Chapter II.

STEINHAUS TYPE THEOREMS

The generalizations of Steinhaus's theorem and its Baire category analogy, Piccard's theorem, play a key role in several later sections as well as certain applications. This chapter deals with these generalizations.

3. GENERALIZATIONS OF A THEOREM OF STEINHAUS

A famous theorem of Steinhaus [189] from 1920 asserts that, for any measurable set $A \subset \mathbb{R}$ with positive Lebesgue measure the set $A - A$ contains an interval. More generally, if $A, B \subset \mathbb{R}^n$ are measurable sets with positive Lebesgue measure, $A + B$ has an interior point; see for example the paper of Kemperman [123].

The proof can be based on Weil's idea [206] that the convolution of the characteristic functions χ_A and χ_B (in case A and B has finite measure) is a continuous function, hence the function

$$t \mapsto \lambda\big(A \cap (t - B)\big) = \int \chi_B(t - y)\chi_A(y)\, d\lambda(y)$$

is continuous and as follows from Fubini's theorem, not everywhere zero. This means that $A + B$ contains a nonvoid open set. This proof works directly when λ is a Haar measure on a locally compact group.

This theorem allows various generalizations and modifications. In the generalizations the following problem is treated: if we replace the addition by a binary operation $F(x, y)$, under what conditions on F can we prove that $F(A, B)$ contains a nonvoid open set? The first step was done by Erdős and Oxtoby [52] proving in the case $x, y \in \mathbb{R}$ that, if F is a continuously differentiable function with nonvanishing partial derivatives, then $F(A, B)$ contains a nonvoid open set.

Further generalizations detail the case when x and y are from different topological measure spaces and F satisfies certain solvability conditions in x and y. See in this direction Kuczma [130] and Sander [178]. Sander has pointed out that one of the sets A, B may be nonmeasurable. These results apply to the case when $x, y \in \mathbb{R}^n$ and F is a continuously differentiable function of which the partial derivatives are nonsingular.

In this § we will treat a generalization for function F with more then two variables. Of course, if F maps $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ we obtain a problem already solved by the theorem of Erdős and Oxtoby. To obtain a really interesting new problem, we have to consider a function with values in $\mathbb{R}^2$. The condition about the nonvanishing partial derivatives will be substituted with the condition that the null space of the derivative (as linear mapping) is in general position. Theorem 3.11 and Remark 3.13 are such generalizations of Steinhaus' theorem. As a corollary we obtain the well-known special case $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

Our proof depends on a very general version of the theorem about the continuity of the convolution formulated in Theorem 3.4. As corollary we obtain a result about the continuity of a mapping

$$t \mapsto \nu\Big(\bigcap_{i=1}^{n} g_{i,t}^{-1}(A_i)\Big) \tag{1}$$

where the functions $g_{i,t}$ do not map sets with positive measure into zero sets and depend smoothly on the parameter t. The investigation of this mapping occurs in an implicit way in Járai [80] and in an explicit way in Krausz [125].

As we explained in the introduction, the lower semicontinuity of mapping (1) and several other variants of Steinhaus' theorem have applications in the theory of functional equations. The results of this § are used in §§ 5, 6, 8, 10, and when discussing applications in § 23.

More detailed references about earlier results may be found in Kuczma [130], Kuczma and Kuczma [131], Sander [178], and Grosse-Erdmann [61].

Another kind of refinement of Steinhaus' theorem using one-sided lower densities in $\mathbb{R}$ is given by Raikoff [168] and by Matkowski and Świątkowski [156]. The results of this § are published in the paper Járai [99].

We start with the investigation of an important condition.

3.1. Condition. In this § and in §§ 8, 10, and 19 a measure theoretical condition will play an important rôle. In general form this may be formulated as follows.

Let X and Y be sets with measures μ and ν, respectively, let T be a set, $D \subset T \times Y$, and $g : D \to X$ a function. Our condition is the following:

(1) For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $B \subset D_t$, $\nu(B) \geq \varepsilon$, $t \in T$, then $\mu\big(g_t(B)\big) \geq \delta$.

Concerning this condition we summarize some simple properties. We treat this condition in 3.10 in the most important case when X and Y are open subsets of Euclidean spaces. Further results can be found in Járai [80].

3.2. Remarks. *With the notation of the previous point,*

(1) if $D_1 \subset D_2 \subset T \times Y$, $g_2 : D_2 \to X$, $g_1 = g_2 | D_1$, and g_2 satisfies condition 3.1.(1), then g_1 too;

(2) if $D = D_1 \cup D_2$, $g_1 = g|D_1$, $g_2 = g|D_2$, moreover g_1 and g_2 satisfies condition 3.1.(1), then g too;

(3) if for all $\varepsilon > 0$ there exists a decomposition $D = D_1 \cup D_2$ such that $g|D_2$ satisfies condition 3.1.(1) and $\mu\big(g_t(D_{1,t})\big) < \varepsilon$ for all $t \in T$, then g also satisfies condition 3.1.(1);

(4) condition 3.1.(1) is equivalent with the following one: for all $\varepsilon > 0$ there exists a $\delta > 0$, such that if $A \subset X$, $\mu(A) < \delta$, then $\nu\left(g_t^{-1}(A)\right) < \varepsilon$ for all $t \in T$;

(5) if μ is a regular measure, then it is enough if (4) is satisfied for all measurable A;

(6) if X and Y are Hausdorff spaces, μ and ν are Radon measures, all g_t is continuous on the Borel set D_t and the condition 3.1.(1) is satisfied, then $g_t^{-1}(A)$ is measurable for all σ-finite measurable set $A \subset X$ whenever $t \in T$;

(7) under conditions of (6) if $B \subset A$ and A is a μ hull of B, then $g_t^{-1}(A)$ is a ν hull of $g_t^{-1}(B)$ whenever $t \in T$;

Proof. (1)–(5) are trivial. To prove (6) let us represent A as the union of a Borel set B and a set C having measure zero. Then $g_t^{-1}(A) = g_t^{-1}(B) \cup$

$g_t^{-1}(C)$ where $g_t^{-1}(B)$ is a Borel set and $g_t^{-1}(C)$ has measure zero. To prove (7) suppose indirectly that for some measurable set $E \subset Y$ we have

$$\nu\left(E \cap g_t^{-1}(A)\right) > \nu\left(E \cap g_t^{-1}(B)\right).$$

Then by the measurability of $g_t^{-1}(A)$ and since ν is a Radon measure there exists a compact set $C \subset E \cap g_t^{-1}(A \setminus B)$ such that $\nu(C) > 0$; but by $g_t(C) \subset A \setminus B$ we should have $\mu\big(g_t(C)\big) = 0$.

3.3. Lemma. *Let T be a topological space, X a uniform space, C a compact uniform space, $D \subset T \times C$, $t_0 \in T$, $\{t_0\} \times C \subset D$, and let $g : D \to X$ be a continuous function. Then for any relation α from the uniformity of X there exists a relation β from the uniformity of C and a neighborhood V of t_0 such that if $t, t' \in V$, the points y, y' are β-near in C and $(t, y), (t', y') \in D$, then the points $g(t, y)$ and $g(t', y')$ are α-near in X.*

Proof. Let δ be a symmetric relation from the uniformity of X for which $\delta \circ \delta \subset \alpha$. For all $y \in C$ there exists a neighborhood V_y of t_0 and a symmetric relation γ_y such that if $t \in V_y$ and the point y' is γ_y-near to y, then $g(t, y')$ and $g(t_0, y)$ are δ-near in X. Let us choose for each γ_y a symmetric relation β_y for which $\beta_y \circ \beta_y \subset \gamma_y$. The open kernels of the sets $\beta_y(y)$ give an open cover of C. Let us choose a finite subcover and let $\beta = \bigcap_{i=1}^n \beta_{y_i}$, $V = \bigcap_{i=1}^n V_{y_i}$. If the points y and y' are β-near in C, then there exists a y_i such that y and y_i, moreover y' and y_i are γ_{y_i}-near, hence $g(t, y)$ and $g(t_0, y_i)$ moreover $g(t', y')$ and $g(t_0, y_i)$ are δ-near. This means that $g(t, y)$ and $g(t', y')$ are α-near in X.

3.4. Theorem. *Let T be a topological space, Y a Hausdorff space, X_i $(i = 1, 2, \ldots, n)$ be completely regular spaces, and let Z, Z_i $(i = 1, 2, \ldots, n)$ be Banach spaces. Let ν and μ_i be finite Radon measures over Y and X_i, respectively. Consider the functions $f_i : X_i \to Z_i$, $g_i : T \times Y \to X_i$, $h : Z_1 \times \cdots \times Z_n \to Z$. Suppose that the following conditions hold:*

(1) h maps bounded subsets into bounded subsets and is continuous;

(2) f_i is Lusin μ_i measurable and is in $\mathcal{L}^\infty(\mu_i)$ $\quad (i = 1, 2, \ldots, n)$;

(3) g_i is continuous and for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu_i(g_{i,t}(B)) \geq \delta$ whenever $B \subset Y$, $\nu(B) \geq \varepsilon$, $t \in T$, and $1 \leq i \leq n$.

Then the function

$$f(t) = \int_Y h\Big(f_1\big(g_1(t, y)\big), \ldots, f_n\big(g_n(t, y)\big)\Big)\, d\nu(y)$$

is continuous on T.

Proof. First we prove that the integral exists. We may replace each f_i by a bounded Borel function defined on all of X_i, which is almost equal to f_i, and is separable valued. This switch does not change the integral, because, by (3), the set of points y for which the value of $f_i\big(g_i(t,y)\big)$ are changed, has measure 0. Hence we may assume that the functions f_i are separable valued bounded Borel functions. By (1) the function

$$y \mapsto h\Big(f_1\big(g_1(t,y)\big), \dots, f_n\big(g_n(t,y)\big)\Big) \tag{4}$$

is a separable valued Borel function whenever $t \in T$ is fixed, and its image is a bounded subset of Z. Hence the integral exists whenever $t \in T$.

Now let $\varepsilon > 0$ and $t_0 \in T$. Let us choose a real number $M > 0$, for which the image of (4) is contained in the closed ball with center 0 and radius M. By (3), there exists a $\delta > 0$ such that $B \subset Y$, $\nu(B) \geq \varepsilon' = \varepsilon/(16Mn)$, $t \in T$, and $1 \leq i \leq n$ implies $\mu_i\big(g_{i,t}(B)\big) \geq \delta$. Let us choose a compact set $C \subset Y$ for which $\nu(Y \setminus C) < \varepsilon/(8M)$. There exists a compact set C_i in X_i for which $\mu_i(X_i \setminus C_i) < \delta$ and $f_i|C_i$ is continuous. Let us choose uniformities on the spaces $C, X_1, \dots, X_n$ compatible with their topology. By (1) there exists an $\alpha > 0$ such that

$$\big|h(z_1, \dots, z_n) - h(z_1', \dots, z_n')\big| < \frac{\varepsilon}{2\nu(Y)}$$

whenever $z_i, z_i' \in f_i(C_i)$ and $|z_i - z_i'| < \alpha$. Because of the uniform continuity of $f_i|C_i$ there exists a reflexive symmetric relation β_i in the uniformity of X_i such that

$$\big|f_i(x_i) - f_i(x_i')\big| < \alpha$$

whenever $x_i, x_i' \in X_i$ and x_i and x_i' are β_i-near, that is, $(x_i, x_i') \in \beta_i$. By Lemma 3.3 there exists an open neighborhood V of t_0 in T and a reflexive symmetric relation γ in the uniformity of C such that $g_i(t_0, y)$ and $g_i(t, y')$ are β_i-near in X_i whenever $t \in V$ and y and y' are γ-near in C. Now let t be an element of V and let

$$K = \bigcap_{i=1}^{n} g_{i,t_0}^{-1}(C_i) \cap \bigcap_{i=1}^{n} g_{i,t}^{-1}(C_i) \cap C.$$

Then

$$Y \setminus K = Y \setminus C \cup \left(\bigcup_{i=1}^{n} g_{i,t_0}^{-1}(X_i \setminus C_i)\right) \cup \left(\bigcup_{i=1}^{n} g_{i,t}^{-1}(X_i \setminus C_i)\right),$$

and hence (using (3) and that $\mu_i(X_i \setminus C_i) < \delta$),

$$\nu(Y \setminus K) < \frac{\varepsilon}{8M} + n\frac{\varepsilon}{16Mn} + n\frac{\varepsilon}{16Mn} = \frac{\varepsilon}{4M}. \tag{5}$$

Using this, we have with the notation

$$H(t,y) = h\Big(f_1\big(g_1(t,y)\big), \dots, f_n\big(g_n(t,y)\big)\Big)$$

that

$$\begin{aligned} \big|f(t) - f(t_0)\big| &\le \int_Y \big|H(t,y) - H(t_0,y)\big|\, d\nu(y) \\ &= \int_{Y\setminus K} \big|H(t,y) - H(t_0,y)\big|\, d\nu(y) + \int_K \big|H(t,y) - H(t_0,y)\big|\, d\nu(y). \end{aligned}$$

By (5) the first term on the right side is not greater than $2M\varepsilon/(4M) = \varepsilon/2$. By the choice of K, α, $\beta_1, \dots, \beta_n$, γ, and V, the second term on the right side is not greater than $\nu(Y)\varepsilon/\big(2\nu(Y)\big) = \varepsilon/2$.

3.5. Corollary. *Let T, Y, X_i, ν, μ_i, and g_i be the same as in the previous theorem. Suppose that condition (3) of the previous theorem is satisfied, and let A_i be a subset of X_i. Suppose that A_i is μ_i measurable if $2 \le i \le n$. Then the function*

$$f(t) = \nu\Big(\bigcap_{i=1}^{n} g_{i,t}^{-1}(A_i)\Big) \quad \text{if} \quad t \in T$$

is continuous on T.

Proof. Condition (3) of the previous theorem by 3.2.(7) implies that the set $g_{1,t}^{-1}(B_1)$ is a ν hull of $g_{1,t}^{-1}(A_1)$ whenever B_1 is a μ_1 hull of A_1. Hence

$$f(t) = \int_Y \chi_{B_1}\big(g_1(t,y)\big)\chi_{A_2}\big(g_2(t,y)\big)\cdots\chi_{A_n}\big(g_n(t,y)\big)\, d\nu(y),$$

where χ_{A_i} is the characteristic function of A_i and χ_{B_1} is the characteristic function of B_1.

3.6. Theorem. *Let T be a topological space, X and Y be Hausdorff spaces with Radon measures μ and ν, respectively. Let D be an open subset of $T \times Y$, $g : D \to X$, $t_0 \in T$, and let $K \subset X$ be a compact set. Suppose, that*

(1) the mappings g and $(t, x) \mapsto g_t^{-1}(x)$ are continuous and g_t is a homeomorphism of D_t onto X if $t \in T$;

(2) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu\big(g_t(B)\big) > \delta$ whenever $B \subset Y$, $\nu(B) \geq \varepsilon$, and $t \in T$.

Then

$$\nu\left(g_t^{-1}(K) \,\triangle\, g_{t_0}^{-1}(K)\right) \to 0 \quad \textit{if} \quad t \to t_0.$$

Proof. Let $\varepsilon > 0$ and let us choose a $\delta > 0$ for $\varepsilon/2$. Let $C = g_{t_0}^{-1}(K)$, and let W be an open subset of D_{t_0} containing C for which $\nu(W \setminus C) < \varepsilon/2$. Similarly, let U be an open subset of X containing K for which $\mu(U \setminus K) < \delta$. Let us choose an open neighborhood V of t_0 such that if $t \in V$, $y \in C$, then $g_t(y)$ is defined and $g_t(y) \in U$; moreover if $t \in V$, $x \in K$, then $g_t^{-1}(x) \in W$. Then by $g_t^{-1}(K) \setminus g_{t_0}^{-1}(K) \subset V \setminus C$ we have $\nu\left(g_t^{-1}(K) \setminus g_{t_0}^{-1}(K)\right) < \varepsilon/2$. Moreover $\nu\left(g_{t_0}^{-1}(K) \setminus g_t^{-1}(K)\right) < \varepsilon/2$ because the mapping g_t maps the set $g_{t_0}^{-1}(K) \setminus g_t^{-1}(K)$ into $U \setminus K$, for which $\mu(U \setminus K) < \delta$. Summarizing, $\nu\left(g_t^{-1}(K) \,\triangle\, g_{t_0}^{-1}(K)\right) < \varepsilon$ if $t \in V$.

3.7. Corollary. *Let G be a locally compact group and let λ be a left Haar measure on G. Let A_i, $i = 1, 2, \ldots, n$ be subsets of G with finite measure. Suppose that A_i is λ measurable if $2 \leq i \leq n$. Then the mapping*

$$(t_1, \ldots, t_n) \mapsto \lambda(t_1 A_1 \cap \cdots \cap t_n A_n)$$

of G^n into $\mathbb{R}$ is continuous.

Proof. Since the replacement of A_1 by a λ hull does not change this function, we may suppose that A_1 is measurable too. Let $\varepsilon > 0$ and $T = G^n$. Let us choose the compact sets K_i such that $K_i \subset A_i$ and $\lambda(A_i \setminus K_i) < \varepsilon$ are satisfied. Let $Y = X_1 = X_2 = \cdots = X_n = G$ and

$$g_i(t, y) = t_i^{-1} y \quad \text{if} \quad (t_1, \ldots, t_n) \in T \text{ and } y \in Y.$$

Since for any measurable subsets B, C with finite measure

$$\big|\lambda(B) - \lambda(C)\big| \leq \lambda(B \,\triangle\, C)$$

and

$$\Big(\bigcap_{i=1}^n B_i\Big) \,\triangle\, \Big(\bigcap_{i=1}^n C_i\Big) \subset \bigcup_{i=1}^n (B_i \,\triangle\, C_i),$$

by the previous theorem we obtain

$$\left|\lambda(t_1K_1 \cap t_2K_2 \cap \cdots \cap t_nK_n) - \lambda(t_1^0K_1 \cap t_2^0K_2 \cap \cdots \cap t_n^0K_n)\right| \\ \leq \sum_{i=1}^{n} \lambda(t_iK_i \bigtriangleup t_i^0K_i) \to 0 \quad \text{whenever} \quad t_i \to t_i^0,$$

that is

$$t \mapsto \lambda(t_1K_1 \cap t_2K_2 \cap \cdots \cap t_nK_n)$$

is continuous on T. But

$$0 \leq \lambda(t_1A_1 \cap t_2A_2 \cap \cdots \cap t_nA_n) - \lambda(t_1K_1 \cap t_2K_2 \cap \cdots \cap t_nK_n) \\ \leq \sum_{i=1}^{n} \lambda(t_iA_i \setminus t_iK_i) \leq n\varepsilon.$$

Hence

$$(t_1, \ldots, t_n) \mapsto \lambda(t_1A_1 \cap \cdots \cap t_nA_n)$$

is the uniform limit of continuous functions, and so itself is continuous.

In the following lemma which will be needed for the proof of our main result, we give sufficient conditions for the validity of condition (3) in Theorem 3.4.

3.8. Lemma. *Let Y be an open subset of $\mathbb{R}^k$, let T be a topological space, $y_0 \in Y$, and $t_0 \in T$. Let $g : T \times Y \to \mathbb{R}^r$ be a continuous function and suppose that $\frac{\partial g}{\partial y}$ is continuous and*

$$\operatorname{rank}\left(\frac{\partial g}{\partial y}(t_0, y_0)\right) = r.$$

Then there exist open neighborhoods Y^* and T^* of y_0 and t_0, respectively, and there exists a constant $0 < C < \infty$ such that $Y^* \subset Y$, $T^* \subset T$, and

$$\lambda^k(B) \leq \lambda^r\big(g_t(B)\big)C(\operatorname{diam} B)^{k-r} \tag{1}$$

whenever $B \subset Y^*$ and $t \in T^*$. (Here $\operatorname{diam} B$ denotes the diameter of the set B.)

Proof. Let $q = k-r$ and let us divide the coordinates of $y = (y_1, \ldots, y_k)$ into two groups $y' = (y_1', \ldots, y_q')$ and $y'' = (y_1'', \ldots, y_r'')$ so that the condition

$$\det\left(\frac{\partial g}{\partial y''}(t_0, y_0)\right) = \det\left(\frac{\partial g}{\partial y''}(t_0, y_0', y_0'')\right) \neq 0$$

is satisfied. Let us introduce the notation

$$L(t,y') = \frac{\partial g}{\partial y''}(t,y',y_0'').$$

Using the proof of the inverse function theorem (see Rudin [172], theorem 9.24) we obtain that, if Y'' is an open ball with center y_0'' in $\mathbb{R}^r$, $t \in T$, $(y',y'') \in Y$, and

$$\left\| \frac{\partial g}{\partial y''}(t,y',y'') - L(t,y') \right\| < \frac{1}{2\left\|L(t,y')^{-1}\right\|} \tag{2}$$

whenever $y'' \in Y''$, then $g_{t,y'}$ is a homeomorphic mapping of Y'' onto an open subset $U(t,y')$ of $\mathbb{R}^r$. Now let

$$0 < \beta < \frac{1}{2\left\|L(t_0,y_0')^{-1}\right\|}$$

and

$$0 < \gamma < \left|\det \frac{\partial g}{\partial y''}(t_0,y_0',y_0'')\right|. \tag{3}$$

Using the continuity of the expressions in (2) and (3) we can choose an open ball Y'' with center y_0'' and open sets Y' and T^* such that $t_0 \in T^*$, $y_0' \in Y'$, $Y^* = Y' \times Y'' \subset Y$, moreover $t \in T^*$, $y' \in Y'$, $y'' \in Y''$ implies that

$$\left\| \frac{\partial g}{\partial y''}(t,y',y'') - L(t,y') \right\| < \beta;$$
$$\beta < \frac{1}{2\left\|L(t,y')^{-1}\right\|};$$
$$\gamma < \left|\det \frac{\partial g}{\partial y''}(t,y',y'')\right|.$$

Let $\alpha(q)$ denote the λ^q measure of the q dimensional unit ball ($\alpha(0) = 1$). We are going to prove that

$$\lambda^k(B) \le \lambda^r\left(g_t(B)\right)\frac{\alpha(q)}{\gamma}(\operatorname{diam} B)^{k-r}$$

whenever $B \subset Y^*$ and $t \in T^*$. Let $R = \operatorname{diam} B$. Then there exists a closed ball V with radius R in $\mathbb{R}^q$ such that $B \subset (V \cap Y') \times Y''$. Suppose to the contrary that there exists a $t \in T^*$ for which

$$\lambda^k(B) > \lambda^r\big(g_t(B)\big)CR^q,$$

where $C = \alpha(q)/\gamma$. Then we can choose an open set U for which $g_t(B) \subset U$ and

$$\lambda^k(B) > \lambda^r(U)CR^q.$$

Let

$$B^* = g_t^{-1}(U) \cap \big((V \cap Y') \times Y''\big).$$

Then $B \subset B^*$, B^* is a Borel set and $g_t(B^*) \subset U$, that is

$$\lambda^r\big(g_t(B^*)\big)CR^q < \lambda^k(B) \le \lambda^k(B^*).$$

We are going to prove that this is impossible. Let

$$B^*_{y'} = \{y'' : (y', y'') \in B^*\} \quad \text{if } y' \in V \cap Y'.$$

Using the theorem concerning transformation of integrals we have that

$$\lambda^r\big(g_t(B^*)\big) \ge \lambda^r\left(g_{t,y'}(B^*_{y'})\right) = \int_{B^*_y} \left|\det \frac{\partial g}{\partial y''}(t, y', y'')\right| d\lambda^r(y'') \ge \gamma\lambda^r(B^*_{y'})$$

whenever $y' \in V \cap Y'$. By Fubini's theorem

$$\begin{aligned}\lambda^k(B^*) &= \int_{V \cap Y'} \lambda^r(B^*_{y'})\, d\lambda^q(y') \\ &\le \frac{\lambda^r\big(g_t(B^*)\big)}{\gamma}\lambda^q(V) = \lambda^r\big(g_t(B^*)\big)CR^q\end{aligned}$$

which is a contradiction. Hence the proof is complete.

3.9. Lemma. *Under the conditions of the previous lemma, if a subset D of $\mathbb{R}^r$ has density 1 in the point $g(t_0, y_0)$, then $g_{t_0}^{-1}(D) \cap Y^*$ has density 1 in the point y_0.*

Proof. By the continuity of $\dfrac{\partial g}{\partial y}(t_0, y)$ the function g_{t_0} satisfies the Lipschitz condition on a neighborhood of y_0. Hence there exist a $\gamma > 0$ and an $0 < M < \infty$ such that $y \in Y^*$ and $\big|g(t_0, y) - g(t_0, y_0)\big| \le M|y - y_0|$ whenever

$|y - y_0| < \gamma$. Let $\alpha(k)$ and $\alpha(r)$ denote the λ^k and λ^r measures of the k and r dimensional unit balls, respectively. Let $\varepsilon > 0$, and let

$$0 < \delta < \frac{\varepsilon\alpha(k)}{CM^r 2^{k-r}\alpha(r)},$$

where C is the constant from the previous lemma.

Let us choose a $\beta > 0$ such that, whenever V is a closed ball with center $g(t_0, y_0)$ and radius less than β, then $\lambda^r(V \cap D) \geq (1-\delta)\lambda^r(V)$. We prove that if W is a closed ball in Y^* with center y_0 and radius less than γ and β/M, then

$$\lambda^k\big(W \cap g_{t_0}^{-1}(D)\big) \geq (1-\varepsilon)\lambda^k(W).$$

Suppose to the contrary that for such a W with radius R,

$$\lambda^k\big(W \cap g_{t_0}^{-1}(D)\big) < (1-\varepsilon)\lambda^k(W).$$

Then there exists a compact subset $B \subset W \setminus g_{t_0}^{-1}(D)$ for which $\lambda^k(B) > \varepsilon\lambda^k(W)$. Hence by the previous lemma,

$$\varepsilon R^k\alpha(k) < \lambda^k(B) \leq C2^{k-r}R^{k-r}\lambda^r\big(g_{t_0}(B)\big).$$

But $g_{t_0}(B)$ is a compact subset of $V \setminus D$, where V is the closed ball in $\mathbb{R}^r$ with center $g(t_0, y_0)$ and radius $MR < \beta$. Since $\lambda^r(V \setminus D) < \delta\lambda^r(V)$ we get

$$\varepsilon R^k\alpha(k) < C2^{k-r}\delta\lambda^r(V) = C2^{k-r}R^{k-r}M^r R^r\alpha(r)\delta,$$

which contradicts the choice of δ.

3.10. Lemma. *Let Y be an open subset of $\mathbb{R}^k$, T a topological space, D an open subset of $T \times Y$, and $(t_0, y_0) \in D$. Suppose that the function $g : D \to \mathbb{R}^r$ is continuous and continuously differentiable with respect to y. If the rank of the matrix $\frac{\partial g}{\partial y}(t_0, y_0)$ is r, then there exist such neighborhoods T^* and Y^* of t_0 and y_0, respectively for which*

(1) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda^r\big(g_t(B)\big) \geq \delta$ whenever $t \in T^*$, $B \subset Y^*$ $\lambda^k(B) \geq \varepsilon$;

(2) if A is a λ^r measurable subset of $\mathbb{R}^r$, then $g_t^{-1}(A) \cap Y^*$ is a λ^k measurable subset of Y for each $t \in T^*$.

Proof. Let us use Lemma 3.8. Shrinking the so obtained set Y^* we may suppose that it is bounded. Now from the statement of Lemma 3.8 and from 3.2.(4) we obtain (1) and from 3.2.(7) we obtain (2).

3.11. Theorem. *Let X be an r-dimensional Euclidean space, and let $X_1, \dots, X_n$ be orthogonal subspaces of X with dimensions $r_1, \dots, r_n$. Suppose that $r_i \geq 1$ $(1 \leq i \leq n)$ and $\sum_{i=1}^{n} r_i = r$. Let U be an open subset of X and $F : U \to \mathbb{R}^m$ be a continuously differentiable function. For each $x \in U$ let N_x denote the null space of $F'(x)$. Let A_i be a subset of X_i $(i = 1, \dots, n)$ and suppose that A_i is λ^{r_i} measurable for $2 \leq i \leq n$. Let $a \in U$, $\dim N_a = r - m$. Let p_i denote the orthogonal projection of X onto X_i. Suppose that $p_i(N_a) = X_i$ and A_i has density 1 in the point $p_i(a)$ whenever $1 \leq i \leq n$. Then $F(A_1 \times \cdots \times A_n)$ is a neighborhood of $F(a)$.*

Proof. Let $k = r - m$. Since $x \mapsto \operatorname{rank} F'(x)$ is lower semicontinuous, and $\operatorname{rank}\big(F'(a)\big) = m$, we may suppose that $\operatorname{rank}\big(F'(x)\big) = m$ whenever $x \in U$. Similarly, choosing a smaller U if necessary, we may suppose that $p_i(N_x) = X_i$ whenever $x \in U$ and $1 \leq i \leq n$; to prove this, suppose to the contrary that there exists an i and for each natural number j there exists an $x_j \in U$ and there exist orthonormal vectors $e_1^{(j)}, \dots, e_{k-r_i+1}^{(j)}$ in N_{x_j} such that $x_j \to a$ and

$$p_i(e_s^{(j)}) = 0 \quad \text{whenever } j = 1, 2, \dots \text{ and } 1 \leq s \leq k - r_i + 1.$$

Using the compactness of the unit sphere we can pass to a subsequence and suppose that

$$e_s^{(j)} \to e_s \qquad \text{if } j \to \infty.$$

But this proves that the vectors e_s are orthonormal in N_a and $p_i(e_s) = 0$ whenever $1 \leq s \leq k - r_i + 1$, which is a contradiction.

Now, choosing a smaller U if necessary and using the rank theorem (see Dieudonné [49], 10.3.1), we have that there exist mappings u, p, and v and an open neighborhood V of $b = F(a)$ in $\mathbb{R}^m$ with the following properties: u maps U onto the open cube I^r, where $I = \,]-1, 1[$, u is invertible and moreover u and u^{-1} are continuously differentiable; v maps I^m onto V, v is invertible and moreover v and v^{-1} are continuously differentiable; p is the projection

$$p : (x_1, \dots, x_r) \mapsto (x_1, \dots, x_m)$$

of I^r onto I^m; and finally $F = v \circ p \circ u$. We may write I^r as $I^r = T \times Y$ where $T = I^m$ and $Y = I^k$. Let $u(a) = (t_0, y_0) \in T \times Y$. Now let us use some facts from differential geometry (see Dieudonné [49], mainly 16.8.8).

$U \cap F^{-1}\big(v(t)\big)$ is a closed submanifold of U whenever $t \in T$. The tangent space of this submanifold in a point $x \in U \cap F^{-1}\big(v(t)\big)$ is equal to the subspace N_x of X. Clearly u^{-1} is a diffeomorphism of the closed submanifold $\{t\} \times Y$

of $T \times Y$ onto $U \cap F^{-1}(v(t))$. Let $g_i = p_i \circ u^{-1}$ if $1 \le i \le n$. By the choice of U, p_i is a submersion of $U \cap F^{-1}(v(t))$ into X_i. Hence the mapping $g_{i,t} : Y \to X_i$ is a submersion too, that is, its derivative has rank r_i whenever $y \in Y$ and $t \in T$.

Now, by Lemma 3.8, there exist open sets T^* and Y^* and there exists a $0 < K < \infty$ such that $t_0 \in T^* \subset T$, $y_0 \in Y^* \subset Y$, and

$$\lambda^k(B) \le K\lambda^{r_i}(g_{i,t}(B))$$

whenever $B \subset Y^*$, $t \in T^*$. Let $X_i^* = X_i$, $A_i^* = A_i$, and g_i^* the restriction of g_i onto $T^* \times Y^*$. Applying Corollary 3.5 to the sets and functions marked by stars we have that the function

$$f(t) = \lambda^k\Big(\bigcap_{i=1}^{n} g_{i,t}^{*-1}(A)\Big) \quad \text{if} \quad t \in T^*$$

is continuous on T^*. By Lemma 3.9, $g_{i,t_0}^{*-1}(A_i)$ has density 1 in the point y_0. Since $g_{i,t_0}^{*-1}(A_i) \cap Y^*$ is measurable by Lemma 3.10 if $2 \le i \le n$, we have that

$$\bigcap_{i=1}^{n} g_{i,t_0}^{*-1}(A_i)$$

has density 1 in the point y_0. Hence $f(t_0) > 0$ and we have that there exists a neighborhood V of t_0 for which $f(t) > 0$ if $t \in V$. Clearly $v(V)$ is a neighborhood of b in $\mathbb{R}^m$. If $z \in v(V)$, then $t := v^{-1}(z) \in V$ and hence the set

$$\bigcap_{i=1}^{n} g_{i,t}^{*-1}(A_i)$$

is nonvoid. If y is an element of this set, then $F(u^{-1}(t,y)) = v(p(t,y)) = v(t) = z$ and $x_i = p_i(u^{-1}(t,y)) = g_{i,t}^*(y) \in A_i$ if $1 \le i \le n$. This means that $F(x_1, \dots, x_n) = z$.

3.12. Corollary. *Let U be an open subset of $\mathbb{R}^r \times \mathbb{R}^r$ and $F : (x,y) \mapsto F(x,y)$ a continuously differentiable mapping of U into $\mathbb{R}^r$. Let $A, B \subset \mathbb{R}^r$ and suppose that B is λ^r measurable. If $(a,b) \in U$,*

$$\det \frac{\partial F}{\partial x}(a,b) \ne 0, \qquad \det \frac{\partial F}{\partial y}(a,b) \ne 0,$$

A has density 1 in the point a and B has density 1 in the point b, then $F(A,B)$ is a neighborhood of $F(a,b)$.

Proof. By Theorem 3.11 we have to prove only that $p_1(N_{a,b}) = \mathbb{R}^r$ and $p_2(N_{a,b}) = \mathbb{R}^r$ where $N_{a,b}$ is the null space of $F'(a,b)$. Let $(x,y) \in N_{a,b}$. If $p_1(x,y) = 0$, then $x = 0$. Hence

$$0 = F'(a,b)(x,y) = \frac{\partial F}{\partial y}(a,b)(y).$$

But $\det \dfrac{\partial F}{\partial y}(a,b) \neq 0$, hence $y = 0$. This proves that $p_1 : N_{a,b} \to \mathbb{R}^r$ is a one-to-one mapping, that is, $p_1(N_{a,b}) = \mathbb{R}^r$. Similarly $p_2(N_{a,b}) = \mathbb{R}^r$.

3.13. Remark. *The previous theorem may be stated in the following global form: If N_x is in general position, i.e., $\dim N_x = r-m$ and $p_i(N_x) = X_i$ for all $x \in U$ $(i = 1,2,\dots,n)$, moreover $A_1 \times A_2 \times \cdots \times A_n \subset U$, $\lambda^{r_i}(A_i) > 0$ $(i = 1,2,\dots,n)$, and the set A_i is λ^{r_i} measurable for $2 \le i \le n$, then $F(A_1 \times \cdots \times A_n)$ contains a nonvoid open set.*

Proof. Let us choose a point $a \in U$ for which A_i has density 1 at $p_i(a)$ whenever $1 \le i \le n$, and let us apply the previous theorem.

4. GENERALIZATIONS OF A THEOREM OF PICCARD

Piccard's result [167] analogous to the theorem of Steinhaus states that the sum of two Baire sets having second Baire category has an inner point. Very strong generalizations exist; in this case also addition can be replaced by a two variable function with weak solvability conditions. These results are useful in the proof of "Baire property implies continuity" and "Baire property implies boundedness" type regularity theorems for functional equations. We refer the reader to the papers Sander [179], [181], [183], Kominek [124], Járai [86], Grosse-Erdmann [61], Lindberg and Szymanski [139], and the references cited in them.

The purpose of this section is to give a generalization of Piccard's theorem analogous to the results of the previous section. These results were published in Járai [103].

The following theorem is an abstract version of our generalization of the theorem of Piccard.

4.1. Theorem. *Let T, Y, and X_i be topological spaces, $g_i : T \times Y \to X_i$ continuous functions, and suppose that $g_{i,t}(B)$ has second Baire category whenever $B \subset Y$ is a subset of Y with second Baire category. Suppose that $A_i \subset X_i$ and A_i is a Baire set whenever $1 \le i \le n$. Then the set V of points $t \in T$ for which*

$$\bigcap_{i=1}^{n} g_{i,t}^{-1}(A_i)$$

is of second category is an open subset of T.

Proof. The sets A_i can be written in the form $A_i = E_i \triangle M_i$, where E_i is open and M_i is of first category. Suppose that $t_0 \in V$, and let $K = \cap_{i=1}^{n} g_{i,t_0}^{-1}(A_i)$, $K' = K \cap \left(\cap_{i=1}^{n} g_{i,t_0}^{-1}(E_i)\right)$. Since $g_{i,t_0}^{-1}(M_i)$ is of first category in Y whenever $i = 1, 2, \dots, n$, we have that

$$K \setminus K' \subset \bigcup_{i=1}^{n} g_{i,t_0}^{-1}(M_i)$$

is of first category, hence K' is of second category in Y. Let y_0 be a point of K' for which $W \cap K'$ is of second category in Y for each open neighborhood W of y_0. Clearly $g_i(t_0, y_0) \in E_i$ if $i = 1, 2, \dots, n$. Since the sets E_i are open and the functions g_i are continuous, the sets $g_i^{-1}(E_i)$ are open and contain the point (t_0, y_0), hence there exist open sets V' and W', such that $t_0 \in V'$, $y_0 \in W'$, and $V' \times W' \subset \cap_{i=1}^{n} g_i^{-1}(E_i)$. We will prove that

$$W' \cap \Bigl(\bigcap_{i=1}^{n} g_{i,t}^{-1}(A_i)\Bigr)$$

is of second category for each $t \in V'$. Were this not true, the sets

$$W' \setminus g_{i,t}^{-1}(A_i), \qquad i = 1, 2, \dots, n$$

would cover — except for a set of first category — the set W'. If we prove that these sets are of first category, then we have a contradiction. But this follows from the inclusion

$$W' \setminus g_{i,t}^{-1}(A_i) \subset g_{i,t}^{-1}(M_i \cap E_i),$$

which is a consequence of $W' \subset g_{i,t}^{-1}(E_i)$.

4.2. Remark. *If we suppose that Y is a complete separable metric space and X_1 is metrizable, then we may omit the condition that the set A_1 has the Baire property.*

To prove this, let C_1 denote the set of all points $x_1 \in X_1$ such that for each neighborhood U_1 of x_1 the set $U_1 \cap A_1$ is of second Baire category. It is known that C_1 is a closed set and $A_1 \setminus C_1$ is of first category. Let B_1 denote the set of inner points of C_1. Then B_1 is open and $A_1 \setminus B_1$ is also of first category. As in the previous proof we obtain that $W' \setminus g_{1,t}^{-1}(B_1)$ and $W' \setminus g_{i,t}^{-1}(A_i)$, $2 \le i \le n$ are of first category. It is enough to prove that $W' \cap g_{1,t}^{-1}(A_1)$ is of second category, because then it follows that

$$W' \cap \left(\cap_{i=1}^{n} g_{i,t}^{-1}(A_i)\right)$$

cannot be of first category.

Suppose that $W' \setminus g_{1,t}^{-1}(A_1)$ is of first category. Then, using that $W' \cap g_{1,t}^{-1}(B_1)$ is an open set of second category, we obtain that

$$\left(W' \cap g_{1,t}^{-1}(B_1)\right) \setminus g_{1,t}^{-1}(A_1) = \left(W' \cap g_{1,t}^{-1}(B_1)\right) \setminus \left(W' \cap g_{1,t}^{-1}(A_1)\right)$$

is a Baire set of second category. Let G be a $\mathcal{G}_\delta$ subset of second category of the set above. Then $g_{1,t}(G)$ is of second category as a subset of X_1. By Bourbaki [38], IX, §6, Exercise 10, $g_{1,t}(G)$ is a Baire set in X_1. Clearly $g_{1,t}(G) \subset B_1 \setminus A_1$. Writing $g_{1,t}(G) = U \triangle F$ where U is open and F is of first category, we see that $U \cap B_1$ is a nonvoid open set for which the intersection with A_1 is of first category. This contradicts the definition of B_1.

In Laczkovich [135], II.9.9 it is proved that the continuous image of a Polish space in a Hausdorff space is a Baire set. This shows that it is enough to suppose that X_1 is Hausdorff.

The following lemma allows us to use derivatives to verify that the conditions on the functions g_i in the previous theorem are satisfied.

4.3. Lemma. *Let Y be an open subset of $\mathbb{R}^k$, T a topological space, D an open subset of $T \times Y$, and $(t_0, y_0) \in D$. Suppose that the function $g : D \to \mathbb{R}^r$ is continuous and has continuous partial derivative with respect to y. If the rank of $\dfrac{\partial g}{\partial y}(t_0, y_0)$ is r, then there exist open neighborhoods T^* and Y^* of t_0 and y_0, respectively, such that*

(1) *if B has second category in Y^*, then $g_t(B)$ has second category in $\mathbb{R}^r$ for each $t \in T^*$;*

(2) *if A is a Baire set in $\mathbb{R}^r$, then $g_t^{-1}(A) \cap Y^*$ is a Baire set in Y for each $t \in T^*$.*

Proof. We have proved in the proof of Lemma 3.8, that there exist open sets T^* and Y' and an open ball Y'' centered at y_0'' such that $t_0 \in T^*$, $y_0' \in Y'$, $T^* \times Y' \times Y'' \subset D$, and $g_{t,y'}$ is a homeomorphism of Y'' onto an open subset $U(t, y')$ of $\mathbb{R}^r$ whenever $t \in T^*$ and $y' \in Y'$.

Suppose that there exists a subset B of $Y^* = Y' \times Y''$ of second category, and a $t \in T^*$, such that $g_t(B)$ is of first category in $\mathbb{R}^r$. Let us choose a Borel set U of first category in $\mathbb{R}^r$, for which $g_t(B) \subset U \subset g_t(Y^*)$ and let $B^* = g_t^{-1}(U) \cap Y^*$. This set B^* is a Baire set and is of second category in Y^*, but $g_t(B^*)$ is of first category in $\mathbb{R}^r$. By the Kuratowski–Ulam theorem the set of all points $y' \in Y'$ for which $B^*_{y'}$ is of second category is a set of second category. On the other hand, by the same theorem, the set of all points $y' \in Y'$ for which $B^*_{y'}$ is not a Baire set, is of first category. From this it follows that there exists $y' \in Y'$ for which $B^*_{y'}$ is a Baire set of second category in Y''. Since $g_{t,y'}$ is a homeomorphism of Y'' onto $U(t, y')$, the set $g_{t,y'}(B^*_{y'})$ is of second category in $\mathbb{R}^r$. This is a contradiction, because $g_{t,y'}(B^*_{y'}) \subset g_t(B^*)$. Hence (1) is proved.

To prove (2) suppose that A is a Baire set in $\mathbb{R}^r$, and let us choose a Borel set B for which $A \subset B$ and $B \setminus A$ is of first category. Then

$$g_t^{-1}(A) \cap Y^* = \big(g_t^{-1}(B) \cap Y^*\big) \setminus \big(g_t^{-1}(B \setminus A) \cap Y^*\big).$$

Using that $g_t^{-1}(B)$ is a Borel set and $g_t^{-1}(B \setminus A) \cap Y^*$ has first category by (1), we have that $g_t^{-1}(A) \cap Y^*$ is a Baire set.

Now we are prepared to prove the local version of our generalization of the theorem of Piccard for a function from an open subset of $\mathbb{R}^r$ into $\mathbb{R}^m$. The condition of the following theorem means, roughly speaking, that the null space of the derivative is large enough and is in general position.

4.4. Theorem. *Let X be the r-dimensional Euclidean space, and let $X_1, \dots, X_n$ be orthogonal subspaces of X with dimensions $r_1, \dots, r_n$, respectively. Suppose that $r_i \geq 1$ whenever $1 \leq i \leq n$ and $\sum_{i=1}^n r_i = r$. Let U be an open subset of X and $F : U \to \mathbb{R}^m$ a continuously differentiable function. For each $x \in U$ let N_x denote the null space of $F'(x)$. Let A_i be a Baire subset of X_i $(i = 1, 2, \dots, n)$, and suppose that $a \in U$ and*

$\dim N_a = r - m$. *Let* p_i *denote the orthogonal projection of* X *onto* X_i. *Suppose, that* $p_i(N_a) = X_i$ *and* $p_i(a)$ *has a neighborhood* U_i *such that* $U_i \setminus A_i$ *is of first category if* $1 \leq i \leq n$. *Then* $F(A_1 \times \cdots \times A_n)$ *is a neighborhood of* $F(a)$.

Proof. Let us define t_0, y_0, T, Y, and g_i as in the proof of 3.11. By the above lemma there exist sets T^* and Y^* such that $t_0 \in T^* \subset T$, $y_0 \in Y^* \subset Y$, and $g_{i,t}(B)$ is of second category whenever $B \subset Y^*$ has second category and $t \in T^*$. Let $X_i^* = X_i$, $A_i^* = A_i$, and g_i^* be the restriction of g_i to $T^* \times Y^*$. Applying the theorem above to the sets marked by a star we have that the set V^* of points t for which

$$\bigcap_{i=1}^{n} g_{i,t}^{-1}(A_i)$$

is of second category is open in T^*. Since g_{i,t_0}^* maps Y^* onto an open neighborhood of $p_i(a)$ and $g^*(t_0, y_0) = p_i(a)$, there exists an open neighborhood W of y_0 in Y^* such that $W \setminus g_{i,t_0}^{*-1}(A_i)$ is of first category if $1 \leq i \leq n$. This proves that $t_0 \in V^*$. Clearly $v(V^*)$ is an open neighborhood of b in $\mathbb{R}^m$. If $z \in v(V^*)$, then $v^{-1}(z) \in V^*$, and hence the set

$$\bigcap_{i=1}^{n} g_{i,t}^{*-1}(A_i)$$

is nonvoid. If y is an element of this set, then $u^{-1}(t, y) \in F^{-1}(z)$ and $x_i = p_i\left(u^{-1}(t, y)\right) \in A_i$ whenever $1 \leq i \leq n$. This implies $F(x_1, \ldots, x_n) = z$, which is enough since $v(V^*)$ is an open neighborhood of $b = F(a)$.

4.5. Corollary. *Let* W *be an open subset of* $\mathbb{R}^r \times \mathbb{R}^r$, *and let* $F : (x, y) \mapsto F(x, y)$ *be a continuously differentiable mapping of* W *into* $\mathbb{R}^r$. *Suppose, that* $A, B \subset \mathbb{R}^r$ *and* A, B *are Baire sets. If* $(a, b) \in W$,

$$\det \frac{\partial F}{\partial x}(a, b) \neq 0, \qquad \det \frac{\partial F}{\partial y}(a, b) \neq 0,$$

and there exist neighborhoods U *and* V *of* a *and* b *respectively such that* $U \setminus A$ *and* $V \setminus B$ *is of first category, then* $F(A, B)$ *contains a neighborhood of* $F(a, b)$.

Proof. Similar to Corollary 3.12 of the previous section.

The above theorem can be formulated in the following global form:

4.6. Theorem. *Let X be the r-dimensional Euclidean space, and let $X_1, \dots, X_n$ be orthogonal subspaces of X with dimensions $r_1, \dots, r_n$, respectively. Let p_i denote the orthogonal projection of X onto X_i. Suppose that $r_i \geq 1$ whenever $1 \leq i \leq n$ and $\sum_{i=1}^{n} r_i = r$. Let U be an open subset of X and $F : U \to \mathbb{R}^m$ a continuously differentiable function. For each $x \in U$ let N_x denote the null space of $F'(x)$. If $\dim N_x = r - m$ and $p_i(N_x) = X_i$ whenever $x \in U$ and $i = 1, 2, \dots, n$, moreover $A_1 \times \cdots \times A_n \subset U$ and A_i is a Baire set having second category if $1 \leq i \leq n$, then $F(A_1 \times \cdots \times A_n)$ contains a nonvoid open set.*

Proof. A Baire set A_i can be written in the form $U_i \triangle F_i$ where U_i is a nonvoid open set and F_i is of first category. For any $a \in U_1 \times U_2 \times \cdots \times U_n$ we may apply the previous theorem.

4.7. Remark. *Using the previous Remark, we may omit the condition that A_1 is a Baire set.*