1
Overview

1.1 Historical and phenomenological aspects

The Schrödinger equation was invented at a time when electrons, protons and neutrons were considered to be the elementary particles. It was extremely successful in what is now called atomic and molecular physics, and it has been applied with great success to baryons and mesons, especially those made of heavy quark–antiquark pairs.

While before World War II approximation methods were developed in a heuristic way, it is only during the post-war period that rigorous results on the energy levels and the wave functions have been obtained and these approximation methods justified. Impressive global results, such as the proof of the 'stability of matter', were obtained as well as the properties of the two-body Hamiltonians including bounds on the number of bound states. The discovery of quarkonium led to a closer examination of the problem of the order of energy levels from a rigorous point of view, and a comparison of that order with what happens in cases of accidental degeneracy such as the Coulomb and harmonic oscillator potentials. Comparison of these cases also leads to interesting results on purely angular excitations of two-body systems.

Who among us has not written the words 'Schrödinger equation' or 'Schrödinger function' countless times? The next generation will probably do the same, and keep his name alive.

Max Born

Born's prediction turned out to be true, and will remain true for atomic and molecular physics, and — as we shall see — even for particle physics.

When Schrödinger found his equation, after abandoning the relativistic version (the so-called Klein–Gordon equation) because it did not agree with experiments, there was no distinction between atomic, nuclear and particle physics. The wonderful property of the Schrödinger equation is that it can be generalized to many-particle systems and, when combined with the Pauli principle, allows one to calculate, any atom, any molecule, any crystal, whatever their size — at least in principle. The Dirac equation, as beautiful as it may be, is a one-particle equation, and any attempt to generalize it to N-particle systems will have severe limitations and may lead to contradictions if pushed too far, unless one accepts working in the broader framework of quantum field theory.

Because of the capacity of the Schrödinger equation for treating N-body systems it is not astonishing that in the period before World War II all sorts of approximation methods were developed and used, such as the Thomas–Fermi approximation, the Hartree and Hartree–Fock approximations, and the Born–Oppenheimer approximation.

However, except for the fact that it was known that variational trial functions gave upper bounds to the ground-state energies of a system (together with the less well-known min-max principle, which allows one to get an upper bound for the n-th energy level of a system), there was no serious effort to make rigorous studies of the Schrödinger equation. Largely under the impulsion of Heisenberg, simple molecules and atoms were calculated, making chemistry, at least in simple cases, a branch of physics. Also, as was pointed out by Gamow, the Schrödinger equation could be applied to nuclei, which were shown by Rosenblum using an $\propto$ spectrometer to have discrete energy levels.

It was not until after World War II that systematic studies of the rigorous properties of the Schrödinger equation were undertaken. In the 1950s Jost [1], Jost and Pais [2], Bargmann [3], and Schwinger [4] and many others obtained beautiful results on the two-body Schrödinger equation. Then N-body systems were studied, and we shall single out the most remarkable success, namely the proof of the 'stability of matter', first given by Dyson and Lenard [5] and then simplified and considerably improved in a quantitative way by Lieb and Thirring [6], and which is still subject to further study [7]. 'Stability of matter' would be better called the extensive character of the energy and volume of matter: i.e., the fact that NZ electrons and N nuclei of charge Z have a binding energy and occupy a volume proportional to N. Other systems whose behaviour has been clarified in this period are those of particles in pure gravitational interaction [8, 9]. These latter systems do not exhibit the above-mentioned 'stability'; the absolute value of binding energy grows like a higher power of N.

In particle physics, during the 1960s, it seemed that the Schrödinger equation was becoming obsolete, except perhaps in calculating the energy levels of muonic or pionic atoms, or the medium-energy nucleon–nucleon scattering amplitude from a field theoretical potential [10]. It was hoped that elementary particle masses could be obtained from the bootstrap mechanism [11], or with limited but spectacular success from symmetries [12].

When the quark model was first formulated very few physicists considered quarks as particles and tried to calculate the hadron spectrum from them. Among those who did we could mention Dalitz [13], almost 'preaching in the desert' at the Oxford conference in 1965, and Gerasimov [14]. The situation changed drastically after the '1974 October Revolution'. As soon as the J/ψ [15, 16] and the ψ' [17] had been discovered, the interpretation of these states as charm–anticharm bound states was universally accepted and potential models using the Schrödinger equation were proposed [18, 19].

In fact, the whole hadron spectroscopy was reconsidered in the framework of the quark model and QCD in the crucial paper of De Rujula, Georgi and Glashow [20], and the independent papers of Zeldovitch and Sakharov, Sakharov [21], and Federman, Rubinstein and Talmi [22]. Impressive fits of baryon spectra (including those containing light quarks) were obtained, in particular by Stanley and Robson [23, 24], Karl and Isgur [25], Richard and Taxil [26], Ono and Schöberl [27], and Basdevant and Boukraa [28].

We would like to return now to the case of quarkonium — i.e., mesons made of a heavy quark–antiquark pair. By heavy quark, we mean the c and b quarks of effective masses $\sim$ 1.8 and 5 GeV, and also the strange quark, for which the effective mass turns out to be 0.5 GeV. The strange quark occupies a borderline position and can be considered either as heavy, as it is here, or light, as in SU_3 flavour symmetry. In this list one would like to include the top quark, which is certainly heavier than 131 GeV, from the DO experiment [29]. From fits of experimental results by the standard model, including, for instance, masses and widths of the W and Z^0 particles and their partial decays as well as low-energy neutrino experiments (with a standard Higgs), the top quark was predicted to have a mass larger than 150 GeV [30]. Since its mass is heavier than 110 GeV, the notion of toponium becomes doubtful, because the width due to single quark decay, $t \to b + w$, exceeds the spacing between the 1S and the 2S states [31]. In fact the existence of the top quark, is now established with a mass of 175 ± 9 GeV [32].

Figures 1.1 and 1.2 give a summary of the experimental situation for the $c\bar{c}$ (J/ψ *etc.*) and $b\bar{b}$ (Υ *etc.*) bound states, respectively.

There are, of course, many potential models used to describe the $c\bar{c}$ and $b\bar{b}$ spectra. The first was a potential

$$V = -\frac{a}{r} + br, \tag{1.1}$$

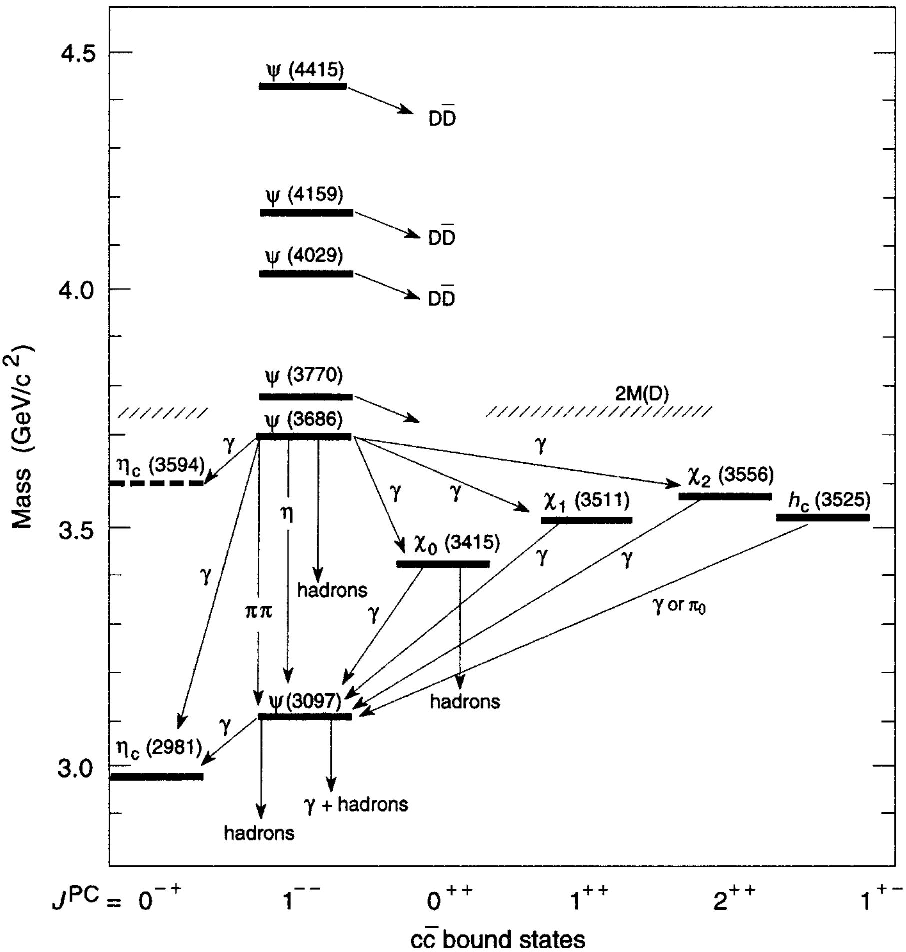

Fig. 1.1. Experimental data on $c\bar{c}$ bound states.

in which the first term represents a one-gluon exchange, analogous to a one-photon exchange, and the second, confinement by a kind of string.

We shall restrict ourselves to two extreme cases of fits. The first, by Buchmüller *et al.* [33], is a QCD-inspired potential in which asymptotic freedom is taken into account in the short-distance part of the potential. The second is a purely phenomenological fit [34] that one of us (A.M.) made with the central potential

$$V = A + Br^{\alpha}. \tag{1.2}$$

Figure 1.3 represents the excitation energies of the $c\bar{c}$ and $b\bar{b}$ systems. The full lines represent the experimental results (for the triplet P states we give only the spin-averaged energies). The dashed lines represent the

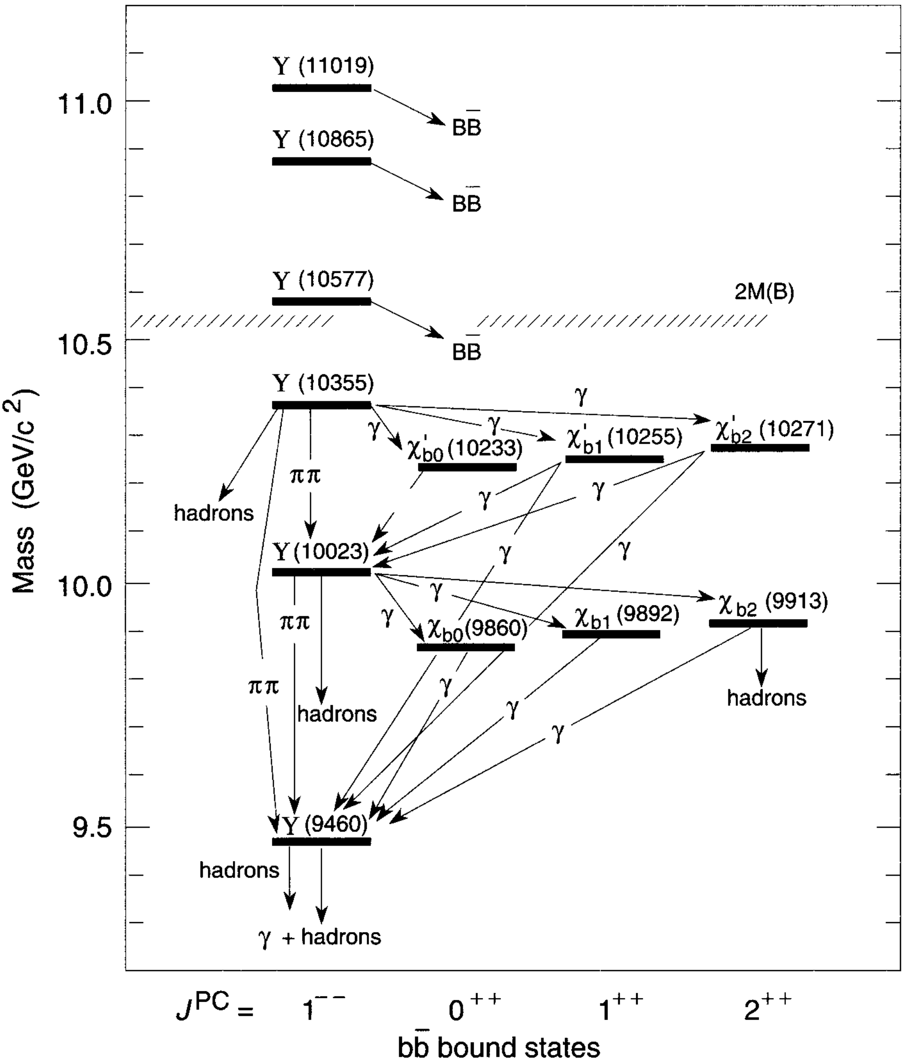

Fig. 1.2. Experimental data on $b\bar{b}$ bound states.

Buchmüller result and the dotted lines result from the potential of Eq. (1.2), to which a zero-range spin–spin interaction $C\delta^3(x)(\vec{\sigma}_1 \cdot \vec{\sigma}_2)/m_1 m_2$ has been added, where m_1 and m_2 are the quark masses, and C was adjusted to the $J/\psi - \eta_c$ separation. The central potential is given by Ref. [34]:

$$V = -8.064 + 6.870 r^{0.1}, \tag{1.3}$$

where the units are powers of GeV, and quark masses $m_b = 5.174$, $m_c = 1.8$ and, as we shall see, $m_s = 0.518$. The smallness of the exponent, $\alpha = 0.1$, means that we are very close to a situation in which the spacing of energy

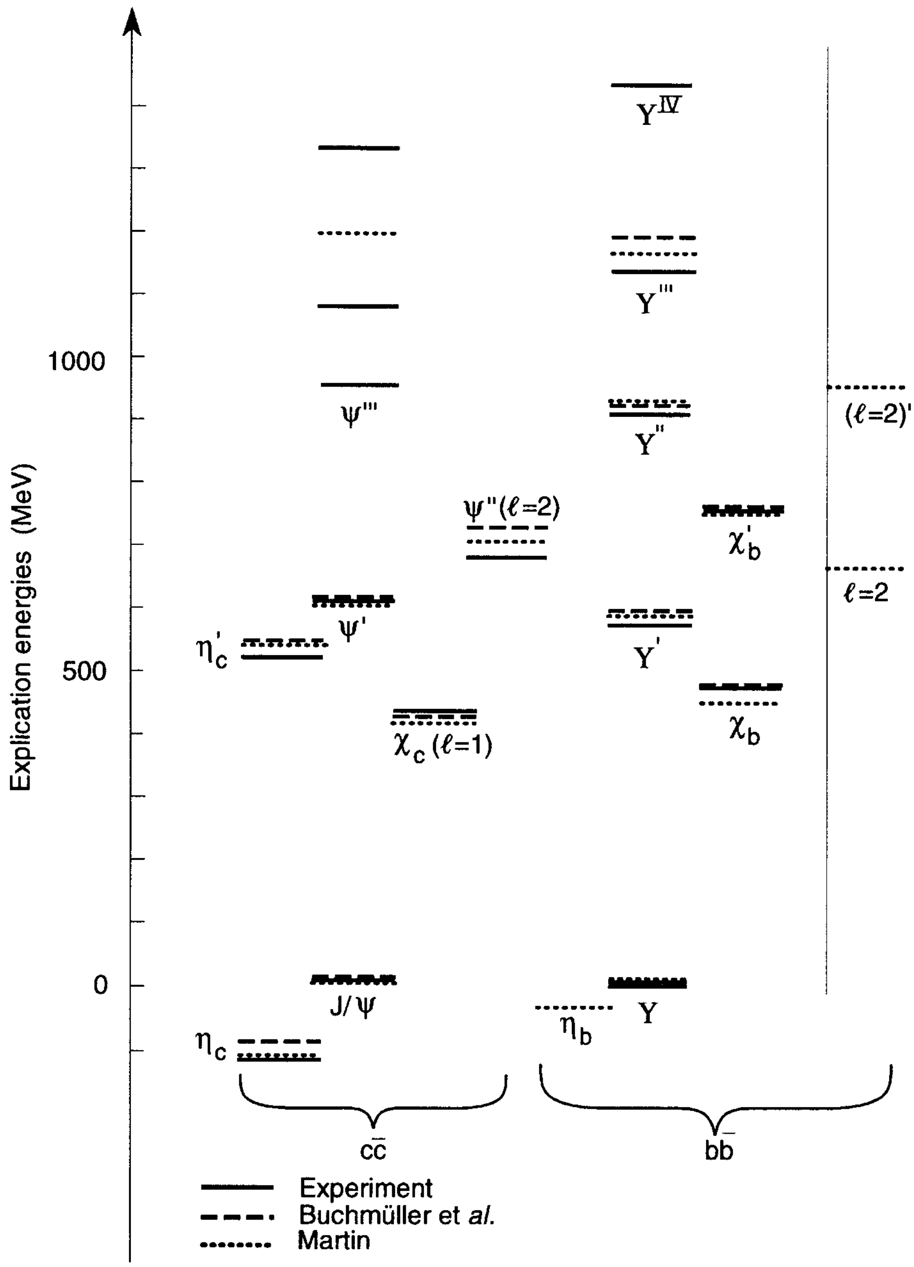

Fig. 1.3. Comparison of the excitation energies of the $c\bar{c}$ and $b\bar{b}$ systems with two theoretical models.

Table 1. Relative leptonic widths.

	Experiment	Buchmüller	Martin
ψ'	0.46 ± 0.6	0.46	0.40
ψ'''	0.16 ± 0.02	0.32	0.25
Υ'	0.44	0.44	0.51
Υ''	0.33	0.32	0.35
Υ'''	0.20	0.26	0.27

levels is independent of the mass of the quarks and the case for a purely logarithmic potential.

Table 1 represents the relative leptonic widths — i.e., the ratios of the leptonic width of a given $\ell = 0$ state to the leptonic width of the ground state. 'Theory' uses the so-called Van Royen-Weisskopf formula.

We see that both the fits are excellent. The QCD-inspired fit reproduces somewhat better the low-energy states, in particular the separation between the $\ell = 0$ and $\ell = 1$ states for the $b\bar{b}$ system. This is presumably due to the fact that the QCD-inspired potential has a correct short-range behaviour while the phenomenological potential has not (there is a discrepancy of 40 MeV, which would have been considered negligible before 1974, but with the new standards of accuracy in hadron spectroscopy can no longer be disregarded). On the other hand, the phenomenological potential gives a better fit for higher excitations, those close to the dissociation threshold into meson pairs $D\bar{D}$, $B\bar{B}$. This may be due to the fact that the optimal $\alpha = 0.1$ takes into account the lowering of the energies of confined channels $c\bar{c}$, $b\bar{b}$, due to their coupling to open channels.

In the list of parameters of the phenomenological potential, we have already indicated the strange-quark effective mass $m_s = 0.518$ GeV. This is because, following the suggestion of Gell-Mann, we have pushed, the phenomenological model beyond its limit of validity! Remarkably, one gets a lot of successful predictions. $M_\phi = 1.020$ GeV is an input but $M_{\phi'} = 1.634$ GeV agrees with experiment (1.650 GeV). At the request of De Rujula the masses of the $c\bar{s}$ states have also been calculated. One gets

$$M_{D_s} = 1.99 \quad (\text{exp } 1.97, \text{previously } 2.01)$$

$$M_{D_s^*} = 2.11 \quad (\text{exp } 2.11)\,,$$

and in 1989 Argus [35] observed what is presumably a $\ell = 1$ $c\bar{s}$ state, which could be $J^P = 1^+$ or 2^+ of mass 2.536 GeV. The spin-averaged mass of such a state was calculated, without changing any parameter of the model, and

$$M_{D_s^{**}}\ (\ell = 1) = 2.532$$

was obtained [36]. One could conclude that the state observed by Argus is no more than 30 MeV away from the centre of gravity.

More recently, a B_s meson was observed, both at LEP and at Fermilab. The least square fit to the mass turns out to be 5369 ± 4 MeV [37], while the theoretical prediction of the model is 5354–5374 MeV [38]. It is impossible not to be impressed by the success of these potential models. But why are they successful? The fact that the various potentials work is understood: different potentials agree with each other in the relevant range of distances, from, say, 0.1 fermi to 1 fermi. However, relativistic effects are not small; for the $c\bar{c}$ system, v^2/c^2 is calculated *a posteriori* to be of the order of $1/4$.

The sole, partial explanation we have to propose is that the potential is simply an effective potential associated with an effective Schrödinger equation. For instance, one can expand $\sqrt{p^2 + m^2}$, the relativistic kinetic + mass energy, around the average $\langle p^2 \rangle$ instead of around zero. For a purely logarithmic potential the average kinetic energy is independent of the excitation, and it happens that the potential is not far from being logarithmic. Anyway, we must take the pragmatic attitude that potential models work and try to push the consequences as far as we can.

Concerning baryons, we shall be more brief. Baryons made purely of heavy quarks, such as *bbb* and *ccc*, have not yet been found, though they must exist. Bjorken [39] advocates the study of *ccc*, which possesses remarkable properties: it is stable against strong interactions and has a lifetime which is a fraction of 10^{-13} seconds. Its lowest excitations are also stable or almost stable. If one accepts that the quark–quark potential inside a baryon is given by [40]

$$V_{QQ} = \frac{1}{2} V_{Q\bar{Q}}\,, \tag{1.4}$$

one can calculate all the properties of *ccc* from a successful $c\bar{c}$ potential. Bjorken thinks that such a state can be produced at a rate not-too-small to be observed.

In the meantime we should remember that the strange quark can be regarded as heavy. J.-M. Richard, using the fit (1.3) of quarkonium and rule (1.4), has obtained a mass for the Ω^- baryon *sss* of 1.665 GeV [41] while experiment [42] gives 1.673 GeV.

For baryons made of lighter quarks, following the pioneering work of Dalitz came the articles of De Rujula, Georgi, Glashow [20], Zeldovitch and Sakharov, Sakharov alone [21], and Federman, Rubinstein and Talmi [22]. In these works, the central potential is taken to be zero or constant

Table 2. Masses for $V = A + B\tau^{0.1}$.

	Theory	Experiment
N	input	0.939
Δ	input	1.232
$\Lambda^\bullet$	1.111	1.115
Σ	1.176	1.193
Ξ	1.304	1.318
Σ^*	1.392	1.383
Ξ^*	1.538	1.533
Ω^-	input	1.672
Λ_c	input	2.282
Σ_c	2.443	2.450
Σ_c^*	2.542	
$\Xi_c = \mathrm{A}$	2.457	2.460
S	2.558	
S*	2.663	

— i.e., incorporated in the quark masses and the dominant feature is given by the spin–spin forces 'derived' from QCD, which lead to remarkable results, in particular the first explanation of the $\Sigma - \Lambda$ mass difference. In this approach, which is zero order in the central potential, the calculation of excited states is excluded.

The next step is to add a soft central potential and try to solve accurately the three-body Schrödinger equation. This has been done by many people. Stanley and Robson [24] were among the first, and Karl, Isgur, Capstick and collaborators [25, 43] were among the most systematic.

Here we would like to limit ourselves to the study of ground states, which has been done, for instance, by Ono and Schöberl [27] and Richard and Taxil [26]. For example, we would like to show, in Table 2, the results of Richard and Taxil with a potential $V = A + Br^{0.1}$ and a spin–spin Hamiltonian

$$C\,\frac{\vec{\sigma}_i\vec{\sigma}_j}{m_i m_j}\,\delta\left(\vec{r}_i - \vec{r}_j\right) \,. \tag{1.5}$$

Although the results are nice, it is not completely obvious whether the rigorous treatment of the central potential does lead to a real improvement. To demonstrate this, we have taken some ratios, which in the De Rujula, Georgi and Glashow model [20] have simple values, and have shown in Table 3 a comparison of the calculated and experimental values.

Perhaps it is worth noting that the equal-spacing rule of the SU_3 flavour

Table 3. Ratio of mass differences including the Gell-Man–Okubo predictions (G.M.O) compared to experiment.

	De Rujula Georgi Glashow Sakharov Zeldovitch Federman Rubinstein Talmi	Richard Taxil	Experiment
$(M_{\Xi^*} - M_{\Xi})/(M_{\Sigma^*} - M_{\Sigma})$	1	1.08	1.12
$(2M_{\Sigma^*} + M_{\Sigma} - 3M_{\Lambda})/2\,(M_{\Delta} - M_{N})$	1	1.07	1.05
$(3M_{\Lambda} + M_{\Sigma})/(2M_{N} + 2M_{\Xi})$ GMO OCTET	1	1.005	1.005
$(M_{\Sigma^*} - M_{\Delta})/(M_{\Xi^*} - M_{\Sigma^*})$ GMO	1	1.10	1.03
$(M_{\Xi^*} - M_{\Sigma^*})/(M_{\Omega^-} - M_{\Xi^*})$ DECUPLET	1	1.09	1.08

decuplet, which was the triumph of Gell-Mann, enters here in a rather unusual way. Naturally, if the spin–spin forces and the central potential are neglected, the equal-spacing rule is absolutely normal, since the mass of the state is obtained by merely adding the quark masses: therefore the mass is a linear function of strangeness.

However, Richard and Taxil [44] discovered by numerical experiments that if one takes a 'reasonable', flavour-independent, two-body central potential the masses of the decuplet are concave functions of strangeness. In other words,

$$M\,(ddd) + M\,(dss) < 2M\,(dds)\,. \tag{1.6}$$

1.2 Rigorous results

This is perhaps a good point to turn to the main part of the book, which concerns rigorous results on the Schrödinger equation stimulated by potential models.