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Physics of Space Storms

From the Solar Surface to the Earth

Bearbeitet von
Hannu Koskinen

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2. Physical Foundations

Physics of space storms is founded on physics of hot tenuous space plasmas. While the reader is assumed to be familiar with the basic concepts of plasma physics and master the classical electrodynamics, the motivation for this chapter is to review some of the main concepts, to introduce definitions and the notation to be used elsewhere in the book, and to highlight some aspects that are specific to space plasma physics.

2.1 What is Plasma?

There is no rigorous way to define the plasma state. A good practical description for our purposes is:

> Plasma is *quasi-neutral* gas with so many *free charges* that *collective electromagnetic phenomena* are important to its physical behavior.

In this treatise we discuss quasi-neutral plasmas only. This means that in a given plasma element there is an equal amount of positive and negative charges. There is no clear threshold for the required degree of ionization. Roughly 0.1% ionization already makes the gas look like plasma, and 1% is sufficient for almost perfect conductivity.

Plasma is sometimes called the fourth state of matter because it arises as the next natural step in the sequence from solid to liquid to gas, when the temperature is increased. There are two natural ways to produce plasma in space. The most common is to heat the gas to a high enough temperature. Usually 10^5–10^6 K (10–100 eV) is sufficient (1 eV $\leftrightarrow$ 11 600 K). Also ionizing radiation is important because it creates and sustains the photospheric and ionospheric plasmas at lower temperatures where the electrons and ions recombine if the radiation stops. The transition from gas to plasma is gradual and thus different from, e.g., the phase transition from liquid to gas. The collective electromagnetic behavior gives plasma liquid-like properties. We speak of *fluid* description of plasmas when dealing with macroscopic plasma properties.

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Three key concepts *Debye shielding*, *plasma oscillations*, and *gyro motion* of charged particles in the magnetic field, lie at the heart of plasma physics. Let us review them briefly.

2.1.1 Debye shielding

The electrostatic Coulomb potential of charge q is $\varphi = q/(4\pi\varepsilon_0 r)$. In a fully ionized plasma individual particles either attract or repel each other by the force due to the gradient of this potential.

Quasi-neutrality implies that in equilibrium there is no net charge in a "large enough" volume. If we introduce an extra test charge q_T into the equilibrium plasma, the charges must be redistributed to maintain the quasi-neutrality within certain volume around q_T. Let us denote the different plasma populations (e.g., ions and electrons) by α and assume that each population is in a *Boltzmann equilibrium*

$$n_\alpha(\mathbf{r}) = n_{0\alpha} \exp\left(-\frac{q_\alpha \varphi}{k_B T_\alpha}\right) , \tag{2.1}$$

where k_B is the *Boltzmann constant* ($k_B = 1.38 \times 10^{-23}\,\mathrm{JK}^{-1}$) and T_α is the temperature of population α. The potential of q_T becomes the *shielded potential*

$$\varphi = \frac{q_T}{4\pi\varepsilon_0 r} \exp\left(-\frac{r}{\lambda_D}\right) , \tag{2.2}$$

where

$$\lambda_D^{-2} = \frac{1}{\varepsilon_0} \sum_\alpha \frac{n_{0\alpha} q_\alpha^2}{k_B T_\alpha} \tag{2.3}$$

defines the *Debye length* λ_D. The rearrangement of the charges is called *Debye shielding* and it is the most fundamental manifestation of the collective behavior of the plasma. Intuitively λ_D is the limit beyond which the thermal speed of the plasma particles is high enough to escape from the Coulomb potential of q_T. Often the electron and ion Debye lengths are given separately. Numerically the electron Debye length is

$$\lambda_D(\mathrm{m}) \approx 7.4 \sqrt{\frac{T(\mathrm{eV})}{n(\mathrm{cm}^{-3})}} . \tag{2.4}$$

Using the Debye length we can redefine the plasma state in a slightly more quantitative way. That the collective properties really dominate the plasma behavior there must be a large number of particles in the *Debye sphere* of radius λ_D, i.e., $(4\pi/3)n_0\lambda_D^3 \gg 1$. The factor $4\pi/3$ is often neglected and we call $\Lambda = n_0\lambda_D^3$ the *plasma parameter*. Because plasma must also be quasi-neutral, its size $L = V^{1/3}$ must be larger than λ_D. Thus for a plasma

$$\frac{1}{\sqrt[3]{n_0}} \ll \lambda_D \ll L . \tag{2.5}$$

Note that many sources [e.g., Boyd and Sanderson, 2003] call $g = 1/n_0\lambda_D^3$ plasma parameter.

Train your brain

Derive (2.2) for the shielded potential of a test charge q_T in a plasma with Boltzmann's density distribution.

Hints:
(i) Use $e^{-x} \simeq 1 - x$ when substituting the densities into Coloumb's law and make use of quasi-neutrality.
(ii) Make also use of spherical symmetry to write

$$\nabla^2\varphi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) .$$

(iii) After solving the differential equation require that the solution approaches the Coulomb potential of q_T when $r \to 0$ and remains finite at all distances.

2.1.2 Plasma oscillations

If plasma equilibrium is disturbed by a small perturbation, plasma starts to oscillate. Much of space plasma physics concerns the great variety of plasma responses to perturbations. The most fundamental example is the *plasma oscillation*.

Considering freely moving cold ($T_e = 0$) electrons and fixed background ions it is an easy exercise to show that a small perturbation in the electron density causes the plasma oscillation at the *plasma frequency*

$$\omega_{pe}^2 = \frac{n_0 e^2}{\varepsilon_0 m_e} . \tag{2.6}$$

Note that both the angular frequency ω_{pe} and the corresponding oscillation frequency $f_{pe} = \omega_{pe}/2\pi$ are usually called plasma frequency. So, be careful!

Plasma frequency is inversely proportional to the square root of the mass of the moving particles, here electrons. Thus the ion plasma frequency is a much smaller quantity than the electron plasma frequency. When we speak of plasma frequency, we usually mean the electron plasma frequency. A useful rule of thumb is

$$f_{pe}(\mathrm{Hz}) \approx 9.0\sqrt{n(\mathrm{m}^{-3})} .$$

The plasma oscillation determines a natural length scale in the plasma known as the *electron inertial length* c/ω_{pe}, where c is the speed of light. Physically it gives the attenuation length scale of an electromagnetic wave with the frequency ω_{pe} when it penetrates to plasma (wave propagation in plasmas will be discussed in detail in Chaps. 4 and 5). It is analogous to the *skin depth* in classical electromagnetism defined in (4.26) and is thus often called *electron skin depth*.

Similarly, the *ion plasma frequency* is defined by

$$\omega_{pi}^2 = \frac{n_0 e^2}{\varepsilon_0 m_i} . \tag{2.7}$$

The corresponding *ion inertial length* is c/ω_{pi}. It is associated with damping of fluctuations near the ion plasma frequency.

2.1.3 Gyro motion

Space plasmas are practically always embedded in a magnetic field. The magnetic field may be due to external or internal current systems. The known magnetic flux densities in space vary by more than 20 orders of magnitude. The interstellar magnetic field is typically less than 1 nT, the magnetic field of the solar wind at the distance of the Earth ($1\,AU$) is a few nT, the field on the surface of the Earth varies 3–6$\times 10^{-5}$ T (0.3–0.6 gauss) and in large fusion devices the fields are several teslas. The largest known fields, exceeding 10^8 T, are found at the rapidly rotating neutron stars (pulsars). Observations of slowly decelerating pulsars emitting X- and soft gamma rays indicate even stronger magnetic fields, exceeding 10^{11} T.

A charged particle in a magnetic field performs a circular motion perpendicular to the field. The angular frequency of this gyro motion for particle species α is

$$\omega_{c\alpha} = \frac{|q_\alpha| B}{m_\alpha} . \tag{2.8}$$

This is called the *gyro frequency* (or cyclotron frequency, Larmor frequency). The corresponding oscillation frequencies $f_{c\alpha} = \omega_{c\alpha}/(2\pi)$ of electrons and protons are given by

$$\begin{aligned} f_{ce}(\mathrm{Hz}) &\approx 28\,B(\mathrm{nT}) \\ f_{cp}(\mathrm{Hz}) &\approx 1.5 \times 10^{-2}\,B(\mathrm{nT}) . \end{aligned}$$

Again the same term is used for both ω_c and f_c.

As discussed later in this chapter the gyro motion determines another important length scale, the *electron* or *ion gyro radius*, also known as cyclotron, or Larmor radius

$$r_{L\alpha} = \frac{v_{\perp\alpha}}{|q_\alpha|} , \tag{2.9}$$

where $v_{\perp\alpha}$ is the speed of the particle perpendicular to the magnetic field.

2.1.4 Collisions

Most of the volume where space storms take place is filled by fully ionized plasmas that behave in a "collisionless" manner. However, there are two important exceptions: in the solar photosphere and in the ionosphere collisions between charged particles and neutrals have a strong influence on the plasma properties, determining, e.g., the ionospheric Ohm's law. Furthermore, the charge exchange collisions between charged particles and the Earth's ring current are important to the dynamics of storms in the inner magnetosphere (Chap. 14).

For the collisionless behavior of fully ionized plasmas the *Coulomb interaction* (Coulomb collisions) between charged particles is essential. In a plasma the finite Debye length limits the Coulomb interaction within the Debye sphere, but yet each particle sees Λ other charges. If we can calculate the collisional *cross-section* σ, we can determine the *mean free path*

$$l_{mfp} = 1/(n\sigma) \tag{2.10}$$

of the particles and their *collision frequency*

$$\nu_c = n\sigma\langle v\rangle \, , \tag{2.11}$$

where $\langle v\rangle$ is the average speed of the particles.

For Coulomb collisions it is sufficient to consider small-angle collisions, in which the particles are just slightly deflected. The reason for this is that each particle interacts with a large number of particles at long distance, whereas the probability for nearby collisions with large deflection angles is much smaller. The rigorous calculation of collisional cross-sections is rather challenging. For electron–ion collisions $\sigma \propto v_0^{-4}$ and

$$\nu_c = \nu_{ei} = \frac{2n_0(Ze^2)^2 \ln\Lambda}{\varepsilon_0^2 m_e^2 v_0^3} \, , \tag{2.12}$$

where v_0 is the particle speed far from the collision and $\ln\Lambda$ is called the *Coulomb logarithm*. Typical values of the Coulomb logarithm are in the range 10–20.

When the temperature of the plasma increases or the density decreases, $g = \Lambda^{-1}$ decreases. At the limit $g \to 0$ plasma becomes *collisionless*. Physically this means that the time between individual collisions, or the mean free path, becomes longer than the temporal or spatial scales of the problems under study. *This does not mean* that the electromagnetic interaction between plasma particles would become negligible. At the collisionless limit it is, however, sufficient to consider the effect of average electromagnetic fields on the particles instead of individual collisions.

Train your brain

Show that in a fully ionized plasma the frequency of small-angle Coulomb collisions is much larger than the frequency of large-angle collisions. To what plasma parameter the ratio of these frequencies is related?

Feed your brain

Derive Equation (2.12). The derivation can be found in many textbooks, including some listed in the References section of this book.

2.2 Basic Electrodynamics

In this section we review some of the basic concepts of classical electrodynamics that are most important in plasma physics.

2.2.1 Maxwell's equations

In plasma physics we usually write Maxwell's equations in the vacuum form

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \tag{2.13}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.14}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.15}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\,, \tag{2.16}$$

where the source terms *charge density* ρ and *current density* $\mathbf{J}$ are determined by the particle distribution functions (Sect. 2.3.3). We call $\mathbf{E}$ the *electric field* ([E] = $\mathrm{V\,m^{-1}}$) and $\mathbf{B}$ *magnetic field* ([B] = $\mathrm{V\,s\,m^{-2}} \equiv \mathrm{T}$). It would be more orthodox to call $\mathbf{B}$ magnetic induction, or more descriptively magnetic flux density, as the *magnetic flux* through a surface S is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}\,. \tag{2.17}$$

The SI units of the source terms in Maxwell's equations are $[\rho]$ = $\mathrm{A\,s\,m^{-3}}$ = $\mathrm{C\,m^{-3}}$ and [J] = $\mathrm{A\,m^{-2}}$. The natural constants in SI units are

$$\begin{aligned}
\varepsilon_0 &\approx 8.854 \times 10^{-12}\ \mathrm{A\,s\,V^{-1}\,m^{-1}}\,, && \text{vacuum permittivity}\\
\mu_0 &= 4\pi \times 10^{-7}\ \mathrm{V\,s\,A^{-1}\,m^{-1}}\,, && \text{vacuum permeability}\\
c &= 1/\sqrt{\varepsilon_0\mu_0} \;=\; 299\,792\,458\ \mathrm{m\,s^{-1}} && \text{definition of the speed of light.}
\end{aligned}$$

In studies of electromagnetic media the *electric displacement* $\mathbf{D}$ and the *magnetic field intensity* $\mathbf{H}$ (the "magnetic field" of engineering physics) are useful and Maxwell's equations are written as

$$\nabla \cdot \mathbf{D} = \rho_f \tag{2.18}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.19}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.20}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\,, \tag{2.21}$$

where ρ_f and $\mathbf{J}_f$ are the source terms due to "free" charges. If the properties of the medium can be described in terms of *electric polarization* $\mathbf{P}$ and *magnetization* $\mathbf{M}$, fields $\mathbf{D}$ and $\mathbf{H}$ are given by the *constitutive equations*

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{2.22}$$

$$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}\,. \tag{2.23}$$

In plasma physics the use of $\mathbf{D}$ and $\mathbf{H}$ is sometimes convenient notation, but the constitutive relations may pose a problem. There is no unique way to define the polarization field in a medium of free charges, although sometimes a useful $\mathbf{P}$ can be introduced formally (e.g., Eq. 9.73). However, the change of polarization is a real plasma phenomenon and the corresponding *polarization current*

$$\mathbf{J}_P = \frac{\partial \mathbf{P}}{\partial t} \tag{2.24}$$

is well-defined (see., e.g., Sect. 3.5.1). Also the *magnetization current*

$$\mathbf{J}_M = \nabla \times \mathbf{M} \tag{2.25}$$

is a useful concept in plasma physics.

The Maxwell equations form a set of 8 partial differential equations. If we know the source terms, we have more than enough equations to calculate the six unknown field components. If we, however, want to treat all 10 variables ($\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$, ρ) self-consistently, we need more equations. In a conductive medium it is customary to use *Ohm's law*

$$\mathbf{J} = \sigma \cdot \mathbf{E}\,, \tag{2.26}$$

where the *conductivity* σ ($[\sigma] = \mathrm{A\,(V\,m)^{-1}} = (\Omega\,\mathrm{m})^{-1}$) is, in general, a tensor and may also depend on $\mathbf{E}$ and $\mathbf{B}$.

Recall that Ohm's law is not a fundamental law in the same sense as Maxwell's equations but merely an empirical relationship to describe the conductivity of the medium similarly to the constitutive relations $\mathbf{D} = \varepsilon \cdot \mathbf{E}$ and $\mathbf{B} = \mu \cdot \mathbf{H}$ where ε and μ are, in general, tensors. The medium is called *linear* if ε, μ, and σ are scalars and constant in space and time. Note that also in the linear media they usually are functions of the wave number and frequency of electromagnetic fields penetrating into the medium. Much of plasma physics deals with the properties of $\varepsilon(\omega, \mathbf{k})$.

2.2.2 Lorentz force

Experimental determination of **E** and **B** is based on the *Lorentz force*

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.27}$$

on a particle with charge q and velocity $\mathbf{v}$. Close to a body with strong gravity (e.g., the Sun) also the gravitational force ($m\mathbf{g}$) must be taken into account. In principle, a complete description of plasma would mean solving the equation of motion (with gravitation if needed) for all plasma particles. In practice, this is impossible.

Often it is useful, and in many problems sufficient, to trace the motion of individual charges in a given electromagnetic field. Examples of this are the motion of cosmic rays, or high-energy particles in the Earth's radiation belts. These problems are often relativistic

$$\mathbf{F} = \frac{d}{dt}(\gamma m \mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \,, \tag{2.28}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ is the *Lorentz factor* with $\beta = v/c$. The time component of the underlying four-force gives the power

$$\frac{dW}{dt} = \frac{d}{dt}(\gamma m c^2) = q\mathbf{E} \cdot \mathbf{v} \,. \tag{2.29}$$

Because the magnetic part of the Lorentz force is perpendicular to $\mathbf{v}$, only the electric field performs work (W). Thus any "magnetic" acceleration of charged particles requires the change in the magnetic field, which induces an electric field in the frame of reference where the acceleration is observed.

2.2.3 Potentials

Equation $\nabla \cdot \mathbf{B} = 0$ implies that there is a *vector potential* $\mathbf{A}$, for which $\mathbf{B} = \nabla \times \mathbf{A}$. Inserting $\mathbf{A}$ into Faraday's law we find

$$\nabla \times (\mathbf{E} + \partial \mathbf{A}/\partial t) = 0 \tag{2.30}$$

$\Rightarrow$

$$\mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla \varphi \,, \tag{2.31}$$

where φ is the *scalar potential*.

Thus we have expressed six variables ($\mathbf{E}$, $\mathbf{B}$) using four functions ($\mathbf{A}$, φ). For this we needed four components of Maxwell's equations. The remaining four equations are now

$$\nabla^2 \varphi + \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = -\rho/\varepsilon_0 \tag{2.32}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t}) = -\mu_0 \mathbf{J} \,. \tag{2.33}$$

At first these look more complicated than the original equations, but they are much easier to solve analytically. The point is that **E** and **B** are derivatives of the scalar and vector potentials and there is quite a lot of freedom to transform the potentials keeping their derivatives unchanged. Such transformations are called *gauge transformations*. There are several *gauge functions* Ψ to define the transformations

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Psi \tag{2.34}$$

$$\varphi \to \varphi' = \varphi - \partial\Psi/\partial t\ . \tag{2.35}$$

The *Lorenz*[1] *gauge* is defined by

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \varphi'}{\partial t} = 0\ . \tag{2.36}$$

This gauge always exists but is not unique. It transforms the Maxwell equations to inhomogeneous wave equations

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\varphi = -\rho/\varepsilon_0 \tag{2.37}$$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\mathbf{A} = -\mu_0\mathbf{J}\ . \tag{2.38}$$

The solutions of which are the *retarded potentials*

$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t-R/c)}{R}\,d^3r' \tag{2.39}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t-R/c)}{R}\,d^3r'\ , \tag{2.40}$$

where $R = |\mathbf{r} - \mathbf{r}'|$ and integrations are over the volume where the source terms are not zero. Thus we have solved Maxwell's equations for given ρ and **J**.

In terms of special relativity the wave equations are actually the time and space components of the wave equation for the four-vector $A^\alpha(\varphi/c, \mathbf{A})$

$$\partial^2 A^\alpha \equiv \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A^\alpha = -\mu_0 j^\alpha\ , \tag{2.41}$$

where $j^\alpha = (c\rho,\ \mathbf{J})$ is the four-current.

Feed your brain by deriving the expressions for the retarded potentials

[1] This is not a spelling error. The first person to apply this method was *Ludvig V. Lorenz* (1829–1891) in 1867, not the much more famous *Hendrik A. Lorentz* (1853–1928).

Example: The radiation terms of the electromagnetic fields

Denote the retarded quantities by brackets $[f] = f(\mathbf{r}', t - R/c)$ and calculate the fields from the potentials. This results in

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\left\{ \int \frac{[\rho]\mathbf{R}}{R^3} d^3r' + \frac{1}{c}\int \left(\frac{2[\mathbf{J}]\cdot\mathbf{RR}}{R^4} - \frac{[\mathbf{J}]}{R^2} \right) d^3r' \right. \\ &\left. + \frac{1}{c^2}\int \left(\frac{([\dot{\mathbf{J}}]\times\mathbf{R})\times\mathbf{R}}{R^3} \right) d^3r' \right\} \qquad (2.42)\\ \mathbf{B} &= \frac{\mu_0}{4\pi}\left\{ \int \frac{[\mathbf{J}]\times\mathbf{R}}{R^3} d^3r' + \frac{1}{c}\int \frac{[\dot{\mathbf{J}}]\times\mathbf{R}}{R^2} d^3r' \right\} , \qquad (2.43)\end{aligned}$$

where the dot above $\mathbf{J}$ denotes the time derivative. Far from the sources ($R \to \infty$) the *radiation terms* dominate

$$\begin{aligned}\mathbf{E}_{rad} &= \frac{1}{4\pi\varepsilon_0 c^2}\int \frac{([\dot{\mathbf{J}}]\times\mathbf{R})\times\mathbf{R}}{R^3} d^3r' \qquad (2.44)\\ \mathbf{B}_{rad} &= \frac{1}{4\pi\varepsilon_0 c^3}\int \frac{[\dot{\mathbf{J}}]\times\mathbf{R}}{R^2} d^3r' . \qquad (2.45)\end{aligned}$$

$\mathbf{E}_{rad}$ and $\mathbf{B}_{rad}$ vanish as $1/R$. The fields due to static currents and charges vanish as $1/R^2$ or faster. Radiation requires temporal variation of $\mathbf{J}$ and a charge moving with a constant velocity does not radiate. We will discuss the electromagnetic radiation in more detail in Chap. 9.

Another important gauge is the *Coulomb gauge*

$$\nabla \cdot \mathbf{A}' = 0 . \qquad (2.46)$$

The vector potential is found by transformation

$$\nabla^2 \Psi = -\nabla \cdot \mathbf{A} , \qquad (2.47)$$

which defines Ψ uniquely (to an additive constant) when $\mathbf{A}$ and $\varphi \to 0$ for $r \to \infty$.

Now the scalar potential

$$\varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}', t)}{R} d^3r' \qquad (2.48)$$

is *not retarded* but determined by the instantaneous value of ρ everywhere. Thus the Coulomb gauge is not Lorentz[2] covariant and one must be careful when transforming between moving coordinate systems.

---

[2] Now the credit goes to the right Lorentz

The vector potential is obtained from the wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c^2}\nabla\frac{\partial \varphi}{\partial t} - \mu_0 \mathbf{J} \, . \tag{2.49}$$

The first term on the RHS is curl-free. Applying the Helmholtz theorem of vector calculus we can divide the current to curl-free and source-free components

$$\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t \; ; \;\; \nabla \times \mathbf{J}_l = 0 \; ; \;\; \nabla \cdot \mathbf{J}_t = 0 \, ,$$

where l stands for *longitudinal* (curl-free) and t for *transversal* (source-free). The continuity equation $\partial\rho/\partial t + \nabla \cdot \mathbf{J} = 0$ reduces (2.49) to

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}_t \, . \tag{2.50}$$

Consequently, the Coulomb gauge is called *transversal gauge*. It is also called *radiation gauge* because the vector potential calculated from the transversal current

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_t(\mathbf{r}', t - R/c)}{R}\, d^3 r' \tag{2.51}$$

is sufficient for the calculation of the radiation fields. The Coulomb gauge separates the electric field to its static (s) and inductive (i) parts

$$\mathbf{E}_s = -\nabla\varphi \; ; \; \mathbf{E}_i = -\partial \mathbf{A}/\partial t \, , \tag{2.52}$$

but this separation is not Lorentz covariant.

The Coulomb gauge is technically easier to use than the Lorenz gauge. It is particularly useful when no sources are present. Then $\varphi = 0$ and

$$\mathbf{E} = -\partial \mathbf{A}/\partial t \; ; \; \mathbf{B} = \nabla \times \mathbf{A} \, . \tag{2.53}$$

This is sometimes called the *temporal gauge*. It is useful, e.g., in studies of Alfvén waves and wave–wave interactions.

For specific purposes there are several other useful potential presentations. Plasmas are often embedded in a background magnetic field created by external currents ($\nabla \times \mathbf{B} = 0$, e.g., the intrinsic magnetic field of a planet). Then the magnetic field can be expressed in terms of the magnetic scalar potential as

$$\mathbf{B} = -\nabla\psi \, . \tag{2.54}$$

Because $\nabla \cdot \mathbf{B} = 0$, ψ can be solved from the Laplace equation

$$\nabla^2 \psi = 0 \tag{2.55}$$

using familiar potential theory methods.

Another representation of the magnetic field is in terms of *Euler potentials* (α, β, χ) as

$$\mathbf{A} = \alpha \nabla \beta + \nabla \chi \tag{2.56}$$

$\Rightarrow$

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times (\alpha \nabla \beta + \nabla \chi) = \nabla \alpha \times \nabla \beta\ . \tag{2.57}$$

Note that $\mathbf{B}$ is perpendicular to both $\nabla \alpha$ and $\nabla \beta$, and α and β are constants along the magnetic field. Thus the magnetic field line can be visualized as the intersection line of $\alpha =$ const. and $\beta =$ const. This presentation is particularly useful in problems where tracing of magnetic field lines is required.

2.2.4 Energy conservation

The energy conservation of electromagnetic fields is expressed by the *Poynting theorem*. In a linear medium the energy densities of electric and magnetic fields are given by

$$w_E = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} \tag{2.58}$$

$$w_M = \frac{1}{2}\, \mathbf{H} \cdot \mathbf{B} = \frac{1}{2}\, \mathbf{J} \cdot \mathbf{A}\ . \tag{2.59}$$

Define the *Poynting vector* as $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. From Maxwell's equations we find

$$\nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\ . \tag{2.60}$$

The Poynting theorem is the integral of this expression over volume $\mathscr{V}$

$$-\int_{\mathscr{V}} \mathbf{J} \cdot \mathbf{E}\, d^3 r = \int_{\mathscr{V}} \nabla \cdot \mathbf{S}\, d^3 r + \int_{\mathscr{V}} \frac{\partial}{\partial t} (w_E + w_M)\, d^3 r\ . \tag{2.61}$$

The LHS is the work performed by the electromagnetic field per unit time (i.e., power) in volume $\mathscr{V}$. The first term on the RHS is $\oint_{\partial \mathscr{V}} \mathbf{S} \cdot d\mathbf{a}$, i.e., the energy flux per unit time through the surface $\partial \mathscr{V}$. Thus the Poynting vector gives the flux of electromagnetic energy density. The last term on the RHS expresses the rate of change of the electromagnetic energy in volume $\mathscr{V}$.

In the following we often assume that the fields have harmonic time or space dependence ($\propto\ \exp(-i\omega t), \exp(i\mathbf{k} \cdot \mathbf{r})$), or both in the case of *plane waves*. For complex fields one must be careful with products. We interpret the real part of the complex vector as the physical field. For example, consider an electric field with harmonic time dependence

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r}) \exp(-i\omega t)\} = \frac{1}{2} \left[\mathbf{E}(\mathbf{r}) \exp(-i\omega t) + \mathbf{E}^*(\mathbf{r}) \exp(i\omega t)\right]\ .$$

Denote the complex conjugate by cc. The product of $\mathbf{E}$ and $\mathbf{J}$ is

$$\begin{aligned}\mathbf{J}\cdot\mathbf{E} &= \frac{1}{4}\left[\mathbf{J}(\mathbf{r})\exp(-i\omega t)+\mathrm{cc}\right]\cdot\left[\mathbf{E}(\mathbf{r})\exp(-i\omega t)+\mathrm{cc}\right] \\ &= \frac{1}{2}\mathrm{Re}\{\mathbf{J}^*(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})+\mathbf{J}(\mathbf{r})\cdot\mathbf{E}^*(\mathbf{r})\exp(-2i\omega t)\}\ . \end{aligned} \tag{2.62}$$

The time average of this is

$$\langle \mathbf{J}\cdot\mathbf{E}\rangle = \frac{1}{2}\mathrm{Re}\{\mathbf{J}^*\cdot\mathbf{E}\}\ . \tag{2.63}$$

The Poynting theorem now reads as

$$\frac{1}{2}\int_{\mathscr{V}}\mathbf{J}^*\cdot\mathbf{E}\,d^3r+\oint_{\partial\mathscr{V}}\mathbf{S}\cdot d\mathbf{a}+2i\omega\int_{\mathscr{V}}(w_E+w_M)\,d^3r=0\ . \tag{2.64}$$

Note that $\mathbf{S}=\frac{1}{2}\mathbf{E}\times\mathbf{H}^*$; $w_E=\frac{1}{4}\mathbf{E}\cdot\mathbf{D}^*$; $w_M=\frac{1}{4}\mathbf{H}\cdot\mathbf{B}^*$.

Using the Poynting vector we can express the momentum density of the electromagnetic field as

$$\hat{\mathbf{p}}=\mathbf{D}\times\mathbf{B}=\mu_0\varepsilon_0\mathbf{S} \tag{2.65}$$

when the momentum of the field is

$$\mathbf{p}_{field}=\int_{\mathscr{V}}\mathbf{D}\times\mathbf{B}\,d^3r\ . \tag{2.66}$$

The elements of the *Maxwell stress tensor* are

$$T_{ij}=E_iD_j+B_iH_j-\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})\delta_{ij}\ . \tag{2.67}$$

With this we can express the conservation of momentum as

$$\frac{d}{dt}(\mathbf{p}_{mech}+\mathbf{p}_{field})_i=\sum_j\int_{\mathscr{V}}\frac{\partial}{\partial x_j}T_{ij}\,d^3r=\oint_{\partial\mathscr{V}}\sum_j T_{ij}n_j\,da; \tag{2.68}$$

where the mechanical force is the Lorentz force

$$\frac{d\mathbf{p}_{mech}}{dt}=\int_{\mathscr{V}}(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})\,d^3r\ . \tag{2.69}$$

2.2.5 Charged particles in electromagnetic fields

In a homogeneous static magnetic field in absence of an electric field the *equation of motion* of a charged particle

$$m\frac{d\mathbf{v}}{dt}=q(\mathbf{v}\times\mathbf{B}) \tag{2.70}$$

has a solution with constant speed along the magnetic field and circular motion around the magnetic field line with the angular frequency

$$\omega_c=\frac{qB}{m}\ . \tag{2.71}$$

The radius of the circular motion (*Larmor radius*, cyclotron radius, gyro radius) is

$$r_L = \frac{v_\perp}{|\omega_c|} = \frac{mv_\perp}{|q|B} , \tag{2.72}$$

where $v_\perp = \sqrt{v_x^2 + v_y^2}$ is the velocity perpendicular to the magnetic field. The gyro period is

$$\tau_L = \frac{2\pi}{|\omega_c|} . \tag{2.73}$$

Looking along the magnetic field, the particle rotating clockwise has a negative charge. In plasma physics this is *the convention of right-handedness*.

This way we have decomposed the velocity to a constant speed $v_\|$ along the field and circular velocity $v_\perp$ perpendicular to the field. The sum of these components is a helical motion with the *pitch angle* α defined as

$$\tan\alpha = v_\perp / v_\| . \tag{2.74}$$

Hannes Alfvén realized that this decomposition is convenient even in temporally and spatially varying fields if the variations are slow compared to the gyro motion. The method is called *guiding center approximation*. The center of the gyro motion is the *guiding center* (GC) and the frame of reference where $v_\| = 0$ is the *guiding center system* (GCS).

In the GCS the charge gives rise to a current $I = q/\tau_L$ with the associated *magnetic moment*

$$\mu = I\pi r_L^2 = \frac{1}{2}\frac{q^2 r_L^2 B}{m} = \frac{1}{2}\frac{mv_\perp^2}{B} = \frac{W_\perp}{B} . \tag{2.75}$$

The magnetic moment is actually a vector

$$\boldsymbol{\mu} = \frac{1}{2} q\mathbf{r}_L \times \mathbf{v}_\perp , \tag{2.76}$$

which is always *opposite to the ambient magnetic field*. Charged particles tend to weaken the magnetic field and thus plasma can be considered as a *diamagnetic* medium.

If there is also a constant electric field, the GC drifts perpendicular to both the electric and magnetic fields with the velocity

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} . \tag{2.77}$$

This is called *electric drift* or E×B drift. The drift velocity is independent of the charge and mass of the particle.

The E×B drift corresponds to the Lorentz transformation to the frame co-moving with the GC

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} . \tag{2.78}$$

In this frame **E**’ = 0 $\Rightarrow \mathbf{E} = -\mathbf{v} \times \mathbf{B}$, from which we find the solution (2.77) for **v**. This coordinate transformation is possible for all sufficiently weak forces $\mathbf{F}_\perp$ resulting in a general expression for the drift velocity

$$\mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2} \; . \tag{2.79}$$

This requires $F/qB \ll c$. If $F \gtrsim qcB$, the GC approximation cannot be used.

From (2.79) we readily find the *gravitational drift* velocity

$$\mathbf{v}_g = \frac{m\mathbf{g} \times \mathbf{B}}{qB^2} \quad \propto \frac{m}{q} \; . \tag{2.80}$$

Gravity separates particles according to their m/q , not in the direction of the gravitational force but perpendicular to it and to $\mathbf{B}$.

The same formalism applies to a slowly time varying electric field if we assume the magnetic field to be constant. This results in the *polarization drift*

$$\mathbf{v}_P = \frac{1}{\omega_c B} \frac{d\mathbf{E}_\perp}{dt} \; . \tag{2.81}$$

We will discuss inhomogeneous magnetic fields and rapidly time varying electric fields in Chap. 3.

2.3 Tools of Statistical Physics

Plasma physics is sometimes considered as applied electrodynamics. Equally well it could be characterized as statistical physics of charged particles. The computation of the motion of all plasma particles from Maxwell's equations and the Lorentz force is an impossible task. Fortunately, we do not always need to know the details of individual particles, but we are interested in the macroscopic properties of the gas or fluid (density, flux, flow velocity, temperature, pressure, heat flux, etc.) and their evolution in space and time. To handle this we need tools of statistical physics.

2.3.1 Plasma in thermal equilibrium

There are different ways to find the fundamental plasma equations. Here we start from *equilibrium statistical mechanics*. Let there be N particles in the plasma ($N/2$ electrons, $N/2$ singly-charged ions). Assume that the plasma is in *thermal equilibrium* at the temperature T. The probability of finding the particles in locations $(\mathbf{r}_1, ..., \mathbf{r}_N)$ is given by the *Gibbs distribution*

$$D(\mathbf{r}_1, ..., \mathbf{r}_N) = \frac{1}{Z} \exp\left(-\frac{\sum_k \sum_{i>k} W_{ik}}{k_B T} \right) \; , \tag{2.82}$$

where

$$W_{ik} = \frac{q_i q_k}{4\pi\varepsilon_0 |\mathbf{r}_i - \mathbf{r}_k|} + \varphi_{ext}$$

and

$$Z = \int \exp\left(-\frac{\sum_k \sum_{i>k} W_{ik}}{k_B T}\right) d^3r_1 ... d^3r_N \ .$$

Z is the *partition function* and φ_{ext} describes the potential energy of all external fields.

The probability of finding particle 1 at $\mathbf{r}_1$ is

$$F_1(\mathbf{r}_1) = \int D\, d^3r_2 ... d^3r_N \ . \qquad (2.83)$$

If there are no external forces, $F_1 = 1/\mathcal{V}$ ($\mathcal{V}$ is the volume). Correspondingly, the probability of finding particle 1 at $\mathbf{r}_1$ and particle 2 at $\mathbf{r}_2$ is

$$F_2(\mathbf{r}_1, \mathbf{r}_2) = \int D\, d^3r_3 ... d^3r_N \qquad (2.84)$$

and so on

$$F_s(\mathbf{r}_1, ..., \mathbf{r}_s) = \int D\, d^3r_{s+1} ... d^3r_N \ . \qquad (2.85)$$

Functions $F_1, ..., F_s$ are called *reduced distributions*. At the limit of non-interacting particles ($W_{ik} \to 0$)

$$F_s \to F_1(\mathbf{r}_1) F_1(\mathbf{r}_2) \cdots F_1(\mathbf{r}_s) = 1/\mathcal{V}^s \ . \qquad (2.86)$$

The reduced distributions can be written using the *Mayer cluster expansion* (we use the notation: $\mathbf{r}_1 \to 1$ when there is no risk of confusion):

$$\begin{aligned} F_2(1,2) &= [1 + P_{12}(1,2)] F_1(1) F_1(2) \\ F_3(1,2,3) &= [1 + P_{12}(1,2) + P_{12}(2,3) + P_{12}(1,3) + T_{123}(1,2,3)] \times \\ &\quad F_1(1) F_1(2) F_1(3) \end{aligned} \qquad (2.87)$$

and so on. P_{12} is the *two-particle* (or *pair*) *correlation function* and T_{123} is the *three-particle correlation function*. At the plasma limit ($\Lambda \gg 1$) the Coulomb interaction is weak and $T_{123} \ll P_{12} \ll 1$. Thus it is usually sufficient to consider pair correlations only. Note that P is symmetric: $P_{12}(1,2) = P_{12}(|\mathbf{r}_1 - \mathbf{r}_2|)$.

The complete Gibbs distribution depends also on velocity:

$$D^*(\mathbf{r}_1, ..., \mathbf{r}_N, \mathbf{v}_1, ..., \mathbf{v}_N) = \frac{1}{Z^*} \exp\left(-\frac{\sum_k \sum_{i>k} W_{ik}}{k_B T}\right) \exp\left(-\frac{\sum_i \frac{1}{2} m_i v_i^2}{k_B T}\right) \ . \qquad (2.88)$$

In this book we will consider non-relativistic plasmas only and can neglect the velocity correlations. The relativistic particles encountered in radiation belts or in solar energetic particle events can be treated as test particles and are not assumed to have significant effects on the macroscopic quantities. Of course, there are relativistic plasmas in the universe. For example, in the magnetospheres of pulsars not only relativistic but also quantum effects become important. Quantum fluctuations produce electron–positron pairs, which annihilate and radiate 511-keV gamma rays.

Differentiating F_s, setting $s = 2$, and assuming $T_{123} \ll P_{12}$ we can derive the equation for P_{12}

$$\frac{\partial P_{12}}{\partial \mathbf{r}_1} + \frac{1}{4\pi\varepsilon_0 k_B T}\frac{\partial}{\partial \mathbf{r}_1}\left(\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\right) + \tag{2.89}$$
$$\frac{1}{4\pi\varepsilon_0 k_B T}\sum_\alpha \frac{N_\alpha}{V}\int [P_{12}(2,\alpha) + P_{12}(1,\alpha)]\frac{\partial}{\partial \mathbf{r}_1}\left(\frac{q_1 q_\alpha}{|\mathbf{r}_1 - \mathbf{r}_\alpha|}\right) d^3 r_\alpha = 0\,,$$

where α indexes the particle species. This equation can be solved by Fourier transformation. The result is

$$P_{12}(|\mathbf{r}_1 - \mathbf{r}_2|) = -\frac{q_1 q_2}{4\pi\varepsilon_0 k_B T}\frac{\exp(-|\mathbf{r}_1 - \mathbf{r}_2|/\lambda_D)}{|\mathbf{r}_1 - \mathbf{r}_2|}\,, \tag{2.90}$$

where we again encounter the Debye shielding. The assumption $P_{12} \ll 1$ is valid if $|\mathbf{r}_1 - \mathbf{r}_2| > \lambda_D$. The Mayer expansion is valid also inside the Debye sphere, where $P_{12} \propto 1/|\mathbf{r}_1 - \mathbf{r}_2|$ as long as the distance $|\mathbf{r}_1 - \mathbf{r}_2|$ remains larger than the average distance between particles in temperature T.

From this description it is possible to derive equilibrium thermodynamic properties of the plasma. For example, in the plasma approximation ($\Lambda \gg 1$) the equation of state is practically that of the *ideal gas*

$$P = nk_B T + O\left(\frac{1}{\Lambda}\right)\,. \tag{2.91}$$

Unfortunately, due to the small collision rates space plasmas seldom are in thermal equilibrium and we must look for a more general approach.

2.3.2 Derivation of Vlasov and Boltzmann equations

There are two main roads to the Boltzmann equation for a plasma. Consider first the *Klimontovich approach*. It starts from the exact density of particles in the six-dimensional phase space $(\mathbf{r},\mathbf{v})$. Consider a single particle whose orbit in this space is $(\mathbf{R}_1(t),\mathbf{V}_1(t))$. The "density" of this particle is

$$N(\mathbf{r},\mathbf{v},t) = \delta[\mathbf{r} - \mathbf{R}_1(t)]\delta[\mathbf{v} - \mathbf{V}_1(t)]\,, \tag{2.92}$$

where δ is Dirac's delta function.[3]

Summing over all particles of a given species α we get the density function N_α for the species. Writing the equation of motion under the Lorentz force for each particle and summing over particles of a given species leads to the *Klimontovich equation* for N_α

$$\frac{\partial N_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial N_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\frac{\partial N_\alpha}{\partial \mathbf{v}} = 0\,. \tag{2.93}$$

[3] Dirac's delta is not really a function, being infinite at one point and zero elsewhere, but we prefer to use in this context the sloppy language of physicists.

This is still a very detailed equation containing exact information of the orbits of all particles. N_α is composed of sums of δ-functions, which makes practical calculations cumbersome. Because we are not interested in the orbits of individual particles, we can take *ensemble averages* of N_α and of equation (2.93). Denoting the average of $N_\alpha(\mathbf{r},\mathbf{v},t)$ by $f_\alpha(\mathbf{r},\mathbf{v},t)$ and neglecting the particle collisions, the ensemble averaging of (2.93) leads to the *Vlasov equation* for f_α

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} = 0\,. \tag{2.94}$$

Another route is the *Liouville approach*. It starts from distribution functions and avoids δ-functions and ensemble averaging. Consider a general distribution of N particles $F(\mathbf{r}_1,...,\mathbf{r}_N;\mathbf{v}_1,...,\mathbf{v}_N;t)$, which is normalized as $\int F\,d^3r_1\cdots d^3r_N\,d^3v_1\cdots d^3v_N = 1$. For a plasma of $N/2$ ions and $N/2$ electrons in thermodynamic equilibrium $F = D$, where D is the Gibbs distribution of the previous section.

The penalty of avoiding δ-functions is to deal with a $6N$-dimensional phase space. F contains information of all particles and is again much too detailed for practical use. A set of *reduced distribution functions* can be defined as follows. The *one-particle distribution function* $f_\alpha^{(1)}$ for species α is

$$f_\alpha^{(1)}(\mathbf{r}_1,\mathbf{v}_1,t) = \mathscr{V}\int F\,d^3r_2\cdots d^3r_N d^3v_2\cdots d^3v_N\,. \tag{2.95}$$

$\mathscr{V}$ is the finite spatial volume where F is nonzero for all $\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N$. The *two-particle distribution function* is

$$f_{\alpha\beta}^{(2)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{v}_1,\mathbf{v}_2,t) = \mathscr{V}^2\int F\,d^3r_3\cdots d^3r_N d^3v_3\cdots d^3v_N \tag{2.96}$$

and so on. Statistical physics tells us that F fulfills the *Liouville equation*

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial F}{\partial \mathbf{r}_i}\cdot\mathbf{v}_i + \frac{\partial F}{\partial \mathbf{v}_i}\cdot\mathbf{a}_i^T\right) = 0\,, \tag{2.97}$$

where $\mathbf{a}_i^T$ is the acceleration by all interactions, including collisions.

The equation of motion for $f_\alpha^{(1)}$ is found by integrating (2.97) over all coordinates except $(\mathbf{r}_1,\mathbf{v}_1)$

$$\frac{\partial f_\alpha^{(1)}}{\partial t} + \mathbf{v}_1\cdot\frac{\partial f_\alpha^{(1)}}{\partial \mathbf{r}_1} + \mathscr{V}\int \mathbf{a}_i^T\cdot\frac{\partial F}{\partial \mathbf{v}_1}\,d^3r_2\cdots d^3r_N\,d^3v_2\cdots d^3v_N = 0\,. \tag{2.98}$$

Here the total number of particles was assumed to be conserved.

If there are external forces ($\mathbf{a}_1^E$) only, we again get the Liouville equation

$$\frac{\partial f_\alpha^{(1)}}{\partial t} + \mathbf{v}_1\cdot\frac{\partial f_\alpha^{(1)}}{\partial \mathbf{r}_1} + \mathbf{a}_1^E\cdot\frac{\partial f_\alpha^{(1)}}{\partial \mathbf{v}_1} = 0\,. \tag{2.99}$$

Denote the interactions between particles by $\mathbf{a}_{ij}$. Now the third term of (2.98) reduces to

$$\mathbf{a}_1^E \cdot \frac{\partial f_\alpha^{(1)}}{\partial \mathbf{v}_1} + \sum_\beta \int \mathbf{a}_{1\beta} \cdot \frac{\partial}{\partial \mathbf{v}_1} f_{\alpha\beta}^{(2)}(\mathbf{r}_1, \mathbf{r}_\beta, \mathbf{v}_1, \mathbf{v}_\beta, t) d^3 r_\beta d^3 v_\beta \ .$$

Note that (2.98) is not a closed equation for $f^{(1)}$, as it depends on $f^{(2)}$. We could write a similar equation for $f^{(2)}$, which then depends on $f^{(3)}$, and so on. This is called the *BBGKY hierarchy* (after Bogoliubov, Born, Green, Kirkwood, and Yvon). In higher orders this hierarchy becomes intractable and the series must be truncated with physical arguments. We do it by approximating $f^{(2)}$.

If the interactions between particles were *strong* and of *short-range* (as in ordinary gases) we would end up with the *Boltzmann equation*

$$\frac{d f_\alpha^{(1)}}{dt} \equiv \frac{\partial f_\alpha^{(1)}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_\alpha^{(1)}}{\partial \mathbf{r}_1} + \mathbf{a}_1^E \cdot \frac{\partial f_\alpha^{(1)}}{\partial \mathbf{v}_1} = \left(\frac{\partial f_\alpha^{(1)}}{\partial t} \right)_c \ . \tag{2.100}$$

However, in a plasma the dominating interaction is the *long-range* Coulomb force, which is, in this context, *weak*. Fortunately, in a plasma the combined effect of remote charges is, on the average, stronger than the acceleration due to the nearest neighbor. The average acceleration $\langle \mathbf{a}^{int} \rangle$ is from the viewpoint of a single particle the same as the acceleration by the external Coulomb force $\mathbf{a}^E$. Thus we can replace $\mathbf{a}_1 = \mathbf{a}_1^E + \langle \mathbf{a}^{int} \rangle$. The effect of binary collisions is

$$\left(\frac{\partial f_\alpha^{(1)}}{\partial t} \right)_c = -\sum_\beta \int \left(\mathbf{a}_{1\beta} - \langle \mathbf{a}_{1\beta}^{int} \rangle \right) \cdot \frac{\partial}{\partial \mathbf{v}_1} f_{\alpha\beta}^{(2)} d^3 r_\beta d^3 v_\beta \ . \tag{2.101}$$

Assuming that the only external force is the Lorentz force we have the Boltzmann equation for plasma

$$\frac{\partial f_\alpha^{(1)}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_\alpha^{(1)}}{\partial \mathbf{r}_1} + \frac{q_\alpha}{m_\alpha} \langle \mathbf{E} + \mathbf{v}_1 \times \mathbf{B} \rangle \cdot \frac{\partial f_\alpha^{(1)}}{\partial \mathbf{v}_1} = \left(\frac{\partial f_\alpha^{(1)}}{\partial t} \right)_c \ , \tag{2.102}$$

where the average fields $\langle \mathbf{E} \rangle$ and $\langle \mathbf{B} \rangle$ fulfill the average Maxwell equations

$$\nabla \cdot \langle \mathbf{E} \rangle = \frac{\rho}{\varepsilon_0} \ ; \ \nabla \times \langle \mathbf{B} \rangle = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \langle \mathbf{E} \rangle}{\partial t} \ . \tag{2.103}$$

Note that the normalization of $f_\alpha^{(1)}$ is different from the normalization of the distribution function f_α in the Vlasov equation (2.94). We retain the same plasma kinetic equation by substitution $f_\alpha = (N_\alpha / \mathscr{V}) f_\alpha^{(1)}$.

A thorough treatment of the collision term is a substantial task. The interested reader is encouraged to consult advanced text-books on Balescu–Lenard and Fokker–Planck equations. We will discuss some elements of the Fokker–Planck theory in Chap. 10. Note that the interparticle collisions may be of very variable nature. They may be elastic, but the kinetic energy of a colliding plasma particle may also be transferred to internal energy of neutral particles or molecular ions of the plasma. Furthermore, there are collisions leading to recombination, ionization, and charge exchange, which are important processes associated with space storms.

A simple and often sufficient first approximation for the collision term is the *relaxation time approximation*, also called the *Krook model* where the average collision frequency is approximated by a constant ν_c and

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = -\nu_c(f - f_0)\,. \qquad (2.104)$$

where f_0 is the equilibrium distribution and $|f - f_0| \ll f_0$. Note that the equilibrium here is a wider concept than a Maxwellian distribution. It is enough that f_0 is a stable solution of the Vlasov equation.

2.3.3 Macroscopic variables

The Vlasov and Boltzmann equations are equations of motion for the *single particle distribution function* $f(\mathbf{r},\mathbf{v},t)$. The function expresses the number density of particles in a volume element $dx\,dy\,dz\,dv_x\,dv_y\,dv_z$ of a six-dimensional phase space ($\mathbf{r}$,$\mathbf{v}$) at the time t (thus the SI units of f are $\mathrm{m}^{-6}\,\mathrm{s}^3$). In the following we use the normalization

$$\int_{\mathscr{V}}\int_{\mathbf{v}} f(\mathbf{r},\mathbf{v},t) d^3r d^3v = N\,, \qquad (2.105)$$

where N is the number of all particles in the phase space volume considered.

The average density in volume $\mathscr{V}$ is $\langle n\rangle = N/\mathscr{V}$. However, the *particle density* is usually a function of space and time. It is defined as the zero order *velocity moment* of the distribution function

$$n(\mathbf{r},t) = \int f(\mathbf{r},\mathbf{v},t)\, d^3v\,. \qquad (2.106)$$

We define the *macroscopic* quantities as velocity moments of the distribution function

$$\int f\, d^3v \;\;;\;\; \int \mathbf{v} f\, d^3v \;\;;\;\; \int \mathbf{v}\mathbf{v} f\, d^3v\,.$$

In a plasma different particle populations (labeled by α) may have different distributions and thus have different velocity moments ($n_\alpha(\mathbf{r},t)$, etc.). If the particles of a species are charged with charge q_α, the *charge density* of the species

$$\rho_\alpha = q_\alpha n_\alpha\,. \qquad (2.107)$$

The first-order moment yields the *particle flux*

$$\Gamma_\alpha(\mathbf{r},t) = \int \mathbf{v} f_\alpha(\mathbf{r},\mathbf{v},t)\, d^3v\,. \qquad (2.108)$$

Dividing this by particle density we get the *average velocity*

$$\mathbf{V}_\alpha(\mathbf{r},t) = \frac{\int \mathbf{v} f_\alpha(\mathbf{r},\mathbf{v},t)\, d^3v}{\int f_\alpha(\mathbf{r},\mathbf{v},t)\, d^3v}\,, \tag{2.109}$$

from which we can further determine the *current density*

$$\mathbf{J}_\alpha(\mathbf{r},t) = q_\alpha \Gamma_\alpha = q_\alpha n_\alpha \mathbf{V}_\alpha\,. \tag{2.110}$$

In the second order we find the *pressure tensor*

$$\mathscr{P}_\alpha(\mathbf{r},t) = m_\alpha \int (\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha) f_\alpha(\mathbf{r},\mathbf{v},t)\, d^3v\,, \tag{2.111}$$

which in a spherically symmetric case reduces to the *scalar pressure*

$$P_\alpha = \frac{m_\alpha}{3}\int (\mathbf{v}-\mathbf{V}_\alpha)^2 f_\alpha(\mathbf{r},\mathbf{v},t)\, d^3v = n_\alpha k_B T_\alpha\,. \tag{2.112}$$

Here we introduce the concept of *temperature* T_α. In the frame moving with the velocity $\mathbf{V}$ the temperature is given by

$$\frac{3}{2}k_B T_\alpha(\mathbf{r},t) = \frac{m_\alpha}{2}\frac{\int v^2 f_\alpha(\mathbf{r},\mathbf{v},t)\, d^3v}{\int f_\alpha(\mathbf{r},\mathbf{v},t)\, d^3v}\,, \tag{2.113}$$

which for a *Maxwellian distribution* is the temperature of classical thermodynamics. In collisionless plasmas equilibrium distributions may be far from Maxwellian. Thus temperature is a non-trivial concept in plasma physics.

Train your brain

Show that a spherically symmetric (in the velocity space) distribution function $f_\alpha(\mathbf{r},v,t)$ yields an isotropic pressure $P_{\alpha ij} = p_\alpha \delta_{ij}$. What kind of distribution function yields the diagonal gyrotropic form

$$P_{\alpha ij} = p_\perp \delta_{ij} + (p_\parallel - p_\perp)\delta_{3i}\delta_{3j} \ \ ?$$

What is the value of scalar pressure p in this case? Here the "parallel" direction (e.g., the direction of background magnetic field) is assumed to be in the direction of the axis number 3.

The relation between the particle pressure and *magnetic pressure* (*magnetic energy density*) is the *plasma beta*

$$\beta = \frac{2\mu_0 \sum_\alpha n_\alpha k_B T_\alpha}{B^2}\,. \tag{2.114}$$

If $\beta > 1$, plasma governs the evolution of the magnetic field. If $\beta \ll 1$, the magnetic field determines the plasma dynamics. Values of beta are very different and highly variable in various landscapes of space storms. In the solar photosphere beta varies from 1 to 100. In the lower corona it is of the order of 10^{-4}–10^{-2} and higher up it starts rising again to be around 1 in the solar wind, but also there with large variations. In the Earth's magnetosphere the lowest beta values ($\beta \sim 10^{-6}$) are found in the auroral region magnetic field lines at altitudes of a few Earth radii. In the tail plasma sheet $\beta \sim 1$, but in the tail lobes it is some 4 orders of magnitude smaller.

The chain of moments continues to higher orders. The third order introduces the *heat flux* , i.e., temperature multiplied by velocity. It can usually be neglected in the magnetosphere but is very important at the solar end of space storms.

2.3.4 Derivation of macroscopic equations

Next we derive macroscopic equations by taking velocity moments of the Boltzmann equation. For the needs of many space applications we could start from the Vlasov equation, but retaining the collision term gives us a more complete macroscopic theory. When not needed, the collision effects can be dropped at the macroscopic level.

We start from the Boltzmann equation for species α

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_c . \tag{2.115}$$

Zeroth moment

We first integrate (2.115) over the velocity space. For physical distributions $f_\alpha \to 0$, when $|v| \to \infty$, and the force term vanishes in the integration. If there are no ionizing nor recombining collisions, or charge-exchange collisions between ions and neutrals, the zero-order moment of the collision term is also zero. The integral of the first term of (2.115) yields the time derivative of density. The second term is of the first order in velocity

$$\int \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} \, d^3v = \nabla \cdot \int \mathbf{v} f_\alpha \, d^3v = \nabla \cdot (n_\alpha \mathbf{V}_\alpha) \tag{2.116}$$

and we have found the *equation of continuity*

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{V}_\alpha) = 0 . \tag{2.117}$$

Continuity equations for charge or mass densities are obtained by multiplying (2.117) by q_α or m_α, respectively. The equation of continuity is an example of the general form of a *conservation law*

$$\frac{\partial F}{\partial t} + \nabla \cdot \mathbf{G} = 0 , \tag{2.118}$$

where F is the density of a physical quantity and $\mathbf{G}$ the associated flux.

First moment

Multiply (2.115) by $m_\alpha \mathbf{v}$ and integrate over $\mathbf{v}$. This yields the *momentum transport equation*, which actually is the macroscopic *equation of motion*

$$n_\alpha m_\alpha \frac{\partial \mathbf{V}_\alpha}{\partial t} + n_\alpha m_\alpha \mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha - n_\alpha q_\alpha \langle \mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B} \rangle + \nabla \cdot \mathscr{P}_\alpha$$
$$= m_\alpha \int \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c d^3 v \,. \tag{2.119}$$

Train your brain

Make a careful derivation of Eq. (2.119). You need to apply the continuity equation.

The average electric and magnetic fields in (2.119) are determined by both internal and external sources and fulfill the average Maxwell equations

$$\nabla \cdot \langle \mathbf{E} \rangle = \sum_\alpha \frac{n_\alpha q_\alpha}{\varepsilon_0} + \rho_{ext}/\varepsilon_0 \tag{2.120}$$

$$\nabla \times \langle \mathbf{B} \rangle = \frac{1}{c^2} \frac{\partial \langle \mathbf{E} \rangle}{\partial t} + \mu_0 \sum_\alpha n_\alpha q_\alpha \mathbf{V}_\alpha + \mu_0 \mathbf{J}_{ext} \,. \tag{2.121}$$

Because collisions transport momentum between different plasma populations, the collision integral does not vanish, except for collisions between the same type of particles. The collision term is a complicated function of velocity. A useful approximation related to the Krook model (2.104) is

$$m_\alpha \int \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c d^3 v = -\sum_\beta m_\alpha n_\alpha (\mathbf{V}_\alpha - \mathbf{V}_\beta) \langle \nu_{\alpha\beta} \rangle \,, \tag{2.122}$$

where $\langle \nu_{\alpha\beta} \rangle$ is the average collision between particles of type α and β.

The second-order contributions $\mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha$ and $\mathscr{P}_\alpha$ arise from terms containing products $\mathbf{vv}$ or $\mathbf{v} \cdot \mathbf{v}$. The divergence of $\mathscr{P}_\alpha$ contains information of inhomogeneity and viscosity of the plasma. Note that $\mathscr{P}_\alpha$ is not independent of the collisions. For example, if the collisions are frequent enough, the pressure tensor becomes diagonal, or even isotropic in which case $\nabla \cdot \mathscr{P} \to \nabla P$.

Second moment

The second velocity moment yields the *energy* or *heat transport equation* (conservation law of energy). We can write the equation in the form

$$\frac{3}{2} n_\alpha k_B \left(\frac{\partial T_\alpha}{\partial t} + \mathbf{V}_\alpha \cdot \nabla T_\alpha \right) + P_\alpha \nabla \cdot \mathbf{V}_\alpha =$$
$$-\nabla \cdot \mathbf{H}_\alpha - (\mathscr{P}'_\alpha \cdot \nabla) \cdot \mathbf{V} + \frac{\partial}{\partial t} \left(\frac{n_\alpha m_\alpha V_\alpha^2}{2} \right)_c \,, \tag{2.123}$$

where the isotropic part of the pressure P_α is written on the LHS and the non-isotropic part $\mathscr{P}'_\alpha$ on the RHS. The relation between the scalar pressure P_α and temperature T_α is assumed to be that of an ideal gas $P_\alpha = n_\alpha k_B T_\alpha$.

The third-order term $\mathbf{H}_\alpha$ describes the *heat flux*. An equation for it is found by taking the third moment. This contains fourth-order contributions, and so on. The chain of equations must again be truncated at some point, just as was done in the case of kinetic equations. In many practical problems this is made in the second order, either by neglecting the heat flux, or by substituting the energy equation by an equation of state. Here physical insight is essential. Krall and Trivelpiece [1973] state this: "The fluid theory, though of great practical use, relies heavily on the cunning of its user". In collisional and Maxwellian plasmas the truncation may be easy to motivate, but in collisionless space plasmas it is a more subtle issue.

2.3.5 Equations of magnetohydrodynamics

Now we have macroscopic equations for each plasma species. In a real plasma several species co-exist; in addition to electrons and protons, there may be a variety of heavier ions, as well as neutral particles, which may contribute to plasma dynamics through collisions, including charge-exchange processes (e.g., Sect. 14.1.4). Sometimes it is also necessary to consider different species of the same type of particles; e.g., in the same spatial volume there may be two electron populations of widely different temperatures or average velocities. Such situations often give rise to plasma instabilities to be discussed in Chap. 7.

As the first step toward a single-fluid theory it is useful to consider all electrons as one fluid and all ions as another. This is called a *two-fluid model*. The separate fluid components interact through collisions and electromagnetic interaction. In the following derivation of the single-fluid theory, it may be practical to think only two components although we have written the expressions for an arbitrary number of species.

Magnetohydrodynamics (MHD) is probably the most widely known plasma theory. In MHD the plasma is considered as a single fluid in the center-of-mass (CM) frame. This is a well-motivated approach in collision-dominated plasmas, where the collisions constrain the plasma particles to follow each other closely and thermalize the distribution toward a Maxwellian, which makes the interpretation of velocity moments straightforward. MHD works also remarkably well in collisionless tenuous space plasmas. However, great care should be exercised both with interpretation and approximations.

The *single-fluid* variables are defined as:

mass density

$$\rho_m(\mathbf{r},t) = \sum_\alpha n_\alpha m_\alpha \,, \tag{2.124}$$

charge density

$$\rho_q(\mathbf{r},t) = \sum_\alpha n_\alpha q_\alpha \tag{2.125}$$

$(= e(n_i - n_e)$ for singly charged ions and electrons$)$, macroscopic velocity

$$\mathbf{V}(\mathbf{r},t) = \frac{\sum_\alpha n_\alpha m_\alpha \mathbf{V}_\alpha}{\sum_\alpha n_\alpha m_\alpha}\,, \tag{2.126}$$

current density

$$\mathbf{J}(\mathbf{r},t) = \sum_\alpha n_\alpha q_\alpha \mathbf{V}_\alpha\,, \tag{2.127}$$

and pressure tensor in the CM frame

$$\mathscr{P}_\alpha^{CM}(\mathbf{r},t) = m_\alpha \int (\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V}) f_\alpha\, d^3v\,, \tag{2.128}$$

from which we get the total pressure

$$\mathscr{P}(\mathbf{r},t) = \sum_\alpha \mathscr{P}_\alpha^{CM}(\mathbf{r},t)\,. \tag{2.129}$$

Summing the individual continuity and momentum transport equations over particle species yields the continuity equations

$$\frac{\partial \rho_m}{\partial t} + \nabla\cdot(\rho_m \mathbf{V}) = 0 \tag{2.130}$$

$$\frac{\partial \rho_q}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{2.131}$$

and the momentum transport equation

$$\rho_m\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = \rho_q\mathbf{E} + \mathbf{J}\times\mathbf{B} - \nabla\cdot\mathscr{P}\,. \tag{2.132}$$

The momentum equation corresponds to the Navier–Stokes equation of hydrodynamics (6.2) where the viscosity terms are written explicitly (here they are hidden in $\nabla\cdot\mathscr{P}$). At macroscopic level the deviations from charge neutrality are small and $\rho_q\mathbf{E}$ is usually negligible. The magnetic part of the Lorentz force $\mathbf{J}\times\mathbf{B}$ (sometimes called Ampère's force) is, however, essential in the theory of magnetic fluids.

Ohm's law in fluid description is a more complicated issue. In the particle picture the plasma current is the sum of all charged particle motions. In a single-fluid theory the current transport equation is derived by multiplying the momentum transport equations of each particle population by q_α/m_α and summing over all populations. In the two-fluid case (e,i) we get

$$\begin{aligned}
&\frac{\partial \mathbf{J}}{\partial t} + \nabla\cdot(\mathbf{V}\mathbf{J} + \mathbf{J}\mathbf{V} - \mathbf{V}\mathbf{V}\rho_q) = \sum_\alpha \frac{n_\alpha q_\alpha^2}{m_\alpha}\mathbf{E} \\
&+\left(\frac{e^2}{m_e} + \frac{e^2}{m_i}\right)\frac{\rho_m \mathbf{V}\times\mathbf{B}}{m_e+m_i} - \left(\frac{em_i}{m_e} - \frac{em_e}{m_i}\right)\frac{\mathbf{J}\times\mathbf{B}}{m_e+m_i} \\
&-\frac{e}{m_e}\nabla\cdot\left(\mathscr{P}_i^{CM}\frac{m_e}{m_i} - \mathscr{P}_e^{CM}\right) + \sum_\alpha \int q_\alpha \mathbf{v}\left(\frac{\partial f_\alpha}{\partial t}\right)_c d^3v\,,
\end{aligned} \tag{2.133}$$

where the products $\mathbf{VJ}$, etc., are cartesian tensors (dyads) with elements $V_i J_k$, and the divergence of a dyad is a vector, e.g., with components $\sum_i \partial_i V_i J_k$. This equation expresses the relationship between the electric current and the electric field. Thus it can be called *generalized Ohm's law*.

The first step to simplify (2.133) is to approximate the collision integral introducing a constant collision frequency ν

$$\sum_\alpha \int q_\alpha \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c d^3 v = -\nu \mathbf{J} \,. \tag{2.134}$$

Defining the *conductivity* by $\sigma = ne^2/\nu m_e$ and neglecting all derivatives and the magnetic field in (2.133) we get the familiar form of Ohm's law $\mathbf{J} = \sigma \mathbf{E}$.

Not all terms in the generalized Ohm's law are equally important. There are some that clearly are smaller than the others (e.g. $\propto m_e/m_i$). Furthermore, the derivatives of the second-order terms $\mathbf{VJ}$, $\mathbf{JV}$ and $\mathbf{VV}$ can usually be neglected. At this level we have the generalized Ohm's law in the form that contains the most important terms for space plasmas:

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{\mathbf{J}}{\sigma} + \frac{1}{ne} \mathbf{J} \times \mathbf{B} - \frac{1}{ne} \nabla \cdot \mathscr{P}_e + \frac{m_e}{ne^2} \frac{\partial \mathbf{J}}{\partial t} \,. \tag{2.135}$$

Assume further so *slow temporal changes and large spatial gradient scales* that $|\mathbf{J} \times \mathbf{B}|$, $|\partial \mathbf{J}/\partial t|$, and $|\nabla \cdot \mathscr{P}|$ are all smaller than $|\mathbf{V} \times \mathbf{B}|$. This leaves us with the standard form of Ohm's law in MHD

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \,, \tag{2.136}$$

which already familiar from elementary electrodynamics in cases when moving frames are taken into account. Here the moving frame is attached to the fluid flow with the velocity $\mathbf{V}$. If the conductivity is very large, we find Ohm's law of the *ideal MHD*

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \,. \tag{2.137}$$

The road from the Liouville or Klimontovich equations to this simple equation is long and there are several potholes on the road. For example, while the ideal MHD is a reasonable starting point, it is not at all clear that the next term to take into account should be $\mathbf{J}/\sigma$. In many space applications the *Hall term* $\mathbf{J} \times \mathbf{B}/ne$ and the pressure term $\nabla \cdot \mathscr{P}/ne$ are more important.

There are effects that originate at the microscopic level, which are not due to actual interparticle collisions, but which may lead to "effective" resistivity or viscosity at the macroscopic level. Various wave–particle interactions and microscopic instabilities tend to inhibit the current flow. Often the macroscopic effect of these processes looks analogous to finite ν and is called *anomalous resistivity*.[4]

Another issue is that plasma does not need to exhibit a local Ohm's law at all. In tenuous space plasmas it may happen that there are not enough current carriers to satisfy $\nabla \cdot \mathbf{J} = 0$ without extra acceleration of the charges. An example is the magnetic field-aligned po-

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[4] This is one more example of unfortunate terminology. There is nothing anomalous in the physics behind the non-collisional resistivity.

tential drop above the discrete auroras. The coupling between the ionosphere and magnetosphere requires more upward field-aligned current to be drawn through this region than there are electrons readily available from the magnetosphere. The *global plasma system* reacts to this by setting up an upward-directed electric field to accelerate electrons to so high velocities that the current continuity is maintained. This results in a *global current–voltage relationship*, which Knight [1973] derived into the form

$$J_{\parallel} = -en\sqrt{\frac{k_B T_e}{2\pi m_e}}\frac{B_I}{B_E}\left[1-\left(1-\frac{B_E}{B_I}\right)\exp\left(-\frac{e\triangle\varphi}{k_B T_e(B_I/B_E-1)}\right)\right] . \tag{2.138}$$

Here B_I is the magnetic field in the ionosphere, B_E in the equatorial plane in the magnetosphere and $\triangle\varphi$ the potential difference between them. At the limit $e\triangle\varphi/k_B T \ll (B_I/B_E-1)$ this reduces to

$$J_{\parallel} = K\left(\triangle\varphi + \frac{k_B T_e}{e}\right) , \tag{2.139}$$

which is often approximated as the direct linear relationship between the current and voltage of the form

$$J_{\parallel} = K\triangle\varphi . \tag{2.140}$$

This last form is known as the *Knight relation*. The coefficient K is a function of plasma parameters and thus not a universal constant.

Feed your brain

The current–voltage relationship is actually not quite as simple as given above. Read carefully the paper by Janhunen and Olsson [1998] and fill in the gaps in their derivations.

The next equation in the velocity moment chain is the *energy transport equation*. After some tedious but straightforward calculation the energy equation can be written in the conservation form

$$\frac{\partial}{\partial t}\left[\rho_m\left(\frac{V^2}{2}+w\right)+\frac{B^2}{2\mu_0}\right] = -\nabla\cdot\mathbf{H} . \tag{2.141}$$

Here w is the *enthalpy* that is related to the the internal free energy (per unit mass) of the plasma u by $w = u + P/\rho_m$. The RHS is the divergence of the heat flux vector $\mathbf{H}$, which is a third-order moment. After some reasonable approximations it can be written as

$$\begin{aligned}\mathbf{H} = &\left(\frac{V^2}{2}+u+\frac{P+B^2/\mu_0}{\rho_m}\right)\rho_m\mathbf{V} - \frac{\mathbf{B}}{\mu_0}\left(\mathbf{V}+\frac{\mathbf{J}}{ne}\right)\cdot\mathbf{B}\\ &-\frac{\mathbf{J}\times\mathbf{B}}{\sigma\mu_0}+\frac{\mathbf{J}B^2}{\mu_0 ne}+\frac{m_e\mathbf{B}}{\mu_0 ne^2}\times\frac{\partial\mathbf{J}}{\partial t} .\end{aligned} \tag{2.142}$$

When integrated over a finite volume $\mathscr{V}$ the LHS of (2.141) describes the temporal change of the energy of the MHD plasma in that volume and the RHS the the energy flux through the boundary $\partial\mathscr{V}$ and energy losses due to resistivity. Thus we have found the MHD equivalent of Poynting's theorem of elementary electrodynamics.

Because the energy equation depends on third-order terms, we do not get a closed set of MHD equations without some further approximations. Often the chain is cut by *selecting* an equation of state. After this the energy equation can be written in a simpler form. Another frequently adopted approach is to assume an isotropic pressure. We can start from the ideal gas law $P = nk_BT$ and use some of the following equations of state depending on what kind of processes we are considering:

- *adiabatic process*

$$T = T_0\left(\frac{n}{n_0}\right)^{\gamma-1} \quad ; \quad P = P_0\left(\frac{n}{n_0}\right)^{\gamma}, \tag{2.143}$$

  where the *polytropic index* $\gamma = c_p/c_v$ is $5/3$ in a three-dimensional plasma and c_p and c_v are the specific heat constants for constant pressure and constant volume, respectively.

- *isothermal process*
  the above with $\gamma = 1 \;\Rightarrow\; P = nk_BT_0$

- *isobaric process*
  the above with $\gamma = 0$, i.e., constant pressure

- *isometric process*
  the above with $\gamma = \infty$, i.e., $P \approx 0$, e.g. the case of $\beta \ll 1$.

Using the equation of state we can write the equations of MHD in the form

$$\frac{\partial \rho_m}{\partial t} + \nabla\cdot(\rho_m\mathbf{V}) = 0 \tag{2.144}$$

$$\rho_m\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} + \nabla P - \mathbf{J}\times\mathbf{B} = 0 \tag{2.145}$$

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = \mathbf{J}/\sigma \tag{2.146}$$

$$P = P_0\left(\frac{n}{n_0}\right)^{\gamma} \tag{2.147}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \tag{2.148}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}\,. \tag{2.149}$$

2.3.6 Double adiabatic theory

Due to the presence of the magnetic field the particle distributions in space plasmas are not always isotropic and the pressure tensor does not even need to be diagonal. To fully appreciate the anisotropic effects we need to refer to some concepts to be investigated in Chap. 3, but their macroscopic consequences are useful to introduce here for completeness of the present discussion.

Consider the ideal MHD equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0 \tag{2.150}$$

$$\rho_m \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} + \nabla \cdot \mathscr{P} - \mathbf{J} \times \mathbf{B} = 0 \tag{2.151}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \tag{2.152}$$

and assume that the pressure tensor is diagonal and *gyrotropic*

$$\mathscr{P} = \begin{pmatrix} P_\perp & 0 & \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\| \end{pmatrix} . \tag{2.153}$$

Assume further that both the parallel and perpendicular pressures behave adiabatically and fulfill the ideal gas equation of state

$$P_\| = n k_B T_\| \tag{2.154}$$

$$P_\perp = n k_B T_\perp . \tag{2.155}$$

There are one parallel and two perpendicular dimensions. From thermodynamics we know that the polytropic index depends on the number of dimensions d as $\gamma = (d+2)/d$. Setting $\gamma_\perp = 2$ and $\gamma_\| = 3$ is, however, *wrong* because the magnetic field not only breaks the symmetry of the pressure tensor but also couples the perpendicular motion to the parallel motion in inhomogeneous plasma (e.g, the mirror force, see Chap. 3).

Assume that the motion of the individual particles is adiabatic, which means that the magnetic moment $\mu = W_\perp / B$ is constant. Then the average magnetic moment $\langle \mu \rangle = k_B T_\perp / B = P_\perp / nB$ is also constant. This yields the perpendicular equation of state

$$\frac{d}{dt} \left(\frac{P_\perp}{\rho_m B} \right) = 0 . \tag{2.156}$$

The parallel direction is more difficult. Chew, Goldberger, and Low developed a theory [Chew et al, 1956] *assuming* that the heat flux parallel to the magnetic field is negligible. This leads to the equation of state

$$\frac{d}{dt} \left(\frac{P_\perp^2 P_\|}{\rho_m^5} \right) = \frac{d}{dt} \left(\frac{P_\| B^2}{\rho_m^3} \right) = 0 . \tag{2.157}$$

This anisotropic version of MHD is called *double adiabatic theory* or *CGL theory*. Now the pressure tensor is of the form $\mathscr{P} = P_\perp \mathscr{I} + (P_\| - P_\perp)\mathbf{bb}$, where $\mathbf{b} = \mathbf{B}/B$ and $\mathscr{I}$ is the unit tensor. The momentum equation separates into two equations

$$\rho_m \left(\frac{d\mathbf{V}}{dt} \right)_\perp + \nabla_\perp \left(P_\perp + \frac{B^2}{2\mu_0} \right) - \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{\mu_0} \left(\frac{P_\perp - P_\|}{B^2/\mu_0} + 1 \right) = 0 \tag{2.158}$$

$$\rho_m \left(\frac{d\mathbf{V}}{dt} \right)_{\|} + \nabla_{\|} P_{\|} + (P_{\perp} - P_{\|}) \left(\frac{\nabla B}{B} \right)_{\|} = 0\,. \tag{2.159}$$

In the CGL theory the parallel and perpendicular polytropic indices are not constant numbers. Assuming that $p_{\|} \propto n^{\gamma_{\|}}$ and $p_{\perp} \propto n^{\gamma_{\perp}}$ the following relations are found

$$\gamma_{\perp} = 1 + \frac{\ln(B/B_0)}{\ln(n/n_0)} \tag{2.160}$$

$$\gamma_{\|} = 3 - 2\frac{\ln(B/B_0)}{\ln(n/n_0)}\,, \tag{2.161}$$

from which

$$\gamma_{\|} + 2\gamma_{\perp} = 5\,. \tag{2.162}$$

While being related to each other, $\gamma_{\perp}$ and $\gamma_{\|}$ are spatially varying functions in an inhomogeneous plasma.

In space physics the CGL equations (2.158, 2.159) are sometimes useful, e.g., in the studies of firehose and mirror instabilities (Chap. 7) related to shock waves. However, one has to be careful with the validity of the approach. For example, the CGL theory predicts that the temperature depends on the magnetic field as

$$T_{\perp} \propto B \;\; ; \;\; T_{\|} \propto (n/B)^2\,. \tag{2.163}$$

For example, direct observations in the magnetic dipole field geometry above the auroral ionosphere show that the perpendicular temperature does not scale as $T_{\perp} \propto B$. Here, and in many other practical examples, the CGL heat flux argument is not valid. In the auroral case the particles precipitate to the upper atmosphere carrying energy (heat) with them. This is actually one of the major sinks of energy associated with space storms in the magnetosphere, as will be discussed in Chap. 13.