2

CONSTRAINTS AND LAGRANGIAN COORDINATES

1 Constrained Trajectories

Constraints are limitations imposed on the geometrical or kinematic configuration of a mechanical system. For example, in a rigid motion any two points are required to be at constant mutual distance. This is a *rigidity constraint*. A system with one of its points constrained on a surface is an example of a constrained mechanical system. Assume that a point P moves, being constrained to a surface $\mathcal{S} \subset \mathbb{R}^3$. Such a surface can be represented, at least locally, as the level set of a regular function f defined in a domain $G \subset \mathbb{R}^3$, i.e.,

$$\mathcal{S} = \{P \mid [f(P) = 0] \text{ and } \|\nabla f(P)\| > 0\}, \quad P \in G. \tag{1.1}$$

By the implicit function theorem, one of the coordinates, say for example x_3,

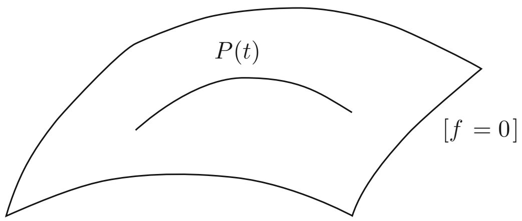

Fig. 1.1.

may be represented explicitly in terms of the remaining two. This provides a local parameterization of $\mathcal{S}$ in terms of (x_1, x_2). Such a parameterization is not unique. Indeed, choosing any pair of parameters $q = (q_1, q_2)$, the surface $\mathcal{S}$ can be represented, at least locally, by

$$\mathcal{S} = \begin{cases} x_1 = x_1(q_1, q_2), \\ x_2 = x_2(q_1, q_2), \\ x_3 = x_3\big(x_1(q), x_2(q)\big), \end{cases} \qquad \text{provided} \qquad \det\left(\frac{\partial(x_1, x_2)}{\partial(q_1, q_2)}\right) \neq 0.$$

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_2,

If P moves on $\mathcal{S}$, the Cartesian representation of its motion is determined by the function of time $t \to q(t) = \big(q_1(t), q_2(t)\big)$, through the composition

$$P(q) = \big(x_1(q), x_2(q), x_3(q)\big).$$

Conversely, the Cartesian representation $t \to P(t)$ of the motion of P permits one, by inversion, to determine $t \to q(t)$. The velocity of P is given by

$$\dot{P} = \frac{dP}{\partial q_h}\dot{q}_h = \nabla_q P \cdot \dot{q}.$$

The parameters $q = (q_1, q_2)$ are the *Lagrangian coordinates*, whereas $\dot{q} = (\dot{q}_1, \dot{q}_2)$ are the *Lagrangian velocities* of P [101]. The system has two degrees of freedom. Assume now that P is constrained on a regular curve $\gamma \subset \mathbb{R}^3$.

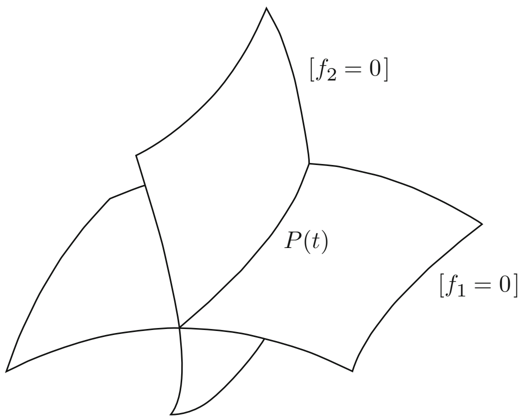

Fig. 1.2.

Such a curve can be represented, at least locally, as the intersection of the level sets of two smooth functions f_1 and f_2 defined in a domain $G \subset \mathbb{R}^3$, i.e.,

$$\gamma = \left\{ \begin{array}{c|c} P & [f_1(P) = 0] \cap [f_2(P) = 0] \\ & (\nabla f_1, \nabla f_2) \text{ of rank } 2 \end{array} \right\}, \qquad P \in G. \tag{1.2}$$

Then, at least locally, two of the coordinate variables, say for example x_2 and x_3, can be represented explicitly in terms of x_1. This provides a local parametric representation of γ in terms of the parameter x_1. At times it might be more convenient to introduce directly a parameter q and parameterize γ as $q \to P(q)$. Such a parameterization can be recast, at least locally, in terms of x_1, or any other parameter, provided $\|P'(q)\| > 0$, which we assume. The geometric trajectory of P is γ, and the motion is determined by the function of time $t \to q(t)$. The velocity is $\dot{P} = (dP/dq)\dot{q}$. The parameter q is the

Lagrangian coordinate of P, whereas $\dot{q}$ is its Lagrangian velocity. A moving point P constrained on γ has one degree of freedom.

The choice of the Lagrangian coordinates is not unique. In the applications it often occurs that one may introduce them directly, as suggested by the mechanical problem at hand, with no reference to the Cartesian representations of the constraints. The constraints in (1.1)–(1.2) are independent of time and are called *fixed* or *workless*. It is conceivable that P might move on a surface, or a curve, itself depending on time. As an example consider a point P moving in a horizontal plane $x_3 = 0$ and constrained by

$$x_1 \sin \omega t - x_2 \cos \omega t = 0, \qquad \omega \in \mathbb{R}.$$

This is a time-dependent restriction on the configurations of P. For a fixed t, the constraint is a straight line. As t changes, the constraint contributes to the determination of the geometric trajectory and the time-law of motion. Taking $q = \|P - O\|$ as Lagrangian coordinate, we have

$$P(q; t) = (q \cos \omega t, q \sin \omega t).$$

Therefore the position of P depends on q, and *explicitly* also on time. By differentiation,

$$\dot{P} = \frac{dP}{dq}\dot{q} + \frac{\partial P}{\partial t}.$$

The first of these vectors is the velocity of P on the constraint as if the constraint were independent of time. The second is the transport velocity due to the movement of the constraint. Constraints of this kind are *moving*.

2 Constrained Mechanical Systems

Consider n points $P_\ell = (x_{\ell,1}, x_{\ell,2}, x_{\ell,3})$, $\ell = 1, \dots, n$, subject to m constraints

$$f_j(P_1, \dots, P_n; t) = 0, \qquad j = 1, \dots, m, \tag{2.1}$$

where f_j are m smooth functions of their arguments. They are defined in $G \times I$, where G is an open subset of $\mathbb{R}^{3n}$ and I is an interval of $\mathbb{R}$. We assume that

$$\mathcal{S}(t) = \bigcap_{j=1}^{m} [f_j(\cdot; t) = 0] \neq \emptyset \quad \text{ for all } \ t \in I$$

and that for all $P \in S(t)$, the Jacobian matrix

$$\left(\frac{\partial f_j(P;t)}{\partial x_{\ell,1}} \quad \frac{\partial f_j(P;t)}{\partial x_{\ell,2}} \quad \frac{\partial f_j(P;t)}{\partial x_{\ell,3}} \right) \tag{2.2}$$

has maximum rank for all $t \in I$. For example, if $m < 3n$, such a matrix has rank m and the system has $3n - m$ degrees of freedom. This defines, at least locally, a $(3n-m)$-dimensional moving manifold $\mathcal{S}(t)$. Such a manifold can be parameterized, for all $t \in I$, in terms of $3n-m$ Lagrangian coordinates $q \in \mathbb{R}^{3n-m}$, i.e.,

$$\mathcal{S}(t) \ni P_\ell = P_\ell(q;t) \qquad \text{rank of} \left(\frac{\partial P_\ell}{\partial q}\right) = 3n-m \quad \forall t \in I.$$

If the points P_ℓ move on their constraints, their motion is determined by the $3n-m$ functions $t \to q(t)$. Their velocity is

$$\dot{P}_\ell = \frac{\partial P_\ell}{\partial q_h}\dot{q}_h + \frac{\partial P_\ell}{\partial t}.$$

The first is the instantaneous velocity of P_ℓ as moving on $\mathcal{S}(t)$, as if this surface were instantaneously fixed. The second is the velocity of transport of the constraint $\mathcal{S}(t)$. The constraints in (2.1) are in general moving constraints. If they do not depend on time, they are fixed, or workless. In such a case $\mathcal{S}$ and its parameterization, in terms of Lagrangian coordinates, are independent of t. The points $P_\ell = P_\ell(q)$ are represented only in terms of q and have no explicit dependence on time. Therefore

$$\frac{\partial P_\ell}{\partial t} = 0 \quad \text{and} \quad \dot{P}_\ell = \frac{\partial P_\ell}{\partial q_h}\dot{q}_h \qquad \text{(fixed constraints)}.$$

2.1 Actual and Virtual Displacements

An elemental displacement of the n points P_ℓ,

$$\big(P_1, \dots, P_n; t\big) \longrightarrow \big(P_1 + dP_1, \dots, P_n + dP_n; t + dt\big),$$

is said to be *actual* or *admissible* if it is compatible with the constraints in (2.1) along their time evolutions, i.e.,

$$\begin{aligned} f_j(P_1, \dots, P_n; t) &= 0, \\ f_j\big(P_1 + dP_1, \dots, P_n + dP_n; t + dt\big) &= 0, \end{aligned} \qquad j = 1, \dots, m.$$

From these we obtain

$$df_j = \frac{\partial f_j}{\partial P_\ell} dP_\ell + \frac{\partial f_j}{\partial t} dt = 0, \qquad j = 1, \dots, m. \tag{2.3}$$

An elemental displacement of the n points P_ℓ of the form

$$(P_1, \dots, P_n; t) \longrightarrow (P_1 + \delta P_1, \dots, P_n + \delta P_n; t)$$

is said to be *virtual* if it is compatible with the constraints (2.1) regarded as fixed at time t, i.e.,

$$\begin{aligned} f_j(P_1, \dots, P_n; t) &= 0, \\ f_j(P_1 + \delta P_1, \dots, P_n + \delta P_n; t) &= 0, \end{aligned} \qquad j = 1, \dots, m.$$

These imply

$$\frac{\partial f_j}{\partial P_\ell} \cdot \delta P_\ell = \nabla_{P_\ell} f_j \cdot \delta P_\ell = 0, \qquad j = 1, \ldots, m, \tag{2.4}$$

where the symbol δ denotes an elemental *virtual differential*. If the constraints in (2.1) are fixed, then virtual and actual displacements coincide.

2.2 Holonomic Constraints

A constraint, fixed or moving, is *holonomic* if it imposes restrictions only on the geometrical configuration of the points P_ℓ, and imposes no restriction on their time variations $\dot{P}_\ell$, $\ddot{P}_\ell$, etc. The constraints in (2.1) are holonomic. Consider two configurations $\mathcal{E} = (P_1, \ldots, P_n; t)$ and $\mathcal{E}' = (P_1', \ldots, P_n'; t')$ of the n points P_ℓ. These are compatible with the constraints (2.1) if they both satisfy the equations of the constraints. However, no restriction is placed on the displacements of the system needed to move $\mathcal{E}$ into $\mathcal{E}'$. For this reason, constraints of the type of (2.1), moving or fixed, are called also *configurational* constraints. A constraint that would impose restrictions on how $\mathcal{E}$ has to move into $\mathcal{E}'$ is not holonomic. For example, a constraint that would impose restrictions of the curvature of the trajectories of the points P_ℓ is not holonomic.

2.3 Unilateral Constraints

A point $P = (x_1, x_2, x_3)$ subject to the limitation $x_3 \geq x_1^2 + x_2^2$ is constrained to move within a paraboloid, possibly up to its boundary. Similarly, the constraint $\|P\| \leq 1$ forces P to move within the unit ball about the origin of $\mathbb{R}^3$, possibly up to its boundary.

Let $f \in C^1(G)$ be such that $\|\nabla f\| > 0$ in G. A point P is said to be subject to a *unilateral* constraint if it is required to satisfy $f(P) \leq 0$. If P is in the open set $[f < 0]$, its elemental displacements δP, starting at P, are unrestricted. Suppose now that $P \in [f = 0]$ and undergoes an elemental displacement δP, starting from this configuration. Since $\nabla f(P)$ points outside the set $[f \leq 0]$, the displacement δP will be compatible with the constraint only if the angle between ∇f and δP is right or obtuse. Therefore elemental displacements δP from boundary configurations $P \in [f = 0]$ are admissible only if $\nabla f(P) \cdot \delta P \leq 0$. It follows that elemental displacements of a point P subject to a unilateral constraint are, in general, not reversible.

A system of n points P_ℓ is subject to a unilateral constraint if

$$\{P_1, \ldots, P_n\} \in \bigcap_{j=1}^{m} \big[f_j \leq 0\big], \qquad f_j \in C^1(G \times I).$$

It is assumed that such an intersection is not empty and that (2.2) is in force. If a point P_ℓ is in the interior of its constraint then its elemental displacements δP_ℓ are unrestricted. If P_ℓ belongs to one of the surfaces $[f_j = 0]$, its virtual displacements δP_ℓ must satisfy $\nabla f_j \cdot \delta P_\ell \leq 0$.

3 Intrinsic Metrics and First Fundamental Form

A surface $\mathcal{S} \subset \mathbb{R}^3$ is the image of a smooth vector-valued function

$$G \ni (u,v) \longrightarrow P(u,v) = \big(x_1(u,v), x_2(u,v), x_3(u,v)\big),$$

defined in a connected open set $G \subset \mathbb{R}^2$, such that the matrix

$$\begin{pmatrix} \dfrac{\partial P}{\partial u} \\[2ex] \dfrac{\partial P}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \\[2ex] \dfrac{\partial x_1}{\partial v} & \dfrac{\partial x_2}{\partial v} & \dfrac{\partial x_3}{\partial v} \end{pmatrix}$$

has maximum rank. Set

$$\begin{aligned} A &= \left(\frac{\partial P}{\partial u}\right)^2 = \frac{\partial P}{\partial u} \cdot \frac{\partial P}{\partial u} = \sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial u}\right)^2, \\ B &= \frac{\partial P}{\partial u} \cdot \frac{\partial P}{\partial v} = \sum_{i=1}^{3} \frac{\partial x_i}{\partial u}\frac{\partial x_i}{\partial v}, \\ C &= \left(\frac{\partial P}{\partial v}\right)^2 = \frac{\partial P}{\partial v} \cdot \frac{\partial P}{\partial v} = \sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial v}\right)^2, \end{aligned}$$

and consider the quadratic form

$$\begin{aligned} (\xi\ \eta) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= A\xi^2 + 2B\xi\eta + C\eta^2 \\ &= \frac{1}{A}\left[(A\xi + B\eta)^2 + (AC - B^2)\eta^2\right], \end{aligned} \tag{3.1}$$

where $(\xi, \eta) \in \mathbb{R}^2$ is arbitrary. By the Cauchy inequality, $AC - B^2 \geq 0$. Therefore the quadratic form in (3.1) is positive definite. It is called the *first fundamental form* of the surface $\mathcal{S}$. An elemental variation (du, dv) of the parameters (u, v) induces an infinitesimal variation dP on $\mathcal{S}$, whose modulus is

$$\begin{aligned} (ds)^2 = dP \cdot dP &= \left(\frac{\partial P}{\partial u}du + \frac{\partial P}{\partial v}dv\right)^2 \\ &= A(du)^2 + 2Bdudv + C(dv)^2. \end{aligned} \tag{3.2}$$

This is the *intrinsic metric* on $\mathcal{S}$. It is intrinsic, since it depends only on the geometry of $\mathcal{S}$, and it is independent of its parameterization (see §3c of the Complements). To a regular curve $\gamma \subset G$, parameterized by t, there corresponds a curve $\Gamma_\gamma \subset \mathcal{S}$ by the correspondence

$$\gamma = \big\{t \to \big(u(t), v(t)\big)\big\} \quad \Longleftrightarrow \quad \Gamma_\gamma = \big\{t \to P\big(u(t), v(t)\big)\big\}. \tag{3.3}$$

If the elemental variation (du, dv) occurs along γ, (3.2) gives an elemental arc length on the corresponding Γ_γ. If γ is parameterized by v or respectively by u, the elemental arc length of Γ_γ is computed from (3.2) as

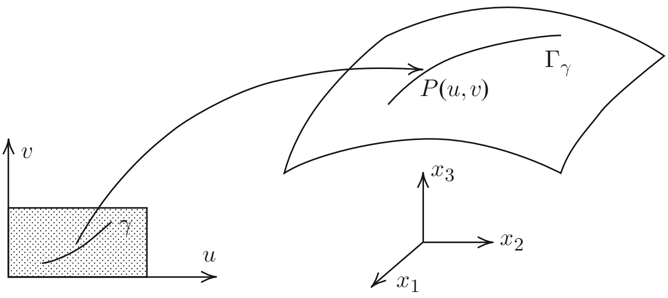

Fig. 3.1.

$$
\begin{aligned}
ds &= \sqrt{A(du)^2 + 2Bdudv + C(dv)^2} \\
&= \sqrt{A\left(\frac{du}{dv}\right)^2 + 2B\left(\frac{du}{dv}\right) + C}\; dv \\
&= \sqrt{A + 2B\left(\frac{dv}{du}\right) + C\left(\frac{dv}{du}\right)^2}\; du.
\end{aligned}
\tag{3.4}
$$

4 Geodesics

Consider now those curves in G for which one of the two parameters is fixed:

$$
\begin{aligned}
\gamma_u &= \{u \to (u, v_o),\ v_o = \text{const}\} \quad \Longleftrightarrow \quad \Gamma_u = \{u \to P(u, v_o)\}\,, \\
\gamma_v &= \{v \to (u_o, v),\ u_o = \text{const}\} \quad \Longleftrightarrow \quad \Gamma_v = \{v \to P(u_o, v)\}\,.
\end{aligned}
$$

The vector $\partial P/\partial u$ is tangent to Γ_u and $\partial P/\partial v$ is tangent to Γ_v. These two vectors are linearly independent, since the matrix $(\partial P/\partial u, \partial P/\partial v)$ has maximum rank. They permit one to compute the elemental area $d\sigma$ on $\mathcal{S}$ and its normal unit vector ν by the formulas

$$
d\sigma = \left\| \frac{\partial P}{\partial u} \wedge \frac{\partial P}{\partial v} \right\| du\, dv, \qquad \nu = \frac{\dfrac{\partial P}{\partial u} \wedge \dfrac{\partial P}{\partial v}}{\left\| \dfrac{\partial P}{\partial u} \wedge \dfrac{\partial P}{\partial v} \right\|}.
$$

A curve $\Gamma_\gamma \subset \mathcal{S}$ is a *geodesic* if its normal $\mathbf{n}$ is parallel to ν at any of its points, i.e., if $\mathbf{n} \wedge \nu = 0$. This occurs if

$$
\mathbf{n} \cdot \frac{\partial P}{\partial u} = 0 \quad \text{and} \quad \mathbf{n} \cdot \frac{\partial P}{\partial v} = 0.
$$

If $\mathit{\Gamma}_\gamma$ is parameterized as in (3.3), set

$$\mathit{\Delta} = A\dot{u}^2 + 2B\dot{u}\dot{v} + C\dot{v}^2 = \dot{s}^2.$$

Then the unit tangent $\mathbf{t}$ and the unit normal $\mathbf{n}$ to $\mathit{\Gamma}_\gamma$ are

$$\mathbf{t} = \frac{1}{\sqrt{\mathit{\Delta}}}\left(\frac{\partial P}{\partial u}\dot{u} + \frac{\partial P}{\partial v}\dot{v}\right), \qquad \kappa\mathbf{n} = \frac{d}{dt}\left[\frac{1}{\sqrt{\mathit{\Delta}}}\left(\frac{\partial P}{\partial u}\dot{u} + \frac{\partial P}{\partial v}\dot{v}\right)\right]\frac{dt}{ds}.$$

Imposing now that $\mathbf{n}$ be normal to $\partial P/\partial u$ and discarding the factor dt/ds gives

$$\begin{aligned}
0 &= \frac{\partial P}{\partial u}\frac{d}{dt}\left[\frac{1}{\sqrt{\mathit{\Delta}}}\left(\frac{\partial P}{\partial u}\dot{u} + \frac{\partial P}{\partial v}\dot{v}\right)\right] \\
&= \frac{d}{dt}\left[\frac{1}{\sqrt{\mathit{\Delta}}}\left(\frac{\partial P}{\partial u}\right)^2\dot{u} + \frac{\partial P}{\partial u}\frac{\partial P}{\partial v}\dot{v}\right] - \frac{1}{\sqrt{\mathit{\Delta}}}\left(\frac{\partial P}{\partial u}\dot{u} + \frac{\partial P}{\partial v}\dot{v}\right)\frac{d}{dt}\frac{\partial P}{\partial u} \\
&= \frac{d}{dt}\left[\frac{1}{\sqrt{\mathit{\Delta}}}(A\dot{u} + B\dot{v})\right] - \frac{1}{\sqrt{\mathit{\Delta}}}\left(\frac{\partial P}{\partial u}\frac{\partial^2 P}{\partial u^2}\dot{u}^2 + \frac{\partial P}{\partial v}\frac{\partial^2 P}{\partial u^2}\dot{u}\dot{v}\right) \\
&\quad - \frac{1}{\sqrt{\mathit{\Delta}}}\left(\frac{\partial P}{\partial u}\frac{\partial^2 P}{\partial u \partial v}\dot{u}\dot{v} + \frac{\partial P}{\partial v}\frac{\partial^2 P}{\partial u \partial v}\dot{v}^2\right) \\
&= \frac{d}{dt}\frac{\partial}{\partial \dot{u}}\sqrt{\mathit{\Delta}} - \frac{1}{2\sqrt{\mathit{\Delta}}}\left(\frac{\partial A}{\partial u}\dot{u}^2 + 2\frac{\partial B}{\partial u}\dot{u}\dot{v} + \frac{\partial C}{\partial u}\dot{v}^2\right) \\
&= \frac{d}{dt}\frac{\partial}{\partial \dot{u}}\sqrt{\mathit{\Delta}} - \frac{\partial}{\partial u}\sqrt{\mathit{\Delta}}.
\end{aligned}$$

Proposition 4.1 *A curve $\mathit{\Gamma}_\gamma \subset \mathcal{S}$ parameterized as in (3.3) is a geodesic if and only if the functions $t \to u(t), v(t)$ are solutions of the system of second-order differential equations*

$$\frac{d}{dt}\frac{\partial}{\partial \dot{u}}\sqrt{\mathit{\Delta}} - \frac{\partial}{\partial u}\sqrt{\mathit{\Delta}} = 0, \qquad \frac{d}{dt}\frac{\partial}{\partial \dot{v}}\sqrt{\mathit{\Delta}} - \frac{\partial}{\partial v}\sqrt{\mathit{\Delta}} = 0. \tag{4.1}$$

These may be rewritten in terms of u or v taken as local parameters. For example, taking u as local parameter, one may express locally $v = v(u)$ as a function of u. Then the second equality of (4.1) takes the form

$$\frac{d}{du}\frac{\partial}{\partial v'}\sqrt{\mathit{\Delta}} - \frac{\partial}{\partial v}\sqrt{\mathit{\Delta}} = 0. \tag{4.2}$$

Remark 4.1 The same equations arise by regarding the geodesics as the curves of least path on $\mathcal{S}$ between any two of its points (§1.4c of Chapter 9).

5 Examples of Geodesics

5.1 Geodesics in a Plane

Let $\mathcal{S}$ be the plane of equation $a_i x_i = b$. Assuming $a_3 \neq 0$, we may take $u = x_1$ and $v = x_2$ and compute

$$A = \frac{a_1^2 + a_3^2}{a_3^2}, \qquad B = \frac{a_1 a_2}{a_3^2}, \qquad C = \frac{a_2^2 + a_3^2}{a_3^2}.$$

In view of (4.2), the problem reduces to solving

$$\frac{d}{du} \frac{Cv' + B}{\sqrt{A + 2Bv' + Cv'^2}} = 0,$$

which implies $v'' = 0$. Therefore the geodesics in a plane are line segments.

5.2 Geodesics on a Sphere

From the parameterization of the sphere of radius R in terms of polar coordinates, we have

$$A = R^2 \sin^2 v, \quad B = 0, \quad C = R^2; \qquad \Delta = R^2 \left(\sin^2 v + v'^2\right).$$

Therefore, by (4.2), the geodesics $v = v(u)$ on a sphere are solutions of

$$\frac{d}{du} \frac{v'}{\sqrt{\sin^2 v + v'^2}} - \frac{\sin v \cos v}{\sqrt{\sin^2 v + v'^2}} = 0.$$

Multiplying by v' and performing elementary manipulations yields

$$\frac{d}{du} \left(\frac{v'^2}{\sqrt{\sin^2 v + v'^2}} - \sqrt{\sin^2 v + v'^2} \right) = 0.$$

It follows that for a constant $c \in (0,1)$,

$$\begin{aligned} u(v) &= c \int \frac{dv}{\sqrt{\sin^4 v - c^2 \sin^2 v}} = \int \frac{1}{\sin^2 v} \frac{dv}{\sqrt{\left(\frac{1-c^2}{c^2}\right) - \cot^2 v}} \\ &= -\arcsin\left(\frac{c}{\sqrt{1-c^2}} \cot v \right) + \bar{c}. \end{aligned}$$

This finally implies

$$(\sin \bar{c}) R \sin v \cos u - (\cos \bar{c}) R \sin v \sin u = \frac{c}{\sqrt{1-c^2}} R \cos v.$$

Therefore the geodesics are the intersection of the sphere with the planes through the origin,

$$(\sin \bar{c}) x_1 - (\cos \bar{c}) x_2 = \frac{c}{\sqrt{1-c^2}} x_3.$$

The meridians on a sphere are geodesics, whereas none of the parallels, except the equator, is a geodesic.

5.3 Geodesics on Surfaces of Revolution

If $\mathcal{S}$ is a surface of revolution, after a possible rotation and relabeling of the coordinate axes, it can be parameterized as

$$x_1 = u\cos v, \quad x_2 = u\sin v, \quad x_3 = f(u). \tag{5.1}$$

Here f is a smooth function of the variable u, and $\mathcal{S}$ is interpreted as a surface obtained by rotating the graph of $x_3 = f(x_1)$ about the x_3 axis. From such a

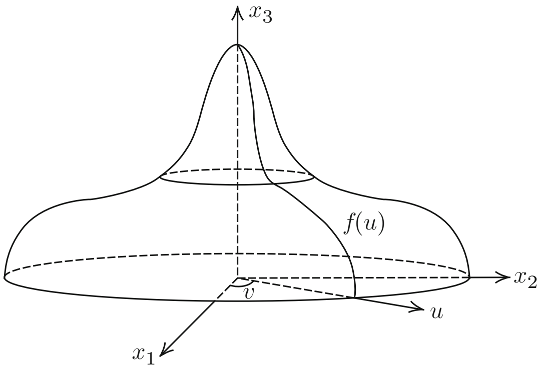

Fig. 5.1.

parametric representation, we compute

$$A = 1 + f'^2(u), \quad B = 0, \quad C = u^2; \qquad \Delta = 1 + f'^2(u) + u^2 v'^2.$$

If $\Gamma_\gamma \subset \mathcal{S}$ is a geodesic parameterized by u, then by (4.2),

$$\frac{d}{du}\frac{\partial}{\partial v'}\sqrt{1 + f'^2(u) + u^2 v'^2} = \frac{d}{du}\frac{u^2 v'}{\sqrt{1 + f'^2(u) + u^2 v'^2}} = 0.$$

Therefore, for a constant $c \in \mathbb{R}$ and $u > c$,

$$u^4 v'^2 = c^2\left[1 + f'^2(u) + u^2 v'^2\right] \implies v' = \pm c\frac{\sqrt{1 + f'^2(u)}}{u\sqrt{u^2 - c^2}}.$$

From these one finds the implicit equation of the geodesics in the form

$$v - v_o = \pm c\int_{u_o}^{u} \frac{\sqrt{1 + f'^2(\eta)}}{\eta\sqrt{\eta^2 - c^2}}\,d\eta.$$

Recalling the geometric meaning of Δ, we obtain

$$\frac{du}{ds} = \frac{1}{\sqrt{1 + f'^2(u) + u^2 v'^2}},$$

where ds is the elemental arc length on the geodesic. With this symbolism, the previous differential equations of a geodesic can be rewritten in the form

$$\frac{d}{ds}\left(u^2\frac{dv}{ds}\right)=0, \qquad \frac{dv}{ds}=\pm c\frac{\sqrt{1+f'^2(u)}}{u\sqrt{u^2-c^2}}\frac{du}{ds}. \tag{5.2}$$

If $c=0$, then $dv/ds=0$ and therefore the curves $v=$ const are geodesics. These are the meridians traced on $\mathcal{S}$. The parallels, i.e., the curves $u=$ const on $\mathcal{S}$, in general are not geodesics. From (5.2) it follows that for a parallel to be a geodesic, we must have $dv/ds=\text{const}\neq 0$, and moreover,

$$c\frac{du}{ds}=\pm\frac{u\sqrt{u^2-c^2}}{\sqrt{1+f'^2(u)}}\frac{dv}{ds}, \qquad \text{i.e.,} \qquad f'^2(u)=\infty.$$

Geometrically, a parallel $\mathcal{P}\subset\mathcal{S}$ is a geodesic if $\mathcal{S}$ is tangent along $\mathcal{P}$ to a right circular cylinder with vertical axis. Thus for a sphere, only the equatorial parallel is a geodesic. For a cylinder the geodesics are curves normal to a generator at each of their points.

Problems and Complements

1c Constrained Trajectories

A class of Lagrangian coordinates arises by a change of variables in $\mathbb{R}^3$ as indicated by the following examples.

1.1c Elliptic Coordinates

Let ℓ be a fixed positive parameter and define

$$\begin{cases} x_1=\ell\sinh u\cos\varphi\sin\theta, & u\in\mathbb{R}^+,\\ x_2=\ell\sinh u\sin\varphi\sin\theta, & \varphi\in[0,2\pi),\\ x_3=\ell\cosh u\cos\theta, & \theta\in[0,\pi]. \end{cases} \tag{1.1c}$$

From these we obtain

$$\frac{x_1^2+x_2^2}{\ell^2\sinh^2 u}+\frac{x_3^2}{\ell^2\cosh^2 u}=1, \qquad \frac{x_3^2}{\ell^2\cos^2\theta}-\frac{x_1^2+x_2^2}{\ell^2\sin^2\theta}=1.$$

Therefore the surfaces $u=\text{const}>0$ are ellipsoids of revolution about the x_2-axis. The semiaxes are $a_1=a_2=\ell\sinh u$ and $a_3=\ell\cosh u$. The surfaces $\theta=$ const are hyperboloids of two sheets. They are of revolution about the x_3-axis. Determine the "coordinate planes" $\varphi=$ const.

The velocity of a moving point P expressed in elliptic coordinates is

$$\dot{P} = \ell \begin{pmatrix} \dot{u}\cosh u \cos\varphi \sin\theta - \dot{\varphi}\sinh u \sin\varphi \sin\theta + \dot{\theta}\sinh u \cos\varphi \cos\theta \\ \dot{u}\cosh u \sin\varphi \sin\theta + \dot{\varphi}\sinh u \cos\varphi \sin\theta + \dot{\theta}\sinh u \sin\varphi \cos\theta \\ \dot{u}\sinh u \cos\theta - \dot{\theta}\cosh u \sin\theta \end{pmatrix},$$

and its modulus squared is

$$\|\dot{P}\|^2 = \ell^2\big(\dot{u}^2 + \dot{\theta}^2\big)\big(\sinh^2 u + \sin^2\theta\big) + 2\ell^2\dot{\varphi}^2\sinh^2 u \sin^2\theta.$$

If $u = \text{const} > 0$, one obtains from these the expressions of the velocity and its modulus for a point constrained on an ellipsoid of revolution about the x_3-axis. Set

$$\mathcal{A}(u,\theta) = \sinh^2 u + \sin^2\theta$$

and compute from (1.1c)

$$u_{x_1} = \frac{\cosh u \cos\varphi \sin\theta}{\ell\,\mathcal{A}(u,\theta)}, \qquad \theta_{x_1} = \frac{\sinh u \cos\varphi \cos\theta}{\ell\,\mathcal{A}(u,\theta)},$$

$$u_{x_2} = \frac{\cosh u \sin\varphi \sin\theta}{\ell\,\mathcal{A}(u,\theta)}, \qquad \theta_{x_2} = \frac{\sinh u \sin\varphi \cos\theta}{\ell\,\mathcal{A}(u,\theta)},$$

$$u_{x_3} = \frac{\sinh u \cos\theta}{\ell\,\mathcal{A}(u,\theta)}, \qquad \theta_{x_3} = \frac{-\cosh u \sin\theta}{\ell\,\mathcal{A}(u,\theta)},$$

$$\varphi_{x_1} = \frac{-\sin\varphi}{\ell\sinh u \sin\theta}, \qquad \varphi_{x_2} = \frac{\cos\varphi}{\ell\sinh u \sin\theta}, \qquad \varphi_{x_3} = 0.$$

From these, the Jacobian of the transformation from Cartesian coordinates into elliptic coordinates is

$$J_{\text{Cart}\to\text{ell}} = \ell^3\mathcal{A}(u,\theta)\sinh u \sin\theta.$$

Let $x \to f(x)$ be a smooth function in a domain $G \subset \mathbb{R}^3$. Set

$$F(u,\varphi,\theta) = f\big(\ell\sinh u \cos\varphi \sin\theta, \ell\sinh u \sin\varphi \sin\theta, \ell\cosh u \cos\theta\big)$$

and verify that

$$\|\nabla_x f\|^2 = \frac{F_u^2}{\ell^2\mathcal{A}(u,\theta)} + \frac{F_\theta^2}{\ell^2\mathcal{A}(u,\theta)} + \frac{F_\varphi^2}{\ell^2\sinh^2 u \sin^2\theta}.$$

1.2c Parabolic Coordinates

Set

$$\begin{cases} x_1 = \sqrt{uv}\cos\varphi, & u, v \geq 0, \\ x_2 = \sqrt{uv}\sin\varphi, & \varphi \in [0, 2\pi), \\ x_3 = \dfrac{u-v}{2}. \end{cases} \tag{1.2c}$$

For $u > 0$ fixed, compute v from the first two equalities and put it in the last to get

$$x_3 = \frac{u^2 - r^2}{2u}, \qquad r = \sqrt{x_1^2 + x_2^2}.$$

For $u = \text{const} > 0$ these are paraboloids with vertex at $(0, 0, \frac{1}{2}u)$. Analogously, keeping v constant, a similar calculation gives the paraboloids

$$x_3 = \frac{r^2 - v^2}{2v}, \qquad v > 0.$$

Therefore the generalized coordinate surfaces $u = \text{const}$ or $v = \text{const}$ are paraboloids. For this reason the variables (u, v, φ) are called *parabolic* coordinates. Describe the surfaces $\varphi = \text{const}$.

The velocity of a point P in terms of parabolic coordinates is

$$\dot{P} = \begin{pmatrix} \frac{1}{2}\dot{u}\sqrt{\frac{v}{u}}\cos\varphi + \frac{1}{2}\dot{v}\sqrt{\frac{u}{v}}\cos\varphi - \dot{\varphi}\sqrt{uv}\sin\varphi \\ \\ \frac{1}{2}\dot{u}\sqrt{\frac{v}{u}}\sin\varphi + \frac{1}{2}\dot{v}\sqrt{\frac{u}{v}}\sin\varphi + \dot{\varphi}\sqrt{uv}\cos\varphi \\ \\ \frac{1}{2}(\dot{u} - \dot{v}) \end{pmatrix},$$

and its modulus squared is

$$\|\dot{P}\|^2 = \frac{u+v}{4}\left(\frac{\dot{u}^2}{u} + \frac{\dot{v}^2}{v}\right) + \dot{\varphi}^2 uv.$$

From the first two equalities of (1.2c) we have $uv = r^2$. From this and the third equality of (1.2c),

$$\begin{aligned} u_{x_i} &= \frac{2x_i}{u+v}, & v_{x_i} &= \frac{2x_i}{u+v}, \qquad i = 1, 2, \\ u_{x_3} &= \frac{2u}{u+v}, & v_{x_3} &= \frac{-2v}{u+v}, \\ \varphi_{x_1} &= -\frac{x_2}{x_1^2}\cos^2\varphi, & \varphi_{x_2} &= \frac{1}{x_1}\cos^2\varphi. \end{aligned}$$

From these the Jacobian of the transformation from Cartesian coordinates to parabolic coordinates is

$$J_{\text{Cart}\to\text{parab}} = \tfrac{1}{4}(u+v).$$

Let $x \to f(x)$ be a smooth function in a domain $G \subset \mathbb{R}^3$. Set

$$F(u, v, \varphi) = f\big(\sqrt{uv}\cos\varphi, \sqrt{uv}\sin\varphi, \tfrac{1}{2}uv\big)$$

and verify that

$$\|\nabla_x f\|^2 = \frac{4}{u+v}\left(F_u^2 u + F_v^2 v\right) + \frac{1}{uv}F_\varphi^2.$$

1.3c Spherical Coordinates

Compute the velocity of a point $P \in \mathbb{R}^3$ in terms of spherical coordinates

$$\begin{cases} x_1 = \rho\cos\varphi\sin\theta, & \rho \geq 0, \\ x_2 = \rho\sin\varphi\sin\theta, & \varphi \in [0, 2\pi), \\ x_3 = \rho\cos\theta, & \theta \in [0, \pi]. \end{cases} \tag{1.3c}$$

Verify that

$$\dot{P} = \begin{pmatrix} \dot\rho\cos\varphi\sin\theta - \dot\varphi\rho\sin\varphi\sin\theta + \dot\theta\rho\cos\varphi\cos\theta \\ \dot\rho\sin\varphi\sin\theta + \dot\varphi\rho\cos\varphi\sin\theta + \dot\theta\rho\sin\varphi\cos\theta \\ \dot\rho\cos\theta - \dot\theta\rho\sin\theta \end{pmatrix}$$

and

$$\|\dot{P}\|^2 = \dot\rho^2 + \dot\varphi^2\rho^2\sin^2\theta + \dot\theta^2\rho^2.$$

Compute the expressions of the velocity of a point constrained to move in the cavity of a sphere (spherical pendulum).

1.4c Cylindrical Coordinates

Compute the velocity of a moving point $P \in \mathbb{R}^3$ in terms of cylindrical coordinates

$$\begin{cases} x_1 = r\cos\varphi, & r \geq 0, \\ x_2 = r\sin\varphi, & \varphi \in [0, 2\pi), \\ x_3 = x_3, & y_3 \in \mathbb{R}. \end{cases} \tag{1.4c}$$

Verify that

$$\dot{P} = \begin{pmatrix} \dot r\cos\varphi - \dot\varphi r\sin\varphi \\ \dot r\sin\varphi + \dot\varphi r\cos\varphi \\ \dot x_3 \end{pmatrix}, \qquad \|\dot{P}\|^2 = \dot r^2 + \dot\varphi^2 r^2 + \dot x_3^2.$$

Compute the expression of the velocity of a point moving on a right circular cylinder with vertical axis.

2c Constrained Mechanical Systems

2.1c Holonomic and Nonholonomic Constraints

Let a mechanical system be described by N independent Lagrangian parameters $q = (q_1, \ldots, q_N)$. A holonomic constraint on the system is of the form $f(q;t) = \text{const}$, or by taking derivatives in t,

$$f_{q_h}\dot q_h + f_t = 0 \qquad \text{along the motion.}$$

On the other hand, a constraint of the type

$$A_h(q;t)\dot{q}_h + A_o(q;t) = 0$$

is in general not holonomic, since it imposes limitations on the Lagrangian configurations q and the Lagrangian velocities $\dot{q}$. However, if there exists a smooth function $f(q;t)$ such that

$$A_h(q;t) = f_{q_h}(q;t), \quad i = 1, \ldots, N, \quad A_o(q;t) = f_t(q;t),$$

then such a constraint can be rewritten as $\dot{f}(q;t) = 0$ or $f(q;t) = \text{const}$ and is therefore holonomic. As an example consider a point P constrained by $\dot{P} = \mathbf{u}$, where $\mathbf{u}$ is a fixed vector. Such a constraint is holonomic, since it requires only that the trajectory be a straight line.

The constraint $\|\dot{P}\| = c$, where c is a given positive constant, restricts the modulus of the velocity, and it cannot be reduced to a holonomic constraint by integration. Notice that, in contrast to the previous example, such a constraint does not restrict the configurations of P. In particular, P might go from P_1 to P_2 along an arbitrary path, provided the motion occurs at constant speed.

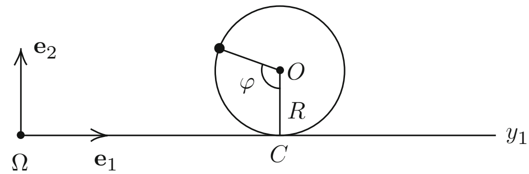

Fig. 2.1c.

2.2c Disk Rolling without Slipping on a Line

A disk of center O and radius R is constrained to move on a linear horizontal guide while remaining in a fixed vertical plane, as in **Figure 2.1c**. The system has two degrees of freedom, and we may choose as Lagrangian coordinates the angle φ between $C-O$ and a fixed radius. Requiring that the disk roll without slipping means to impose on the contact point C, regarded as part of the rigid motion of the disk, to have zero velocity,

$$\dot{C} = \dot{O} - \dot{\varphi}\mathbf{e}_3 \wedge (C - O) = 0.$$

This can be written in the form $\dot{y}_1 - R\dot{\varphi} = 0$, which is equivalent to

$$y_1 - R\varphi = \text{const}.$$

Therefore for a disk on a guide, the constraint of “rolling without slipping” is holonomic. Assume that the disk moves on a parabola, an ellipse, or a cycloid, remaining on a fixed vertical plane. Write down the analytical expression of the constraint “rolling without slipping” and conclude that in all cases, the constraint is holonomic.

2.3c Sphere Rolling without Slipping in a Plane

A sphere of center O and radius R is required to roll without slipping in a horizontal plane, as in **Figure 2.2c**. As Lagrangian coordinates take the Cartesian coordinates y_1, y_2 of the center O and the Euler angles φ, ψ, θ, formed by a moving triad S with origin O and fixed with the sphere, with a fixed triad Σ. The constraint of “rolling without slipping” translates into

$$\dot{C} = \dot{y}_1 \mathbf{e}_1 + \dot{y}_2 \mathbf{e}_2 + \boldsymbol{\omega} \wedge (C - O) = 0.$$

Using the expression of the vector $\boldsymbol{\omega}$ in terms of the Euler angles (formula

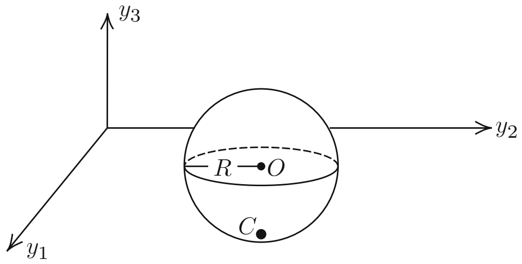

Fig. 2.2c.

(9.5) of Chapter 1), this can be rewritten as

$$\begin{aligned} \dot{y}_1 + R(\dot{\psi} \sin\theta \cos\varphi - \dot{\theta} \sin\varphi) = 0, \\ \dot{y}_2 + R(\dot{\psi} \sin\theta \sin\varphi + \dot{\theta} \cos\varphi) = 0. \end{aligned}$$

Such a constraint cannot be integrated, that is, cannot be expressed as

$$f\big(y_1, y_2, \theta, \varphi, \psi; t\big) = \text{const}$$

for some smooth function f. If such an f were to exist, it would have to satisfy

$$f_{y_1}\dot{y}_1 + f_{y_2}\dot{y}_2 + f_\theta \dot{\theta} + f_\varphi \dot{\varphi} + f_\psi \dot{\psi} + f_t = 0. \tag{$*$}$$

Put into this the previous expressions of $\dot{y}_1$ and $\dot{y}_2$ to get

$$\begin{aligned} -f_{y_1} R(\dot{\psi} \sin\theta \cos\varphi - \dot{\theta} \sin\varphi) - f_{y_2} R(\dot{\psi} \sin\theta \sin\varphi + \dot{\theta} \cos\varphi) \\ + f_\theta \dot{\theta} + f_\varphi \dot{\varphi} + f_\psi \dot{\psi} + f_t = 0. \end{aligned}$$

Taking the derivative with respect to $\dot{\varphi}$ gives $\partial f/\partial\varphi = 0$. Therefore f is independent of φ. Taking now the derivative with respect to φ, and keeping in mind that f is independent of φ, yields

$$\sin\theta(f_{y_1}\sin\varphi - f_{y_2}\cos\varphi)\dot{\psi} + (f_{y_1}\cos\varphi + f_{y_2}\sin\varphi)\dot{\theta} = 0.$$

Since the displacements $d\theta$ and $d\varphi$ are arbitrary, this generates the algebraic homogeneous linear system

$$\begin{aligned} f_{y_1}\sin\varphi - f_{y_2}\cos\varphi &= 0, \\ f_{y_1}\cos\varphi + f_{y_2}\sin\varphi &= 0, \end{aligned}$$

in the unknowns f_{y_i}. The system admits only the trivial solution $f_{y_i} = 0$, $i = 1, 2$. Therefore f is independent of y_1 and y_2. The independence of φ, y_1, y_2 permits one to rewrite $(*)$ as

$$f_\theta\dot{\theta} + f_\psi\dot{\psi} + f_t = 0.$$

Taking now the derivative with respect to $\dot{\theta}$ gives $f_\theta = 0$, and analogously we also have $f_\psi = 0$. Therefore f is independent of θ and ψ. Finally, it is also independent of t. The contradiction implies that no such f exists. Therefore for a sphere moving in a plane, the constraint of "rolling without slipping" is not holonomic. The nonexistence of f means that the Lagrangian parameters $(y_1, y_2, \theta, \varphi, \psi)$ are not restricted, i.e., the sphere might take any configuration in the plane. Thus the constraints must act by limiting the Lagrangian velocities.

2.4c Rigid Rod with Constrained Extremities

The extremities A and B of a rigid rod of length h are constrained to move in two orthogonal planes π_1 and π_2 as in **Figure 2.3c**. One of the extremities, say B, is connected to a point $C \in \pi_2$ through a rigid rod BC of length ℓ. The other extremity, A, is connected to a point $O \in \pi_1 \cap \pi_2$ by a rigid rod OA, of length ℓ. The point C is at distance ℓ from $\pi_1 \cap \pi_2$. Take a Cartesian system with origin in O, and x-axis as $\pi_1 \cap \pi_2$, oriented so that $C = (-\ell, 0, \ell)$.

(a). Determine the number of degrees of freedom of the system. Write down the equations of the constraints and form their Jacobian matrix.

(b). Compute the determinant of all minors of maximum rank and find conditions on h and ℓ for the Jacobian matrix to have maximum rank.

The system has one degree of freedom and the constraints are

$$z_A = 0, \quad y_B = 0, \quad x_A^2 + y_A^2 = \ell^2, \quad (x_B + \ell)^2 + (z_B - \ell)^2 = \ell^2,$$

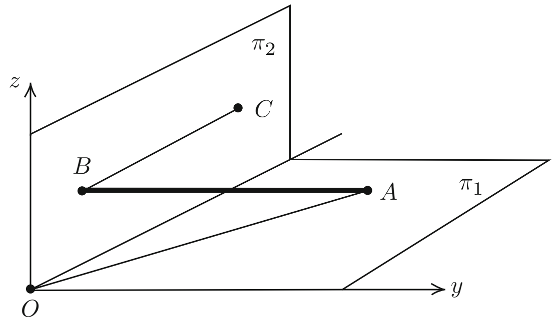

Fig. 2.3c.

and in addition, $|B - A| = h$. Using the third and fourth equations of the constraints, this can be rewritten as

$$2x_A x_B + 2\ell\,(x_B - z_B) + h^2 = 0.$$

From these one computes the Jacobian matrix

$$J = \begin{pmatrix} 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ x_A & y_A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_B + \ell & 0 & z_B - \ell \\ x_B & 0 & 0 & x_A + \ell & 0 & -\ell \end{pmatrix}.$$

The minors of order 5 with nonzero determinant must contain the third and fifth columns. Therefore the problem reduces to extracting the nontrivial non minors of order three out of the last three rows. These are

$$D_1 = \begin{pmatrix} x_A & y_A & 0 \\ 0 & 0 & x_A + \ell \\ x_B & 0 & x_A + \ell \end{pmatrix}, \qquad D_2 = \begin{pmatrix} x_A & y_A & 0 \\ 0 & 0 & z_B - \ell \\ x_B & 0 & -\ell \end{pmatrix},$$

$$D_3 = \begin{pmatrix} x_A & 0 & 0 \\ 0 & x_B + \ell & z_B - \ell \\ x_B & x_A + \ell & -\ell \end{pmatrix}, \qquad D_4 = \begin{pmatrix} y_A & 0 & 0 \\ 0 & x_B + \ell & z_B - \ell \\ 0 & x_A + \ell & -\ell \end{pmatrix}.$$

By direct calculation,

$$\begin{aligned} \det D_1 &= y_A x_B (x_B + \ell), \\ \det D_2 &= y_A x_B (z_B - \ell), \\ \det D_3 &= -x_A [(x_B + \ell)\ell + (z_B - \ell)(x_A + \ell)], \\ \det D_4 &= -y_A\, [(x_B + \ell)\ell + (z_B - \ell)(x_A + \ell)]. \end{aligned}$$

For J to have maximum rank we must have $\sum_{i=1}^4 \det^2 D_i > 0$. Using the last two equations of the constraints, we obtain

$$\sum_{i=1}^{4} \det^2 D_i = y_A^2 x_B^2 \ell^2 + [(x_B + \ell)\ell + (z_B - \ell)(x_A + \ell)]^2\, \ell^2.$$

Therefore J is not of maximum rank if

$$y_A x_B = 0 \quad \text{and} \quad (x_B + \ell)\ell = -(z_B - \ell)(x_A + \ell). \tag{$*$}$$

If $x_B = 0$, then $z_B = \ell = 0$ and $h^2 = 2\ell^2$. If $x_B \neq 0$ and $y_A = 0$, then $x_A = \pm\ell$. If $x_A = -\ell$, then also $x_B = -\ell$ and $h^2 = 2\ell^2$. Examine the remaining cases.

3c Intrinsic Metrics and First Fundamental Form

A new parameterization of $\mathcal{S}$ is a smooth invertible transformation

$$G \ni (u,v) \quad \left\{ \begin{matrix} u = u(u',v') & v = v(u',v') \\ u' = u'(u,v) & v' = v'(u,v) \end{matrix} \right\}, \qquad (u',v') \in G',$$

from G into a domain $G' \subset \mathbb{R}^2$. The matrix is invertible if the Jacobian determinant is nonzero, i.e., if

$$J = \begin{pmatrix} \dfrac{\partial u}{\partial u'} & \dfrac{\partial u}{\partial v'} \\ \\ \dfrac{\partial v}{\partial u'} & \dfrac{\partial v}{\partial v'} \end{pmatrix}, \qquad \det J \neq 0.$$

The surface may be then parameterized by

$$G' \ni (u',v') \longrightarrow Q(u',v') = P\big(u(u',v'), v(u',v')\big),$$

and one computes $(du, dv) = (du', dv')J^t$. From these,

$$\begin{aligned} ds^2 = (du,dv)\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} &= (du',dv')J^t\begin{pmatrix} A & B \\ B & C \end{pmatrix} J \begin{pmatrix} du' \\ dv' \end{pmatrix} \\ = (du',dv')\begin{pmatrix} A' & B' \\ B' & C' \end{pmatrix}\begin{pmatrix} du' \\ dv' \end{pmatrix} &= ds'^2, \end{aligned}$$

where A', B', C' are the coefficients of the first fundamental form, relative to the new parameterization of $\mathcal{S}$.

3.1c A Parameterization of the Torus

A torus is the surface obtained by a rigid revolution about the x_3-axis of a circumference of center $(x_o, 0, 0)$ and radius $R \in (0, x_o)$. A parameterization of the torus is

$$\begin{aligned} x_1(u,v) &= (x_o + R\cos u)\cos v, \quad & u \in [0, 2\pi], \\ x_2(u,v) &= (x_o + R\cos u)\sin v, \quad & v \in [0, 2\pi], \\ x_3(u,v) &= R\sin u. \end{aligned}$$

Prove that the surface is nondegenerate, i.e., its first fundamental form is positive definite at each of its points.

4c Geodesics

Let $\mathit{\Gamma}_\gamma \subset \mathcal{S}$ be as in (3.3). The condition for $\mathit{\Gamma}_\gamma$ to be a geodesic may be expressed using the intrinsic parameterization in terms of the arc length. Denoting by κ the curvature of $\mathit{\Gamma}_\gamma$, we have

$$\begin{aligned}\mathbf{t} &= \frac{dP}{ds} = \frac{\partial P}{\partial u}\frac{du}{ds} + \frac{\partial P}{\partial v}\frac{dv}{ds},\\ \kappa\mathbf{n} &= \frac{d^2P}{ds^2} = \frac{\partial^2 P}{\partial u^2}\left(\frac{du}{ds}\right)^2 + 2\frac{\partial^2 P}{\partial u\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial^2 P}{\partial v^2}\left(\frac{dv}{ds}\right)^2\\ &\qquad + \frac{\partial P}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial P}{\partial v}\frac{d^2 v}{ds^2}.\end{aligned}$$

Imposing the condition that $\mathit{\Gamma}_\gamma$ be a geodesic yields the differential system

$$\frac{1}{2}\frac{\partial A}{\partial u}\left(\frac{du}{ds}\right)^2 + \frac{\partial A}{\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial P}{\partial u}\frac{\partial^2 P}{\partial v^2}\left(\frac{dv}{ds}\right)^2 + A\frac{d^2u}{ds^2} + B\frac{d^2v}{ds^2} = 0,$$

$$\frac{\partial P}{\partial v}\frac{\partial^2 P}{\partial u^2}\left(\frac{du}{ds}\right)^2 + \frac{\partial A}{\partial u}\frac{du}{ds}\frac{dv}{ds} + \frac{1}{2}\frac{\partial C}{\partial v}\left(\frac{dv}{ds}\right)^2 + B\frac{d^2u}{ds^2} + C\frac{d^2v}{ds^2} = 0.$$

Observing that

$$\frac{\partial P}{\partial u}\frac{\partial^2 P}{\partial v^2} = \frac{\partial B}{\partial v} - \frac{1}{2}\frac{\partial C}{\partial u}, \qquad \frac{\partial P}{\partial v}\frac{\partial^2 P}{\partial u^2} = \frac{\partial B}{\partial u} - \frac{1}{2}\frac{\partial A}{\partial v},$$

this system can be rewritten as

$$\begin{aligned}Au'' + Bv'' &= -\tfrac{1}{2}[A_u u'^2 + 2A_v u'v' + (2B_v - C_u)v'^2],\\ Bu'' + Cv'' &= -\tfrac{1}{2}[(2B_u - A_v)u'^2 + 2A_u u'v' + C_v v'^2].\end{aligned} \tag{4.1c}$$

Solving it, we arrive at the system in normal form:

$$\begin{aligned}u'' &= -\left(c^1_{11}u'^2 + 2c^1_{12}u'v' + c^1_{22}v'^2\right),\\ v'' &= -\left(c^2_{11}u'^2 + 2c^2_{12}u'v' + c^2_{22}v'^2\right).\end{aligned} \tag{4.2c}$$

The coefficients c^k_{ij}, $i, j, k = 1, 2$, are called the *Christoffel symbols*, and can be computed explicitly from (4.1c).

Prove that (4.1c) is equivalent to (4.1). Compute the Christoffel symbols in the case that $\mathcal{S}$ is a plane, a sphere, or a surface of revolution.

5c Examples of Geodesics

5.1c The Clairaut Theorem [29]

Assume that the constant c in (5.2) is not zero, thereby excluding that the geodesic is a meridian. From the parametric representation (5.1) it follows that the unit vector tangent to a parallel ($u = \text{const}$) is $\mathbf{u} = (-\sin v, \cos v, 0)$. Given now a geodesic that is not a meridian, compute its unit tangent at the generic point of curvilinear coordinate s. From (5.1), written in terms of the parameter s,

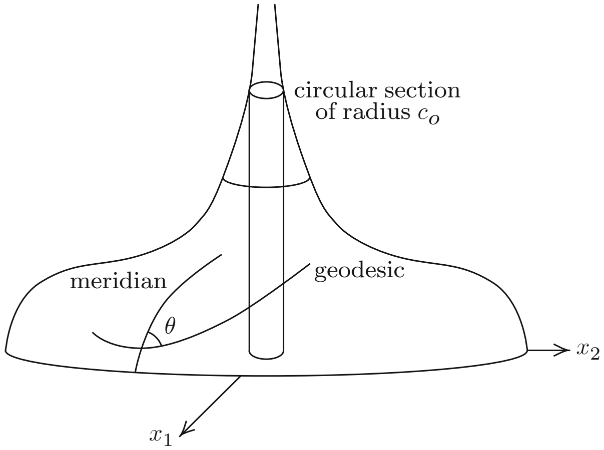

Fig. 5.1c.

$$\mathbf{t} = \left(\cos v \frac{du}{ds} - u \sin v \frac{dv}{ds}, \sin v \frac{du}{ds} + u \cos v \frac{dv}{ds}, f'(u) \frac{du}{ds} \right).$$

Let $\theta(s)$ be the angle formed by the geodesic at s, with the meridian passing through the same point. From the expression of $\mathbf{u}$ and $\mathbf{t}$,

$$\mathbf{u} \cdot \mathbf{t} = \sin \theta = u \frac{dv}{ds}.$$

Combining this with the first equality of (5.2) gives the Clairaut theorem [29]

$$u \sin \theta = \text{const},$$

along geodesics that are not meridians. Give a geometric interpretation of this fact in the particular case when $f(u) \nearrow \infty$ as $u \searrow 0$.