extraction of a cube root is similar. The first digit of the root, a, is obtained by estimation. The following numbers are then successively calculated in the dividend: $M = N - a^3$. The second quotient a' is obtained by dividing M by $3a^2$ to one digit, giving the remainder $N' = M - 3a^2a'$; then $M' = N' - 3aa'^2 - a'^3$. The third quotient a'' is obtained by dividing M' by $3\,(a + a')^2$, with remainder $N'' = M' - 3(a + a')^2 a''$; then $M'' = N'' - 3a''^2\,(a + a') - a''^3$, and so forth.
22. Let us point out the difficulties posed by translating the character *hō*, which according to the situation can mean the side of a square, a square, or a square shape (as opposed to a round one).
23. Observe that in Liu Hui's commentary to the *Nine Chapters*, one finds volumes similar to those considered by Imamura, but designated by different names. The rule that Imamura gives is the following:

    > "We now have the side of the 'side-diagonal.' The rule for knowing the volume [literally, the number of *tsubo*] is the following:
    > – Automultiply twice [that is, take the cube] and divide by three.
    > – Automultiply twice the number of *shaku* [that is, the length] of the side,
    >
    > then divide by three the number of *tsubo* [that is, the volume] obtained. The division by three that we use is called the operator of the side-diagonal." (Imamura (2), p. 149)

24. See for instance Isomura (2), pp. 115–140.
25. Yoshida (1), Problem 9 of the second book.
26. To each regular polygon studied (from three to ten sides), Imamura associates, according to the parity of the number of sides, two or three distinguished quantities. In all cases, the side is called *hō*.
    - When the number of sides is even, one defines the *kakukō* (base of the angles) or segment connecting two opposite vertices, i.e., the diameter of the circumscribed circle, and the *heikō* (flat base) or segment connecting the centers of two opposite sides, i.e., the diameter of the inscribed circle.
    - When the number is odd, only the *kō* (base) connecting a vertex with the middle of the opposite side is defined.

    The rules give, for example, the method for calculating the *kakukō* (resp. *heikō* or *kō*) starting from the side, or the area starting from any of these dimensions. These rules are based on approximate numerical coefficients whose origin is not given.

Example: in the case of the pentagon, one has $h = 1.5457\,a$; $A = 1.119\,ah$; $A = 1.73\,a^2$; $A = 0.7241\,h^2$



where a, h and A denote the side, the *kō* and the area of the pentagon, respectively.

27. This interpretation is reinforced by Isomura Yoshinori's use of this method in the *Sanpō Ketsugishō* (1660). See *MNS*, vol. 1, pp. 300–301.
28. See the preface of the *Jugairoku* in Section 2.
29. The "coefficient of the circumference" (the number by which one should multiply the diameter to obtain the circumference) used by all Japanese mathematicians between 1620 and 1660 is 3.16 (the more "precise" value 3.162 is also employed), this value being linked – as Imamura's formula for the arc suggests – to the number $\sqrt{10}$. We have to await the *Sanso* (1663) by Muramatsu Shigekiyo (see Chapter 3) to see this value improved. See Shimodaira (1'), pp. 1–2.
30. Isomura, *op. cit.*, p. 112.
31. More precisely, this interpolation method appeared for the first time in the *Chongxuanli* calendar, applied in China at the end of the 9th and the beginning of the 10th centuries. But we should specify that this calendar was never applied in Japan, where the *Xuanmingli*, a calendar dating from an earlier period, was adopted until the 17th century, that is, for more than eight hundred years.
32. The interpolation was always done by "functions" associating to the variable 0 the value 0. For the second degree interpolation occurring here, the function is therefore sought from the start in the form $ax + bx^2$.
33. The division of the circle is a method for measuring the area of the circle by the successive inscription of regular polygons. The oldest example of its use in China goes back to the commentary to the *Nine Chapters* by the mathematician Liu Hui (3rd century). See Chapter 8 of the present study.
34. Cheng Dawei (1), Chapter 3, *fangtian*.
35. *MNS*, vol. 1, p. 229.
36. According to a phrase of the *Conversations* by Confucius (Book II, "On man," No. 18): "Listen to all that is said around you, and put in parentheses the doubtful points so as to speak with prudence only of what is certain." Confucius (1'), p. 36.