2

Renewal Processes and Random Walks

2.1 Introduction

In the first chapter we stated and proved various limit theorems for stopped random walks. These limit theorems shall, in subsequent chapters, be used in order to obtain results for random walks stopped according to specific stopping procedures as well as for the families of stopping times (random indices) themselves. However, before doing so we shall, in this chapter, survey some of the basic facts about random walks.

Our emphasis will be on that part of the theory which is most relevant for this book. Classical fluctuation theory, the combinatorial formulas, Wiener–Hopf factorization etc. are therefore excluded in our presentation; we refer the reader to the existing literature cited below. We furthermore assume that the reader already has some familiarity with much of the material, so its character will rather be a review than a through exposition. As a consequence proofs will not always be given; in general only when they are short or in the spirit of the present treatise.

We begin our survey by considering an important class of random walks which has attended special interest; the class of *renewal processes*. Their special feature is that they are concentrated on $[0, \infty)$, that is, the steps are non-negative. A similar theory exists, of course, for random walks concentrated on $(-\infty, 0]$. Sections 2.2–2.7 are devoted to the study of renewal processes.

In the remaining sections of the chapter we treat random walks on $(-\infty, \infty)$ (and such that they are not concentrated on either half axis). We give a characterization of the three possible kinds of random walks, introduce ladder variables, the partial maxima and present some general limit theorems.

We close this section by introducing some notation. Throughout $(\Omega, \mathscr{F}, P)$ is a probability space on which everything is defined, $\{S_n, n \geq 0\}$ is a random walk with i.i.d. increments $\{X_k, k \geq 1\}$ and $S_0 = 0$, or, equivalently, $\{X_k, k \geq 1\}$ is a sequence of i.i.d. random variables with partial sums $S_n = \sum_{k=1}^{n} X_k, \ n \geq 0$. We let F denote the (common) distribution of

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the increments (summands). To avoid trivialities we assume throughout that $P(X_1 \neq 0) > 0$.

Depending on the support of the distribution function F we shall distinguish between two kinds of random walks (renewal processes). We say that the random walk (renewal process) is *arithmetic* if F has its support on $\{0, \pm d, \pm 2d, \dots\}$ ($\{0, d, 2d, \dots\}$) for some $d > 0$. The largest d with this property is called the *span*. A random walk or renewal process which is *arithmetic with span d* will also be called *d-arithmetic*. If F is not of this kind (for any d) we say that the random walk or renewal process is *nonarithmetic*.

2.2 Renewal Processes; Introductory Examples

Example 2.1. Consider some electronic device, in particular, one specific component. As soon as the component breaks down it is automatically and instantly replaced by a new identical one, which, when it breaks down, is replaced similarly etc. Let $\{X_k, k \geq 1\}$ denote the successive lifetimes and set $S_n = \sum_{k=1}^{n} X_k,\ n \geq 0$. With this setup S_n denotes the (random) total lifetime of the n first components. If, for example, $\{X_k, k \geq 1\}$ is a sequence of i.i.d. exponentially distributed random variables, then S_n has a gamma distribution.

In practice, however, it is more likely that one is interested in the *random number of components* required to keep the device alive during a *fixed time interval* rather than being interested in the accumulated *random lifetime* for a *fixed number* of components.

In the exponential case, the stochastic process which counts the number of components that are replaced during fixed time intervals is the well known Poisson process.

Example 2.2. In a chemical substance the movement of the molecules can be modeled in such a way that each molecule is subject to repeated (i.i.d.) random displacements at random times. In analogy with the previous example, rather than being interested in the accumulated displacement of a molecule after a fixed number of steps, it is more natural to consider the location of a molecule at a fixed time point.

Example 2.3 (The Bernoulli random walk). Here $\{X_k, k \geq 1\}$ are i.i.d. Be(p)-distributed random variables, that is, $P(X_k = 1) = 1 - P(X_k = 0) = p$, and $S_n = \sum_{k=1}^{n} X_k,\ n \geq 0$, denotes the random number of "successes" after n trials, which has a Binomial distribution. If, instead, we consider the random number of performances required to obtain a given number of successes we are lead to the Negative Binomial process.

A common feature in these examples thus is that rather than studying the random value of a sum of a fixed number of random variables one investigates the random number of terms required in order for the sum to attain a certain (deterministic) value. For nonnegative summands we are lead to the part of probability theory called *renewal theory* and the summation process $\{S_n, n \geq 0\}$ is called a *renewal process.* Examples 2.1 and 2.3 are exactly of this kind and, under appropriate additional assumptions, also Example 2.2.

We now proceed to give stringent definitions and present a survey of the most important results for renewal processes.

Some fundamental papers on renewal theory are Feller (1949), Doob (1948) and Smith (1954, 1958). The most standard book references are Feller (1968), Chapter XIII for the arithmetic case (where the treatment is in the context of recurrent events) and Feller (1971), Chapters VI and XI for the nonarithmetic case. Some further references are Prabhu (1965), Chapter 5, Cox (1967), Çinlar (1975), Chapter 9, Jagers (1975), Chapter 5 and Asmussen (2003).

2.3 Renewal Processes; Definition and General Facts

Let $\{X_k, k \geq 1\}$ be i.i.d. nonnegative random variables and let $\{S_n, n \geq 0\}$ be the partial sums. The sequence $\{S_n, n \geq 0\}$ thus defined is a *renewal process.*

We further let F denote the common distribution function of the summands and let F_n be the distribution function of $S_n,\ n \geq 0$. Thus

$$F_0(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0, \end{cases} \tag{3.1a}$$

$$F_1(x) = F(x), \tag{3.1b}$$

$$F_n(x) = F^{n*}(x) \qquad (n \geq 1), \tag{3.1c}$$

that is, F_n equals the n-fold convolution of F with itself.

To avoid trivialities we assume throughout that $P(X_1 > 0) > 0$.

The main object of interest in the study of renewal processes (in renewal theory) is the *renewal counting process* $\{N(t), t \geq 0\}$, defined by

$$N(t) = \max\{n\colon S_n \leq t\}. \tag{3.2}$$

An alternative interpretation is $N(t) =$ the number of renewals in $(0, t] =$ $\text{Card}\{n \geq 1\colon S_n \leq t\}$.

Remark 3.1. A case of particular importance is when $F(x) = 1 - e^{-\lambda x},\ x \geq 0$, that is, when the lifetimes are exponentially distributed. In this case $\{N(t), t \geq 0\}$ is a *Poisson process* with intensity λ (recall Example 2.1). We also observe that the renewal process defined in Example 2.1 is nonarithmetic. The renewal process defined in Example 2.3 is, however, arithmetic with span 1 (1-arithmetic).

Remark 3.2. The definition (3.2) is not the only existing definition of the renewal counting process. Some authors prefer $\text{Card}\{n \geq 0\colon S_n \leq t\}$, thus counting $n = 0$ as a renewal.

We are now ready for our first result.

Theorem 3.1.

(i) $P(N(t) < \infty) = 1$;
(ii) $E(N(t))^r < \infty$ *for all* $r > 0$;
(iii) *There exists* $s_0 > 0$ *such that* $Ee^{sN(t)} < \infty$ *for all* $s < s_0$.

Proof. By assumption there exists $x_0 > 0$ such that $P(X_1 \geq x_0) > 0$. Since scaling does not affect the conclusions we may, without loss of generality, assume that $x_0 = 1$. Now, define, for $k \geq 1$,

$$\bar{X}_k = \begin{cases} 0, & \text{if } X_k < 1, \\ 1, & \text{if } X_k \geq 1, \end{cases} \tag{3.3}$$

$\bar{S}_n = \sum_{k=1}^n \bar{X}_k,\ n \geq 0$, and $\bar{N}(t) = \max\{n\colon \bar{S}_n \leq t\}$. Then, clearly, $\bar{X}_k \leq X_k,\ k \geq 1,\ \bar{S}_n \leq S_n,\ n \geq 0$, and, hence, $\bar{N}(t) \geq N(t),\ t \geq 0$.

But, $\{\bar{N}(t), t \geq 0\}$ is the Negative Binomial process from Example 2.3 (cf. also Remark 3.1), for which the theorem is well known, and the conclusions follow. □

An important relation upon which several proofs are based is the inverse relationship between renewal processes and counting processes, namely

$$\{N(t) \geq n\} = \{S_n \leq t\}. \tag{3.4}$$

As an immediate example we have

$$EN(t) = \sum_{n=1}^{\infty} P(N(t) \geq n) = \sum_{n=1}^{\infty} P\{S_n \leq t\} = \sum_{n=1}^{\infty} F_n(t). \tag{3.5}$$

Next we define the *renewal function* $U(t) = \sum_{n=1}^{\infty} F_n(t)$ and conclude, in view of (3.5), that

$$U(t) = \sum_{n=1}^{\infty} F_n(t) = EN(t). \tag{3.6}$$

Remark 3.3. Just as there is no unique way of defining counting processes (see Remark 3.2) there is also some ambiguity concerning the renewal function. If one defines counting processes in such a way that $n = 0$ is also counted as a renewal, then it is more natural to define the renewal function as $\sum_{n=0}^{\infty} F_n(t)$ (for example, in order for the analog of (3.6) to remain true). In fact, some authors *begin* by defining $U(t)$ this way and *then* define $N(t)$ so that $U(t) = EN(t)$, that is, such that $N(t) = \text{Card}\{n \geq 0\colon S_n \leq t\}$.

By using the first equality in (3.6) twice we obtain

$$\begin{aligned} U(t) &= F_1(t) + \sum_{n=1}^{\infty} F_{n+1}(t) = F(t) + \sum_{n=1}^{\infty} (F_n * F)(t) \\ &= F(t) + (U * F)(t), \end{aligned}$$

which is one half of the following theorem.

Theorem 3.2 (The Integral Equation for Renewal Processes). *The renewal function $U(t)$ satisfies the integral equation*

$$U(t) = F(t) + (U * F)(t) \tag{3.7a}$$

or, equivalently,

$$U(t) = F(t) + \int_0^t U(t-s)dF(s). \tag{3.7b}$$

Moreover, $U(t)$ is the unique solution of (3.7), which is bounded on finite intervals.

Remark 3.4. If the renewal process is d-arithmetic, then

$$U(nd) = u_0 + u_1 + \cdots + u_n \qquad (n \geq 0), \tag{3.8a}$$

where

$$u_k = \sum_{j=1}^{\infty} P(S_j = kd) \qquad (k \geq 0). \tag{3.8b}$$

Moreover, with $f_k = P(X_1 = kd),\ k \geq 0$, we obtain the discrete convolution formula

$$u_n = f_n + \sum_{k=0}^{n} u_{n-k} f_k. \tag{3.9}$$

Remark 3.5. By defining indicator variables $\{I_j, j \geq 1\}$ by $I_j = I\{S_j = kd\}$ we note that $\sum_{j=1}^{\infty} I\{S_j = kd\}$ equals the *actual number* of partial sums equal to $kd,\ k \geq 0$, and that u_k equals the *expected number* of partial sums which are equal to $kd,\ k \geq 0$. (Observe that S_0 is not included in the count).

A mathematically important fact is that $N(t)$ *is not a stopping time* (with respect to the renewal process); for the definition of a stopping time we refer to the end of Section A.2. To see this intuitively, we note that we cannot determine whether or not the event $\{N(t) = n\}$ has occurred without looking into the future. It is therefore convenient to introduce the *first passage time process* $\{\nu(t), t \geq 0\}$, defined by

$$\nu(t) = \min\{n\colon S_n > t\}, \tag{3.10}$$

because $\nu(t)$ *is a stopping time* for all $t > 0$. Moreover, it turns out that first passage time processes are the natural processes to consider in the context of renewal theory for random walks (see Chapter 3 below).

We now observe that

$$\nu(t) = N(t) + 1, \tag{3.11}$$

from which it, for example, follows that the conclusions of Theorem 3.1 also hold for the first passage times.

The following formula, corresponding to (3.6) also follows.

$$E\nu(t) = 1 + EN(t) = 1 + \sum_{n=1}^{\infty} F_n(t) = \sum_{n=0}^{\infty} F_n(t). \tag{3.12}$$

However, whereas $0 \leq S_{N(t)} \leq t$, that is, $S_{N(t)}$ has moments of all orders we cannot conclude that $S_{\nu(t)}$ has any moments without further assumptions. We have, in fact,

$$E(S_{\nu(t)})^r < \infty \iff E(X_1)^r < \infty \qquad (r > 0). \tag{3.13}$$

This follows from Theorems 3.1, 1.5.1 and 1.5.2. We give no details, since such an equivalence will be established for general random walks with $EX_1 = \mu > 0$ (and $r \geq 1$) in Chapter 3; see Theorem 3.3.1.

In the following section we present some asymptotic results for renewal counting processes, so-called renewal theorems, which, due to their nature and in view of (3.11), also hold for first passage time processes. In fact, the proofs of some of them are normally first given for the latter processes (because of the stopping time property) after which one uses (3.11).

2.4 Renewal Theorems

Much of the early work on renewal processes was devoted to the study of the renewal function $U(t)$, in particular to asymptotics. In this section we present some of these limit theorems.

Theorem 4.1 (The Elementary Renewal Theorem). *Let* $0 < \mu = EX_1 \leq \infty$. *Then*

$$\frac{U(t)}{t} \to \frac{1}{\mu} \quad as \quad t \to \infty, \tag{4.1}$$

the limit being 0 *when* $\mu = +\infty$.

Proof. Suppose first that $0 < \mu < \infty$. Since $N(t)$ is not a stopping time we consider $\nu(t)$ in order to apply Theorem 1.5.3. It follows that

$$U(t) = EN(t) = E\nu(t) - 1 = \frac{1}{\mu} ES_{\nu(t)} - 1$$

$$= \frac{t}{\mu} + \frac{1}{\mu} E(S_{\nu(t)} - t) - 1,$$

and, hence, that

$$\frac{U(t)}{t} = \frac{1}{\mu} + \frac{E(S_{\nu(t)} - t)}{\mu t} - \frac{1}{t}. \tag{4.2}$$

Since $S_{\nu(t)} - t \geq 0$ we obtain

$$\liminf_{t\to\infty} \frac{U(t)}{t} \geq \frac{1}{\mu}. \tag{4.3}$$

Next we note that

$$S_{\nu(t)} - t \leq S_{\nu(t)} - S_{N(t)} = X_{\nu(t)}. \tag{4.4}$$

Suppose that $P(X_k \leq M) = 1$ for some $M > 0,\ k \geq 1$. Then (4.2) and (4.4) together imply that

$$\frac{U(t)}{t} \leq \frac{1}{\mu} + \frac{M}{t} \tag{4.5}$$

and thus that

$$\limsup_{t\to\infty} \frac{U(t)}{t} \leq \frac{1}{\mu}, \tag{4.6}$$

which, together with (4.3), proves (4.1) for that case.

For arbitrary $\{X_k, k \geq 1\}$ the conclusion follows by a truncation procedure. We define a new renewal process $\{S'_n, n \geq 0\}$, by defining $X'_k = X_k I\{X_k \leq M\},\ k \geq 1$. By arguing as in the proof of Theorem 3.1 we now obtain $U(t) \leq U'(t)$ and, hence, that

$$\limsup_{t\to\infty} \frac{U(t)}{t} \leq \frac{1}{\mu'} = \frac{1}{EX_1 I\{X_1 \leq M\}}. \tag{4.7}$$

The conclusion follows by letting $M \to \infty$.

Finally, if $\mu = +\infty$ we truncate again and let $M \to \infty$. □

The following result is a refinement of Theorem 4.1. The arithmetic case is due to Kolmogorov (1936) and Erdős, Feller and Pollard (1949) and the nonarithmetic case is due to Blackwell (1948).

Theorem 4.2.

(i) *For nonarithmetic renewal processes we have*

$$U(t) - U(t-h) \to \frac{h}{\mu} \quad as \quad t \to \infty. \tag{4.8}$$

(ii) *For d-arithmetic renewal processes we have*

$$u_n = \sum_{k=1}^{\infty} P(S_k = nd) \to \frac{d}{\mu} \quad as \quad n \to \infty. \tag{4.9}$$

The limits are 0 *when* $\mu = +\infty$.

Theorem 4.2 is closely related to the integral equation (3.7). The following theorem gives a connection.

Theorem 4.3 (The Key Renewal Theorem).

(i) *Suppose that the renewal process is nonarithmetic. If* $G(t),\ t \geq 0$, *is a bounded, nonnegative, nonincreasing function, such that* $\int_0^\infty G(t)dt < \infty$, *then*

$$\int_0^t G(t-s)dU(s) \to \frac{1}{\mu}\int_0^\infty G(s)ds \quad \text{as} \quad t \to \infty. \tag{4.10}$$

(ii) *Suppose that the renewal process is d-arithmetic. If* $G(t),\ t \geq 0$, *is nonnegative and* $\sum_{n=0}^\infty G(nd) < \infty$, *then*

$$\sum_{k=0}^{n} G(nd-kd)u_{kd} \to \frac{d}{\mu}\sum_{k=0}^{\infty} G(kd) \quad \text{as} \quad n \to \infty. \tag{4.11}$$

If $\mu = +\infty$ *the limits in* (i) *and* (ii) *are* 0.

If, in particular, $\{X_k, k \geq 1\}$ are exponentially distributed with mean λ^{-1}, that is, if $\{N(t), t \geq 0\}$ is a Poisson process with intensity λ (recall Example 2.1 and Remark 3.1), then $\mu = \lambda^{-1}$ and it follows, in particular, that $U(t) = EN(t) = \lambda t = t/\mu$ and that $U(t) - U(t-h) = h/\mu$. Thus, there is *equality* in Theorems 4.1 and 4.2 for all t in this case.

The renewal process in Example 2.3 (which is arithmetic with span 1) runs as follows: First there is a geometric (with mean q/p) number of zeroes, then a one, then a geometric (with mean q/p) number of zeroes followed by a one and so on. It follows that the actual number of partial sums equal to k (recall Remark 3.5) equals $1 +$ a geometric (with mean q/p) number. To see this we observe that the first partial sum that equals n is obtained for S_k where k is such that $S_{k-1} = n - 1$ and $X_k = 1$ (and thus, $S_k = n$). After this there is a geometric number of zeroes before the next one (which brings the sum to $n + 1$) appears. We thus obtain $u_n = 1 + (q/p) = 1/p = 1/\mu$ (since $\mu = p$). Thus, formula (4.9) is exact for all n in this case.

Remark 4.1. In the classical proofs of Theorems 4.2 and 4.3 a lemma due to Choquet and Deny (1960) plays an important role. In Doob, Snell and Williamson (1960) a proof, based on martingale theory, is given; see also Meyer (1966), pp. 192–193 or Jagers (1975), p. 107.

Remark 4.2. In the 1970s an old method due to Doeblin, called coupling enjoyed a big revival. We only mention some references where proofs of renewal theorems using coupling can be found; Lindvall (1977, 1979, 1982, 1986), Arjas, Nummelin and Tweedie (1978), Athreya, McDonald and Ney (1978), Ney (1981) and Thorisson (1987).

Remark 4.3. The key renewal theorem (Theorem 4.3) has been generalized by Mohan (1976) in the nonarithmetic case and by Niculescu (1979) in the d-arithmetic case. They assume that G varies regularly with exponent α ($0 \leq \alpha < 1$) and conclude (essentially) that the LHS's in (4.10) and (4.11) vary regularly with exponent α. For a slightly different generalization, see Erickson (1970).

2.5 Limit Theorems

In this section we present the strong law and the central limit theorem for renewal counting processes. The strong law is due to Doob (1948) (a.s. convergence) and Feller (1941) and Doob (1948) (convergence of the mean) and Hatōri (1959) (convergence of moments of order > 1). The central limit theorem is due to Feller (1949) in the arithmetic case and to Takács (1956) in the nonarithmetic case. Here we present a proof based on Anscombe's theorem (Theorem 1.3.1). The asymptotic expansions of mean and variance are due to Feller (1949) in the arithmetic case and Smith (1954) in the nonarithmetic case.

Let us, however, first observe that, by (3.4), we have

$$N(t) \to +\infty \quad \text{as} \quad t \to \infty. \tag{5.1}$$

By (5.1) and (3.11) we also have $\nu(t) \to \infty$ as $t \to \infty$.

Theorem 5.1 (The Strong Law for Counting Processes). *Let* $0 < \mu = EX_1 \leq \infty$. *Then*

(i) $$\frac{N(t)}{t} \xrightarrow{\text{a.s.}} \frac{1}{\mu} \quad \textit{as} \quad t \to \infty;$$

(ii) $$E\left(\frac{N(t)}{t}\right)^r \to \frac{1}{\mu^r} \quad \textit{as} \quad t \to \infty \quad \textit{for all } r > 0.$$

For $\mu = +\infty$ *the limits are* 0.

Proof. Suppose that $0 < \mu < \infty$.
(i) The relation

$$S_{N(t)} \leq t < S_{\nu(t)} \tag{5.2}$$

and (3.11) together yield

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{\nu(t)}}{\nu(t)} \cdot \frac{N(t)+1}{N(t)}. \tag{5.3}$$

An application of Theorem 1.2.3 now shows that the extreme members in (5.3) both converge almost surely to μ as $t \to \infty$, which shows that

$$\frac{t}{N(t)} \xrightarrow{\text{a.s.}} \mu \quad \text{as} \quad t \to \infty \tag{5.4}$$

and (i) follows.
(ii) We wish to show that

$$\left\{ \left(\frac{N(t)}{t} \right)^r, t \geq 1 \right\} \quad \text{is uniformly integrable for all } r > 0, \tag{5.5}$$

because (ii) then follows from (i), (5.5) and Theorem A.1.1.

We prefer, however, to prove the equivalent fact (recall (3.11)) that

$$\left\{ \left(\frac{\nu(t)}{t} \right)^r, t \geq 1 \right\} \quad \text{is uniformly integrable for all } r > 0. \tag{5.6}$$

This result is, essentially, a consequence of the subadditivity of first passage time processes. Namely, let $t, s > 0$ and consider $\nu(t+s)$. In order to reach the level $t+s$ we must first reach the level t. When this has been done the process starts afresh. Since $S_{\nu(t)} > t$ the remaining distance for the process to climb is at most equal to s, and thus, the required number of steps to achieve this is majorized by a random quantity distributed as $\nu(s)$. More formally,

$$\begin{aligned} \nu(t+s) &\leq \nu(t) + \min\{k - \nu(t) \colon S_k - S_{\nu(t)} > s\} \\ &= \nu(t) + \nu_1(s), \end{aligned} \tag{5.7}$$

where $\nu_1(s)$ is distributed as $\nu(s)$.

Now, let $n \geq 1$ be an integer. By induction we have

$$\nu(n) \leq \nu_1(1) + \cdots + \nu_n(1), \tag{5.8}$$

where $\{\nu_k(1), k \geq 1\}$ are distributed as $\nu(1)$. This, together with Minkowski's inequality (see e.g. Gut (2007), Theorem 3.2.6) and Theorem 3.1(ii), shows that

$$\|\nu(n)\|_r \leq n \|\nu(1)\|_r < \infty. \tag{5.9}$$

Finally, since $\nu(t) \leq \nu([t]+1)$, we have, for $t \geq 1$,

$$\frac{\nu(t)}{t} \leq 2 \frac{\nu([t]+1)}{[t]+1} \tag{5.10}$$

and thus, in view of (5.9), that

$$\|\nu(t)/t\|_r \leq 2\|\nu([t]+1)/([t]+1)\|_r \leq 2\|\nu(1)\|_r < \infty. \tag{5.11}$$

Since the bound is uniform in t it follows that $\{(\nu(t)/t)^p, t \geq 1\}$ is uniformly integrable for all $p < r$. Since r was arbitrary, (5.6) (and hence (5.5)) follows.

This concludes the proof for the case $0 < \mu < \infty$. For $\mu = +\infty$ the conclusion follows by the truncation procedure used in the proof of Theorem 4.1. We omit the details. □

Remark 5.1. Theorem 5.1 also holds for $\{\nu(t), t \geq 0\}$. This is immediate from Theorem 5.1, (3.11) and (5.6).

Theorem 5.2 (The Central Limit Theorem for Counting Processes). *Suppose that* $0 < \mu = EX_1 < \infty$ *and* $\sigma^2 = \operatorname{Var} X_1 < \infty$. *Then*

(i)
$$\frac{N(t) - t/\mu}{\sqrt{\frac{\sigma^2 t}{\mu^3}}} \xrightarrow{d} N(0,1) \quad \textit{as} \quad t \to \infty,$$

(ii) *If the renewal process is nonarithmetic, then*

$$EN(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(1) \quad \textit{as} \quad t \to \infty \tag{5.12}$$

$$\operatorname{Var} N(t) = \frac{\sigma^2 t}{\mu^3} + o(t) \quad \textit{as} \quad t \to \infty. \tag{5.13}$$

If the renewal process is d-arithmetic, then

$$EN(nd) = \frac{nd}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + \frac{d}{2\mu} + o(1) \quad \textit{as} \quad n \to \infty \tag{5.14}$$

$$\operatorname{Var} N(nd) = \frac{\sigma^2 nd}{\mu^3} + o(n) \quad \textit{as} \quad n \to \infty. \tag{5.15}$$

Proof. (i) We first observe that

$$\frac{S_{N(t)} - N(t)\mu}{\sigma\sqrt{t/\mu}} \leq \frac{t - N(t)\mu}{\sigma\sqrt{t/\mu}} < \frac{S_{\nu(t)} - \nu(t)\mu}{\sigma\sqrt{t/\mu}} + \frac{\mu}{\sigma}\sqrt{\frac{\mu}{t}} \tag{5.16}$$

by (3.11) and (5.2). In view of Theorem 5.1(i) and Remark 5.1 we now apply Anscombe's theorem (Theorem 1.3.1(ii)) with $\theta = \mu^{-1}$ to the extreme members of (5.16) and conclude that they both converge in distribution to the standard normal distribution. Thus

$$\frac{t - N(t)\mu}{\sigma\sqrt{t/\mu}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t \to \infty. \tag{5.17}$$

which, in view of the symmetry of the normal distribution, proves (i).

(ii) Formulas (5.12) and (5.13) follow, essentially, from repeated use of the key renewal theorem (Theorem 4.3(i)). Formulas (5.14) and (5.15) follow similarly. □

Remark 5.2. The classical proofs of Theorem 5.2(i) are based on the ordinary central limit theorem and the inversion formula (3.4).

Remark 5.3. Theorem 5.2 (with obvious modifications) also holds for $\{\nu(t), t \geq 0\}$ in view of (3.11).

To prove moment convergence in Theorem 5.1 we proceeded via uniform integrability and Theorem A.1.1 (and the remarks there). For Theorem 5.2 we referred to proofs based on direct computations. Now, in order to conclude, from Theorem 5.2 that

$$\left\{\left(\frac{N(t) - t/\mu}{\sqrt{t}}\right)^2, t \geq 1\right\} \quad \text{is uniformly integrable} \tag{5.18}$$

we observe that this does not follow immediately from Theorem A.1.1 since we are concerned with a *family* of random variables; recall Remark A.1.2. It does, however, follow that

$$\left\{\left(\frac{N(n) - n/\mu}{\sqrt{n}}\right)^2, n \geq 1\right\} \quad \text{is uniformly integrable,} \tag{5.19}$$

which, together with the monotonicity of $\{N(t), t \geq 0\}$ proves (5.18).

Moreover, (5.18) and (3.11) together imply that (5.18) also holds for first passage time processes, that is, that

$$\left\{\left(\frac{\nu(t) - t/\mu}{\sqrt{t}}\right)^2, t \geq 1\right\} \quad \text{is uniformly integrable.} \tag{5.20}$$

Let us now consider the Poisson process and the Negative Binomial process.

In the former case (recall the notation from above) we have $\mu = \lambda^{-1}$, $\sigma^2 = \lambda^{-2}, EN(t) = \lambda t$ and $\operatorname{Var} N(t) = \lambda t$. The validity of the central limit theorem is trivial. As for formulas (5.12) and (5.13) we find that

$$\frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} = \lambda t \quad \text{and that} \quad \frac{\sigma^2 t}{\mu^3} = \lambda t,$$

that is, we have equalities (without remainder) as expected.

In the latter case we have $\mu = p, \sigma^2 = pq$ (and $d = 1$). Here, however, it is easier to see that $E\nu(n) = (n+1) \cdot (1/p)$ and that $\operatorname{Var} \nu(n) = (n+1) \cdot (q/p^2)$, since $\nu(n)$ equals the number of performances required in order to succeed more than n times (that is, $n+1$ times). Since $N(n) = \nu(n) - 1$ it follows that $EN(n) = ((n+1)/p) - 1$ and that $\operatorname{Var} N(n) = (n+1)q/p^2$. (Alternatively $N(n)$ equals the sum of n independent geometric variables with mean $1/p$ and one geometric variable with mean q/p.) Again the central limit theorem presents no problem and it is easy to check that

$$\frac{n}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + \frac{1}{2\mu} = \frac{n+1}{p} - 1.$$

Finally,

$$\frac{\sigma^2 n}{\mu^3} = \frac{pqn}{p^3} = \frac{(n+1)q}{p^2} - \frac{q}{p^2},$$

that is, here we only have asymptotic equality with the leading term (but with a remainder which is much smaller than $o(n)$).

Finally, some remarks for the case when $\operatorname{Var} X_1 = +\infty$.

A generalization of (5.12) and (5.14) under the assumption that $E|X_1|^r < \infty$ for some r $(1 < r < 2)$ has been proved in Täcklind (1944). The remainder then is $o(t^{2-r})$ as $t \to \infty$.

For nonarithmetic renewal processes it has been shown by Mohan (1976) that $EN(t) - (t/\mu)$ $(= U(t) - (t/\mu))$ is regularly varying with exponent $2 - \alpha$ iff F belongs to the domain of attraction of a stable law with index α $(1 < \alpha \leq 2)$. Moreover, a generalization of (5.13) is obtained, see also Teugels (1968).

2.6 The Residual Lifetime

In view of the fact that the sequence $\{X_k, k \geq 1\}$ frequently is interpreted as a sequence of lifetimes it is natural to consider, in particular, the object (component) that is alive at time t. Its total lifetime is, of course, $X_{N(t)+1} = X_{\nu(t)}$. Of special interest is also *the residual lifetime*

$$R(t) = S_{N(t)+1} - t = S_{\nu(t)} - t. \tag{6.1}$$

In this section we present some asymptotic results.

As a first observation we note that, if $\operatorname{Var} X_1 = \sigma^2 < \infty$, then, by Theorem 1.5.3, we have

$$ER(t) = ES_{\nu(t)} - t = \mu\left(E\nu(t) - \frac{t}{\mu}\right). \tag{6.2}$$

By combining this with Theorem 5.2(ii) and (3.11) the following result emerges.

Theorem 6.1. *Suppose that* $\operatorname{Var} X_1 = \sigma^2 < \infty$.

(i) *If the renewal process is nonarithmetic, then*

$$ER(t) \to \frac{\sigma^2 + \mu^2}{2\mu} \quad as \quad t \to \infty. \tag{6.3}$$

(ii) *If the renewal process is d-arithmetic, then*

$$ER(nd) \to \frac{\sigma^2 + \mu^2}{2\mu} + \frac{d}{2} \quad as \quad n \to \infty. \tag{6.4}$$

This result indicates that $R(t)$, under appropriate conditions, may converge without normalization. That this is, indeed, the case is shown next.

Theorem 6.2. *Suppose that* $0 < EX_1 = \mu < \infty$.

(i) *If the renewal process is nonarithmetic, then, for* $x > 0$*, we have*

$$\lim_{t\to\infty} P(R(t) \le x) = \frac{1}{\mu}\int_0^x (1 - F(s))ds. \tag{6.5}$$

(ii) *If the renewal process is d-arithmetic, then, for* $k = 1, 2, 3, \ldots,$ *we have*

$$\lim_{n\to\infty} P(R(nd) \le kd) = \frac{d}{\mu}\sum_{j=0}^{k-1}(1 - F(jd)) \tag{6.6}$$

or, equivalently,

$$\lim_{n\to\infty} P(R(nd) = kd) = \frac{d}{\mu}P(X_1 \ge kd). \tag{6.7}$$

Proof. We only prove (i), the proof of (ii) being similar.

It is more convenient to consider the tail of the distribution. We have

$$\begin{aligned}
P(R(t) > x) &= \sum_{n=1}^{\infty} P(S_{n-1} \le t, S_n > t + x) \\
&= P(X_1 > t + x) + \sum_{n=2}^{\infty}\int_0^t P(X_n > t + x - s)dF_{n-1}(s) \\
&= 1 - F(t + x) + \int_0^t (1 - F(t + x - s))dU(s) \\
&\to 0 + \frac{1}{\mu}\int_0^{\infty}(1 - F(x + s))ds \\
&= \frac{1}{\mu}\int_x^{\infty}(1 - F(s))ds \quad \text{as} \quad t \to \infty.
\end{aligned}$$

For the convergence we use the key renewal theorem (Theorem 4.3(i)) with $G(t) = 1 - F(t + x),\ t > 0$, and the fact that $\int_0^{\infty} G(t)dt = \int_x^{\infty}(1 - F(s))ds \le EX_1 < \infty$. □

Before we proceed, let us, as in Sections 2.4 and 2.5, consider the Poisson process and the Negative Binomial process.

Due to the lack of memory property it is clear that $R(t)$ is exponential with mean $\mu = \lambda^{-1}$ for all t. It is now easy to check that $\mu^{-1}\int_0^x(1 - F(s))ds = 1 - e^{-\lambda x}$ for $x > 0$ and that $(\sigma^2 + \mu^2)/2\mu = \lambda^{-1}$, that is, we have equality for all t in (6.3) and (6.5) as expected.

For the Negative Binomial process $R(n) = 1$ a.s. for all n. Since $\mu = p$ and $\sigma^2 = pq$ we have $\mu^{-1}P(X_1 \geq 1) = 1$ and $(\sigma^2 + \mu^2)/(2\mu) + \frac{1}{2} = 1$, that is, there is equality in (6.7) and (6.4) for all n.

In Theorem 6.1 we have seen how the expected value of the residual lifetime behaves asymptotically. The following result extends Theorem 6.1 to arbitrary moments; however, we confine ourselves to the nonarithmetic case. The proof is, essentially, due to Lai (1976), p. 65.

Theorem 6.3. *Suppose that $EX_1^r < \infty$ for some $r > 1$. If the renewal process is nonarithmetic, then*

$$E(R(t))^{r-1} \to \frac{1}{r\mu} EX_1^r \quad \textit{as} \quad t \to \infty. \tag{6.8}$$

Proof. We have

$$R(t) = \sum_{n=1}^{\infty} I\{S_{n-1} \leq t\} \cdot (S_{n-1} + X_n - t)^+ \qquad (t > 0), \tag{6.9}$$

where all terms but one equal 0. We can thus raise the sum to any power termwise. An elementary computation then shows that

$$E(R(t))^{r-1} = E((X_1 - t)^+)^{r-1} + \int_0^t G(t-s)dU(s), \tag{6.10}$$

where

$$G(y) = \int_y^{\infty} (u-y)^{r-1} dF(u) \tag{6.11}$$

and U is the renewal function. Since

$$\int_0^{\infty} G(y)dy = \frac{1}{r} EX_1^r < \infty, \tag{6.12}$$

an application of the key renewal theorem (Theorem 4.3(i)) yields

$$E(R(t))^{r-1} \to \frac{1}{\mu} \int_0^{\infty} G(y)dy \quad \text{as} \quad t \to \infty, \tag{6.13}$$

which, in view of 6.12, proves the theorem. □

Remark 6.1. Note that, for $r = 2$, we rediscover (6.3).

Another quantity of interest is *the age* of the object that is alive at time t, $A(t) = t - S_{N(t)}$. However, since this object will not be considered in the sequel (it does not carry over in a reasonable sense to random walks), we only mention here that it can be shown that, in the nonarithmetic case, the limit distribution of the age is the same as that of the residual lifetime given in Theorem 6.2 (a minor modification is necessary in the arithmetic case).

We also note that $R(t)$ is, mathematically, a more pleasant object, since it is expressed in terms of a random quantity indexed by a stopping time, whereas $A(t)$ is not.

We finally mention that it is also possible to obtain the asymptotic distribution of the lifetime, $X_{\nu(t)} = A(t) + R(t)$, itself. One can, in fact, show that

$$\lim_{t\to\infty} P(X_{\nu(t)} \leq x) = \frac{1}{\mu}\int_0^x s dF(s). \tag{6.14}$$

2.7 Further Results

An exhaustive exposition of renewal theory would require a separate book. As mentioned in the introduction of this chapter our aim is to present a review of the main results, with emphasis on those which are most relevant with respect to the contents of this book. In this section we shall, however, briefly mention some further results and references.

2.7.1

Just as for the classical limit theorems one can ask for limit theorems for the case when the variance is infinite or even when the mean is infinite. However, one must then make more detailed assumptions about the tail of the distribution function F.

In Feller (1971), pp. 373–74 it is shown that a limit distribution for $N(t)$ exists if and only if F belongs to some domain of attraction—the limit laws are the stable distributions.

Concerning renewal theorems and expansions of $U(t)$ $(= EN(t))$ and Var $N(t)$, suppose first that Var $X_1 = +\infty$. Then Teugels (1968) shows, essentially, that if $1 - F(x) = x^{-\alpha}L(x)$, where $1 < \alpha < 2$ and L is slowly varying, then $U(t) - t/\mu = EN(t) - t/\mu$ varies regularly with exponent $2-\alpha$ and Var $(N(t))$ varies regularly with exponent $3-\alpha$ (see his Section 3). Mohan (1976) improves these results. He also considers the case $\alpha = 2$.

For the case $EX_1 = +\infty$ Teugels (1968) obtains asymptotic expansions for the renewal function under the assumption that the tail $1 - F$ satisfies $1 - F(x) = x^{-\alpha} \cdot L(x)$ as $x \to \infty$, where $0 \leq \alpha \leq 1$ and L is slowly varying; the result, essentially, being that U varies regularly with exponent α (see his Theorem 1). For $L(x) = \text{const}$ and $0 < \alpha < 1$ this was obtained by Feller (1949) in the arithmetic case. Erickson (1970) generalizes the elementary renewal theorem, Blackwell's renewal theorem and the key renewal theorem (Theorems 4.1–4.3 above) to this situation. For a further contribution, see Anderson and Athreya (1987).

Garsia and Lamperti (1962/63) and Williamson (1968) study the arithmetic case, the latter also in higher dimensions.

Finally, if the mean is infinite and F belongs to the domain of attraction of a (positive) stable law with index α, $0 < \alpha < 1$, then the limit distribution for $R(t)/t$ and $A(t)/t$ (the normalized residual lifetime and age, respectively) is given by a so-called generalized arc sine distribution, see Feller (1971), Section XIV.3; see also Dynkin (1955) and Lamperti (1958, 1961). For the case $\alpha = 1$, see Erickson (1970).

2.7.2

There are also other kinds of limit theorems which could be of interest for renewal processes, such as the Marcinkiewicz-Zygmund strong law, the law of the iterated logarithm, convergence of higher moments in the central limit theorem (Theorem 5.2(i)), remainder term estimates in the central limit theorem etc.

It turns out that several such results have not been established separately for renewal processes; the historic development was such that renewal theory had been extended to renewal theory for random walks (on the whole real line) in such a way that proofs for the random walk case automatically also covered the corresponding theorems for renewal processes. In other words, the generalization to random walks came first.

We remark, however, that Feller (1949) presents a law of the iterated logarithm for renewal counting processes in the arithmetic case.

The following Berry–Esseen theorem for renewal counting processes, that is, a remainder term estimate in the central limit theorem, Theorem 5.2(i), is due to Englund (1980).

Theorem 7.1. *Suppose that* $\mu = EX_1, \sigma^2 = \operatorname{Var}\, X_1$ *and* $\gamma^3 = E|X_1 - \mu|^3$ *are all finite. Then*

$$\sup_n \left| P(N(t) < n) - \Phi\left(\frac{(n\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}}\right) \right| \le 4\left(\frac{\gamma}{\sigma}\right)^3 \sqrt{\frac{\mu}{t}}. \tag{7.1}$$

This result is mentioned mainly because it has not yet been extended to the random walk case.

Some large deviation results, that is, limit theorems for the ratio of the tail of the normalized distribution function of $N(t)$ and the tail of the standard normal distribution, are obtained in Serfozo (1974).

2.7.3

In this subsection we briefly mention two generalizations.

Consider Example 2.1. It is completely reasonable to assume that the initial component is not new, but has been used before. This amounts to the assumption that X_1 has a distribution different from F. Such a process is called a *delayed renewal process*. The renewal processes discussed above are then called *pure renewal processes*.

Another generalization is to allow defective distributions, that is, distributions such that $F(\infty) < 1$. The defect, $1 - F(\infty)$, corresponds to the probability of termination. Such processes are called *terminating*, or *transient renewal processes*. There are important applications of such processes, for example in insurance risk theory, see e.g. Asmussen (2000, 2003).

2.7.4

It is possible to develop a renewal theory for random walks (on the whole real line). We shall, in fact, do so in Chapter 3 (for the case $EX_1 > 0$).

2.7.5

Some attention has also been devoted to renewal theory for Markov chains, or Markov renewal theory; see Smith (1955), Çinlar (1975), Chapter 10 and Asmussen (2003). Some further references in this connection are Kemperman (1961), Spitzer (1965), Kesten (1974), Lalley (1984b, 1986) and papers by Alsmeyer and coauthors listed in the bibliography. In some of these references more general state spaces are considered. For a few lines on this topic, see Section 6.13 below.

2.7.6

We conclude by mentioning multidimensional or multivariate renewal theory. Some references are Chung (1952), Farrell (1964, 1966), Bickel and Yahav (1965), Doney (1966), Stam (1968, 1969, 1971), Hunter (1974a,b, 1977), Nagaev (1979), Gafurov (1980), Carlsson (1982) and Carlsson and Wainger (1984).

2.8 Random Walks; Introduction and Classifications

Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. random variables and set $S_n = \sum_{k=1}^{n} X_k,\ n \geq 0$, (where $S_0 = 0$). The sequence $\{S_n, n \geq 0\}$ is called a *random walk*. Throughout we ignore the trivial case $X_k = 0$ a.s., $k \geq 1$.

One of the basic problems concerning random walks is to investigate the asymptotics as $n \to \infty$, that is, to investigate "where" the random walk is after a "long time." All the classical limit theorems provide some answer to this problem. Another way of studying random walks is to investigate "how often" a given point or interval is visited, in particular, if there are finitely many visits or infinitely many visits and if the answer differs for different points or intervals.

Example 8.1. The simple random walk. Here $P(X_k = +1) = p$ and $P(X_k = -1) = q = 1 - p$ $(0 \leq p \leq 1)$.

Set $u_n = P(S_n = 0)$. It is well known (and/or easy to see) that

$$u_{2n} = \binom{2n}{n} p^n q^n \quad \text{and} \quad u_{2n-1} = 0 \quad (n \geq 1). \tag{8.1}$$

Thus, if $p \neq q$, then $\sum_{n=1}^{\infty} u_n < \infty$ and it follows from the Borel–Cantelli lemma that

$$P(S_n = 0 \text{ i.o.}) = 0 \quad \text{for } p \neq q. \tag{8.2}$$

If $p = q = \frac{1}{2}$, then $\sum_{n=1}^{\infty} u_n = +\infty$ and it follows (but *not* from the Borel–Cantelli lemma) that

$$P(S_n = 0 \text{ i.o.}) = 1 \quad \text{for } p = q = \frac{1}{2}. \tag{8.3}$$

A similar computation would show that the same dichotomy holds for returns to any integer point on the real line.

The different behaviors for $p \neq q$ and $p = q$ divide the simple random walks into two separate kinds. An analogous characterization can be made for all random walks. To this end we generalize the renewal function introduced in Section 2.3 as follows. We define the *renewal measure*

$$U\{I\} = \sum_{n=0}^{\infty} P(S_n \in I), \tag{8.4}$$

where $I \subset (-\infty, \infty)$ typically is an interval; here it is more convenient to let the summation start with $n = 0$ (recall Remarks 3.2 and 3.3).

For renewal processes we thus have, in particular, that $U\{[0,t]\} = 1 + U(t) < \infty$ for all t, that is $U\{I\} < \infty$ for all finite I. For random walks, which are not concentrated on one of the half axes, the series may, however, diverge even when I is a finite interval. This is far from obvious in general (see however Example 8.1 with $p = q = \frac{1}{2}$ and $I = \{0\}$) and gives, in fact, rise to the characterization to follow.

First, however, we note the following result.

Theorem 8.1.

(i) *If the random walk is nonarithmetic, then either $U\{I\} < \infty$ for every finite interval I or else $U\{I\} = +\infty$ for all intervals I.*

(ii) *If the random walk is arithmetic with span d, then either $U\{I\} < \infty$ for every finite interval I or else $U\{I\} = +\infty$ for all intervals I containing a point in the set $\{nd\colon n = 0, \pm 1, \pm 2, \dots\}$.*

This naturally leads to the following definition.

Definition 8.1. *A random walk is called transient if $U\{I\} < \infty$ for all finite intervals I and recurrent otherwise.*

Remark 8.1. Define the indicator variables $\{I_n, n \geq 0\}$ such that $I_n = 1$ on $\{S_n \in I\}$ and $I_n = 0$ otherwise and set $\Lambda\{I\} = \sum_{n=0}^{\infty} I_n$. Then $\Lambda\{I\}$ equals the *actual number of visit* to I made by the random walk (including S_0) and $U\{I\} = E\Lambda\{I\}$, that is, $U\{I\}$ equals the *expected number of visits* to I (cf. Remark 3.5).

We now return to the simple random walk from Example 8.1 and note that

$$U\{0\} = \sum_{n=0}^{\infty} u_n. \tag{8.5}$$

It thus follows from our computations there that $U\{0\} < \infty$ ($= \infty$) when $p \neq q$ ($p = q$). In view of Theorem 8.1 and Definition 8.1 it thus follows that the simple random walk is transient when $p \neq q$ and recurrent when $p = q = \frac{1}{2}$.

Moreover, it follows from the strong law of large numbers that $S_n \xrightarrow{\text{a.s.}} +\infty$ as $n \to \infty$ when $p > q$ and that $S_n \xrightarrow{\text{a.s.}} -\infty$ as $n \to \infty$ when $p < q$. In these cases the random walk drifts to $+\infty$ and $-\infty$, respectively. If $p = q = \frac{1}{2}$ it follows from the law of the iterated logarithm that $\limsup_{n\to\infty} S_n = +\infty$ a.s. and that $\liminf_{n\to\infty} S_n = -\infty$ a.s. In this case the random walk oscillates (between $+\infty$ and $-\infty$).

It turns out that these characterizations are not typical for simple random walks only; in fact, most of the above facts remain true for arbitrary random walks and all of them remain valid if, in addition, EX_1 is assumed to exist (that is, if $E|X_1| < \infty$ or $EX_1^- < \infty$ and $EX_1^+ = +\infty$ or $EX_1^- = \infty$ and $EX_1^+ < \infty$). We collect these facts as follows.

Theorem 8.2. *Let $\{S_n, n \geq 0\}$ be a random walk. Then exactly one of the following cases holds:*

(i) *The random walk drifts to $+\infty$; $S_n \xrightarrow{\text{a.s.}} +\infty$ as $n \to \infty$. The random walk is transient;*
(ii) *The random walk drifts to $-\infty$; $S_n \xrightarrow{\text{a.s.}} -\infty$ as $n \to \infty$. The random walk is transient;*
(iii) *The random walk oscillates between $-\infty$ and $+\infty$; $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = +\infty$ a.s. In this case the random walk may be either transient or recurrent.*

Theorem 8.3. *Suppose, in addition, that $\mu = EX_1$ exists.*

(i) *If $0 < \mu \leq +\infty$ the random walk drifts to $+\infty$;*
(ii) *If $-\infty \leq \mu < 0$ the random walk drifts to $-\infty$;*
(iii) *If $\mu = 0$ the random walk oscillates. Moreover, in this case it is recurrent.*

Remark 8.2. Theorem 8.3 (i) and (ii) follow immediately from the strong law of large numbers. The case $EX_1 = 0$ is harder.

The development so far is based on whether or not the measure $U\{I\}$ is finite, that is whether or not the *expected number of visits* to a given interval is finite or not (recall Remark 8.1). However, it was shown in Example 8.1 that, for the simple random walk, the *actual number of visits* to 0 is a.s. finite when $p \neq q$ and a.s. infinite when $p = q$. Now, if, for a general random walk, $U\{I\} = E\Lambda\{I\} < \infty$, where $\Lambda\{I\}$ is as defined in Remark 8.1, then, obviously, we have $\Lambda\{I\} < \infty$ a.s.

However, it is, in fact, possible to show that the converse also holds. Consequently,

$$U\{I\} < \infty \quad \Longleftrightarrow \quad \Lambda\{I\} < \infty \text{ a.s. for all finite } I. \tag{8.6}$$

Theorem 8.1 thus remains true with $U\{I\}$ replaced by $\Lambda\{I\}$ and transience and recurrence can, equivalently, be defined in terms of $\Lambda\{I\}$. Moreover, since $\Lambda\{I\} < \infty$ for all finite I iff $|S_n| \to +\infty$ as $n \to \infty$ the following result emerges.

Theorem 8.4. *The random walk is transient iff* $|S_n| \to \infty$ a.s. *as* $n \to \infty$.

It is also possible to characterize random walks according to the transience or recurrence of *points* (rather than intervals). We refer to Chung and Fuchs (1951) and Chung and Ornstein (1962). Just as for irreducible Markov chains one can show that all points are of the same kind (where "all" has the obvious interpretation in the arithmetic case). Furthermore, one can show that the interval characterization and the point characterization are equivalent.

Remark 8.3. Note that we, in fact, studied the transience/recurrence of the *point* 0 for the simple random walk. However, for arithmetic random walks any finite interval consists of finitely many points of the form $\{nd; d = 0, \pm 1, \dots\}$ so the distinction between points and intervals only makes sense in the nonarithmetic case.

Some book references on random walks are (in alphabetical order) Chung (1974), Feller (1968, 1971), Prabhu (1965) and Spitzer (1976).

2.9 Ladder Variables

Let $\{S_n, n \geq 0\}$ be a random walk with i.i.d. increments $\{X_k, k \geq 1\}$. Set $T_0 = 0$ and define

$$\begin{aligned} T_1 &= \min\{n\colon S_n > 0\}, \\ T_k &= \min\{n > T_{k-1}\colon S_n > S_{T_{k-1}}\} \quad (k \geq 2). \end{aligned} \tag{9.1}$$

If no such n exists we set $T_k = +\infty$. The random variables thus defined are called the (*strong*) (*ascending*) *ladder epochs*. The random variables

$Y_k = S_{T_k}$, $k \geq 0$, are the corresponding (*strong*) (*ascending*) *ladder heights* (with $Y_0 = S_{T_0} = S_0 = 0$).

Further, define, for $k \geq 1$,

$$N_k = T_k - T_{k-1} \quad \text{and} \quad Z_k = Y_k - Y_{k-1} = S_{T_k} - S_{T_{k-1}}, \tag{9.2}$$

provided $T_k < \infty$.

It follows from the construction that $\{(N_k, Z_k), k \geq 1\}, \{N_k, k \geq 1\}$ and $\{Z_k, k \geq 1\}$ are sequences of i.i.d. random variables. Moreover, $\{T_n, n \geq 0\}$ and $\{Y_n, n \geq 0\}$ are (possibly terminating) renewal processes; the former, in fact, being arithmetic with span 1.

Similarly, set $\overline{T}_0 = 0$ and define

$$\overline{T}_1 = \min\{n\colon S_n \leq 0\} \quad \text{and}$$

$$\overline{T}_k = \min\{n > \overline{T}_{k-1}\colon S_n \leq \overline{T}_{k-1}\} \quad (k \geq 2). \tag{9.3}$$

We call $\{\overline{T}_k, k \geq 0\}$ the (*weak*) (*descending*) *ladder epochs*. The sequences $\{\overline{Y}_k, k \geq 0\}, \{\overline{N}_k, k \geq 1\}$ and $\{\overline{Z}_k, k \geq 1\}$ are defined in the obvious manner.

Recall that a random variable is *proper* if it is finite a.s. Otherwise it is *defective*.

Theorem 9.1.

(i) *If the random walk drifts to* $+\infty$, *then* T_1 *and* Y_1 *are proper and* $\overline{T}_1$ *and* $\overline{Y}_1$ *are defective. Moreover,* $ET_1 < \infty$;
(ii) *If the random walk drifts to* $-\infty$, *then* T_1 *and* Y_1 *are defective and* $\overline{T}_1$ *and* $\overline{Y}_1$ *are proper. Moreover,* $E\overline{T}_1 < \infty$;
(iii) *If the random walk oscillates, then* $T_1, Y_1, \overline{T}_1$ *and* $\overline{Y}_1$ *are proper. Moreover,* $ET_1 = E\overline{T}_1 = +\infty$.

Theorem 9.2. *If, in addition,* $\mu = EX_1$ *exists, then (i), (ii) and (iii) correspond to the cases* $0 < \mu \leq \infty$, $-\infty \leq \mu < 0$ *and* $\mu = 0$, *respectively. Moreover,*

$$EY_1 < \infty \quad \textit{and} \quad EY_1 = \mu ET_1 \quad \textit{when } 0 < \mu < \infty \tag{9.4}$$

$$E\overline{Y}_1 > -\infty \quad \textit{and} \quad E\overline{Y}_1 = \mu E\overline{T}_1 \quad \textit{when } -\infty < \mu < 0. \tag{9.5}$$

Remark 9.1. The equations in (9.4) and (9.5) are, in fact, special cases of Theorem 1.5.3.

Remark 9.2. For the case $EX_1 = 0$ we recall Example 1.5.1—the simple symmetric random walk, for which $Y_1 = 1$ a.s. and $ET_1 = +\infty$.

Remark 9.3. If $\mu = EX_1 = 0$ and, moreover, $\sigma^2 = \text{Var } X_1 < \infty$, then $EY_1 < \infty$ and $E\overline{Y}_1 < \infty$. Furthermore,

$$\sigma^2 = -2EY_1 \cdot E\overline{Y}_1 \tag{9.6}$$

and

$$EY_1 = \frac{\sigma c}{\sqrt{2}} \quad \text{and} \quad E\overline{Y}_1 = -\frac{\sigma}{c\sqrt{2}}, \tag{9.7}$$

where $0 < c = \exp\{\sum_{n=1}^{\infty} \frac{1}{n}(\frac{1}{2} - P(S_n > 0))\} < \infty$.

These results are due to Spitzer (1960, 1976). The absolute convergence of the sum was proved by Rosén (1961), see also Gut (2001), page 414.

For further results on the moments of the ladder heights we refer to Lai (1976), Chow and Lai (1979) and Doney (1980, 1982).

The case $EX_1 > 0$ will be further investigated in Chapter 3.

Remark 9.4. Weak ascending and strong descending ladder variables can be defined in the obvious manner.

2.10 The Maximum and the Minimum of a Random Walk

For a random walk $\{S_n, n \geq 0\}$ we define the partial maxima, $\{M_n, n \geq 0\}$, by

$$M_n = \max\{0, S_1, S_2, \ldots, S_n\} \tag{10.1}$$

and the partial minima, $\{m_n, n \geq 0\}$, by

$$m_n = \min\{0, S_1, S_2, \ldots, S_n\}. \tag{10.2}$$

In this section we show how these sequences can be used to characterize a random walk and in Section 2.12 we present some general limit theorems for M_n.

If the random walk drifts to $+\infty$, then, since $M_n \geq S_n$, it follows from Theorem 8.2 that $M_n \xrightarrow{\text{a.s.}} +\infty$ as $n \to \infty$. Furthermore, with probability 1, there is a last (random) epoch at which the random walk assumes a negative value. Thus

$$m = \min_{n \geq 0} S_n > -\infty \text{ a.s.} \tag{10.3}$$

Since $\{m_n, n \geq 0\}$ is nonincreasing it follows that $m_n \to m$ monotonically as $n \to \infty$. These facts, together with a similar (symmetric) argument for the case when the random walk drifts to $-\infty$, yield (i) and (ii) of the following theorem. The proof of (iii) is immediate.

Theorem 10.1.

(i) *If the random walk drifts to* $+\infty$*, then*

$$M_n \xrightarrow{\text{a.s.}} +\infty \quad \textit{and} \quad m_n \xrightarrow{\text{a.s.}} m = \min_n S_n > -\infty \text{ a.s.} \quad \textit{as} \quad n \to \infty;$$

(ii) *If the random walk drifts to* $-\infty$, *then*

$$M_n \xrightarrow{\text{a.s.}} M = \max_{n\geq 0} S_n < \infty \text{ a.s.} \quad \textit{and} \quad m_n \xrightarrow{\text{a.s.}} -\infty \quad \textit{as} \quad n \to \infty;$$

(iii) *If the random walk is oscillating, then*

$$M_n \xrightarrow{\text{a.s.}} +\infty \quad \textit{and} \quad m_n \xrightarrow{\text{a.s.}} -\infty \quad \textit{as} \quad n \to \infty.$$

If, in addition, $\mu = EX_1$ *exists, then (i), (ii) and (iii) correspond to the cases* $0 < \mu \leq +\infty$, $-\infty \leq \mu < 0$ *and* $\mu = 0$, *respectively.*

2.11 Representation Formulas for the Maximum

The sequence of partial maxima $\{M_n, n \geq 0\}$, defined in the previous section, describes the successive record values of a random walk. However, so does the sequence of strong ascending ladder heights. At every strong ascending ladder epoch there is a new record value, that is, the sequence of partial maxima and the sequence of strong ascending ladder heights both jump to a new, common, record value. Thus, each M_n equals some strong ascending ladder height.

To make this argument more stringent, let $\{N(n), n \geq 1\}$ be the counting process of the renewal process $\{T_n, n \geq 0\}$, generated by the strong ladder epochs. Thus,

$$N(n) = \max\{k\colon T_k \leq n\} \quad (n \geq 1), \tag{11.1}$$

or, equivalently, $N(n)$ equals the number of strong ascending ladder epochs in $[0, n]$.

The following lemma, which is due to Prabhu (1980), formalizes the relation between the sequence of partial maxima and the sequence of ladder heights.

Lemma 11.1. *We have*

$$M_n = Y_{N(n)} = \sum_{k=1}^{N(n)} Z_k = \sum_{k=1}^{N(n)} (S_{T_k} - S_{T_{k-1}}). \tag{11.2}$$

Remark 11.1. A completely analogous relation holds between the sequence of partial minima and the sequence of strong descending ladder heights.

The usefulness of this lemma lies in the fact that we have represented M_n as a sum of a random number of i.i.d. positive random variables, which permits us to apply the results obtained in Chapter 1 for such sums in order to prove results for M_n. Note, however, that $N(n)$ is not a stopping time, since $\{N(n), n \geq 1\}$ is a renewal counting process.

There is, however, another and much simpler way to derive some of these limit theorems, namely those concerning convergence in distribution or those which only involve probabilities and here the following lemma is useful.

Lemma 11.2. *We have*

$$(M_n, M_n - S_n) \stackrel{d}{=} (S_n - m_n, -m_n), \tag{11.3}$$

in particular,

$$M_n \stackrel{d}{=} S_n - m_n. \tag{11.4}$$

Remark 11.2. This is Prabhu (1980), Lemma 1.4.1. The basis for the proof is the fact that the original random walk has the same distributional properties as the so-called dual random walk. In particular,

$$(S_0, S_1, S_2, \ldots, S_n) \stackrel{d}{=} (S_n - S_n, S_n - S_{n-1}, \ldots, S_n - S_1, S_n - S_0). \tag{11.5}$$

Lemmas 11.1 and 11.2 are thus used in different ways. Whereas Lemma 11.1 gives an actual *representation of* M_n, Lemma 11.2 (only) defines two quantities which are *equidistributed*. Thus, if we are interested in the "sample function behavior" of $\{M_n, n \geq 0\}$ we must use Lemma 11.1, since two sequences of random variables with pairwise the same distribution need not have the same sample function behavior; for example, one may converge a.s. whereas the other does not. However, if we only need a "weak" result, then Lemma 11.2 is very convenient.

In the following section we shall use Lemmas 11.1 and 11.2 to prove (or indicate the proofs of) some general limit theorems for M_n. In Section 4.4 we shall exploit the lemmas more fully for random walks with positive drift.

We conclude this section by showing that we can obtain a representation formula for $M = \max_{n\geq 0} S_n$ for random walks drifting to $-\infty$ by letting $n \to \infty$ in (11.2).

Thus, suppose that the random walk drifts to $-\infty$; this is, for example, the case when $EX_1 < 0$. We then know from Theorem 10.1(ii) that $M_n \xrightarrow{\text{a.s.}} M = \max_{n\geq 0} S_n$ as $n \to \infty$, where M is a.s. finite. Furthermore, $N(n) \xrightarrow{\text{a.s.}} N < \infty$ a.s. as $n \to \infty$, where N equals the number of strong ascending ladder epochs (note that, in fact, $N(n) = N$ for n sufficiently large). This, together with Theorem 1.2.4, establishes the first part of the following result, see Janson (1986), Lemma 1.

Lemma 11.3. *If the random walk drifts to* $-\infty$, *then*

$$M = \max_{n\geq 0} S_n = Y_N = \sum_{k=1}^{N} Z_k = \sum_{k=1}^{N} (S_{T_k} - S_{T_{k-1}}), \tag{11.6}$$

where N *equals the number of strong ascending ladder epochs and*

$$P(N = n) = p(1-p)^n, \ n \geq 0, \quad \textit{and} \quad p = P(T_1 = +\infty) = P(M = 0) > 0.$$

Moreover, the conditional distribution of Z_k *given that* $N \geq k$ *is independent of* k *and* $Z_1, \ldots, Z_{k-1}$ *and* Z_k *are independent given that* $N \geq k$.

Proof. Formula (11.6) was proved above. For the remaining part of the proof we observe that $\{T_k, k \geq 1\}$ is a sequence of defective stopping times and that the random walk after T_k is independent of the random walk up to T_k on the set $\{T_k < \infty\}$. Thus

$$P(N \geq k+1 | N \geq k) = P(T_{k+1} < \infty | T_k < \infty) = P(T_1 < \infty) = 1 - p. \quad (11.7)$$

By rewriting this as

$$P(N \geq k+1) = P(T_1 < \infty) P(N \geq k), \quad (11.8)$$

it follows that N has a geometric distribution as claimed. Also, since $P(N < \infty) = 1$, we must have $p = P(T_1 = +\infty) > 0$. The final claim about independence follows as above (cf. (11.7)). □

Remark 11.3. A consequence of Lemma 11.3 is that, for a random walk drifting to $-\infty$ we can represent the maximum, M, as follows:

$$M \stackrel{d}{=} \sum_{k=1}^{N} U_k, \quad (11.9)$$

where $\{U_k, k \geq 1\}$ is a sequence of i.i.d. random variables, which, moreover, is independent of N. Furthermore, the distribution of U_1 equals the conditional distribution of Y_1 $(= S_{T_1})$ given that $T_1 < \infty$ and the distribution of N is geometric as above.

Remark 11.4. Results corresponding to Lemmas 11.1 and 11.3 also hold for the *weak* ascending ladder epochs. Since $P(S_{T_1} > 0) \geq P(X_1 > 0) > 0$ it follows that the sums corresponding to (11.2) and (11.6) consist of a geometric number of zeroes followed by a positive term, followed by a geometric number of zeroes etc. Equivalently, the terms in (11.2) and (11.6) consist of the positive terms in the sums corresponding to the weak ascending ladder epochs.

Remark 11.5. A result analogous to Lemma 11.3 also holds for the minimum $m = \min_{n \geq 0} S_n$ if the random walk drifts to $+\infty$.

2.12 Limit Theorems for the Maximum

Having characterized random walks in various ways we shall now proceed to prove some general limit theorems for the partial maxima of a random walk. We assume throughout that the mean exists (in the sense that at least one of the tails has finite expectation). All results for partial maxima have obvious counterparts for the partial minima, which we, however, leave to the reader.

Theorem 12.1 (The Strong Law of Large Numbers). *Suppose that* $-\infty \leq \mu = EX_1 \leq \infty$. *We have*

$$\frac{M_n}{n} \xrightarrow{\text{a.s.}} \mu^+ \quad \textit{as} \quad n \to \infty. \tag{12.1}$$

Proof. Suppose first that $0 \leq \mu \leq \infty$. Since $M_n \geq S_n$ it follows from the strong law of large numbers that

$$\liminf_{n\to\infty} \frac{M_n}{n} \geq \mu \text{ a.s.} \tag{12.2}$$

If $\mu = +\infty$ there is nothing more to prove, so suppose that $0 \leq \mu < \infty$. It remains to prove that

$$\limsup_{n\to\infty} \frac{M_n}{n} \leq \mu \text{ a.s.} \tag{12.3}$$

Now, let $\varepsilon > 0$ and choose $\omega \in \Omega$ such that

$$\limsup_{n\to\infty} \frac{M_n(\omega)}{n} > \mu + \varepsilon. \tag{12.4}$$

Then there exists a subsequence $\{n_k, k \geq 1\}$ tending to infinity such that

$$\frac{M_{n_k}(\omega)}{n_k} > \mu + \varepsilon \quad (k \geq 1). \tag{12.5}$$

Define $\tau_{n_k} = \min\{n\colon S_n = M_{n_k}\}$. Since $\tau_{n_k} \leq n_k$ it follows from (12.4) (suppressing ω) that

$$\frac{S_{\tau_{n_k}}}{\tau_{n_k}} = \frac{M_{n_k}}{\tau_{n_k}} \geq \frac{M_{n_k}}{n_k} > \mu + \varepsilon. \tag{12.6}$$

However, in view of the strong law of large numbers the set of ω such that this is possible must have probability 0, that is, (12.4) is a.s. impossible and (12.3) follows. This completes the proof for the case $0 \leq \mu < \infty$.

Finally, suppose that $-\infty \leq \mu < 0$. We then know from Theorem 10.1 that M_n converges a.s. to an a.s. finite random variable, M. This immediately implies that

$$\frac{M_n}{n} \xrightarrow{\text{a.s.}} 0 = \mu^+ \quad \text{as} \quad n \to \infty \tag{12.7}$$

and we are done. □

Remark 12.1. We just wish to point out that if $0 < EX_1 < \infty$ we can use Lemma 11.1 to give an alternative proof of Theorem 12.1 by arguing as follows.

By the representation (11.2) we have

$$\frac{M_n}{n} = \frac{Z_1 + \cdots + Z_{N(n)}}{n}, \tag{12.8}$$

where, again, $N(n) = \max\{k\colon T_k \leq n\}$.

Now, from Theorem 9.2 we know that $EZ_1 = EY_1 < \infty$. From renewal theory (Theorem 5.1(i)) we thus conclude that

$$\frac{N(n)}{n} \xrightarrow{\text{a.s.}} \frac{1}{ET_1} \quad \text{as} \quad n \to \infty, \tag{12.9}$$

which, together with Theorem 1.2.3(iii), yields

$$\frac{Z_1 + \cdots + Z_{N(n)}}{n} \xrightarrow{\text{a.s.}} EZ_1 \cdot \frac{1}{ET_1} \quad \text{as} \quad n \to \infty. \tag{12.10}$$

The conclusion now follows from Theorem 1.5.3 (or directly from formula (9.4)).

Remark 12.2. The same proof also works when $EX_1 = 0$ provided we also assume that Var $X_1 < \infty$. This is necessary to ensure that $EY_1 = EZ_1 < \infty$.

Next we assume that, in addition, Var $X_1 < \infty$. The resulting limit laws are different for the cases $EX_1 = 0$ and $EX_1 > 0$, respectively, and we begin with the former case.

Theorem 12.2. *Suppose that $EX_1 = 0$ and that $\sigma^2 = Var\ X_1 < \infty$. Then*

$$\frac{M_n}{\sqrt{n}} \xrightarrow{d} |N(0,1)| \quad \textit{as} \quad n \to \infty. \tag{12.11}$$

Sketch of Proof. By Lemma 11.1 we have (cf. (12.8))

$$\frac{M_n}{\sqrt{n}} = \frac{Z_1 + \cdots + Z_{N(n)}}{N(n)} \cdot \frac{N(n)}{\sqrt{n}}. \tag{12.12}$$

The idea now is that, since $EZ_1 < \infty$ (cf. Remark 9.3), it follows from Theorem 1.2.3 that the first factor in the RHS of (12.12) converges a.s. to EZ_1 as $n \to \infty$. Furthermore, one can show that

$$\frac{N(n)}{c\sqrt{2n}} \xrightarrow{d} |N(0,1)| \quad \text{as} \quad n \to \infty. \tag{12.13}$$

where c is the constant in Remark 9.3. The conclusion then follows from Cramér's theorem and the fact that $EZ_1 = EY_1 = \sigma c/\sqrt{2}$ (recall (9.7)). □

We now turn to the case $EX_1 > 0$.

Theorem 12.3 (The Central Limit Theorem). *Suppose that $0 < \mu = EX_1 < \infty$ and that $\sigma^2 = Var\ X_1 < \infty$. Then*

$$\frac{M_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1) \quad \textit{as} \quad n \to \infty. \tag{12.14}$$

Proof. By Lemma 11.2 we have

$$\frac{M_n - n\mu}{\sigma\sqrt{n}} \stackrel{d}{=} \frac{S_n - n\mu}{\sigma\sqrt{n}} - \frac{m_n}{\sigma\sqrt{n}}. \tag{12.15}$$

Since $m_n \xrightarrow{\text{a.s.}} m > -\infty$ a.s. as $n \to \infty$ (Theorem 10.1) it follows that

$$\frac{m_n}{\sqrt{n}} \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty. \tag{12.16}$$

This, together with the ordinary central limit theorem and Cramér's theorem, shows that the RHS (and hence the LHS) of (12.15) converges in distribution to the standard normal distribution as $n \to \infty$, which proves the theorem. □

Theorem 12.1 is part of a more general theorem (see Theorem 4.4.1(i)) due to Heyde (1966). The present proof is due to Svante Janson. Theorems 12.2 and 12.3 are due to Erdős and Kac (1946) and Wald (1947), respectively. Chung (1948) proves both results with a different method (and under additional assumptions). The proofs presented here are due to Prabhu (1980), Section 1.5.

In Section 4.4 we present further limit theorems for the maximum of random walks with positive drift.