1 Introduction

Spherical harmonics are the analogues of trigonometric functions for Fourier expansion theory on the sphere. They were introduced in the 1780s to study gravitational theory (cf. P.S. de Laplace (1785), A.M. Legendre (1785)). Early publications on the theory of spherical harmonics in their original physically motivated meaning as multipoles are, e.g., due to R.F.A. Clebsch (1861), T. Sylvester (1876), E. Heine (1878), F. Neumann (1887), and J.C. Maxwell (1891). Today, the use of spherical harmonics in diverse procedures is a well-established technique in all geosciences, particularly for the purpose of representing scalar potentials. A great incentive came from the fact that global geomagnetic data became available in the first half of the 19^{th} century (cf. C.F. Gauß (1838)). Nowadays, reference models for the Earth's gravitational or magnetic field, for example, are widely known by tables of coefficients of the spherical harmonic Fourier expansion of their potentials. It is characteristic for the Fourier approach that each spherical harmonic, as an 'ansatz-function' of polynomial nature, corresponds to exactly one degree, i.e., in the jargon of signal processing to exactly one frequency. Thus, orthogonal (Fourier) expansion in terms of spherical harmonics amounts to the superposition of summands showing an oscillating character determined by the degree (frequency) of the Legendre polynomial (see Table 1.1). The more spherical harmonics of different degrees are involved in the Fourier (orthogonal) expansion of a signal, the more the oscillations grow in number, and the less are the amplitudes in size.

Concerning the mathematical representation of spherical vector and tensor fields in applied sciences, one is usually not interested in their separation into their (scalar) cartesian component functions. Instead, we have to observe inherent physical constraints. For example, the external gravitational field is curl-free, the magnetic field is divergence-free, and the equations for incompressible Navier–Stokes equations in meteorological applications or the geostrophic formulation of ocean circulation include divergence-free vector solutions. In many cases, certain quantities are related to each other in an obvious manner by vector operators like the surface gradient or the surface curl gradient. In this respect, the gravity field, the magnetic field, the wind field, the field of oceanic currents, or electromagnetic waves generated by surface currents should be mentioned as important examples.

In addition, spherical modeling in terms of spherical harmonics arises naturally in the analysis of the elastic-gravitational free oscillations of a spherically symmetric, non-rotating Earth. Altogether, vector/tensor spherical harmonics are used throughout mathematics, theoretical physics, geo- and astrophysics, and engineering – indeed, wherever one deals with physically based fields.

Table 1.1: Fourier expansion of scalar square-integrable functions on the unit sphere Ω.

Weierstraß approximation theorem: use of homogeneous polynomials $\downarrow$ (geo)physical constraint of harmonicity

spherical harmonics $Y_{n,j}$
as restrictions of homogeneous harmonic polynomials $H_{n,j}$
to the unit sphere $\Omega \subset \mathbb{R}^3$

orthonormality and orthogonal invariance $\downarrow$ addition theorem

one-dimensional Legendre polynomial P_n satisfying

$$P_n(\xi \cdot \eta) = \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta), \ \ \xi, \eta \in \Omega$$

convolution against the Legendre kernel $\downarrow$ Funk–Hecke formula

Legendre transform of F:

$$(P_n * F)(\xi) = \frac{2n+1}{4\pi} \int_\Omega P_n(\xi \cdot \eta) F(\eta) d\omega(\eta), \ \ \xi \in \Omega$$

superposition over frequencies $\downarrow$ orthogonal (Fourier) series expansion

Fourier series of $F \in \mathrm{L}^2(\Omega)$:

$$F(\xi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_\Omega P_n(\xi \cdot \eta) F(\eta) d\omega(\eta), \ \ \xi \in \Omega$$

1.1 Motivation

In the second half of the last century, a physically motivated approach for the decomposition of spherical vector and tensor fields was presented based on a spherical variant of the Helmholtz theorem (see, e.g., P.M. Morse, H. Feshbach (1953), G.E. Backus (1966); G.E. Backus (1967, 1986)). Following this concept, e.g., the tangential part of a spherical vector field is split up into a curl-free and a divergence-free field by use of two differential operators, viz. the already mentioned surface gradient and the surface curl gradient. Of course, an analogous splitting is valid in tensor theory.

Table 1.2: Twofold transition.

Scalar Legendre kernels	$\longrightarrow$	Vector Legendre kernels	$\longrightarrow$	Tensor Legendre kernels
$\downarrow$		$\downarrow$		$\downarrow$
scalar zonal kernels	$\longrightarrow$	vector zonal kernels	$\longrightarrow$	tensor zonal kernels
$\downarrow$		$\downarrow$		$\downarrow$
scalar Dirac kernel	$\longrightarrow$	vector Dirac kernel	$\longrightarrow$	tensor Dirac kernel

In subsequent publications during the second half of the last century, however, the vector spherical harmonic theory was usually written in local coordinate expressions that make mathematical formulations lengthy and hard to read. Tensor spherical harmonic settings are even more difficult to understand. In addition, when using local coordinates within a global spherical concept, differential geometry tells us that there is no representation of vector and tensor spherical harmonics which is free of singularities. In consequence, the mathematical arrangement involving vector and tensor spherical harmonics has led to an inadequately complex and less consistent literature, yet. Coordinate free explicit formulas on vector and/or tensor variants of the Legendre polynomial could not be found in the literature. As an immediate result, the orthogonal invariance based on specific vector /tensor extensions of the Legendre polynomials was not worked out suitably in a unifying scalar/vector/tensor framework. Even more, the concept of zonal (kernel) functions was not generalized adequately to the spherical vector/tensor case. All these new structures concerning spherical functions in mathematical (geo-)physics are successfully developed in this work. Basically two transitions are undertaken in our approach, namely the transition from spherical harmonics via zonal kernel functions to the Dirac kernels on the one hand and the transition from scalar to vector and tensor theory on the other hand (see Table 1.2).

To explain the *transition from the theory of scalar spherical harmonics to its vectorial and tensorial extensions* (see Chapters 3, 4, 5, and 6 for details), our work starts from physically motivated dual pairs of operators (the reference space being always the space of signals with finite energy, i.e., the space of square-integrable fields). The pair $o^{(i)}, O^{(i)}, i \in \{1,2,3\}$, is originated in the constituting ingredients of the Helmholtz decomposition of a vector field (see Chapter 5), while $o^{(i,k)}, O^{(i,k)}, i,k \in \{1,2,3\}$, take the analogous role for the Helmholtz decomposition of tensor fields (see Chapter 6). For example, in vector theory, $o^{(1)}F$ is assumed to be the normal field $\xi \mapsto o_\xi^{(1)}F(\xi) = F(\xi)\xi$, $\xi \in \Omega$, $o^{(2)}F$ is the surface gradient field $\xi \mapsto o_\xi^{(2)}F(\xi) = \nabla_\xi^* F(\xi)$, $\xi \in \Omega$, and $o^{(3)}F$ is the surface curl gradient field $\xi \mapsto o_\xi^{(3)}F(\xi) = \mathrm{L}_\xi^* F(\xi), \xi \in \Omega$, with $\mathrm{L}_\xi^* = \xi \wedge \nabla_\xi^*$ applied to a scalar valued function F, while $O^{(1)}f$ is the normal component $\xi \mapsto O_\xi^{(1)}f(\xi) = f(\xi)\cdot\xi$, $\xi \in \Omega$, $O^{(2)}f$ is the negative surface divergence $\xi \mapsto O_\xi^{(2)}f(\xi) = -\nabla_\xi^*\cdot f(\xi)$, $\xi \in \Omega$, and $O^{(3)}f$ is the negative surface curl $\xi \mapsto O_\xi^{(3)}f(\xi) = -\mathrm{L}^*\cdot f(\xi)$, $\xi \in \Omega$ taken over a vector valued function f. Clearly, the operators $o^{(i,k)}, O^{(i,k)}$ are also definable in orientation to the tensor Helmholtz decomposition theorem (for reasons of simplicity, however, their explicit description is omitted here). It should be noted that, in vector as well as tensor theory, the connecting link from the operators to the Helmholtz decomposition is the Green function with respect to the (scalar) Beltrami operator and its iterations (for more details, the reader is referred to Chapter 4 of this work).

The pairs $o^{(i)}, O^{(i)}$ and $o^{(i,i)}, O^{(i,i)}$ of dual operators lead us to an associated palette of Legendre kernel functions, all of them generated by the classical one-dimensional Legendre polynomial P_n of degree n. To be more concrete, three types of Legendre kernels occur in the vectorial as well as tensorial context (see Table 1.3).

Table 1.3: Legendre kernel functions.

Scalar Legendre polynomial $P_n = \frac{O^{(i)}O^{(i)}\mathbf{p}_n^{(i,i)}}{\mu_n^{(i)}} = \frac{O^{(i,k)}O^{(i,k)}\mathbf{P}_n^{(i,k)}}{\mu_n^{(i,k)}}$	
application of $o^{(i)}$ $\downarrow\uparrow$ application of $O^{(i)}$	application of $o^{(i,k)}$ $\downarrow\uparrow$ application of $O^{(i,k)}$
vector Legendre kernel $p_n^{(i)} = \frac{o^{(i)}P_n}{(\mu_n^{(i)})^{1/2}} = \frac{O^{(i)}\mathbf{p}_n^{(i,i)}}{(\mu_n^{(i)})^{1/2}}$	tensor Legendre kernel (order 2) $\mathbf{p}_n^{(i,k)} = \frac{o^{(i,k)}P_n}{(\mu_n^{(i,k)})^{1/2}} = \frac{O^{(i,k)}\mathbf{P}_n^{(i,k)}}{(\mu_n^{(i,k)})^{1/2}}$
application of $o^{(i)}$ $\downarrow\uparrow$ application of $O^{(i)}$	application of $o^{(i,k)}$ $\downarrow\uparrow$ application of $O^{(i,k)}$
tensor Legendre kernel (order 2) $\mathbf{p}_n^{(i,i)} = \frac{o^{(i)}p_n^{(i)}}{(\mu_n^{(i)})^{1/2}} = \frac{o^{(i)}o^{(i)}P_n}{\mu_n^{(i)}}$	tensor Legendre kernel (order 4) $\mathbf{P}_n^{(i,k,i,k)} = \frac{o^{(i,k)}\mathbf{p}_n^{(i,k)}}{(\mu_n^{(i,k)})^{1/2}} = \frac{o^{(i,k)}o^{(i,k)}P_n}{\mu_n^{(i,k)}}$
vectorial context	*tensorial context*

The Legendre kernels $o^{(i)}P_n$, $o^{(i)}o^{(i)}P_n$ are of concern for the vector approach to spherical harmonics, whereas $o^{(i,i)}P_n$, $o^{(i,i)}o^{(i,i)}P_n, i = 1, 2, 3$, form the analogues in tensorial theory. Corresponding to each Legendre kernel, we are led to two variants for representing square-integrable fields by orthogonal (Fourier) expansion, where the reconstruction – as in the scalar case – is undertaken by superposition over all frequencies.

The Tables 1.3, 1.4, and 1.5 bring together – into a single unified notation – the formalisms for the vector/tensor spherical harmonic theory based on the following principles:

- The vector/tensor spherical harmonics involving the $o^{(i)}, o^{(i,i)}$-operators, respectively, are obtainable as restrictions of three-dimensional homogeneous harmonic vector/tensor polynomials, respectively, that are computable exactly exclusively by integer operations.
- The vector/tensor Legendre kernels are obtainable as the outcome of sums extended over a maximal orthonormal system of vector/tensor spherical harmonics of degree (frequency) n, respectively.

- The vector/tensor Legendre kernels are zonal kernel functions, i.e., they are orthogonally invariant (in vector/tensor sense, respectively) with respect to orthogonal transformations (leaving one point of the unit sphere Ω fixed).
- Spherical harmonics of degree (frequency) n form an irreducible subspace of the reference space of (square-integrable) fields on Ω.
- Each Legendre kernel implies an associated Funk–Hecke formula that determines the constituting features of the convolution of a square-integrable field against the Legendre kernel.
- The orthogonal Fourier expansion of a square-integrable field is the sum of the convolutions of the field against the Legendre kernels being extended over all frequences.

Unfortunately, the vector spherical harmonics generated by the operators $o^{(i)}, O^{(i)}$, $i = 1, 2, 3$, do not constitute eigenfunctions with respect to the Beltrami operator. But it should be mentioned that certain operators $\tilde{o}^{(i)}$, $i = 1, 2, 3$, can be introduced in terms of the operators $o^{(i)}$, $i = 1, 2, 3$, which define alternative classes of vector spherical harmonics that represent eigensolutions to the Beltrami operator. The price to be paid is that the separation of spherical vector fields into normal and tangential parts is lost. More precisely, the operators $\tilde{o}^{(i)}, i \in \{1, 2\}$, generate so-called spheroidal fields, while $\tilde{o}^{(3)}$ generates poloidal fields. In fact, all statements involving orthogonal (Fourier) expansion of spherical fields remain valid for this new class of operators. Moreover, analogous classes of tensor spherical harmonics can be introduced by operators $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}, i, k = 1, 2, 3$, in close analogy to the vector case. In addition, it should be noted that the spherical harmonics based on the $\tilde{o}^{(i)}, \tilde{O}^{(i)}, \tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$-operators play a particular role whenever the Laplace operator comes into play, i.e., in gravitation for representing any kind of harmonic fields (see Chapter 10).

To summarize, the theory of spherical harmonics as presented in this book (see Chapters 3, 4, 5, and 6) is a unifying attempt of consolidating, reviewing and supplementing the different approaches in real scalar, vector, and tensor theory. The essential tools are the Legendre kernels which are shown to be explicitly available and tremendously significant in rotational invariance and in orthogonal Fourier expansions. The work is self-contained: the reader is told how to derive all equations occuring in due course. Most importantly, our coordinate-free setup yields a number of formulas and theorems that previously were derived only in coordinate representation (such as polar coordinates). In doing so, any kind of singularities is avoided at the poles. Finally, our philosophy opens new promising perspectives of constructing important, i.e., zonal classes of spherical trial functions by summing up Legendre kernel expressions, thereby providing (geo-)physical relevance and

Table 1.4: Fourier expansion of (square-integrable) vector fields f.

Vector spherical harmonics $y_{n,j}^{(i)} = (\mu_n^{(i)})^{-1/2} o^{(i)} Y_{n,j}$	
addition theorem $\downarrow$ vectorial variant	addition theorem $\downarrow$ tensorial variant
${}^v\mathbf{p}_n^{(i,i)}(\xi,\eta) = \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi) \otimes y_{n,j}^{(i)}(\eta)$	$p_n^{(i)}(\xi,\eta) = \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi) Y_{n,j}(\eta)$
Funk–Hecke formula $\downarrow$ tensorial variant	Funk–Hecke formula $\downarrow$ vectorial variant
Legendre transform $\frac{2n+1}{4\pi} \times \int_\Omega {}^v\mathbf{p}_n^{(i,i)}(\xi,\eta) f(\eta)\, d\omega(\eta)$	Legendre transform $\frac{2n+1}{4\pi} (\mu_n^{(i)})^{-1/2} \times \int_\Omega p_n^{(i)}(\xi,\eta) O_\eta^{(i)} f(\eta)\, d\omega(\eta)$
superposition $\downarrow$ over frequencies	superposition $\downarrow$ over frequencies
$f(\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \times \int_\Omega {}^v\mathbf{p}_n^{(i,i)}(\xi,\eta) f(\eta)\, d\omega(\eta)$	$f(\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} (\mu_n^{(i)})^{-1/2} \times \int_\Omega p_n^{(i)}(\xi,\eta) O_\eta^{(i)} f(\eta)\, d\omega(\eta)$
rank–2 tensorial approach	*vectorial approach*

increasing local applicability.

To understand the *transition from the theory of spherical harmonics to zonal kernel function up to the Dirac kernel* (for details see Chapters 7, 8, and 9), we have to realize the relative advantages of the classical Fourier expansion method by means of spherical harmonics not only in the frequency domain, but also in the space domain. Obviously, it is characteristic for Fourier techniques that the spherical harmonics as polynomial trial functions admit no localization in space domain, while in the frequency domain

Table 1.5: Fourier expansion of a square-integrable tensor fields $\mathbf{f}$.

Tensor spherical harmonics $\mathbf{y}_{n,j}^{(i,k)} = (\mu_n^{(i,k)})^{-1/2}\mathbf{o}^{(i,k)}Y_{n,j}$	
addition theorem $\downarrow$ rank-4 tensorial variant	addition theorem $\downarrow$ rank-2 tensorial variant
$\mathbf{P}_n^{(i,k,i,k)}(\xi,\eta) = \sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi) \otimes \mathbf{y}_{n,j}^{(i,k)}(\eta)$	${}^t\mathbf{p}_n^{(i,k)}(\xi,\eta) = \sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi) Y_{n,j}(\eta)$
Funk–Hecke formula $\downarrow$ rank-4 tensorial variant	Funk–Hecke formula $\downarrow$ rank-2 tensorial variant
Legendre transform $\frac{2n+1}{4\pi} \times \int_\Omega \mathbf{P}_n^{(i,k,i,k)}(\xi,\eta)\mathbf{f}(\eta)d\omega(\eta)$	Legendre transform $\frac{2n+1}{4\pi}(\mu_n^{(i,k)})^{-1/2} \times \int_\Omega {}^t\mathbf{p}_n^{(i,k)}(\xi,\eta)O_\eta^{(i,k)}\mathbf{f}(\eta)d\omega(\eta)$
superposition $\downarrow$ over frequencies	superposition $\downarrow$ over frequencies
$\mathbf{f}(\xi) = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \times \int_\Omega \mathbf{P}_n^{(i,k,i,k)}(\xi,\eta)\mathbf{f}(\eta)d\omega(\eta)$	$\mathbf{f}(\xi) = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \frac{1}{(\mu_n^{(i,k)})^{1/2}} \times \int_\Omega {}^t\mathbf{p}_n^{(i,k)}(\xi,\eta)O_\eta^{(i,k)}\mathbf{f}(\eta)d\omega(\eta)$
rank-4 tensorial approach	*rank-2 tensorial approach*

(more precisely, momentum domain), they always correspond to exactly one degree, i.e., frequency, and therefore, are said to show ideal frequency localization. Because of the ideal frequency localization and the simultaneous absence of space localization, in fact, local changes of fields (signals) in the space domain affect the whole table of orthogonal (Fourier) coefficients. This, in turn, causes global changes of the corresponding (truncated) Fourier series in the space domain. Nevertheless, the ideal frequency localization usually proves to be helpful for meaningful physical interpretations (e.g., within Meissl schemes in physical geodesy (see, e.g., P.A. Meissl (1971), E.W. Grafarend (2001), H. Nutz (2002) and the references therein) relating – for a frequency being fixed – the different observables of the Earth's gravitational potential to each other.

Taking these aspects on spherical harmonic modeling by Fourier series into account, trial functions which simultaneously show ideal frequency localization as well as ideal space localization would be a desirable choice. In fact, such an ideal system of trial functions would admit models of highest spatial resolution which were expressible in terms of single frequencies. However, the uncertainty principle (see, e.g., F.J. Narcowich, J.D. Ward (1996), W. Freeden (1998), N. Laín Fernández (2003)) – connecting space and frequency localization – tells us that both characteristics are mutually exclusive. Extreme trial functions in the sense of such an uncertainty principle are, on the one hand, the Legendre kernels (no space localization, ideal frequency localization) and, on the other hand, the Dirac kernel (ideal space localization, no frequency localization). In conclusion, Fourier expansion methods are well suited to resolve low and medium frequency phenomena, i.e., the 'trend' of a signal, while their application to obtain high resolution in global or local models is critical. This difficulty is also well known to theoretical physics, e.g., when describing monochromatic electromagnetic waves or considering the quantum-mechanical treatment of free particles. In this case, plane waves with fixed frequencies (ideal frequency localization, no space localization) are the solutions of the corresponding differential equations, but do certainly not reflect the physical reality. As a remedy, plane waves of different frequencies are superposed to so-called wave-packages which gain a certain amount of space localization, while losing their ideal spectral localization. In a similar way, a suitable superposition of polynomial Legendre kernel functions leads to so-called zonal kernel functions, in particular to kernel functions with a reduced frequency, but increased space localization.

Additive clustering of weighted Legendre kernels – the weights are usually said to define the Legendre symbol – generates zonal kernel functions. The uncertainty principle (see Chapter 7) describes a trade-off between two 'spreads' of the zonal kernels, one for the space and the other for the frequency. The main statement is that sharp localization of zonal kernels in space and in frequency is mutually exclusive. The reason for the validity of the uncertainty relation is that the aforementioned operators $o^{(1)}$ and $o^{(3)}$ do not commute. Thus, $o^{(1)}$ and $o^{(3)}$ cannot be sharply defined simultaneously. As already mentioned, extremal members in the space/frequency (momentum) relation are the Legendre kernels and the Dirac kernels (see Table 1.6). More explicitly, the uncertainty principle allows us to give a

Table 1.6: From Legendre kernels via zonal kernels to the Dirac kernel.

Legendre kernels	zonal kernels		Dirac kernel
	general case		
	bandlimited	spacelimited	

quantitative classification in the form of a canonically defined hierarchy of the space/frequency localization properties of zonal kernel functions, be they of scalar, vectorial, or tensorial nature. For simplicity, restricting ourselves to scalar zonal kernels of the form

$$K(\xi \cdot \eta) = \sum_{k=0}^{\infty} \frac{2n+1}{4\pi} K^{\wedge}(n) P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega \tag{1.1}$$

(with $K^{\wedge}(n), n = 0, 1, \ldots,$ being the symbol of the kernel K), we are led to the following conclusion: In view of the amount of space/frequency (momentum) localization, it is remarkable to distinguish bandlimited kernels (i.e., $K^{\wedge}(n) = 0$ for all $n \geq N$) and non-bandlimited ones, for which infinitely many numbers $K^{\wedge}(n)$ do not vanish. Non-bandlimited kernels show a much stronger space localization than their bandlimited counterparts. Empirically, if $K^{\wedge}(n) \approx K^{\wedge}(n+1) \approx 1$ for many successive large integers n, then the support of the series (1.1) in the space domain is small, i.e., the kernel is spacelimited (i.e., in the jargon of approximation theory, locally supported). Assuming the condition $\lim_{n\to\infty} K^{\wedge}(n) = 0$, we are confronted with the situation that the slower the sequence $\{K^{\wedge}(n)\}_{n=0,1,\ldots}$ converges to zero, the lower is the frequency localization, and the higher is the space localization.

Our considerations lead us to the following characterization of trial functions in constructive approximation: Fourier expansion methods with polynomial ansatz functions offer the canonical ‘trend-approximation’ of low-frequent phenomena (for global modeling), while bandlimited kernels can be used for the transition from long-wavelength to short-wavelength phenomena (global to local modeling). Because of their excellent localization properties in the space domain, the non-bandlimited kernels can be used for the modeling of short-wavelength phenomena (local modeling). Using kernels of different scales reflecting the different stages of space/frequency localization (see, e.g., W. Freeden (1998), W. Freeden, V. Michel (1999) and the references therein), the modeling process can be adapted to the localization properties of the physical phenomena (see Table 1.7).

Table 1.7: Multiscale expansion of scalar (square-integrable) spherical functions F.

Sequence of scale-dependent zonal kernels (i.e., scaling functions) Φ_j	$\downarrow$	convolutions against Φ_j
low-pass filtered versions of F $(\Phi_j * F)(\xi) = \int_\Omega \Phi_j(\xi \cdot \eta) F(\eta)\ d\omega(\eta), \quad \xi \in \Omega$		
continuous 'summation' over positions $\eta \in \Omega$	$\downarrow$	'zooming in' ($\Phi_j \to \delta$ as $j \to \infty$)
multiscale expansion of F involving a Dirac family of zonal scalar kernels $F(\xi) = \lim_{j\to\infty} \int_\Omega \Phi_j(\xi \cdot \eta) F(\eta)\ d\omega(\eta), \xi \in \Omega$		

In case of so-called scaling functions, the width of the corresponding frequency bands and, consequently, the amount of space localization is controlled (in continuous and/or discrete way) using a so-called scale-parameter, such that the Dirac kernel acts as limit kernel as the scale-parameter takes its limit. Typically, the generating kernels of scaling functions have the characteristics of low-pass filters, i.e., the zonal kernels *involved* in the convolution of the field against the Legendre kernels are significantly based on low frequencies, while the higher frequencies are attenuated or even completely left out in the summation. Conventionally, the difference between successive members in a scaling function is called a wavelet function. Clearly, it is again a zonal kernel. In consequence, wavelet functions have the typical properties of band-pass filters, i.e., the weighted Legendre kernels of low and high frequency within the wavelet kernel are attenuated or even completely left out. According to their particular construction, wavelet-techniques provide a decomposition of the reference space into a sequence of approximating subspaces – the scale spaces – corresponding to the scale parameter. In each scale space, a filtered version of a spherical field under consideration is calculated as a convolution of the field against the respective member of the scaling function and, thus, leading to an approximation of the field at certain resolutions. For increasing scales, the approximation improves and the information obtained on coarse levels is usually contained in foregoing levels. The difference between two successive bandpass filtered version of the signal is called the detail information and is collected in the so-called detail space. The wavelets constitute the basis functions of the detail spaces and, summarizing our excursion to multiscale modeling, every element of the reference space can be represented as a structured linear combination of scaling functions and wavelets corresponding to different scales and at different positions. That is, using scaling functions und wavelets at different scales, the corresponding multiscale technique can be constructed as to be suitable for the specific local field structure. Consequently, although most fields show a correlation in space as well as in frequency, the zonal kernel functions with their simultaneous space and frequency localization allow for the efficient detection and approximation of essential features by only using fractions of the original information (decorrelation).

The Tables 1.7, 1.8, and 1.9 bring together, into a unified nomenclature, the formalisms for zonal kernel function theory based on the following principles:

- Weighted Legendre kernels are the constituting summands of zonal kernel functions.
- The only zonal kernel that is both band- and spacelimited is the trivial kernel; the Legendre kernel is ideal in frequency localization, the Dirac kernel is ideal in space localization.
- The convolution of a field (signal) against a zonal kernel function provides a filtered version of the original.
- Scaling kernels, i.e., certain sequences of (parameter-dependent) zonal kernels tending to the Dirac kernel, provide better and better approximating low-pass filtered versions of the field (signal) under consideration.

To summarize, the theory of zonal kernels as presented in this book (see Chapters 7, 8, and 9) is a unifying attempt of reviewing, clarifying and supplementing the different additive clusters of weighted Legendre kernels. The kernels exist as bandlimited and non-bandlimited, spacelimited, and non-spacelimited variants. The uncertainty principle determines the frequency/ space window for approximation. A fixed space window is used for the windowed Fourier transform of fields (signals), where the approximation is still taken over the frequencies. The power of the scaling function

Table 1.8: Interrelations between space and frequency localization, kernel type, correlation, integral transform and resolution.

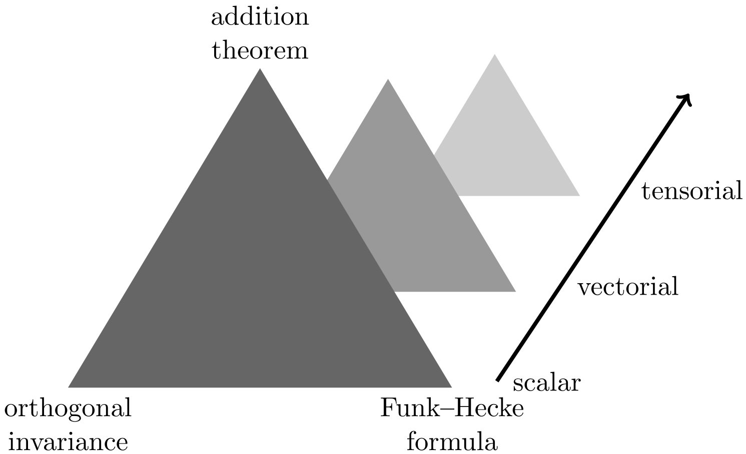
Space localization	no space localization			ideal space localization
frequency localization	ideal frequency localization			no frequency localization
kernel type	Legendre kernel	bandlimited	locally supported	Dirac kernel
correlation	ideal correlation			no correlation
integral transform	Fourier	windowed Fourier		wavelet
resolution	low			high

lies in the fact that zonal kernels with a variable (space localizing) support come into use. The multiscale transform using scaling (kernel) functions is a space-reflected replacement of the Fourier transform, however, giving the dynamical space-varying frequency distribution of a field. Due to the possibility that variable kernel functions (i.e., scaling functions as sequential space localizing reductions) are being applied, a substantial better modeling of the high-frequency 'short wavelength' part of a field (signal) is possible. This finally amounts to the transition from global to (scale-dependent) local approximation (including multiresolution by spherical wavelets).

1.2 Layout

Chapter 2 gives an introduction into spherical nomenclature and settings. Fundamental results of spherical vector analysis are recapitulated. Orthogonal invariance is explained within the scalar, vectorial, and tensorial concept (see Table 1.9).

Table 1.9: The fundamentals of the book.



In Chapter 3, the scalar surface theory of spherical harmonics is formulated based on the work of C. Müller (1952, 1966) and W. Freeden (1979a); W. Freeden (1980b). Important ingredients are the addition theorem of spherical harmonics and the formula of Funk and Hecke. The closure and completeness of scalar spherical harmonics in the space of square-integrable functions is shown by Bernstein or Abel-Poisson summability. Exact generation of linearly independent systems of homogeneous harmonic polynomials only by integer operations is investigated briefly. Fourier (orthogonal) expansions are discussed, (the energy of) a square-integrable function (signal) is split into degree variances in terms of spherical harmonics. The scalar spherical harmonics are recognized to be eigenfunctions of the scalar Beltrami operator on the (unit) sphere. The Legendre polynomial is identified as the only scalar spherical harmonic invariant under orthogonal transformations. Zonal, tesseral, and sectorial spherical harmonics, i.e., associated Legendre harmonics, are introduced by use of associated Legendre functions. Scalar angular derivatives are seen to produce anisotropic operators within the scalar framework.

Chapter 4 presents the theory of Green functions with respect to the scalar Beltrami operator (as proposed by W. Freeden (1979a); W. Freeden (1980b, 1981a)). Its definition is given by formulating four constituting properties, i.e., the Beltrami differential equation relating the Green function to the Dirac function(al), the characteristic logarithmic singularity, the rotational symmetry, and a certain normalization condition to assure uniqueness. Integral formulas are formulated that enable us to estimate the error between a (sufficiently smooth) function and its truncated orthogonal expansion in terms of scalar spherical harmonics. Integral expressions are deduced which act as solutions of the equations involving surface gradient, surface curl gradient, and (iterated) Beltrami differential operators. The results on Green functions are meant to be the preparatory material for decomposition theorems of spherical vector and tensor fields, respectively, in accordance with the Helmholtz approach. Iterated Beltrami equations are solved by integral expressions involving Green functions.

In Chapter 5, the vector theory of spherical harmonics is developed in consistency with its scalar counterpart (based on the work T. Gervens (1989), W. Freeden, T. Gervens (1989, 1991), W. Freeden et al. (1998)). A particular role is played by the Helmholtz decomposition theorem which separates a spherical vector field into three field components, namely a radial part, a tangential divergence-free, and a tangential curl-free part. As already pointed out, an essential tool for representing a spherical vector field is the Green function with respect to the Beltrami operator. The physical background for the Helmholtz decomposition is based on well-known facts of surface vector analysis, viz. the existence of surface potentials and stream functions, and the characterization of tangential vector fields such as surface (curl) gradient fields. To be more concrete, the surface gradient field on the sphere is seen to be generated by a potential function, while the surface curl gradient field is canonically related to a stream function. Vectorial analogues of the Legendre polynomials are introduced, their properties are analyzed in detail. Outstanding keystones in the vectorial framework of vector spherical harmonics are the addition theorem and the formulas of Funk and Hecke. The closure and completeness of vector spherical harmonics for the space of square-integrable vector fields is shown via Bernstein summability. Two different ways of expanding square-integrable fields in terms of (an orthonormal system of) vector spherical harmonics are described alternatively based on a (one-step) tensor-vector multiplication or on a consecutive (two-step) vector-scalar and scalar-vector multiplication.

Chapter 6 deals with the theory of tensor spherical harmonics (in close orientation to M. Schreiner (1994), W. Freeden et al. (1994, 1998)). All essential results known from the scalar and vectorial approach are extended to the tensor case. Orthonormal tensor spherical harmonics are introduced in the space of square-integrable tensor fields on the unit sphere. In particular, the addition theorem for tensor spherical harmonics is formulated and the decomposition theorem for spherical tensor fields is verified by use of the Green function with respect to iterations of the Beltrami operator. The tensor spherical harmonics are characterized as eigenfunctions of a tensorial analogue of the Beltrami operator. Alternative approaches to tensor spherical harmonics are studied. Tensorial versions of the Funk–Hecke formula are described in more detail.

Chapter 7 presents the mathematical classification of zonal kernel functions. The verification and interpretation of an uncertainty principle for fields (with second distributional derivatives) on the the unit sphere is the essential tool for the classification. Frequency as well as space localization are formulated by means of the expectation value and the variance of the surface curl gradient and the radial projection operator, respectively. The results obtained by certain tools of spherical vector analysis are used for a large class of band/spacelimited and non-band/spacelimited zonal kernel functions. The particular role of the Legendre kernel and the Dirac kernel is pointed out. The series expansions of vector/tensor zonal kernel functions in terms of (zonal) Legendre kernels are indicated by the specification of their symbols. All representations are coordinate-free.

Chapter 8 considers two different ways of generating vectorial and tensorial zonal kernel functions (cf. H. Nutz (2002)). In particular, scale-dependent bandlimited and non-bandlimited zonal kernel functions are listed such that the scale parameter acts as regulation for the amount of space/frequency localization. The Funk-Hecke formulas enable us to establish filtered versions of spherical fields by forming convolutions. The sequences of zonal kernel functions tending to the Dirac kernel, i.e., the so-called scaling functions, provide a 'zooming in' approximation of square-integrable fields from global to local features under (geophysically) constraints.

Chapter 9 presents the concept of tensorial zonal kernel functions. Their description is given in parallel to the vectorial case. Particular emphasis is laid on tensor scaling functions.

Finally, Chapter 10 is an application of our spherically oriented approach to geoscientifically relevant gravitation. The essential goal is to present the mathematical concepts, structures, and tools for the understanding of mass balance and mass transport seen in the closely interrelated Earth's gravity field. The key observables in gravitational field determination such as gravity anomalies, gravity disturbances, geoidal undulations, deflections of the vertical, dynamic ocean topography etc are mathematically characterized, both in terms of spherical harmonics and zonal kernel functions. The problems of determining the (geostrophic) ocean circulation, the elastic field from ground displacements, and the density distribution inside the Earth are studied in more detail. Finally, vector and tensor outer harmonic zonal kernels are shown to be the adequate means for 'downward continuation' of vectorial and tensorial gravitational data from satellite orbits to the Earth's surface.

A brief view over the contents of the chapters of this book is given in Table 1.10.

Table 1.10: Contents (in brief).

	Scalar framework	**Vector framework**	**Tensor framework**
Basic settings (differential operators, orthogonal invariance)	Chapter 2	Chapter 2	Chapter 2
Green's functions, integral theorems	Chapter 4		
Spherical harmonics (definition, Legendre functions, addition theorems, Funk–Hecke formulas)	Chapter 3	Chapter 5	Chapter 6
Zonal kernel Functions (definition, classification, scaling functions, Dirac kernel)	Chapter 7	Chapter 8	Chapter 9
Applications (mass distribution interrelated to gravity field quantities)	Chapter 10	Chapter 10	Chapter 10