Chapter 2
Preliminaries of Nonlinear Dynamics and Chaos

Abstract This chapter provides a brief review of some concepts and tools related to the subject of the monograph – chaos suppression, chaos synchronization, and chaotification. After a quick review of the history of 'dynamical systems,' we provide a summary of important definitions and theorems, including equilibrium points, periodic orbits, quasiperiodic orbits, stable and unstable manifolds, attractors, chaotic attractors, Lyapunov stability, orbital stability, and symbolic dynamics, which are all from the theory of ordinary differential equations and ordinary difference equations. The results are summarized both for continuous time and for discrete time. Then, we present three examples for chaotic attractors including the logistic map, the Lorenz attractor, and the Smale horseshoe. At the end of the chapter, we provide some necessary definitions and theorems of functional differential equations (PDEs).

2.1 Introduction

Roughly speaking, a *dynamical system* consists of two ingredients: a rule or 'dynamics,' which is described by a set of equations (difference, differential, integral, functional, or abstract operator equations, or a combination of some of them) and specifies how a system evolves, and an initial condition or 'state' from which the system starts. A *nonlinear dynamical system* is a dynamical system described by a set of nonlinear equations; that is, the dynamical variables describing the properties of the system (for example, position, velocity, acceleration, pressure, etc.) appear in the equations in a nonlinear form. The most successful class of rules for describing natural phenomena are differential equations. All the major theories of physics are stated in terms of differential equations. This observation led the mathematician V. I. Arnol'd to comment, 'consequently, differential equations lie at the basis of scientific mathematical philosophy' [2]. This scientific philosophy began with the discovery of calculus by Newton and Leibniz and continues to the present day. The theory of *dynamical systems* grew out of the qualitative study of differential equations, which in turn began as an attempt to understand and predict the motions that surround us such as the orbits of the planets, the vibrations of a string, the ripples on the surface of a pond, and the forever evolving patterns of the weather. The first two hundred years of this scientific philosophy, from Newton and Euler to Hamilton and Maxwell, produced many stunning successes in formulating the ‘rules of the world,’ but only limited results in finding their solutions.

By the end of the 19th century, researchers had realized that many nonlinear differential equations did not have explicit solutions. Even the case of three masses moving under the laws of Newtonian attraction could exhibit very complicated behavior and its explicit solution was not possible to obtain (e.g., the motion of the sun, the earth, and the moon cannot be given explicitly in terms of known functions). Short-term solutions could be given by power series, but these were not useful in determining long-term behavior. The modern theory of nonlinear dynamical systems began with Poincaré at the end of the 19th century with fundamental questions concerning the stability and evolution of the solar system. Poincaré shifted the focus from finding explicit solutions to discovering geometric properties of solutions. He introduced many ideas in specific examples. In particular, he realized that a deterministic system in which the outside forces are not varying and are not random can exhibit behavior that is apparently random (i.e., chaotic). Poincaré’s point of view was enthusiastically adopted and developed by G. D. Birkhoff. He found many different types of long-term limiting behavior. His work resulted in the book [4] from which the term ‘dynamical systems’ came. Other people, such as Lyapunov, Pontryagin, Andronov, Morser, Smale, Peixoto, Kolmogorov, Arnol’d, Sinai, Lorenz, May, Yorke, Feigenbaum, Ruelle, and Takens, all made important contributions to the theory of dynamical systems. The field of nonlinear dynamical systems and especially the study of chaotic systems has been hailed as one of the important breakthroughs in science in the 20th century. Today, nonlinear dynamical systems are used to describe a vast variety of scientific and engineering phenomena and have been applied to a broad spectrum of problems in physics, chemistry, mathematics, biology, medicine, economics, and various engineering disciplines.

This chapter is a brief review of some concepts and tools related to the subject of the monograph – chaos suppression, chaos synchronization, and chaotification. The goal of the chapter is to provide readers with some necessary background on nonlinear dynamical systems and chaos so as to ease the difficulty when they read subsequent chapters of this book. Readers interested in the complete theory of dynamical systems are recommended to refer to [1, 7, 11, 13, 17].

2.2 Background

Two types of models are extensively studied in the field of dynamical systems: the continuous-time model and the discrete-time model. Most continuous-time nonlinear dynamical systems are described by a differential equation of the form

$$\dot{x} = f(x,t;\mu), \tag{2.1}$$

with $x \in U \subset \mathbb{R}^n$, $t \in \mathbb{R}$, and $\mu \in V \subset \mathbb{R}^p$, where U and V are open sets in $\mathbb{R}^n$ and $\mathbb{R}^p$. Meanwhile, most discrete-time nonlinear dynamical systems are described by an equation of the form

$$x(k+1) = f(x(k),k;\mu), \quad k = 0,1,2,\ldots. \tag{2.2}$$

We refer to (2.1) as a *vector field* or *ordinary differential equation* and to (2.2) as a *map* or *difference equation*. By a solution of (2.1) we mean a map, x, from some interval $I \subset \mathbb{R}$ into $\mathbb{R}^n$, which is denoted by

$$\begin{aligned} x\colon I &\to \mathbb{R}^n, \\ t &\mapsto x(t), \end{aligned}$$

such that $x(t)$ satisfies (2.1), i.e.,

$$\dot{x}(t) = f(x(t),t;\mu).$$

The map x has the geometrical interpretation of a curve in $\mathbb{R}^n$, and (2.1) gives the tangent vector at each point of the curve, hence the reason for referring to (2.1) as a vector field. We will refer to the space of independent variables of (2.1) (i.e., $\mathbb{R}^n$) as the *phase space* or *state space*. One goal of the study of dynamical systems is to understand the geometry of solution curves in the phase space. It is useful to distinguish a solution curve which passes through a particular point in the phase space at a specific time, i.e., for a solution $x(t)$ with $x(t_0) = x_0$. We refer to this as specifying an *initial condition* or *initial value*. This is often included in the expression for a solution by $x(t,t_0,x_0)$. In some situations explicitly displaying the initial condition may be unimportant, in which case we will denote the solution merely as $x(t)$. In other situations the initial time may be always understood to be a specific value, say $t_0 = 0$; in this case we would denote the solution as $x(t,x_0)$. Similarly, it may be useful to explicitly display the parametric dependence of solutions. In this case we would write $x(t,t_0,x_0;\mu)$ or, if we were not interested in the initial condition, $x(t;\mu)$.

Ordinary differential equations that depend explicitly on time (i.e., $\dot{x} = f(x,t;\mu)$) are referred to as *nonautonomous* or *time-dependent* ordinary differential equations or vector fields, and ordinary differential equations that do not depend explicitly on time (i.e., $\dot{x} = f(x;\mu)$) are referred to as *autonomous* or *time-independent* ordinary differential equations or vector fields. The same terminology may be used in the same way for discrete-time systems. It should be noted that a nonautonomous vector field or map can always be made autonomous by redefining time as a new independent variable. This is done as follows. For a vector field $\dot{x} = f(x,t)$, by writing it as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{f(x,t)}{1} \tag{2.3}$$

and using the chain rule, we can introduce a new independent variable s so that (2.3) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} \equiv \dot{x} = f(x,t), \\ \dfrac{\mathrm{d}t}{\mathrm{d}s} \equiv \dot{t} = 1. \end{cases} \tag{2.4}$$

If we define $y = (x,t)^{\mathrm{T}}$ and $\tilde{f}(y) = (f(x,t),1)^{\mathrm{T}}$, we see that (2.4) becomes

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \tilde{f}(y), \;\; y \in \mathbb{R}^{n+1}.$$

For the map $x(k+1) = f(x(k),k)$, if we define $y(k) = (x(k),k)^{\mathrm{T}}$ and $\tilde{f}(y) = (f(x(k),k),k+1)^{\mathrm{T}}$, we get the autonomous system under the new phase space

$$y(k+1) = \tilde{f}(y(k)), \;\; y \in \mathbb{R}^{n+1}.$$

So, it is generally sufficient to consider autonomous systems

$$\dot{x} = f(x), \;\; x \in \mathbb{R}^n, \tag{2.5}$$

and

$$x(k+1) = g(x(k)), \;\; x \in \mathbb{R}^n. \tag{2.6}$$

2.3 Existence, Uniqueness, Flow, and Dynamical Systems

2.3.1 Existence and Uniqueness

Consider the autonomous vector field (2.5). Geometrically, $x(t)$ is a curve in $\mathbb{R}^n$ whose tangent vector $\dot{x}(t)$ exists for all t in its dominant field J and equals $f(x(t))$. For simplicity, we usually take initial time $t_0 = 0$. The main problem in differential equations is to find the solution for any *initial value problem*; that is, to determine the solution of the system that satisfies the initial condition $x(t_0) = x_0$ for each $x_0 \in \mathbb{R}^n$.

Unfortunately, nonlinear differential equations may have no solutions satisfying certain initial conditions.

Example 2.1 ([9]). Consider the following simple first-order differential equation:

$$\dot{x} = \begin{cases} 1, & \text{if } x < 0, \\ -1, & \text{if } x \geq 0. \end{cases}$$

This vector field on $\mathbb{R}$ points to the left when $x \geq 0$ and to the right if $x < 0$. Consequently, there is no solution that satisfies the initial condition $x(0) = 0$. Indeed, such a solution must initially decrease since $\dot{x}(0) = -1$, but, for all negative values of x, solutions must increase. This cannot happen. Note further that solutions are never defined for all time. For example, if $x_0 > 0$, then the solution through x_0 is given by $x(t) = x_0 - t$, but this solution is only valid for $-\infty < t < x_0$ for the same reason as above.

The problem in this example is that the vector field is not continuous at 0; whenever a vector field is discontinuous we face the possibility that nearby vectors may point in 'opposing' directions, thereby causing solutions to halt at these points. □

Example 2.2. Consider the following differential equation:

$$\dot{x} = 3x^{2/3}.$$

The identically zero function $u\colon \mathbb{R} \to \mathbb{R}$ given by $u(t) \equiv 0$ is clearly a solution with initial condition $u(0) = 0$. But $u_0(t) = t^3$ is also a solution satisfying this initial condition. Moreover, for any $\tau > 0$, the function given by

$$u_\tau(t) = \begin{cases} 0, & \text{if } t \le \tau, \\ (t-\tau)^3, & \text{if } t > \tau \end{cases}$$

is also a solution satisfying the initial condition $u_\tau(0) = 0$. While the differential equation in this example is continuous at $x_0 = 0$, the problems arise because $3x^{2/3}$ is not differentiable at this point. □

From these two examples it is clear that, to ensure the existence and uniqueness of solutions, certain conditions must be imposed on the function f. In the first example, f is not continuous at the point 0, while, in the second example, f fails to be differentiable at 0. It turns out that the assumption that f is continuously differentiable is sufficient to guarantee both existence and uniqueness of the solution. In fact, we can furthermore guarantee the existence and uniqueness under a weaker condition, called the Lipschitz condition, on f. We now state several qualitative theorems about the solutions of system (2.5) [7].

Theorem 2.1 (Local Existence and Uniqueness). Let $U \subset \mathbb{R}^n$ be an open subset of real Euclidean space (or of a differentiable manifold M[1]), let $x_0 \in U$, and let $f\colon U \to \mathbb{R}^n$ be a (locally) Lipschitzian map, i.e.,

$$\|f(y) - f(x)\| \le K\|x - y\|$$

for some $K < \infty$. Then, there are some constant $c > 0$ and a unique solution $x(\cdot, x_0)\colon (-c, c) \to U$ satisfying the differential equation described by (2.5) with initial condition $x(0) = x_0$. □

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[1] Roughly speaking, a manifold is a set which locally has the structure of Euclidean space. In applications, manifolds are most often met as m-dimensional surfaces embedded in $\mathbb{R}^n$. If the surface has no singular points, i.e., the derivative of the function representing the surface has maximal rank, then by the implicit function theorem it can locally be represented as a graph. The surface is a C^r manifold if the (local) coordinate charts representing it are C^r.

The local existence theorem becomes global in all cases when we work on *compact* manifolds[2] M instead of open spaces like $\mathbb{R}^n$.

Theorem 2.2 (Global Existence). The differential equation $\dot{x} = f(x)$, $x \in M$, with M compact, and $f \in C^1$, has solution curves defined for all $t \in \mathbb{R}$. □

The local theorem can be extended to show that solutions depend in a 'nice' manner on initial conditions.

Theorem 2.3 (Dependence on Initial Value). Let $U \in \mathbb{R}^n$ be open and suppose that $f : U \to \mathbb{R}^n$ has a Lipschitz constant K. Let $y(t), z(t)$ be solutions of $\dot{x} = f(x)$ on the closed interval $[t_0, t_1]$. Then, for all $t \in [t_0, t_1]$,

$$\|y(t) - z(t)\| \le \|y(t_0) - z(t_0)\| \mathrm{e}^{K(t-t_0)}. \qquad \square$$

2.3.2 Flow and Dynamical Systems

If $x(t)$ is a solution of (2.5), then $x(t+\tau)$ is also a solution for any $\tau \in \mathbb{R}$. So, it suffices to choose a fixed initial time, say, $t_0 = 0$, which is understood and therefore often omitted from the notation. If we denote by $\phi_t(x) = \phi(t,x)$ the state in $\mathbb{R}^m$ reached by the system at time t starting from x, then the totality of solutions of (2.5) can be represented by a one-parameter family of maps $\phi^t : U \to \mathbb{R}^m$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}[\phi(t,x)]\bigg|_{t=\tau} = f[\phi(\tau,x)]$$

for all $x \in U$ and for all $\tau \in I$ for which the solution is defined. The family of maps $\phi_t(x) = \phi(t,x)$ is called the *flow* (or the flow map) generated by the vector field f. The set of points $\{\phi(t,x_0) : t \in I\}$ defines an *orbit* of (2.5), starting from a given point x_0. It is a solution curve in the state space, parameterized by time. The set $\{[t, \phi(t,x_0)] : t \in I\}$ is a *trajectory* of (2.5) and it evolves in the space of motions. However, in applications, the terms 'orbit' and 'trajectory' are often used as synonyms. A simple example of a trajectory in the space of motions $\mathbb{R} \times \mathbb{R}^2$ and the corresponding orbit in the state space $\mathbb{R}^2$ are given in Fig. 2.1. Clearly, the orbit is obtained by projecting the trajectory on to the state space. The flows generated by vector fields form a very important subset of a more general class of maps, characterized by the following definition.

Definition 2.1. A *flow* is a map $\phi : I \subset \mathbb{R} \times X \to X$ where X is a metric space, that is, a space endowed with a distance function, and ϕ has the following properties:

[2] A compact manifold is a manifold that is compact as a topological space, such as the circle (the only one-dimensional compact manifold) and the n-dimensional sphere and torus. For many problems in topology and geometry, it is convenient to study compact manifolds because of their 'nice' behavior. Among the properties making compact manifolds 'nice' are the facts that they can be covered finitely by many coordinate charts and that any continuous real-valued function is bounded on a compact manifold.

(i) $\phi(0,x) = x$ for every $x \in X$ (identity axiom);
(ii) $\phi(t+s,x) = \phi(s,\phi(t,x)) = \phi(t,\phi(s,x)) = \phi(s+t,x)$, that is, time-translated solutions remain solutions;
(iii) for fixed t, $\phi^t := \phi(t,\cdot)$ is a homeomorphism[3] of the phase space on X. □

Remark 2.1. A *distance* on a space X (or, a *metric* on X) is a function $d(\cdot,\cdot)\colon X \times X \to \mathbb{R}^+$ satisfying the following properties for all $x,y \in X$:

(i) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ (symmetry);
(iii) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality).

Notice that there also exist notions of distance which are perfectly meaningful but do not satisfy the definition above and therefore do not define a metric, for example the distance between a point and a set A

$$d(x,A) = \inf_{y \in A} d(x,y)$$

and the distance between two sets A and B

$$d(A,B) = \inf_{x \in A} \inf_{y \in B} d(x,y).$$ □

In the following, we give a formal definition of a 'dynamical system.'

Definition 2.2. A *dynamical system* is a triplet $\{T, X, \phi^t\}$ where T is a time set, X is a state space, and $\phi^t\colon X \to X$ is a flow parameterized by $t \in T$. □

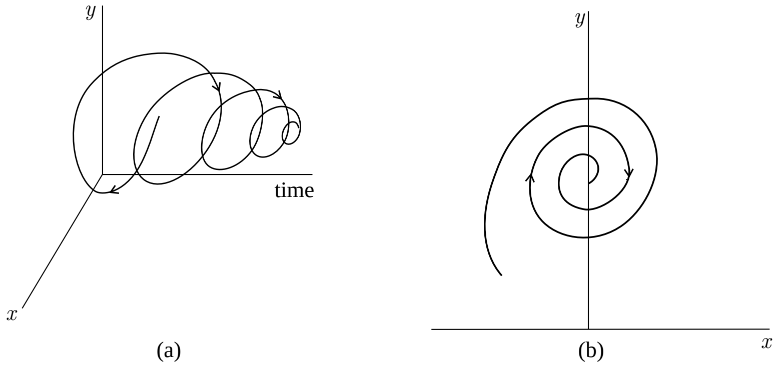

Fig. 2.1 A damped oscillator in $\mathbb{R}^2$: (a) space of motions; (b) state space

[3] $h\colon U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ is said to be a C^r diffeomorphism if both h and h^{-1} are C^r. h is called a homeomorphism if $r = 0$.

2.4 Equilibrium, Periodic Orbit, Quasiperiodic Orbit, and Poincaré Map

2.4.1 Equilibrium of Continuous-Time Systems

Definition 2.3 (Equilibrium of Autonomous Systems). An *equilibrium solution* of (2.5) is a point $\bar{x} \in \mathbb{R}^n$ such that

$$f(\bar{x}) = 0,$$

i.e., a solution which does not change in time. Other terms often substituted for the term 'equilibrium solution' are 'fixed point,' 'stationary point,' 'rest point,' 'singularity,' 'critical point,' or 'steady state.' □

Remark 2.2. What about the notion of equilibria of nonautonomous vector fields? We should note that ideas developed for autonomous systems can lead to incorrect results for nonautonomous systems. For example, consider the following one-dimensional nonautonomous field:

$$\dot{x} = -x + t. \tag{2.7}$$

The solution through the point x_0 at $t = 0$ is given by

$$x(t) = t - 1 + \mathrm{e}^{-t}(x_0 + 1),$$

from which it is clear that all solutions asymptotically approach the solution $t - 1$ as $t \to \infty$. The frozen time or 'instantaneous' fixed points for (2.7) are given by

$$x = t.$$

At a fixed t, this is the unique point where the vector field is zero. However, $x = t$ is not a solution of (2.7). This is different from the case of an autonomous vector field where a fixed point is a solution of the vector field. □

Definition 2.4. Consider the following nonautonomous system:

$$\dot{x} = f(t, x), \tag{2.8}$$

where $f\colon [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point of (2.8) at $t = 0$ if

$$f(t, 0) = 0, \quad \forall t \geq 0.$$ □

For the discrete-time system (2.6), an equilibrium solution is the point $\bar{x} \in \mathbb{R}^n$ such that

$$\bar{x} = g(\bar{x}).$$

An equilibrium solution of a discrete-time system is usually called a fixed point.

2.4.2 Periodic and Quasiperiodic Orbits

Consider again the basic system of differential equation

$$\dot{x} = f(x) \tag{2.9}$$

and the derived flow ϕ. A solution $\phi(t,x^*)$ of system (2.9) through a point x^* is said to be *periodic* with period $T > 0$ if $\phi(T,x^*) = x^*$. The set $L_0 = \{\phi(t,x^*) \colon t \in [0,T)\}$ is a closed curve in the state space and is called a *periodic orbit* or *cycle*. T is called the period of the cycle and measures its time length. It should be emphasized that isolated periodic solutions are possible only for nonlinear differential equations. Moreover, a *limit cycle*[4] can be *structurally stable* in the sense that, if it exists for a given system of differential equations, it will persist under a slight perturbation of the system in the parameter space. On the contrary, linear systems of differential equations in $\mathbb{R}^m$ $(m \geq 2)$ may have a continuum of periodic solutions characterized by a pair of purely imaginary eigenvalues (the case of a *center*, which will be introduced later). But, these periodic solutions can be destroyed by arbitrarily small perturbations of the coefficients. In other words, these periodic solutions are not structurally stable.

For the discrete-time system of $x_{k+1} = G(x_k)$, an n-periodic orbit is defined as the set of points $L_0 = \{x_0, x_1, \ldots, x_{n-1}\}$ with $x_i \neq x_j$ $(i \neq j)$ such that

$$x_1 = G(x_0),\ x_2 = G(x_1), \ldots, x_{n-1} = G(x_{n-2}),\ x_0 = G(x_{n-1}).$$

It should be noted that each point in an n-periodic orbit is an *n-periodic point* since, for $k = 0, \ldots, n-1$,

$$x_k = G^n(x_k) \ \text{ and } \ G^j(x_k) \neq x_k \ \text{ for } \ 0 < j < n.$$

Periodic orbits of continuous-time systems and discrete-time systems are illustrated in Fig. 2.2.

To illustrate what quasiperiodic orbits are we will consider two examples, one for discrete time and one for continuous time.

Example 2.3. Consider the following unit-circle map:

$$M_c \colon S^1 \to S^1,\ \ z_{n+1} = M_c(z_n) = cz_n, \tag{2.10}$$

where $z_n = \mathrm{e}^{i2\pi\theta_n}$, $\theta_n \in \mathbb{R}$, α is a positive constant, and $c = \mathrm{e}^{i2\pi\alpha}$. The map (2.10) describes an anticlockwise jump of a particle on the unit circle S^1. The length of the circular arc between two adjacent jump points is α. If α is rational, that is, $\alpha = p/q$ with p and q integers, then any (initial) point on S^1 is a q-periodic point of the map M_c. If α is irrational, at each iteration a new point is added on the unit

[4] A limit cycle is an isolated periodic solution of an autonomous system. The points on the limit cycle constitute the *limit set*, which is the set of points in the state space that a trajectory repeatedly visits. A limit set is only defined for discrete-time or continuous-time autonomous systems.

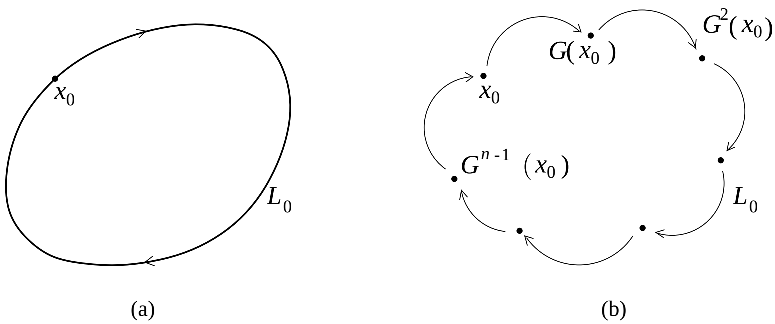

Fig. 2.2 Periodic orbits in: (a) a continuous-time system; (b) a discrete-time system

circle. For any jump point there exist other jump points arbitrarily close to it, but no point is revisited in finite time. This means that the map is topologically transitive (see Definition 2.12). However, points that are close to each other will remain close under the iteration. □

Example 2.4. The simplest example of quasiperiodic motion in continuous time is a system defined by a pair of oscillators of the form

$$\ddot{x}+\omega_1^2 x=0,\ \ \ddot{y}+\omega_2^2 y=0,$$

where $x,y\in\mathbb{R}$ and ω_1 and ω_2 are real constants. The above system can be rewritten in the form of first-order linear differential equations in $\mathbb{R}^4$:

$$\begin{cases}\dot{x}_1=-\omega_1 x_2,\\ \dot{x}_2=\omega_1 x_1,\\ \dot{y}_1=-\omega_2 y_2,\\ \dot{y}_2=\omega_2 y_1,\end{cases} \tag{2.11}$$

where $x_2=x$ and $y_2=y$. Transforming the variables x_1,x_2 and y_1,y_2 into polar coordinates, system (2.11) can be written as

$$\begin{cases}\dot{\theta}_1=\omega_1,\\ \dot{r}_1=0,\\ \dot{\theta}_2=\omega_2,\\ \dot{r}_2=0,\end{cases} \tag{2.12}$$

where θ_i and r_i $(i=1,2)$ denote the angle and the radius, respectively. We can see that the above equations describe a particle rotating on a two-dimensional torus for a given pair (r_1,r_2), $r_i>0$ $(i=1,2)$ (see Fig. 2.3). There are two basic possibilities for the motion:

(i) ω_1/ω_2 is a rational number, in which case there exists a continuum of periodic orbits of period q;

(ii) ω_1/ω_2 is an irrational number, in which case the orbit starting from any initial point on the torus wanders on it, getting arbitrarily near any other point, without ever returning to that exact initial point. The flow generated by (2.12) is topologically transitive on the torus (see Fig. 2.4).

In both cases, points that are close to each other remain close under the action of the flow. □

A general definition of quasiperiodicity of an orbit as a function of time can be given as follows:

Definition 2.5 ([11], p. 128). A function $h\colon \mathbb{R} \to \mathbb{R}^m$ is called quasiperiodic if it can be written in the form of $h(t) = H(\omega_1 t, \omega_2 t, \ldots, \omega_m t)$, where H is periodic of period 2π in each of its arguments, and two or more of the m (positive) frequencies are incommensurable. □

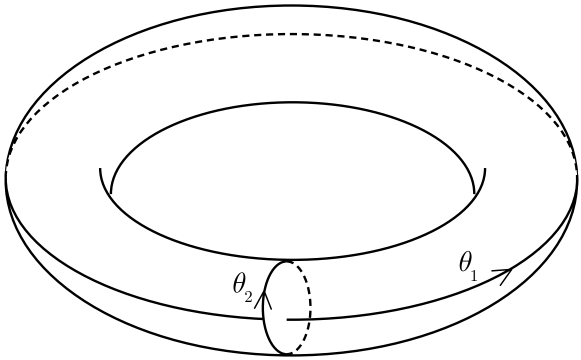

Fig. 2.3 The motion on two-dimensional torus

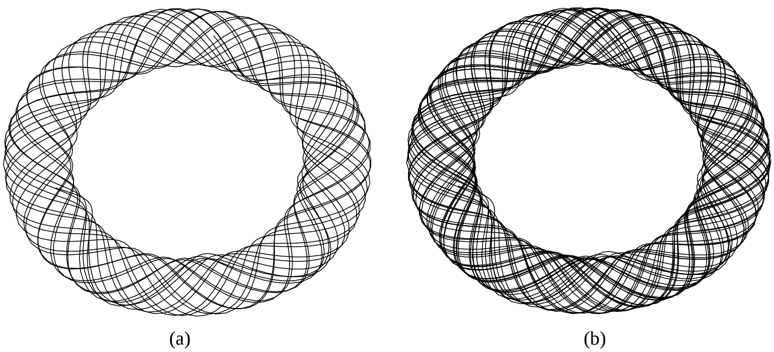

Fig. 2.4 The quasiperiodic motions with different evolution times

2.4.3 Poincaré Map

A *Poincaré map* is a classical technique for analyzing dynamical systems. It replaces the flow of an nth-order continuous-time system with an $(n-1)$st-order discrete-time system. The definition of the Poincaré map ensures that the limit sets of the discrete-time system correspond to the limit sets of the underlying flow. The Poincaré map's usefulness lies in the reduction of order and the fact that it bridges the gap between continuous-time and discrete-time systems.

The definitions of a Poincaré map are different for autonomous systems and nonautonomous systems. We present the two cases separately.

Case 1: In this case we consider an nth-order time-periodic nonautonomous system $\dot{x} = f(t,x)$, with period T. We can convert it into an $(n+1)$st-order autonomous system by appending an extra state $\theta := 2\pi t/T$. Then, the autonomous system is given by

$$\begin{cases} \dot{x} = f(x, \theta T/2\pi), & x(t_0) = x_0, \\ \dot{\theta} = 2\pi/T, & \theta(t_0) = 2\pi t_0/T. \end{cases} \tag{2.13}$$

Since f is time periodic with period T, system (2.13) is periodic in θ with period 2π. Hence, the planes $\theta = 0$ and $\theta = 2\pi$ may be identified and the state space transformed from the Euclidean space $\mathbb{R}^{n+1}$ to the cylindrical space $\mathbb{R}^n \times S^1$, where S^1 is the unit circle. The solution of (2.13) in the cylindrical state space is

$$\begin{pmatrix} x(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \phi_t(x_0, t_0) \\ 2\pi t/T \bmod 2\pi \end{pmatrix}, \tag{2.14}$$

where the modulo function restricts to $0 \le \theta < 2\pi$. Consider the n-dimensional hyperplane $\Sigma \in \mathbb{R}^n \times S^1$ defined by

$$\Sigma := \{(x,\theta) \in \mathbb{R}^n \times S^1 : \theta = \theta_0\}.$$

Every T seconds, the trajectory of (2.14) intersects Σ (see Fig. 2.5). The resulting map $P_\mathrm{N}\colon \Sigma \to \Sigma$ $(\mathbb{R}^n \to \mathbb{R}^n)$ is defined by

$$P_\mathrm{N}(x) := \phi_{t_0+T}(x, t_0).$$

P_N is called the Poincaré map of the nonautonomous system.

Case 2: Consider an nth-order autonomous system with a limit cycle Γ shown in Fig. 2.6. Let x_0 be a point on the limit cycle and let Σ be an $(n-1)$-dimensional hyperplane transversal to Γ at x_0. The trajectory emanating from x_0 will hit Σ at x_0 in T seconds, where T is the minimum period of the limit cycle. Due to the continuity of ϕ_t with respect to the initial condition, the trajectory starting on Σ in a sufficiently small neighborhood of x_0 will, in approximately T seconds, intersect Σ in the vicinity of x_0. Hence, ϕ_t and Σ define a mapping P_A of some neighborhood $U \subset \Sigma$ of x_0 on to another neighborhood $V \subset \Sigma$ of x_0. P_A is a Poincaré map of the autonomous system.

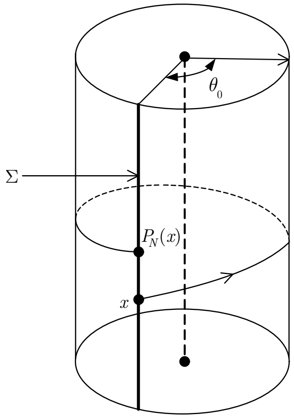

Fig. 2.5 The Poincaré map of a first-order nonautonomous system

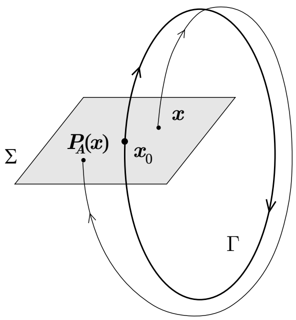

Fig. 2.6 The Poincaré map of a third-order autonomous system

Remark 2.3.

(i) P_A is defined locally, i.e., in a neighborhood of x_0. Unlike the nonautonomous case, it is not guaranteed that the trajectory emanating from any point on Σ will intersect Σ.

(ii) For a Euclidean state space, the point $P_A(x)$ is not the first point where $\phi_t(x)$ intersects Σ; $\phi_t(x)$ must pass through Σ at least once before returning to V. This is in contrast with the cylindrical state space in Fig. 2.5.

(iii) P_A is a diffeomorphism and is, therefore, invertible and differentiable [12]. □

2.5 Invariant and Attracting Sets

Definition 2.6 (Invariant Set). Let $S \subset \mathbb{R}^n$ be a set. Then,

(i) (Continuous time) S is said to be *invariant* under the vector field $\dot{x} = f(x)$ if for any $x_0 \in S$ we have $x(t, 0, x_0) \in S$ for all $t \in \mathbb{R}$, where $x(0, 0, x_0) = x_0$.

(ii) (Discrete time) S is said to be *invariant* under the map $x_{k+1} = g(x_k)$ if for any $x_0 \in S$ we have $g^n(x_0) \in S$ for all $n \in \mathbb{Z}$.

If we restrict ourselves to positive time (i.e., $t \geq 0$, and $n \geq 0$), then we refer to S as a *positively invariant set*, while, for negative time, as a *negatively invariant set*. □

The definition means that trajectories starting in the invariant set remain in the invariant set, for all of their future and all of their past.

Definition 2.7. An invariant set $S \subset \mathbb{R}^n$ is said to be a C^r $(r \geq 1)$ *invariant manifold* if S has the structure of a C^r differentiable manifold. Similarly, a positively (negatively) invariant set $S \subset \mathbb{R}^n$ is said to be a $C^r(r \geq 1)$ *positively* (*negatively*) *invariant manifold* if S has the structure of a C^r differentiable manifold. □

Definition 2.8. Let $\phi(t,x)$ be a flow on a metric space M. Then, a point $y \in M$ is called an *ω-limit point* of $x \in M$ for $\phi(t,x)$ if there exists an infinitely increasing sequence $\{t_i\}$ such that

$$\lim_{i \to \infty} d(\phi(t_i,x),y) = 0.$$

The set of all ω-limit points of x for $\phi(t,x)$ is called the *ω-limit set* and is denoted by $\omega(x)$. □

The definitions of *α-limit point* and *α-limit set* of a point $x \in M$ are obtained just by taking sequences t_i decreasing in i to $-\infty$. The α-limit set of x is denoted as $\alpha(x)$.

Definition 2.9. A point x_0 is called *nonwandering* if the following condition holds. *Flows:* for any neighborhood U of x_0 and $T > 0$, there exists some $|t| > T$ such that

$$\phi(t,U) \cap U \neq \emptyset;$$

Maps: for any neighborhood U of x_0, there exists some $n \neq 0$ such that

$$g^n(U) \cap U \neq \emptyset.$$

The set of all nonwandering points of a flow or map is called the *nonwandering set* of that particular flow or map. □

Definition 2.10. A closed invariant set $A \subset \mathbb{R}^n$ is called an *attracting set* if there is some neighborhood U of A such that
Flows: for any $t \geq 0$, $\phi(t,U) \subset U$ and $\bigcap_{t>0} \phi(t,U) = A$;
Maps: for any $n \geq 0$, $g^n(U) \subset U$ and $\bigcap_{n>0} g^n(U) = A$. □

Definition 2.11. The *basin of attraction* of an attracting set A is given by
Flows: $\bigcup_{t\leq 0} \phi(t,U)$;
Maps: $\bigcup_{n\leq 0} g^n(U)$;
where U is any open set satisfying Definition 2.10. □

Definition 2.12. A closed invariant set A is said to be *topologically transitive* if, for any two open sets $U,V \subset A$,
Flows: there exists a $t \in \mathbb{R}$ such that $\phi(t,U) \cap V \neq \emptyset$;
Maps: there exists an $n \in \mathbb{Z}$ such that $g^n(U) \cap V \neq \emptyset$. □

Definition 2.13. An *attractor* is a topologically transitive attracting set. □

2.6 Continuous-Time Systems in the Plane

In this section and the next two sections we will discuss the types of equilibrium points of planar systems of continuous time and discrete time, respectively. In applications, we very often encounter linear systems described by two first-order differential equations (or a differential equation of second order), either because the underlying model is linear or because it is linearized around an equilibrium point. Systems in two-dimensional space are particularly easy to discuss in full detail and give rise to a number of interesting basic dynamic configurations. Moreover, in practice, it is very difficult or impossible to determine the exact values of the eigenvalues and eigenvectors for matrices of order greater than two. Thus, one can draw inspiration from the discussion about planar systems when studying high-dimensional systems.

The general form of a continuous-time planar system can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.15}$$

where $x, y \in \mathbb{R}$ and a_{ij} are real constants. If $\det(A) \neq 0$, the unique equilibrium, for which $\dot{x} = \dot{y} = 0$, is $x = y = 0$. The characteristic equation is

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0,$$

and the eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}(A) \pm \sqrt{\Delta}\right),$$

where $\Delta \equiv (\mathrm{tr}(A))^2 - 4\det(A)$ is called the *discriminant*. For system (2.15) the discriminant is

$$\Delta = (a_{11} - a_{22})^2 + 4a_{12}a_{21}.$$

The different types of dynamical behavior of (2.15) can be described in terms of the two eigenvalues of the matrix A, which in planar systems can be completely characterized by the trace and determinant of A. In the following we consider nondegenerate equilibria for which λ_1 and λ_2 are both nonzero, when there is no explicit claim. We distinguish behaviors according to the sign of the discriminant.

Case 1: $\Delta > 0$. Eigenvalues and eigenvectors are real. Solutions have the form

$$\begin{cases} x(t) = c_1 e^{\lambda_1 t} u_1^{(1)} + c_2 e^{\lambda_2 t} u_2^{(1)}, \\ y(t) = c_1 e^{\lambda_1 t} u_1^{(2)} + c_2 e^{\lambda_2 t} u_2^{(2)}, \end{cases}$$

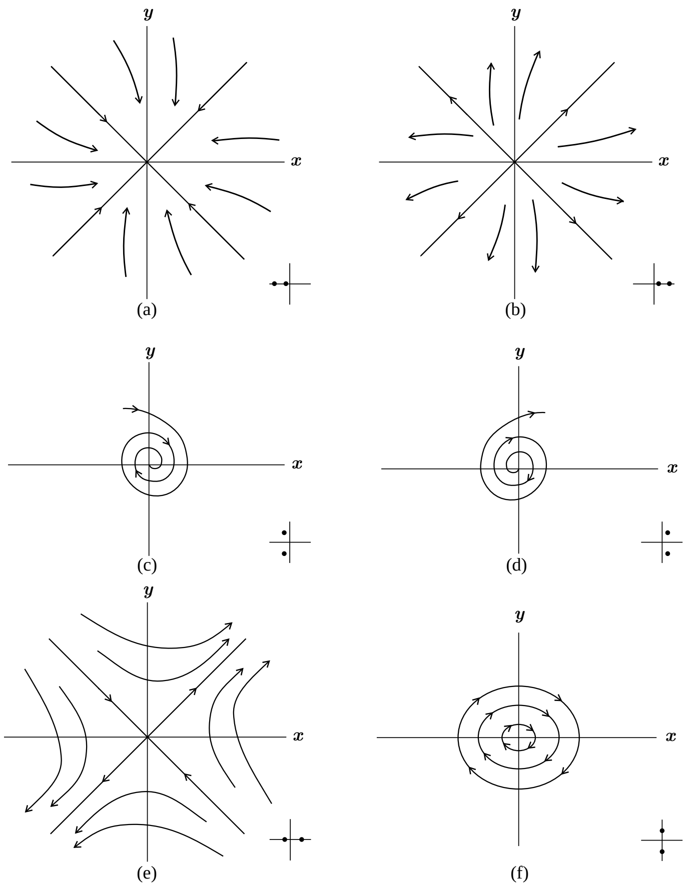

Fig. 2.7 Equilibrium types in the plane

where $u_1 = \left(u_1^{(1)}, u_1^{(2)}\right)^{\mathrm{T}}$ and $u_2 = \left(u_2^{(1)}, u_2^{(2)}\right)^{\mathrm{T}}$ are eigenvectors corresponding to the eigenvalues λ_1 and λ_2, respectively. We have three basic subcases corresponding to Fig. 2.7 (a), (b), and (e), respectively (eigenvalues are plotted in the complex plane).

(i) $\mathrm{tr}(A) < 0$, $\det(A) > 0$. In this case, eigenvalues and eigenvectors are real and both eigenvalues are negative (say, $0 > \lambda_1 > \lambda_2$). The two-dimensional

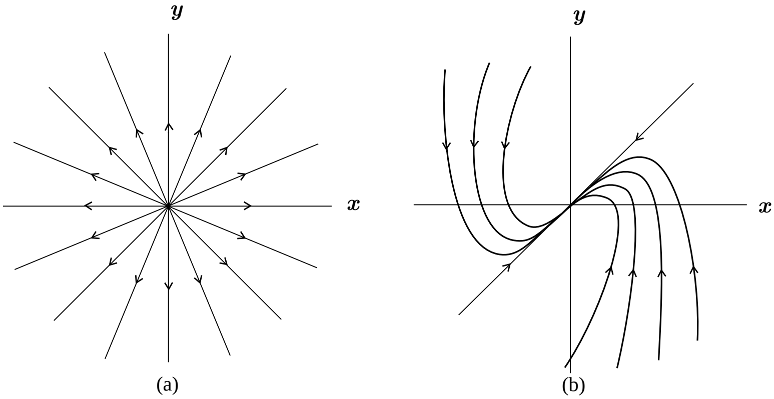

Fig. 2.8 Equilibrium types in the plane with repeated eigenvalue: (a) bicritical node; (b) Jordan node

state space coincides with the stable eigenspace.[5] The equilibrium is called a *stable node*, and the term 'node' refers to the characteristic shape of the ensemble of orbits around the equilibrium.

(ii) $\mathrm{tr}(A) > 0$, $\det(A) > 0$. In this case, eigenvalues and eigenvectors are real, both eigenvalues are positive (say, $\lambda_1 > \lambda_2 > 0$), and the state space coincides with the unstable eigenspace. The equilibrium is called an *unstable node*.

(iii) $\det(A) = 0$. In this case, $\Delta > 0$ independent of the sign of the trace of A. One eigenvalue is positive, and the other is negative (say, $\lambda_1 > 0 > \lambda_2$). There are, then, a one-dimensional stable eigenspace and a one-dimensional unstable eigenspace and the equilibrium is known as a *saddle point*.

Case 2: $\Delta < 0$. The eigenvalues and eigenvectors are complex conjugate pairs and we have

$$(\lambda_1, \lambda_2) = (\lambda, \bar{\lambda}) = \alpha \pm \mathrm{i}\beta$$

with

$$\alpha = \frac{1}{2}\mathrm{tr}(A), \quad \beta = \frac{1}{2}\sqrt{-\Delta}.$$

The solutions have the form

$$\begin{cases} x(t) = C\mathrm{e}^{\alpha t}\cos(\beta t + \phi), \\ y(t) = C\mathrm{e}^{\alpha t}\sin(\beta t + \phi), \end{cases}$$

and the motion is oscillatory. If $\alpha \neq 0$ there is no strict periodicity in the sense that there exists no τ such that $x(t) = x(t+\tau)$. However, a *conditional period* can be defined as the length of time between two successive maxima of a variable, which is equal to $2\pi/\beta$. The frequency is simply the number of oscillations per

[5] An eigenspace is spanned by eigenvectors. A stable eigenspace is spanned by the eigenvectors corresponding to negative eigenvalues, and an unstable eigenspace is spanned by the eigenvectors corresponding to positive eigenvalues.

time unit, that is, $\beta/2\pi$. The amplitude or size of the oscillations depends on the initial condition and $e^{\alpha t}$ (more on this point below). There are three subcases depending on the sign of $\mathrm{tr}(A)$ and therefore of $\mathrm{Re}(\lambda) = \alpha$; see the corresponding illustrations in Figs. 2.7 (c), (d), and (f), respectively.

(i) $\mathrm{tr}(A) < 0$, $\mathrm{Re}(\lambda) = \alpha < 0$. The oscillations are dampened and the system converges to the equilibrium. The equilibrium point is known as a *focus* or, sometimes, a *vortex*, due to the characteristic shape of the orbits around the equilibrium. In this case the focus or vortex is stable and the stable eigenspace coincides with the state space.

(ii) $\mathrm{tr}(A) > 0$, $\mathrm{Re}(\lambda) = \alpha > 0$. The amplitude of the oscillations gets larger with time and the system diverges from the equilibrium. The unstable eigenspace coincides with the state space and the equilibrium point is called an unstable focus or vortex.

(iii) $\mathrm{tr}(A) = 0$, $\mathrm{Re}(\lambda) = \alpha = 0$. In this special case we have a pair of purely imaginary eigenvalues. Orbits neither converge to, nor diverge from, the equilibrium point, but they oscillate regularly around it with a constant amplitude that depends only on initial conditions and the equilibrium point is called a *center*.

Case 3: $\Delta = 0$. The eigenvalues are real and equal to each other, $\lambda_1 = \lambda_2 = \lambda$. In this case, if $A \neq \lambda I$, only one eigenvector can be determined, say $u = \left(u^{(1)}, u^{(2)}\right)^{\mathrm{T}}$, defining a single straight line through the origin. We can write the general solution as

$$\begin{cases} x(t) = (c_1 u^{(1)} + c_2 v^{(1)})e^{\lambda t} + t c_2 u^{(1)} e^{\lambda t}, \\ y(t) = (c_1 u^{(2)} + c_2 v^{(2)})e^{\lambda t} + t c_2 u^{(2)} e^{\lambda t}, \end{cases}$$

with

$$\begin{cases} x(0) = c_1 u^{(1)} + c_2 v^{(1)}, \\ y(0) = c_1 u^{(2)} + c_2 v^{(2)}. \end{cases}$$

The equilibrium type is again a node, sometimes called a *Jordan node*. An example of this type is provided in Fig. 2.8 (b), where it is obvious that there is a single eigenvector. If $A = \lambda I$ the equilibrium is still a node, sometimes called a *bicritical node*. However, all half-lines from the origin are solutions, giving a star-shaped form (see Fig. 2.8 (a)).

Fig. 2.9 provides a very useful geometric representation in the $(\mathrm{tr}(A), \det(A))$ plane of the various cases discussed above. Quadrants III and IV of the plane correspond to saddle points, quadrant II to stable nodes and foci, and quadrant I to unstable nodes and foci. The parabola divides quadrants I and II into nodes and foci (the former below the parabola and the latter above it). The positive part of the $\det(A)$ axis corresponds to centers.

The analysis of systems with $n > 2$ variables can be developed along the same lines although geometric insight will fail when the dimension of the state space is larger than three. In order to give the reader a broad idea of common situations we

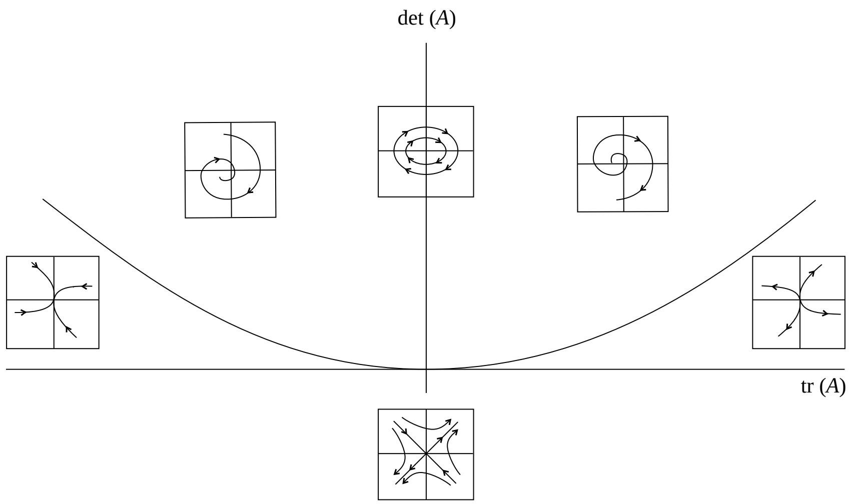

Fig. 2.9 Continuous-time dynamics in $\mathbb{R}^2$

depict sample orbits of three-dimensional systems in Fig. 2.10, which indicates the corresponding eigenvalues in the complex plane. The system in Fig. 2.10 (a) has two positive real eigenvalues and one negative real eigenvalue. The equilibrium point is an unstable saddle. In this case the plane associates with the positive real eigenvalues. All orbits eventually converge to the unstable eigenspace and are captured by the expanding dynamics. The only exceptions are those orbits initiating on the stable eigenspace (defined by the eigenvector associated with the negative eigenvalue) which converge to the plane at the equilibrium point. When A is an $m \times m$ matrix, we divide the eigenvectors (or, in the complex case, the vectors equal to the real and imaginary parts of them) into three groups, according to whether the corresponding eigenvalues have negative, positive, or zero real parts. Then, the subsets of the state space spanned (or generated) by each group of vectors are known as the stable, unstable, and center eigenspaces, respectively, and denoted by $E^{\mathrm{s}}, E^{\mathrm{u}}$, and E^{c}. Notice that the term saddle in $\mathbb{R}^m$ refers to all cases in which there exist some eigenvalues with positive and some with negative real parts. We use the term *saddle node* when eigenvalues are all real, *saddle focus* when some of the eigenvalues are complex. An example of the latter is presented in Fig. 2.10 (b). The two real vectors associated with the complex conjugate pair of eigenvalues, with negative real parts, span the stable eigenspace. Orbits approach asymptotically the unstable eigenspace defined by the eigenvector associated with the positive eigenvalue, along which the dynamics is explosive.

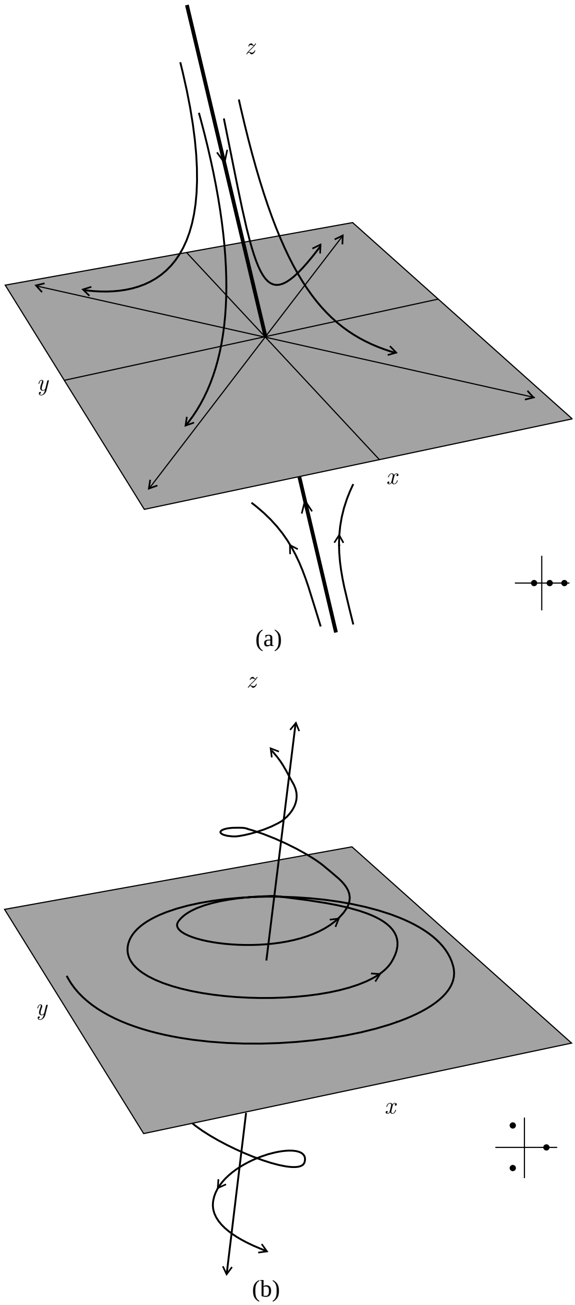

Fig. 2.10 Continuous-time dynamics in $\mathbb{R}^3$: (a) saddle node; (b) saddle focus

2.7 General Solutions of Discrete-Time Linear Systems

Now consider the following linear, discrete-time system described by

$$x(n+1) = Bx(n), \quad x \in \mathbb{R}^m. \tag{2.16}$$

It is easy to see that $x = 0$ is the only equilibrium solution. If κ_i is a real, distinct eigenvalue of the $m \times m$ matrix B and v_i is the corresponding real eigenvector so that $Bv_i = \kappa_i v_i$, it can be verified that

$$x(n) = \kappa_i^n v_i \tag{2.17}$$

is a solution of (2.16). Suppose that we have a pair of eigenvalues of B

$$(\kappa_j, \kappa_{j+1}) = (\kappa_j, \overline{\kappa}_j) = \sigma_j \pm \mathrm{i}\theta_j$$

with a corresponding pair of eigenvectors

$$(v_j, v_{j+1}) = (v_j, \overline{v}_j) = p_j \pm \mathrm{i}q_j,$$

where p_j and q_j are m-dimensional vectors. Then, the pair of functions

$$\begin{cases} x_j(n) = \dfrac{1}{2}\left(\kappa_j^n v_j + \overline{\kappa}_j^n \overline{v}_j\right) = r_j^n[p_j \cos(\omega_j n) - q_j \sin(\omega_j n)], \\ x_{j+1}(n) = -\dfrac{\mathrm{i}}{2}\left(\kappa_j^n v_j - \overline{\kappa}_j^n \overline{v}_j\right) = r_j^n[p_j \sin(\omega_j n) + q_j \cos(\omega_j n)] \end{cases} \tag{2.18}$$

are the jth and $(j+1)$st solutions of (2.16), respectively. Here, we have used the polar coordinate transformations

$$\begin{cases} \sigma_j = r_j \cos\omega_j, \\ \theta_j = r_j \sin\omega_j, \end{cases} \tag{2.19}$$

and a well-known result

$$(\cos\omega \pm \mathrm{i}\sin\omega)^n = \cos(\omega n) \pm \mathrm{i}\sin(\omega n).$$

From (2.19) we have $r_j = \sqrt{\sigma_j^2 + \theta_j^2}$. Then, r_j is simply the modulus of the complex eigenvalues. If $p_j^{(l)}$ and $q_j^{(l)}$ denote the lth elements of p_j and q_j, respectively, then in polar coordinates we have

$$\begin{cases} p_j^{(l)} = C_j^{(l)} \cos(\phi_j^{(l)}), \\ q_j^{(l)} = C_j^{(l)} \sin(\phi_j^{(l)}), \end{cases}$$

where $l = 1, 2, \ldots, m$. (2.18) can be rewritten as m-dimensional vectors whose lth elements have the form

$$\begin{cases} x_j^{(l)}(n) = C_j^{(l)} r_j^n \cos(\omega_j n + \phi_j^{(l)}), \\ x_{j+1}^{(l)}(n) = C_j^{(l)} r_j^n \sin(\omega_j n + \phi_j^{(l)}). \end{cases}$$

Assuming that we have m linearly independent solutions defined by (2.17) and (2.18), by the superposition principle the general solution of (2.16) can be written as a linear combination of the individual solutions, namely

$$x(n) = c_1 x_1(n) + c_2 x_2(n) + \cdots + c_m x_m(n),$$

where c_i are constants depending on the initial conditions.

When eigenvalues are repeated, the general solution becomes

$$x(n) = \sum_{j=1}^{h} \sum_{l=0}^{n_j-1} k_{jl} n^l \kappa_j^n,$$

where $n_j \geq 1$ is the multiplicity of the jth eigenvalue, $h \leq m$ is the number of distinct eigenvalues, and k_{jl} are independent vectors whose values depend on the m arbitrary initial conditions.

Inspection of (2.17) and (2.18) indicates that if the modulus of any of the eigenvalues is greater than 1, solutions tend to $+\infty$ or $-\infty$ as time goes to $+\infty$. On the contrary, if all eigenvalues have modulus smaller than 1, solutions converge asymptotically to the equilibrium point. Analogous to the situation of continuous time, we call the space spanned by the eigenvectors whose corresponding eigenvalues have modulus less than 1 (greater than 1) a stable eigenspace (unstable eigenspace) and call the space spanned by the eigenvectors whose corresponding eigenvalues have modulus equal to 1 a center eigenspace. We also use the same symbols, $E^{\mathrm{s}}, E^{\mathrm{u}}$, and E^{c}, to denote them.

2.8 Discrete-Time Systems in the Plane

The discrete-time autonomous system that is analogous to the continuous-time system (2.15) is described by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = B \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \quad (2.20)$$

We assume that the matrix $I - B$ is nonsingular. Thus, the origin is the unique equilibrium point of (2.20). The characteristic equation is analogous to the continuous case as well,

$$\kappa^2 - \mathrm{tr}(B)\kappa + \det(B) = 0,$$

and the eigenvalues are given by

$$\kappa_{1,2} = \frac{1}{2}\left(\operatorname{tr}(B) \pm \sqrt{\operatorname{tr}(B)^2 - 4\det(B)}\right).$$

We also assume that the equilibria of (2.20) are nondegenerate, i.e., $|\kappa_1|, |\kappa_2| \neq 1$. We will discuss the dynamics of the discrete-time system (2.20) for the following three cases.

Case 1: $\Delta > 0$. The eigenvalues are real and solutions take the form

$$\begin{cases} x(n) = c_1\kappa_1^n v_1^{(1)} + c_2\kappa_2^n v_2^{(1)}, \\ y(n) = c_1\kappa_1^n v_1^{(2)} + c_2\kappa_2^n v_2^{(2)}. \end{cases}$$

(i) If $|\kappa_1| < 1$ and $|\kappa_2| < 1$ the fixed point is a stable node. This means that solutions are sequences of points approaching the equilibrium as $n \to +\infty$. If $\kappa_1, \kappa_2 > 0$ the approach is monotonic; otherwise, there are improper os-

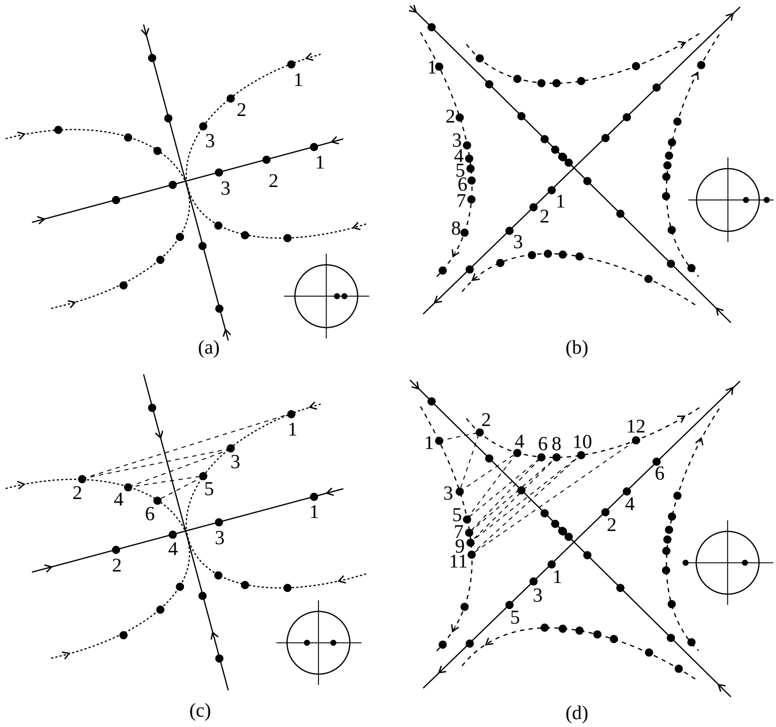

Fig. 2.11 Phase diagrams for real eigenvalues: (a) and (c) stable nodes; (b) and (d) saddle points

cillations[6] (see Figs. 2.11 (a) and (c), respectively). In this case, the stable eigenspace coincides with the state space.

(ii) If $|\kappa_1| > 1$ and $|\kappa_2| > 1$ the fixed point is an unstable node. In this case, solutions are sequences of points approaching equilibrium as $n \to -\infty$. If $\kappa_1, \kappa_2 > 0$ the approach is monotonic; otherwise, there are improper oscillations (as in Figs. 2.11 (a) and (c), respectively, but arrows point in the opposite direction and the time order of points is reversed). In this case, the unstable eigenspace coincides with the state space.

(iii) If $|\kappa_1| > 1$ and $|\kappa_2| < 1$ the fixed point is a saddle point. No sequences of points approach the equilibrium for $n \to \pm\infty$ except for those originating from points on the eigenvectors associated with κ_2. Again, if $\kappa_1,\ \kappa_2 > 0$ orbits move monotonically (see Fig. 2.11 (b)); otherwise they oscillate improperly (see Fig. 2.11 (d)). The stable and unstable eigenspaces are one dimensional.

Case 2: $\Delta < 0$. In this case, $\det(B) > 0$. Eigenvalues are a complex conjugate pair given by

$$(\kappa_1, \kappa_2) = (\kappa, \bar{\kappa}) = \sigma \pm \mathrm{i}\theta$$

and solutions are sequences of points situated on a spiral whose amplitude increases or decreases in time according to the factor r^n, where

$$r = |\sigma \pm \mathrm{i}\theta| = \sqrt{\sigma^2 + \theta^2} = \det(B)$$

is the modulus of the complex eigenvalue pair. Solutions are of the form

$$\begin{cases} x(n) = Cr^n \cos(\omega n + \phi), \\ y(n) = Cr^n \sin(\omega n + \phi). \end{cases}$$

(i) If $r < 1$ solutions converge to equilibrium and the equilibrium point is a stable focus (see Fig. 2.12 (a)).

(ii) If $r > 1$ solutions diverge and the equilibrium point is an unstable focus (as in Fig. 2.12 (a), but arrows point in the opposite direction and the time order of points is reversed).

(iii) If $r = 1$ the eigenvalues lie exactly on the unit circle, an exceptional case. There are two subcases which depend on the frequency of the oscillation $\omega/2\pi, \omega = \arccos[\mathrm{tr}(B)/2]$:

   a. $\omega/2\pi$ is rational and the orbit in the state space is a periodic sequence of points situated on a circle, the radius of which depends on initial conditions (see Fig. 2.12 (b));
   b. $\omega/2\pi$ is irrational and the sequence is nonperiodic or quasiperiodic, that is, starting from any point on the circle, orbits stay on the circle but no

[6] First-order, discrete-time equations (where the order is determined as the difference between the extreme time indices) can also have fluctuating behavior, called improper oscillations, owing to the fact that if their eigenvalue $\beta < 0$, β^n will be positive or negative according to whether n is even or odd. The term *improper* refers to the fact that in this case oscillations of variables have a 'kinky' form that does not properly describe the smoother ups and downs of real variables.

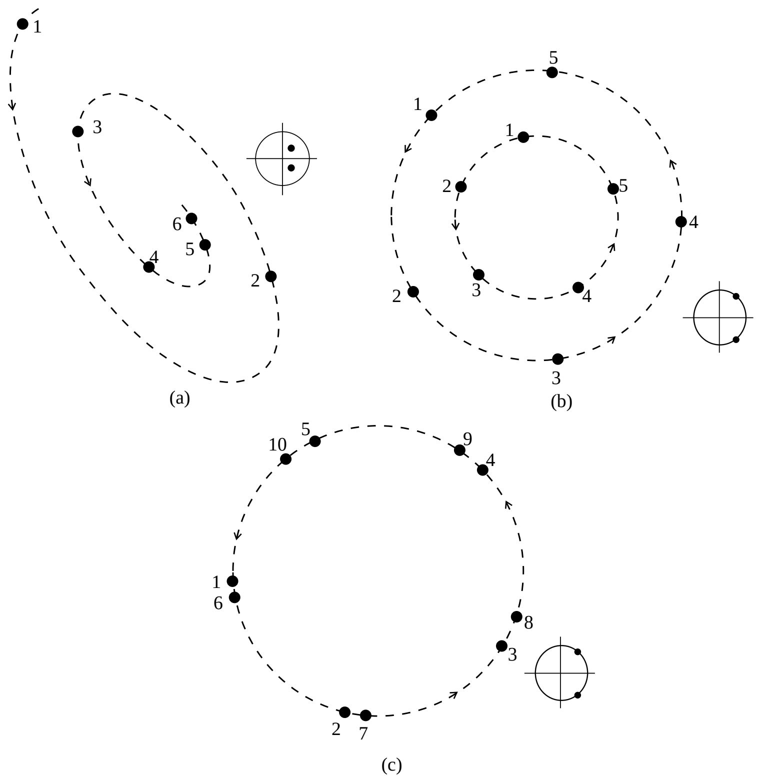

Fig. 2.12 Phase diagrams for complex eigenvalues: (a) a stable focus; (b) periodic cycles; (c) a quasiperiodic solution

sequence returns to the initial point in finite time. Therefore, solutions wander on the circle filling it up, without ever becoming periodic (see Fig. 2.12 (c)).

Case 3: $\Delta = 0$. There is a repeated real eigenvalue $\kappa_1 = \mathrm{tr}(B)/2$. The general form of solutions with a repeated eigenvalue κ is as follows:

$$\begin{cases} x(n) = (c_1 v^{(1)} + c_2 u^{(1)})\kappa^n + nc_2 v^{(1)}\kappa^n, \\ y(n) = (c_1 v^{(2)} + c_2 u^{(2)})\kappa^n + nc_2 v^{(2)}\kappa^n. \end{cases}$$

If $|\kappa| < 1, \lim_{n\to\infty} n\kappa^n = 0$. If the repeated eigenvalue is equal to 1 in absolute value, the equilibrium is unstable (with improper oscillations for $\kappa_1 = -1$). However, the divergence is linear not exponential.

The dynamics of the discrete case can be conveniently summarized by the diagram in Fig. 2.13. (For the sake of simplicity, we represent orbits as continuous rather than dotted curves.) If we replace the greater-than sign with the equal sign in conditions (i)–(iii) of Case 2, we obtain three lines intersecting each other in the (tr(B), det(B)) plane, defining a triangle. Points inside the triangle correspond to stable combinations of the trace and determinant of B.[7] The parabola defined by

$$\mathrm{tr}(B)^2 = 4\det(B)$$

divides the plane into two regions corresponding to real eigenvalues (below the parabola) and complex eigenvalues (above the parabola). Combinations of trace and determinant above the parabola but in the triangle lead to stable foci, combinations below the parabola but in the triangle are stable nodes. All other combinations lead to unstable equilibria.

Fig. 2.14 is an example of a system in $\mathbb{R}^3$. There are a complex conjugate pair with modulus less than one, and one dominant real eigenvalue greater than one. The equilibrium point is a *saddle focus*.

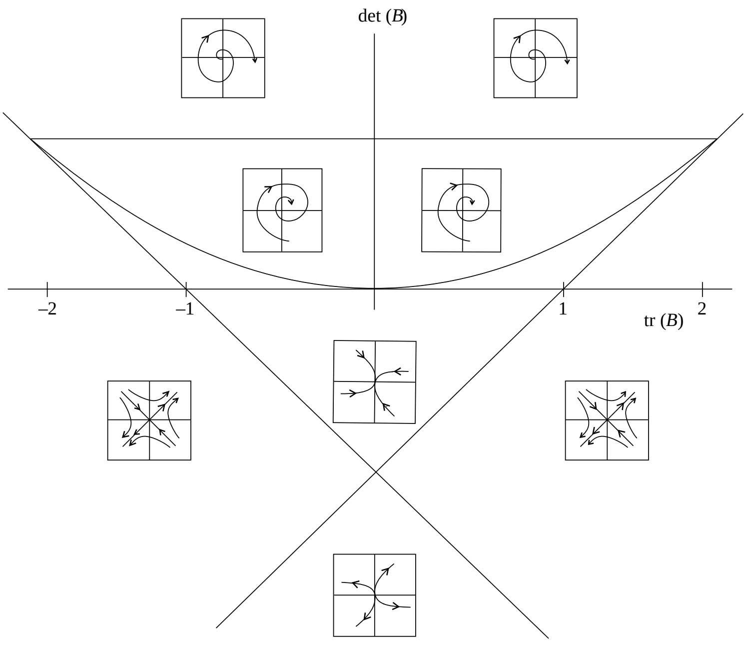

Fig. 2.13 Discrete-time dynamics in $\mathbb{R}^2$

[7] In Case 2, if $1 + \mathrm{tr}(B) + \det(B) = 0$ while (ii) and (iii) hold, one eigenvalue is equal to -1; if $1 - \mathrm{tr}(B) + \det(B) = 0$ while (i) and (iii) hold, one eigenvalue is equal to $+1$; and if $\det(B) = 1$ while (i) and (ii) hold, the two eigenvalues are a complex conjugate pair with modulus equal to 1.

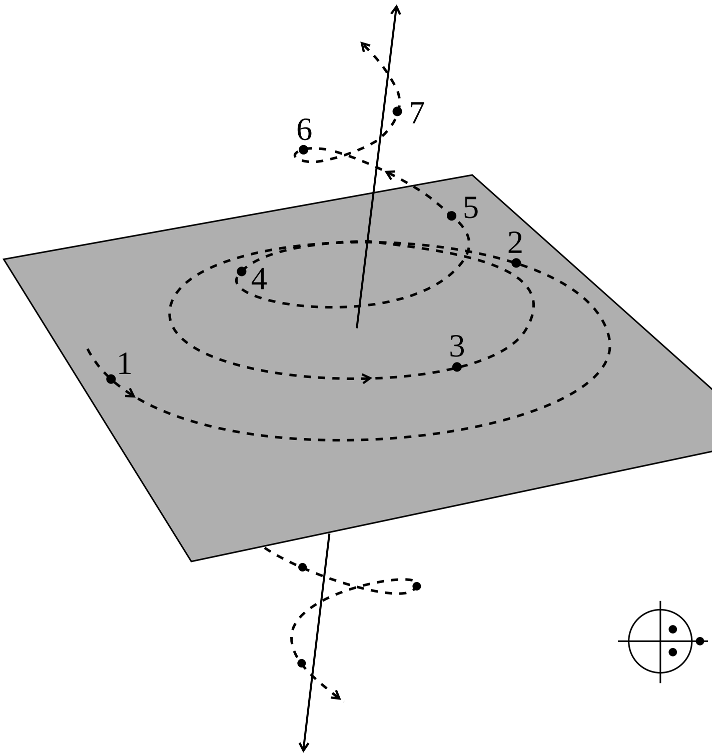

Fig. 2.14 A discrete-time dynamics in $\mathbb{R}^3$

2.9 Stabilities of Trajectories I: The Lyapunov First Method

Stability theory plays a central role in systems theory and engineering. The so-called Lyapunov first method or Lyapunov indirect method is used to study the stability of a system's trajectories through calculating the eigenvalues of a linearized system at the objective trajectories. This means that the Lyapunov first method is essentially a local method which can only be used in a neighborhood of the objective trajectory.

2.9.1 The Definition of Lyapunov Stability

Let $\overline{x}(t)$ be any solution of (2.5). Roughly speaking, $\overline{x}(t)$ is *stable* if solutions starting 'close' to $\overline{x}(t)$ at a given time remain close to $\overline{x}(t)$ for all later times. It is *asymptotically stable* if nearby solutions not only stay close, but also converge to $\overline{x}(t)$ as $t \longrightarrow \infty$.

Definition 2.14 (Lyapunov Stability). $\overline{x}(t)$ is said to be stable (or Lyapunov stable) if, for any given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for any other solution, $y(t)$, of (2.5) satisfying $\|\overline{x}(t_0) - y(t_0)\| < \delta$, we have $\|\overline{x}(t) - y(t)\| < \varepsilon$ for $t > t_0$, $t_0 \in \mathbb{R}$. □

A solution which is not stable is said to be unstable.

Definition 2.15 (Asymptotic Stability). $\overline{x}(t)$ is said to be asymptotically stable if it is Lyapunov stable and, for any other solution, $y(t)$, of (2.5), there exists a constant $b > 0$ such that, if $\|\overline{x}(t_0) - y(t_0)\| < b$, then $\lim_{t\to\infty} \|\overline{x}(t) - y(t)\| = 0$. □

A new stability definition which is different from Lyapunov's definitions is given as follows.

Definition 2.16. An orbit generated by system $\dot{x} = f(x)$ $(x \in \mathbb{R}^n)$, with initial condition x_0 on a compact, ϕ-invariant subset A of the state space (i.e., $\phi(A) \subset A$), is said to be orbitally stable (asymptotically orbitally stable) if the invariant set

$$\Gamma = \{\phi(t, x_0) \colon x_0 \in A,\ t \geq 0\}$$

(the forward orbit of x_0) is stable (asymptotically stable) according to Definition 2.14 (Definition 2.15). □

The analogous definitions of stability for autonomous dynamical systems in discrete time with the general form

$$x(n+1) = G(x(n)), \quad x \in \mathbb{R}^m \tag{2.21}$$

are as follows.

Definition 2.17. The equilibrium point $\overline{x}$ is *Lyapunov stable* (or, simply, stable) if, for every $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that

$$\|x_0 - \overline{x}\| < \delta(\varepsilon) \Rightarrow \|G^n(x_0) - \overline{x}\| < \varepsilon,\ \forall n > 0.$$ □

Definition 2.18. The equilibrium point $\overline{x}$ is *asymptotically stable* if

(i) it is stable and
(ii) $\exists b > 0$ such that

$$\|x_0 - \overline{x}\| < b \Rightarrow \lim_{n\to\infty} \|G^n(x_0) - \overline{x}\| = 0.$$ □

Property (ii) can be replaced by the following equivalent property:

(ii') there exists $b > 0$ and, for each $\varepsilon > 0$, there exists an integer $T = T(b, \varepsilon) > 0$ such that
$$\|x_0 - \overline{x}\| < b \Rightarrow \|G^n(x_0) - \overline{x}\| < \varepsilon,\ \forall n \geq T.$$

Figs. 2.15 and 2.16 are the visualization of Definitions 2.14 and 2.15. Notice that these two definitions imply that we have information on the infinite-time existence of solutions. This is obvious for equilibrium solutions but is not necessary for

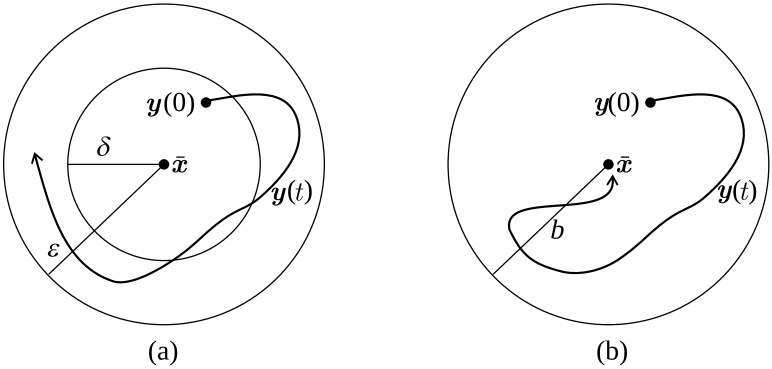

Fig. 2.15 (a) Lyapunov stability; (b) asymptotic stability

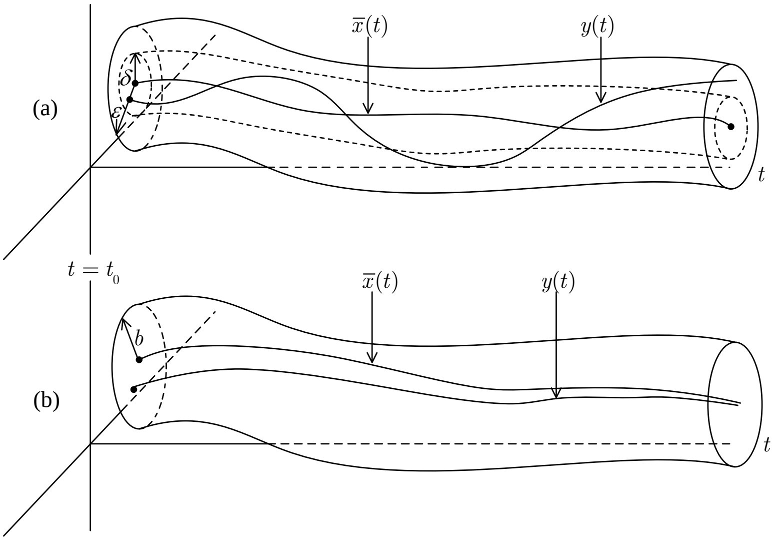

Fig. 2.16 (a) Lyapunov stability; (b) asymptotic stability

nearby solutions. Also, these definitions are for autonomous systems, since in the nonautonomous case it may be that δ and b depend explicitly on t_0.

In order to determine the stability of $\overline{x}(t)$ we must understand the nature of solutions near $\overline{x}(t)$. Let

$$x(t) = \overline{x}(t) + y. \tag{2.22}$$

Substituting (2.22) into (2.5) and performing Taylor expansion about $\overline{x}(t)$ gives

$$\dot{x} = \dot{\overline{x}}(t) + \dot{y} = f(\overline{x}(t)) + Df(\overline{x}(t))y + o(\|y\|^2), \tag{2.23}$$

where Df is the derivative of f called the Jacobian matrix of f, $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$, and $o(\|y\|^2)$ denotes the higher-order infinitesimal term of $\|y\|^2$. Using the fact that $\dot{\bar{x}}(t) = f(\bar{x}(t))$, (2.23) becomes

$$\dot{y} = Df(\bar{x}(t))y + o(\|y\|^2). \tag{2.24}$$

Equation (2.24) describes the evolution of orbits near $\bar{x}(t)$. For stability questions we are concerned with the behavior of solutions arbitrarily close to $\bar{x}(t)$, so it seems reasonable that this question could be answered by studying the associated linear system

$$\dot{y} = Df(\bar{x}(t))y. \tag{2.25}$$

Usually it is difficult to determine the stability of $x(t)$ by (2.25) since there are no general analytical methods for finding the solution of linear ordinary differential equations with time-dependent coefficients. However, if $\bar{x}$ is an equilibrium solution, i.e., $\bar{x}(t) = \bar{x}$, then $Df(\bar{x}(t)) = Df(\bar{x})$ is a matrix with constant entries, and the solution of (2.25) through the point $y_0 \in \mathbb{R}^n$ at $t = 0$ can immediately be written as

$$y(t) = \mathrm{e}^{Df(\bar{x})t} y_0.$$

Thus, $y(t)$ is *asymptotically stable* if all eigenvalues of $Df(\bar{x})$ have negative real parts.

Theorem 2.4. Suppose that all of the eigenvalues of $Df(\bar{x})$ have negative real parts. Then, the equilibrium solution $x = \bar{x}$ of the nonlinear vector field (2.5) is asymptotically stable. □

Definition 2.19. Let $x = \bar{x}$ be a fixed point of $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. Then, $\bar{x}$ is called a *hyperbolic fixed point* if none of the eigenvalues of $Df(\bar{x})$ has zero real part. □

The eigenvalues of the Jacobian matrix $Df(\bar{x})$ are also referred to as the eigenvalues of the fixed point $\bar{x}$.

Theorem 2.5 (Hartman–Grobman). If $\bar{x}$ is a hyperbolic fixed point of (2.5), then there is a homeomorphism h defined on some neighborhood N of $\bar{x}$ in $\mathbb{R}^n$, locally

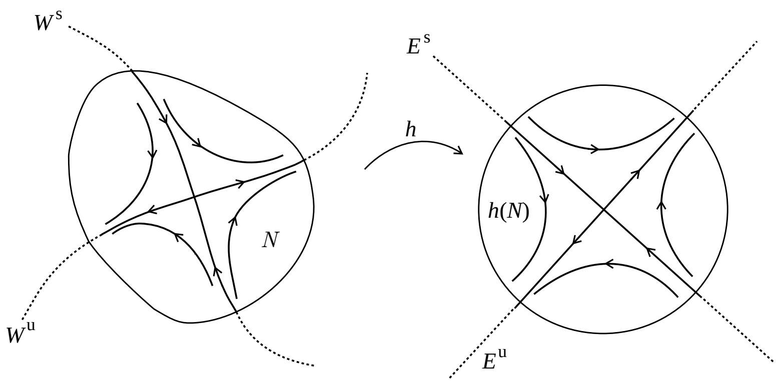

Fig. 2.17 The Hartman–Grobman theorem

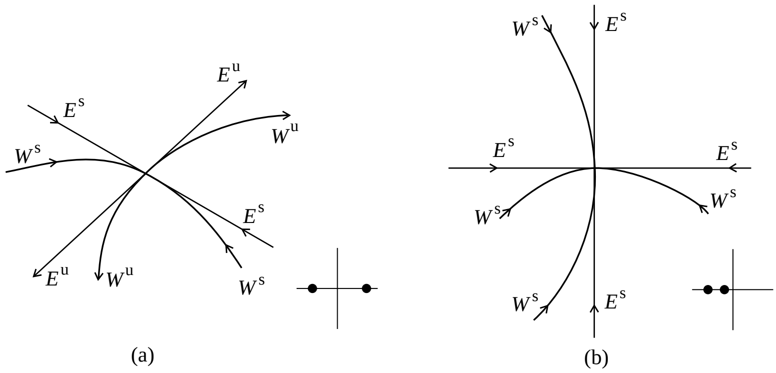

Fig. 2.18 Stable and unstable eigenspaces and manifolds in $\mathbb{R}^2$

mapping orbits of the nonlinear system (2.5) to those of the linear system (2.25). The map h preserves the sense of orbits and can also be chosen so as to preserve parameterization by time. □

If h is a homeomorphism, then from Theorem 2.5 we can deduce that asymptotic stability (or the lack of it) for the linear system (2.25) implies local asymptotic stability of the nonlinear system (2.5) (or the lack of it). However, homeomorphic equivalence does not preserve all the interesting geometric features of a dynamical system. For example, a linear system characterized by an asymptotically stable node is topologically conjugate to another linear system characterized by an asymptotically stable focus.

If the equilibrium point is not hyperbolic, that is to say, if there exists at least one eigenvalue with real part exactly equal to 0, the Hartman–Grobman theorem cannot be applied. The reason is that the linearized system is not sufficiently informative. In particular, the stability properties of the system depend on the higher-order terms of the expansion which have been ignored in the approximation (2.25).

In the above discussion of linear systems we emphasized the importance of certain invariant subspaces, i.e., the eigenspaces, defined by the eigenvectors of the Jacobian matrix. If the nonlinear system (2.5) has an isolated,[8] hyperbolic equilibrium $\bar{x}$, in the neighborhood of $\bar{x}$ there exist certain invariant surfaces, called stable and unstable manifolds, which are the nonlinear counterparts of the stable and unstable eigenspaces. Locally, these manifolds are continuous deformations, respectively, of the stable and unstable eigenspaces of the linear system (2.25) (because $\bar{x}$ is hyperbolic, there is no center eigenspace for (2.25)) and they are tangents to the eigenspaces of the linear system (2.25) at $\bar{x}$. We denote stable manifolds, unstable manifolds, and center manifolds by W^s, W^u, and W^c. Some simple examples of the phase diagrams of nonlinear systems and the corresponding linearized systems in $\mathbb{R}^2$ and $\mathbb{R}^3$ are provided in Figs. 2.17, 2.18, and 2.19.

[8] An equilibrium point is isolated if it has a surrounding neighborhood containing no other equilibrium point.

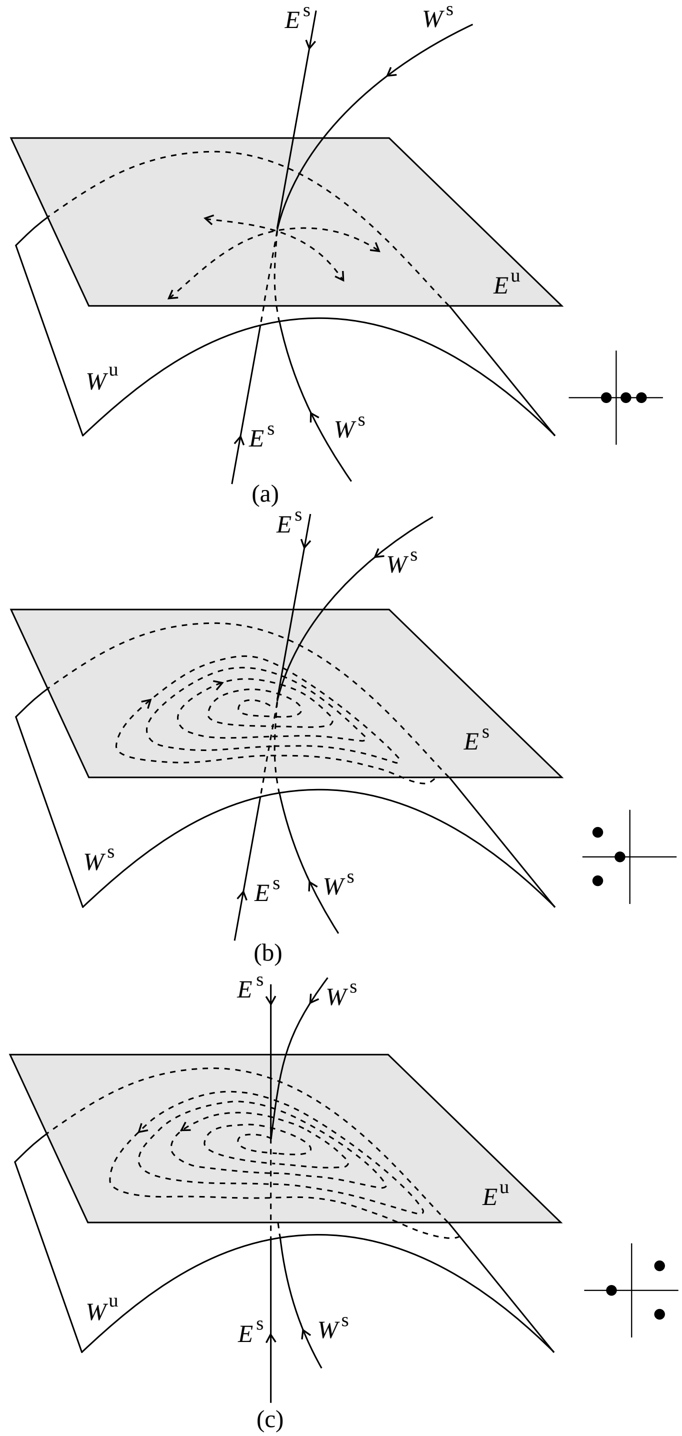

Fig. 2.19 Stable and unstable eigenspaces and manifolds in $\mathbb{R}^3$

The method of linear approximation can be applied in a perfectly analogous manner to nonlinear systems described by difference equations. Consider system (2.21), with a fixed point $\bar{x}$, and assume that G is differentiable. A local linear approximation of (2.21) near $\bar{x}$ is

$$\xi(n+1) = DG(\bar{x})\xi(n), \tag{2.26}$$

where $\xi = x - \bar{x}$ and $DG(\bar{x})$ is the Jacobian matrix of partial derivatives of G, evaluated at $\bar{x}$.

The discrete-time version of the Hartman–Grobman theorem for $x(n+1) = G(x(n))$ is perfectly analogous to that for flows except for the following important differences:

(i) For discrete-time systems, fixed points are hyperbolic if none of the eigenvalues of the Jacobian matrix, evaluated at the equilibrium, is equal to 1 in modulus.

(ii) The map h of the Hartman–Grobman theorem defining the local relationship between the nonlinear system (2.21) and the linearized system (2.26) is a diffeomorphism if the eigenvalues of $DG(\bar{x})$ satisfy a nonresonance condition. In the case of maps this condition requires that no eigenvalue κ_k of $DG(\bar{x})$ satisfies

$$\kappa_k = \prod_{i=1}^{m} \kappa_i^{c_i}$$

for any choice of $c_i \geq 0$ with $\sum_i c_i \geq 2$.

2.9.2 Floquet Theory

Local stability of a periodic solution of the system $\dot{x} = f(x)$ ($x \in \mathbb{R}^n$) can be discussed in terms of eigenvalues of certain matrices. Suppose that the system has a periodic orbit $\Gamma = \{x^*(t) \colon t \in [0,T),\ x^*(t) = x^*(t+T)\}$. Define $\xi := x(t) - x^*(t)$. Linearizing $\dot{\xi}$ about $\xi = 0$, i.e., about the periodic orbit Γ, we obtain

$$\dot{\xi} = A(t)\xi, \tag{2.27}$$

where the matrix $A(t) := Df(x^*(t))$ has periodic coefficients of period T, so that $A(t) = A(t+T)$. Solutions of (2.27) take the general form of

$$B(t)\mathrm{e}^{\lambda t},$$

where the vector $B(t)$ is periodic in time with period T, $B(t) = B(t+T)$. Denote the fundamental matrix of (2.27) as $\Phi(t)$, that is, the $m \times m$ time-varying matrix whose m columns are solutions of (2.27). Thus, $\Phi(t)$ can be written as

$$\Phi(t) = Z(t)\mathrm{e}^{tR},$$

where $Z(t)$ is an $m \times m$, T-periodic matrix and R is a constant $m \times m$ matrix. Moreover, we can always set $\Phi(0) = Z(0) = I$, from which we get

$$\Phi(T) = \mathrm{e}^{TR}.$$

Therefore, the dynamics of orbits near the cycle Γ are determined by the eigenvalues $(\lambda_1, \ldots, \lambda_m)$ of the matrix e^{TR} which are uniquely determined by (2.27).[9] The λs are called *characteristic (Floquet) multipliers* of (2.27), whereas the eigenvalues of R, $(\kappa_1, \ldots, \kappa_m)$, are called *characteristic (Floquet) exponents*.

One of the roots (multipliers), say λ_1, is always equal to 1, so that one of the characteristic exponents, say κ_1, is always equal to 0, which implies that one of the solutions of (2.27) must have the form $B(t) = B(t+T)$. This can be verified by putting $B(t) = \dot{x}^*(t)$ and differentiating it with respect to time. The presence of a characteristic multiplier equal to 1 (a characteristic exponent equal to 0) can be interpreted as that if, starting from a point on the periodic orbit Γ, the system is perturbed by a small displacement in the direction of the flow, it will remain on Γ. What happens for small, random displacements off Γ depends only on the remaining $m-1$ multipliers λ_j $(j = 2, \ldots, m)$ (or the remaining κ_j exponents, $j = 2, \ldots, m$), provided none of the other moduli is equal to 1 (provided none of them is equal to 0). In particular, we have

(i) If all the characteristic multipliers λ_j $(j = 2, \ldots, m)$ satisfy the conditions $|\lambda_j| < 1$, then the periodic orbit is asymptotically (in fact, exponentially) orbitally stable.
(ii) If for at least one of the multipliers, say λ_k, $|\lambda_k| > 1$, then the periodic orbit is unstable.

2.10 Stabilities of Trajectories II: The Lyapunov Second Method

The so-called second or direct method of Lyapunov is one of the greatest landmarks in the theory of dynamical systems and has proved to be an immensely fruitful tool for analysis. The basic idea of the method is as follows. Suppose that there is a vector field in the plane with a fixed point $\bar{x}$, and we want to determine whether it is stable or not. Roughly speaking, according to our previous definitions of stability it would be sufficient to find a neighborhood U of $\bar{x}$ for which orbits starting in U remain in U for all positive time (for the moment we do not distinguish between stability and asymptotic stability). This condition would be satisfied if we could show that the vector field is either tangent to the boundary of U or pointing inward toward $\bar{x}$ (see Fig. 2.20). This situation should remain true even as we shrink U down to $\bar{x}$. Now, Lyapunov's method gives us a way of making this precise; we will show this for vector fields in the plane and then generalize our results to $\mathbb{R}^n$.

[9] The matrix e^{TR} itself is uniquely determined but for a similarity transformation, that is, we can substitute e^{TR} with $P^{-1}\mathrm{e}^{TR}P$ where P is a nonsingular $m \times m$ matrix. This transformation leaves eigenvalues unchanged.

Suppose that we have the vector field

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y), \quad (x,y) \in \mathbb{R}^2, \end{cases} \tag{2.28}$$

which has a fixed point at $(\overline{x},\overline{y})$ (assume that it is stable). We want to show that in any neighborhood of $(\overline{x},\overline{y})$ the above situation holds. Let $V(x,y)$ be a scalar-valued function on $\mathbb{R}^2$, i.e., $V: \mathbb{R}^2 \to \mathbb{R}$ (and at least C^1), with $V(\overline{x},\overline{y}) = 0$, and such that the loci of points satisfying $V(x,y) = C =$ constant form closed curves for different values of C encircling $(\overline{x},\overline{y})$ with $V(x,y) > 0$ in a neighborhood of $(\overline{x},\overline{y})$ (see Fig. 2.21).

Recall that the gradient of V, ∇V, is a vector perpendicular to the tangent vector along each curve $V = C$ which points in the direction of increasing V. So, if the vector field were always to be either tangent or pointing inward for each of these curves surrounding $(\overline{x},\overline{y})$, we would have

$$\nabla V(x,y) \cdot (\dot{x},\dot{y}) \leq 0,$$

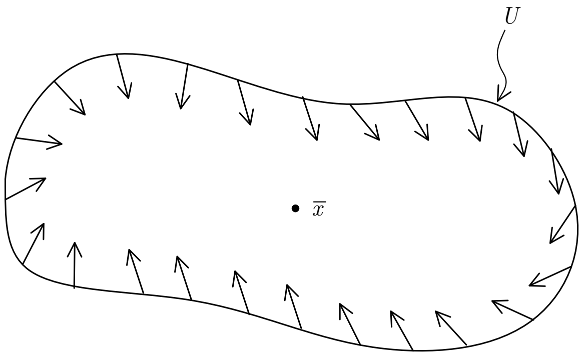

Fig. 2.20 The vector field on the boundary of U

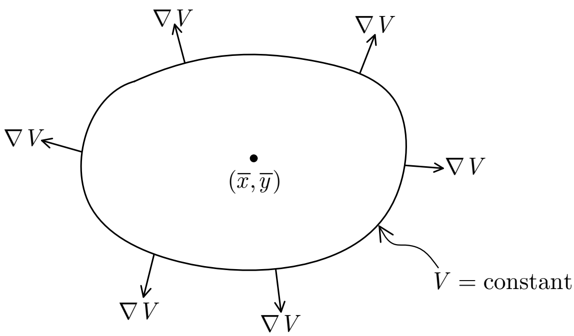

Fig. 2.21 Level set of V and ∇V denotes gradient vector of V at various points on the boundary

where the '$\cdot$' represents the usual vector scalar product. (This is simply the derivative of V along orbits of (2.28), and is sometimes referred to as the *orbital derivative*.) We now state the general theorem which makes these ideas precise.

Theorem 2.6. Consider the following vector field:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{2.29}$$

Let $\bar{x}$ be a fixed point of (2.29) and let $V\colon U \to \mathbb{R}$ be a C^1 function defined on some neighborhood U of $\bar{x}$ such that

(i) $V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$.
(ii) $\dot{V} \leq 0$ in $U - \{\bar{x}\}$.

Then, $\bar{x}$ is stable. Moreover, if

(iii) $\dot{V} < 0$ in $U - \{\bar{x}\}$,

then $\bar{x}$ is asymptotically stable. □

We refer to V as a *Lyapunov function*. If U can be chosen to be all of $\mathbb{R}^n$, then $\bar{x}$ is said to be *globally asymptotically stable*, if (i) and (iii) hold.

Sometimes it is possible to prove asymptotic stability of a fixed point even when the Lyapunov function V in the relevant neighborhood of the point implies that $\dot{V} \leq 0$, but not necessarily $\dot{V} < 0$. For that case we have the following theorem.

Theorem 2.7 (Invariance Principle of LaSalle). Let $\bar{x} = 0$ be a fixed point of $\dot{x} = f(x)$ and V a Lyapunov function such that $\dot{V} \leq 0$ on some neighborhood N of $\bar{x} = 0$. If $x_0 \in N$ has its forward orbit, $\gamma^+(x_0) = \{\phi(t, x_0)\colon t \geq 0\}$, bounded with limit points in N, and M is the largest invariant subset of $E = \{x \in N\colon \dot{V}(x) = 0\}$, then

$$\phi(t, x_0) \to M \ \text{ as } \ t \to \infty.$$ □

According to Theorem 2.7, if a Lyapunov function $V(x)$ can be found such that $\dot{V}(x) \leq 0$ for $x \in N$, among the sets of points with forward orbits in N there exist sets of points defined by

$$V_k = \{x\colon V(x \leq k)\}$$

(k is a finite and positive scalar) which lie entirely in N. Since $\dot{V} \leq 0$, the sets V_k are invariant in the sense that no orbit starting in a V_k can ever move outside of it. If, in addition, it could be shown that the fixed point $\bar{x} = 0$ is the largest (or, for that matter, the only) invariant subset of E, Theorem 2.7 would guarantee its asymptotic stability.

The direct method can also be extended to discrete-time systems. We only state a result analogous to Theorem 2.6 in the following. A discrete-time version of the invariance principle of LaSalle will be introduced in Chap. 5.

Theorem 2.8. Consider the system described by the difference equation given in (2.21). Let $\bar{x} = 0$ again be an isolated equilibrium point at the origin. If there exists a C^1 function $V(x_n)\colon N \to \mathbb{R}$, defined on some neighborhood $N \subset \mathbb{R}^m$ of $\bar{x} = 0$, such that

(i) $V(0) = 0$,
(ii) $V(x) > 0$ in $N - \{0\}$,
(iii) $\Delta V(x_n) := V[G(x_n)] - V(x_n) \leq 0$ in $N - \{0\}$,

then $\bar{x} = 0$ is stable (in the sense of Lyapunov). Moreover, if

(iv) $\Delta V(x) < 0$ in $N - \{0\}$,

then $\bar{x} = 0$ is asymptotically stable. □

2.11 Chaotic Sets and Chaotic Attractors

More complicated invariant, attracting sets and attractors in structure than that of periodic or quasiperiodic sets are called chaotic. A dynamical system (discrete-time or continuous-time) is called chaotic if its typical orbits are aperiodic, bounded, and such that nearby orbits separate fast in time. Chaotic orbits never converge to a stable fixed or periodic point, but exhibit sustained instability, while remaining forever in a bounded region of the state space.

Definition 2.20. A flow ϕ (a continuous map G) on a metric space M is said to possess *sensitive dependence on initial conditions* on M if there exists a real number $\delta > 0$ such that for all $x \in M$ and for all $\varepsilon > 0$, there exist $y \in M$ ($y \neq x$) and $T > 0$ (an integer $n > 0$) such that $d(x,y) < \varepsilon$ and $d[\phi(T,x), \phi(T,y)] > \delta$ ($d[G^n(x), G^n(y)] > \delta$). □

Definition 2.21. A flow ϕ (a continuous map G) is said to be *chaotic* on a compact invariant set[10] A if:

(i) it is topologically transitive on A (Definition 2.12);
(ii) it has sensitive dependence on initial conditions on A. □

Remark 2.4. There is something that should be pointed out.

(i) Condition (i) of Definition 2.21 guarantees that the invariant set is single and indecomposable.
(ii) Condition (ii) of Definition 2.21 can be made sharper (and more restrictive) in two ways. First, the divergence of nearby points taking place at an exponential rate is required. This property can be made more precise by means of Lyapunov exponents which will be introduced later. Second, we may require that the divergence (exponential or otherwise) occurs for *each pair* $x, y \in A$. In this case, the flow ϕ or map G is called *expansive* on A.
(iii) The requirement that A is compact is necessary. Consider the following differential equation:

$$\dot{x} = ax, \; x \in \mathbb{R}, \; a > 0,$$

[10] Roughly speaking, a subset D of $\mathbb{R}^n$ is said to be compact if it can be covered by a finite collection of open sets $\{U_j\}_{j=1}^{l}$.

which is linear and its solution is $\phi(t,x) = x_0 e^{at}$. Therefore, the flow map ϕ is topologically transitive on the open, unbounded (and therefore noncompact) invariant sets $(-\infty, 0)$ and $(0, \infty)$. Also, for any two points $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$ we have

$$|\phi(t,x_1) - \phi(t,x_2)| = e^{at}|x_1 - x_2|$$

and ϕ has sensitive dependence on initial conditions on $\mathbb{R}$. However, the orbits generated by ϕ are not chaotic.

(iv) This definition refers to a 'chaotic flow (or map) on a set A' or, for short, a 'chaotic set A.' It does not imply that all orbits of a chaotic flow (or map) on A are chaotic. In fact, there are many nonchaotic orbits on chaotic sets, in particular, many unstable periodic orbits. They are so important that some researchers add a third condition for chaos, that periodic orbits are dense on A [5]. This is an interesting property and it is automatically satisfied if the chaotic invariant set is hyperbolic [17].

(v) Two quite general results can be used to confirm the close relationship between chaos, as characterized in Definition 2.21, and dense periodic sets. The first result [3] states that for any continuous map on a metric space, transitivity and the presence of a dense set of periodic orbits imply sensitive dependence on initial conditions, that is, chaos. The second result [16] states that for any continuous map on an interval of R, transitivity alone implies the presence of a dense set of periodic orbits and, therefore, in view of the first result, it implies sensitive dependence on initial conditions, and therefore chaos.

(vi) There are several other different definitions of chaos based on orbits rather than sets. For example, in [1] (p. 196, Definition 5.2; p. 235, Definition 6.2; pp. 385–386, Definition 9.6), a chaotic set is defined as the ω-limit set of a chaotic orbit $G^n(x_0)$ which itself is contained in the ω-limit set. In this case, the presence of sensitive dependence on initial conditions (or a positive Lyapunov characteristic exponent) is not enough to characterize chaotic properties of orbits and additional conditions must be added to exclude unstable periodic or quasiperiodic orbits. □

2.12 Symbolic Dynamics and the Shift Map

Symbolic dynamics is a powerful tool for understanding the orbit structure of a large class of dynamical systems. In this section we only provide a brief introduction to this tool.

To establish the tool, three steps are needed. First, we define an auxiliary system characterized by a map, called a *shift map*, acting on a space of infinite sequences called the *symbol space*. Next, we prove some properties of the shift map. Finally, we establish a certain equivalence relation between a map we want to study and the shift map, and show that the relationship preserves the properties in question.

We begin by defining the symbol space and the shift map. Let S be a collection of symbols. In a physical interpretation, the elements of S could be anything, for example letters of an alphabet or discrete readings of some measuring device for the observation of a given dynamical system. To make ideas more clear, we assume here that S consists of only two symbols; let them be 0 and 1. Then, we have $S = \{0,1\}$. Next, we want to construct the space of all possible bi-infinite sequences of 0 and 1, defined as

$$\Sigma_2 := \cdots S \times S \times S \times \cdots .$$

A point in Σ_2, s, is therefore represented as a *bi-infinity-tuple* of elements of S, that is, $s \in \Sigma_2$ means

$$s = \{\dots s_{-n} \dots s_{-1} s_0 s_1 \dots s_n \dots\},$$

where $\forall i$, $s_i \in S$ (i.e., $s_i = 0$ or 1). For example, $s = \{\dots 00010100111 \dots\}$.

We can define a distance function $\bar{d}$ in the space Σ_2

$$\bar{d}(s,\bar{s}) = \sum_{i=-\infty}^{+\infty} \frac{d(s_i,\bar{s_i})}{2^{|i|}}, \tag{2.30}$$

where d is the discrete distance in $S = \{0,1\}$, that is

$$d(s_i,\bar{s_i}) = \begin{cases} 0 \text{ if } & s_i = \bar{s_i}; \\ 1 \text{ if } & s_i \neq \bar{s_i}. \end{cases}$$

This means that two points of Σ_2 are close to each other if their central elements are close, i.e., if the elements whose indexes have small absolute values are close. Notice that, from the definition of $\bar{d}(s_i,\bar{s_i})$, the infinite sum on the right-hand side of (2.30) is less than 3, and, therefore, converges.

Next, we define the *shift map* on Σ_2 as

$$T : \Sigma_2 \to \Sigma_2, \ \ T(s) = s' \text{ and } s'_i = s_{i+1}.$$

The map T maps each entry of a sequence from one place to the left. Similarly, the *one-sided shift map* T_+ can be defined on the space of one-sided infinite sequences, Σ_{2+}, that is, $s \in \Sigma_{2+}$, where $s = \{s_0 s_1 \dots s_n \dots\}$. In this case, we have

$$T_+ : \Sigma_{2+} \to \Sigma_{2+}, \ \ T_+(s) = s' \text{ and } s'_i = s_{i+1},$$

so that

$$T_+(s_0 s_1 s_2 \dots) = (s'_0 s'_1 s'_2 \dots) = (s_1 s_2 s_3 \dots).$$

It is obvious that the T_+ map shifts a one-sided sequence from one place to the left and drops its first element. Although maps T and T_+ have very similar properties, T is invertible whereas T_+ is not. The distance on Σ_{2+} is essentially the same as (2.30) with the difference that the infinite sum will now run from zero to ∞. The map T_+ can be used to prove chaotic properties of certain noninvertible, one-dimensional maps frequently employed in applications.

Theorem 2.9. The shift map T_+ on Σ_{2+} is chaotic according to Definition 2.21. □

Remark 2.5. The shift map T_+ on Σ_{2+} has a property that is stronger than *topological transitivity* called *topological mixing*. In general, we say that a map G is topologically mixing on a set A if for any two open subsets U and V of A there exists a positive integer N_0 such that $G^n(U) \cap V \neq \emptyset$ for all $n \geq N_0$. If a map G is topologically mixing, then for any integer n the map G^n is topologically transitive. □

The importance of the fact that the shift map is chaotic in a precise sense lies in that the chaotic properties of invariant sets of certain one- and two-dimensional maps and three-dimensional flows may sometimes be proved by showing that the dynamics on these sets are *topologically conjugate* to that of a shift map on a symbol space. This indirect argument is often the only available strategy for investigating nonlinear maps (or flows). We have encountered the concept of topological conjugacy in the Hartman–Grobman theorem (Theorem 2.5, which we called homeomorphic equivalence) between a nonlinear map (or flow) and its linearization in a neighborhood of a fixed point. We now provide some formal definitions.

Definition 2.22. Let X and Y be topological spaces, and let $f\colon X \to X$ and $g\colon Y \to Y$ be continuous functions. We say that f is *topologically semiconjugate* to g if there exists a continuous surjection[11] $h\colon Y \to X$ such that $f \circ h = h \circ g$. If h is a homeomorphism, then we say that f and g are topologically conjugate, and we call h a topological conjugation between f and g.

Similarly, a flow ϕ on X is topologically semiconjugate to a flow ψ on Y if there is a continuous surjection $h\colon Y \to X$ such that $\phi(h(y),t) = h(\psi(y,t))$ for each $y \in Y$, $t \in \mathbb{R}$. If h is a homeomorphism, then ϕ and ψ are topologically conjugate. □

Remark 2.6. Topological conjugation defines an *equivalence relation* in the space of all continuous surjections of a topological space to itself, by declaring f and g to be related if they are topologically conjugate. This equivalence relationship is very useful in the theory of dynamical systems, since each class contains all functions which share the same dynamics from the topological viewpoint. In fact, orbits of g are mapped to homeomorphic orbits of f through the conjugation. Writing $g = h^{-1} \circ f \circ h$ makes this fact evident: $g^n = h^{-1} \circ f^n \circ h$. Roughly speaking, topological conjugation is a 'change of coordinates' in the topological sense. □

However, the analogous definition for flows is somewhat restrictive. In fact, we require the maps $\phi(\cdot,t)$ and $\psi(\cdot,t)$ to be *topologically conjugate* for each t, which requires more than simply that orbits of ϕ be mapped to orbits of ψ homeomorphically. This motivates the definition of *topological equivalence*, which also partitions the set of all flows in X into classes of flows sharing the same dynamics, again from the topological viewpoint.

Definition 2.23. We say that two flows ψ and ϕ of a compact manifold M are *topologically equivalent* if there is an homeomorphism $h\colon Y \to X$, mapping orbits of

[11] A function $f\colon X \to Y$ is a surjection if, for every $y \in Y$, there is an $x \in X$ such that $f(x) = y$.

ψ to orbits of ϕ homeomorphically, and preserving orientation of the orbits. This means that

(i) $\{h(\psi(y,t)) : t \in \mathbb{R}\} = \{\phi(h(y),t) : t \in \mathbb{R}\}$ for each $y \in Y$;
(ii) for each $y \in Y$, there is $\delta > 0$ such that, if $0 < |s| < t < \delta$, and if s satisfies $\phi(h(y),s) = h(\psi(y,t))$, then $s > 0$. □

2.13 Lyapunov Exponent

Although sensitive dependence on initial conditions can be verified in some cases, it is not easy to verify for many systems. The *Lyapunov exponent* is a generalization of the eigenvalues at an equilibrium point, and it is used as a measure of exponential divergence of orbits. Suppose that $\phi(t,x_0)$ and $\phi(t,y_0)$ are solutions of an autonomous vector field $\dot{x} = f(x)$ starting from x_0 and y_0, respectively. By using the linear approximation for fixed t, we get

$$\phi(t,y_0) - \phi(t,x_0) \approx D_x\phi(t,x_0)(y_0 - x_0).$$

For any curve starting from initial condition x_s, letting

$$v(t) := \frac{\partial}{\partial s}\phi(t,x_s)\bigg|_{s=0} = D_x\phi(t,x_0)\frac{\partial x_s}{\partial s} = D_x\phi(t,x_0)v_0$$

and $v_0 := \dfrac{\partial x_s}{\partial s}$, then $v(t)$ satisfies the *first variation equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = Df_{(\phi(t,x_0))}v(t).$$

If $v_0 = y_0 - x_0$, then $v(t)$ would give the infinitesimal displacement at time t.

The growth rate of $\|v(t)\|$ is a number χ such that

$$\|v(t)\| \approx C\mathrm{e}^{\chi t},$$

where C is a constant. Taking the logarithm, we have

$$\frac{\ln\|v(t)\|}{t} \approx \frac{\ln(C)}{t} + \chi;$$

therefore,

$$\chi = \lim_{t\to\infty}\frac{\ln\|v(t)\|}{t}.$$

Definition 2.24. Let $v(t)$ be the solution of the first variation equation, starting from x_0 with $v(0) = v_0$. The *Lyapunov exponent* for initial condition x_0 and initial infinitesimal displacement v_0 is defined as

$$\chi(x_0, v_0) = \lim_{t\to\infty} \frac{\ln\|v(t)\|}{t},$$

whenever this limit exists. □

Remark 2.7. For most initial conditions x_0 for which the forward orbit is bounded, the Lyapunov exponents exist for all vectors v. In n-dimensional state space, there are at most n distinct values for $\chi(x_0, v)$ as v varies. If we count multiplicities, then there are exactly n values,

$$\chi_1(x_0) = \chi(x_0, v_1),\ \chi_2(x_0) = \chi_2(x_0, v_2), \ldots, \chi_n(x_0) = \chi(x_0, v_n).$$

We can order these so that

$$\chi_1(x_0) \geq \chi_2(x_0) \geq \cdots \geq \chi_n(x_0).$$ □

Several results on Lyapunov exponents are listed as follows. For detailed proofs please refer to [13].

Theorem 2.10. Assume that x_0 is a *fixed point* of the differential equation $\dot{x} = f(x)$. Then, the Lyapunov exponents at the fixed point are the real parts of the eigenvalues of the fixed point. □

Theorem 2.11. Let x_0 be an initial condition such that $\phi(t, x_0)$ is bounded and $\omega(x_0)$ does not contain any fixed points. Then,

$$\chi(x_0, f(x_0)) = 0.$$ □

Remark 2.8. The above theorem means that there is no growth or decay in the direction of the vector field, $v = f(x_0)$. □

Theorem 2.12. Let x_0 be an initial condition on a periodic orbit of period T. Then, the principal $(n-1)$ Lyapunov exponents are given by $(\ln|\lambda_j|)/T$, where λ_j are the *characteristic multipliers* of the periodic orbit and the eigenvalues of the Poincaré map. □

Theorem 2.13. Assume that $\phi(t, x_0)$ and $\phi(t, y_0)$ are two orbits for the same differential equation, which are bounded and converge exponentially (i.e., there are constants $a > 0$ and $C \geq 1$ such that $\|\phi(t, x_0) - \phi(t, y_0)\| \leq C\mathrm{e}^{-at}$ for $t \geq 0$). Then, the Lyapunov exponents for x_0 and y_0 are the same. So, if the limits defining the Lyapunov exponents exist for one of the points, they exist for the other point. The vectors which give the various Lyapunov exponents can be different at the two points. □

Theorem 2.14. Consider the system of $\dot{x} = f(x)$ in $\mathbb{R}^n$. Assume that x_0 is a point such that the Lyapunov exponents $\chi_1(x_0), \ldots, \chi_n(x_0)$ exist.

(i) Then, the sum of Lyapunov exponents is the limit of the average of the divergence along the trajectory,

$$\sum_{j=1}^{n} \chi_j(x_0) = \lim_{T\to\infty} \frac{1}{T} \int_0^{\mathrm{T}} \nabla \cdot f_{\phi(t,x_0)} \mathrm{d}t.$$

(ii) In particular, if the system has constant divergence δ, then the sum of the Lyapunov exponents at any point must equal δ.

(iii) In the three-dimensional case, assume that the divergence is a constant δ and that x_0 is a point for which the positive orbit is bounded and $\omega(x_0)$ does not contain any fixed points. If $\chi_1(x_0)$ is a nonzero Lyapunov exponent at x_0, then the other two Lyapunov exponents are 0 and $\delta - \chi_1$. □

The definition of the Lyapunov exponent for discrete-time systems is similar to the case of continuous time.

Definition 2.25. The Lyapunov exponent for the maps $x_{n+1} = G(x_n)$ is defined by

$$\chi(x_0, w) = \lim_{n\to\infty} \ln \frac{\|D^n G(x_0) w\|}{\|w\|},$$

where

$$D^n G(x_0) = DG(x_0) DG(x_1) \cdots G(x_{n-1})$$

and w is a vector in the tangent space at x_0. □

2.14 Examples

In this section, we will explore how chaos appears by investigating some examples.

2.14.1 *Tent Map and Logistic Map*

The tent map has the form of

$$G_\Lambda(y) = \begin{cases} 2y, & \text{if } 0 \le y \le \dfrac{1}{2}, \\ 2(1-y), & \text{if } \dfrac{1}{2} < y \le 1, \end{cases} \tag{2.31}$$

which is shown in Fig. 2.22.

Proposition 2.1. The tent map (2.31) is chaotic on $[0,1]$.

Proof. Consider that the graph of the nth iteration of G_Λ consists of 2^n linear pieces, each with slope $\pm 2^n$. Each of these linear pieces of the graph is defined on a subinterval of $[0,1]$ of length 2^{-n}. Then, for any open subinterval J of $[0,1]$, we can find a subinterval K of J of length 2^{-n}, such that the image of K under G_Λ^n covers the entire interval $[0,1]$. Therefore, G_Λ is topologically transitive on $[0,1]$. This fact, and the discussion in point (v) of Remark 2.4 [16], proves the proposition. $\square$

Remark 2.9. From the geometry of the iterated map G_Λ^m, it appears that the graph of G_Λ^m on J intersects the bisector and therefore G_Λ^m has a fixed point in J. This proves that periodic points are dense in $[0,1]$. Also, for any $x \in J$ there exists a $y \in J$ such that $|G_\Lambda^n(x) - G_\Lambda^n(y)| \geq 1/2$ and, therefore, G_Λ has sensitive dependence on initial conditions. $\square$

This result can be used to show that the logistic map

$$G_4 \colon [0,1] \to [0,1], \;\; G_4(x) = 4x(1-x)$$

(see Fig. 2.23) is also chaotic. Consider the map $h(y) = \sin^2(\pi y/2)$. The map h is continuous and, restricted to $[0,1]$, is also one-to-one and onto. Its inverse is continuous and h is thus a homeomorphism. Consider now the diagram

$$\begin{array}{ccc} [0,1] & \xrightarrow{G_\Lambda} & [0,1] \\ {\scriptstyle h(y)}\downarrow & & \downarrow{\scriptstyle h(y)} \\ [0,1] & \xrightarrow[G_4]{} & [0,1] \end{array}$$

where G_Λ is the tent map. Recalling the trigonometric relations:

(i) $\sin^2(\theta) + \cos^2(\theta) = 1$;
(ii) $4\sin^2(\theta)\cos^2(\theta) = \sin^2(2\theta)$;
(iii) $\sin(\pi - \theta) = \sin(\theta)$;

we can see that the diagram is commutative. Hence, the map G_4 is topologically conjugate to G_Λ and, therefore, its dynamics on $[0,1]$ is chaotic.

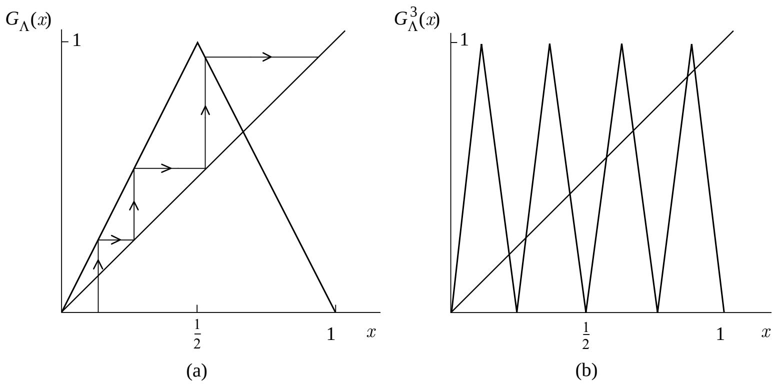

Fig. 2.22 The tent map: (a) G_Λ; (b) G_Λ^3

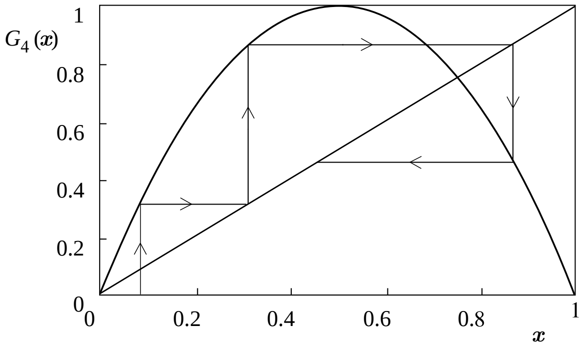

Fig. 2.23 The logistic map G_4

Both the tent map and the logistic map are all from an interval of $\mathbb{R}$ to itself. There are also many other one-dimensional mappings presenting chaotic dynamics. In fact, for one-dimensional mappings from $\mathbb{R}$ to itself we have the following so-called Li–Yorke theorem [13], famous for the phrase 'period three implying chaos,' and Sarkovskii's theorem [13], the generalization of Li–Yorke's result. Since many references include the proof of these two theorems, we will only state the content of the theorems.

Theorem 2.15 (Li–Yorke [13]). Assume that f is a continuous function from $\mathbb{R}$ to itself.

(i) If f has a period-3 point, then it has points of all periods.
(ii) Assume that there is a point x_0 such that either

a. $f^3(x_0) \leq x_0 < f(x_0) < f^2(x_0)$ or
b. $f^3(x_0) \geq x_0 > f(x_0) > f^2(x_0)$.

Then, f has points of all periods. □

This theorem was obtained by Li and Yorke in 1975 and soon after it was shown that Li–Yorke theorem is a special case of Sharkovskii's theorem. Before introducing Sharkovskii's theorem, we first define a new order for natural numbers as follows:

$$3 \triangleright 5 \triangleright 7 \cdots \triangleright 2\cdot 3 \triangleright 2\cdot 5 \triangleright 2\cdot 7 \triangleright \cdots \triangleright 2^k\cdot 3 \triangleright 2^k\cdot 5 \triangleright 2^k\cdot 7 \triangleright \cdots \triangleright 2^k \triangleright 2^{k-1} \triangleright \cdots \triangleright 2^2 \triangleright 2 \triangleright 1.$$

Theorem 2.16 (Sharkovskii's theorem [13]). Let f be a continuous function from $\mathbb{R}$ to itself. Suppose that f has period-n points and $n \triangleright k$. Then, f has period-k points. □

2.14.2 Smale Horseshoe

The Smale horseshoe is the prototypical map possessing a chaotic invariant set. Therefore, a thorough understanding of the Smale horseshoe is absolutely essential for understanding what is meant by the term 'chaos' as it is applied to the dynamics of specific physical systems [10].

Consider the geometrical construction in Fig. 2.24. Take a square S on the plane (Fig. 2.24 (a)). Contract it in the horizontal direction and expand it in the vertical direction (Fig. 2.24 (b)). Fold it in the middle (Fig. 2.24 (c)) and place it so that it intersects the original square S along two vertical strips (Fig. 2.24 (d)). This procedure defines a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. The image $f(S)$ of the square S under this transformation resembles a horseshoe. That is why it is called a *horseshoe map*. The exact shape of the image $f(S)$ is irrelevant; however, for simplicity we assume that both the contraction and expansion are linear and that the vertical strips in the intersection are rectangle. The map f can be made invertible and smooth together with its inverse. The inverse map f^{-1} transforms the horseshoe $f(S)$ back into the square S through stages (d)–(a). This inverse transformation maps the dotted square S shown in Fig. 2.24 (d) into the dotted horizontal horseshoe in Fig. 2.24 (a), which is assumed to intersect the original square S along two horizontal rectangles.

Denote the vertical strips in the intersection $S \cap f(S)$ by V_1 and V_2,

$$S \cap f(S) = V_1 \cup V_2$$

(see Fig. 2.25 (a)). Now make the most important step: perform the *second iteration* of the map f. Under this iteration, the vertical strips V_1 and V_2 will be transformed

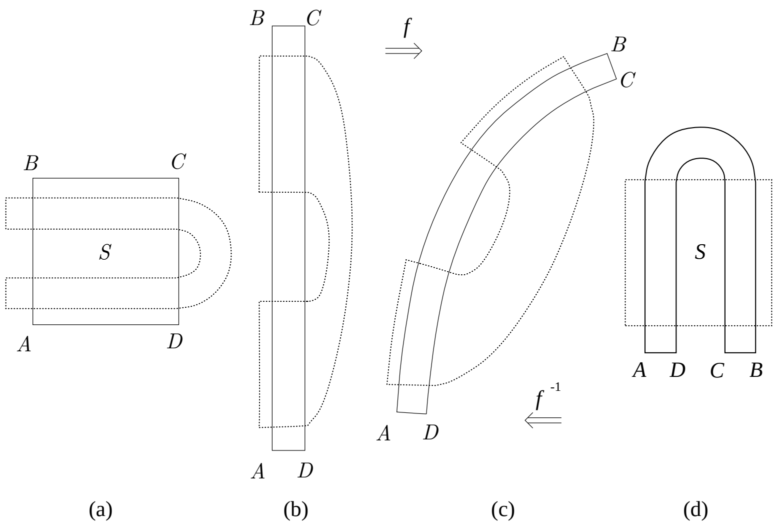

Fig. 2.24 Construction of the horseshoe map

into two 'thin horseshoes' that intersect the square S along four narrow vertical strips: V_{11}, V_{21}, V_{22}, and V_{12} (see Fig. 2.25 (b)). We write this as

$$S \cap f(S) \cap f^2(S) = V_{11} \cup V_{21} \cup V_{22} \cup V_{12}.$$

Similarly,

$$S \cap f^{-1}(S) = H_1 \cup H_2,$$

where H_1 and H_2 are the horizontal strips shown in Fig. 2.25 (c), and

$$S \cap f^{-1}(S) \cap f^{-2}(S) = H_{11} \cup H_{12} \cup H_{22} \cup H_{21},$$

with four narrow horizontal strips H_{ij} (Fig. 2.25 (d)). Notice that $f(H_i) = V_i$, $i = 1, 2$, as well as $f^2(H_{ij}) = V_{ij}$, $i, j = 1.2$ (see Fig. 2.26).

Iterating the map f further, we obtain 2^k vertical strips in the intersection $S \cap f^k(S)$, $k \in \mathbb{N}$. Similarly, iteration of f^{-1} gives 2^k horizontal strips in the intersection $S \cap f^{-k}(S)$, $k \in \mathbb{N}$.

Most points leave the square S under iterations of f or f^{-1}. We consider all remaining points in the square under all iterations of f and f^{-1}:

$$\Gamma = \{x \in S \colon f^k(x) \in S, \ \forall k \in \mathbb{Z}\}.$$

Clearly, if the set Γ is nonempty, it is an *invariant set* of the discrete-time dynamical system defined by f. This set can be alternatively presented as an infinite intersection,

$$\Gamma = \cdots \cap f^{-k}(S) \cap \cdots \cap f^{-1}(S) \cap S \cap f(S) \cap \cdots f^k(S) \cap \cdots .$$

It is clear from this representation that the set Γ has a peculiar shape. Indeed, it should be located within

$$f^{-1}(S) \cap S \cap f(S),$$

which is formed by four small squares (see Fig. 2.27 (a)). Furthermore, it should be located inside

$$f^{-2}(S) \cap f^{-1}(S) \cap S \cap f(S) \cap f^2(S),$$

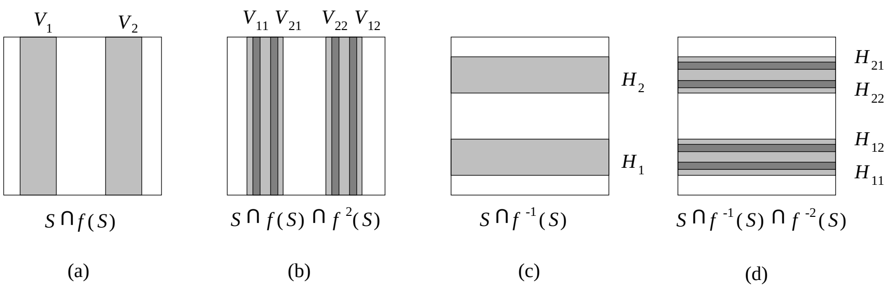

Fig. 2.25 Vertical and horizontal strips

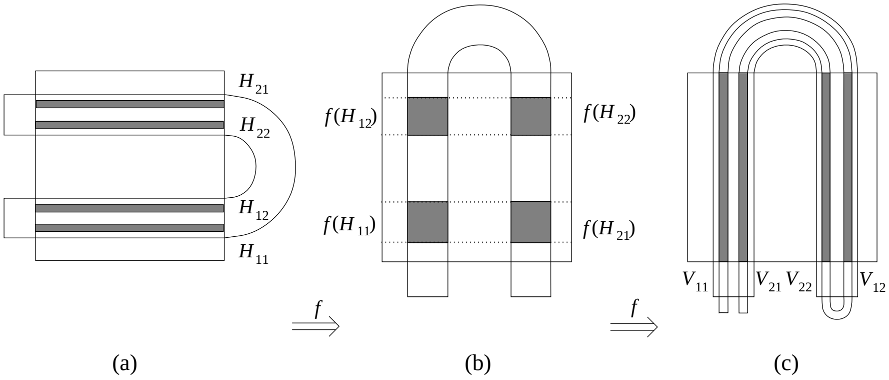

Fig. 2.26 Transformation $f^2(H_{ij}) = V_{ij}$, $i, j = 1, 2$

which is the union of sixteen smaller squares (see Fig. 2.27 (b)), and so forth. In the limit, we get a *Cantor set*. About the horseshoe map, we have the following lemma.

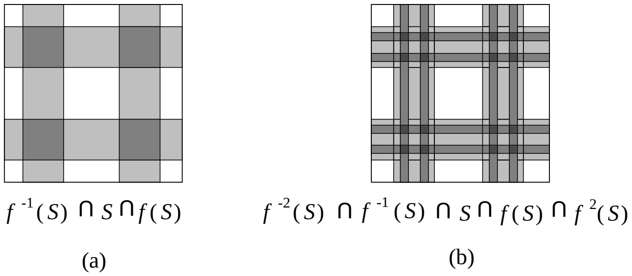

Fig. 2.27 Location of the invariant set

Lemma 2.1. There is a one-to-one correspondence $h\colon \Gamma \to \Sigma_2$ between points of Γ and all bi-infinite sequences of two symbols.

Proof. For any point $x \in \Gamma$, define a sequence of the two symbols $\{1, 2\}$ by

$$\omega = \{\dots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \dots\}$$

by the formula

$$\omega_k = \begin{cases} 1, \text{ if } & f^k(x) \in H_1, \\ 2, \text{ if } & f^k(x) \in H_2, \end{cases} \tag{2.32}$$

for $k \in \mathbb{Z}$. Here, $f^0 = \text{id}$, the identity map. Clearly, this formula defines a map $h\colon \Gamma \to \Sigma_2$, which assigns a sequence to each point of the invariant set. To verify that this map is invertible, take a sequence $\omega \in \Sigma_2$, fix $m > 0$, and consider a set $R_m(\omega)$ of all points $x \in S$, not necessarily belonging to Γ, such that

$$f^k(x) \in H_{\omega_k},$$

for $-m \leq k \leq m-1$. For example, if $m = 1$, the set R_1 is one of the four intersections $V_j \cap H_k$. In general, R_m belongs to the intersection of a vertical and a horizontal strip. These strips are getting thinner and thinner as $m \to \infty$, approaching in the limit a vertical and a horizontal segment, respectively. Such segments obviously intersect at a single point x with $h(x) = \omega$. Thus, $h\colon \Gamma \to \Sigma_2$ is a one-to-one map. It implies that Γ is nonempty. □

Remark 2.10. The map $h\colon \Gamma \to \Sigma_2$ is continuous together with its inverse (a homeomorphism) if we use the standard Euclidean metric in $S \subset \mathbb{R}^2$ and the metric given by (2.30) in Σ_2. □

Consider now a point $x \in \Gamma$ and its corresponding sequence $\omega = h(x)$, where h is the map previously constructed. Next, consider a point $y = f(x)$, that is, the image of x under the horseshoe map f. Since $y \in \Gamma$ by definition, there is a sequence that corresponds to y: $\theta = h(y)$. As one can easily see from (2.32), there is a simple relationship between these sequences ω and θ. Namely,

$$\theta_k = \omega_{k+1}, \quad k \in \mathbb{Z},$$

since

$$f^k(f(x)) = f^{k+1}(x).$$

In other words, the sequence θ can be obtained from the sequence ω by the *shift map* σ:

$$\theta = \sigma(\omega).$$

Therefore, the restriction of f to its invariant set $\Gamma \subset \mathbb{R}^2$ is *equivalent* to the shift map σ on the set of sequences Σ_2. This result can be formulated as the following lemma.

Lemma 2.2. $h(f(x)) = \sigma(h(x))$, for all $x \in \Gamma$. □

This lemma can be written as an even shorter one:

$$f|_\Gamma = h^{-1} \circ \sigma \circ h.$$

Combining Lemmas 2.1 and 2.2 with obvious properties of the shift dynamics on Σ_2, we get a theorem giving a rather complete description of the behavior of the horseshoe map.

Theorem 2.17. The horseshoe map f has a closed invariant set Γ that contains a countable set of periodic orbits of arbitrarily long period, and an uncountable set of nonperiodic orbits, among which there are orbits passing arbitrarily close to any point of Γ. □

Remark 2.11. The limit set Γ of a Smale horseshoe map is unstable and, therefore, not attracting. It follows that the existence of a Smale horseshoe does not imply the existence of a chaotic attractor. The existence of a Smale horseshoe does imply, however, that there is a region in state space that experiences sensitive dependence on initial conditions. Thus, even when there is no strange attractor in the flow, the dynamics of the system can appear chaotic until the steady state is reached. □

Remark 2.12. We can slightly perturb the constructed map f without qualitative changes to its dynamics. Clearly, Smale's construction is based on a sufficiently strong contraction or expansion, combined with a fold. Thus, a (smooth) perturbation $\tilde{f}$ will have similar vertical and horizontal strips, which are no longer rectangles but curvilinear regions. However, provided that the perturbation is sufficiently small, these strips will shrink to *curves* that deviate only slightly from vertical and horizontal lines. Thus, the construction can be carried through word for word, and the perturbed map $\tilde{f}$ will have an invariant set Γ on which the dynamics is completely described by the shift map σ on the sequence space Σ_2. This is an example of structurally stable behavior. □

2.14.3 The Lorenz System

Although plenty of numerical evidence of chaotic behavior arising from a variety of problems in different fields of applications has been provided, apart from the cases of one-dimensional, noninvertible maps, there are few rigorous proofs that specific mathematical models possess chaotic attractors as characterized in one or more of the above definitions, and those proofs are mostly restricted to artificial examples unlikely to arise in typical applications. We now turn to an example of a chaotic attractor derived from a celebrated model first discussed in 1963 by E. Lorenz to provide a mathematical description of atmospheric turbulence. Lorenz's investigation was enormously influential and stimulated a vast literature in the years that followed. Its extraordinary success was in part due to the fact that it showed how computer technology could be effectively used to study nonlinear dynamical systems. Lorenz's work provided strong numerical evidence that a low-dimensional system of differential equations with simple nonlinearities could generate extremely complicated orbits. The original Lorenz model is defined by the following three differential equations:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y, \\ \dot{y} = -xz + rx - y, \\ \dot{z} = xy - bz, \end{cases} \tag{2.33}$$

where $x, y, z \in \mathbb{R}$ and $\sigma, r, b > 0$. System (2.33) is symmetrical under the transformation $(x, y, z) \to (-x, -y, z)$. The three equilibria are

$$\begin{aligned} &E_1 : (0,0,0), \\ &E_2 : (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1), \\ &E_3 : (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1). \end{aligned}$$

In numerical analysis of the Lorenz model, the typical parameter configuration is $\sigma = 10$ and $b = 8/3$. When $r < r_H = 24.74$ there are two symmetric unstable periodic orbits with the Lorenz system. When $r > r_H$, the celebrated Lorenz attractor (the so-called ‘butterfly’) is observed numerically (see Fig. 2.28). The three Lyapunov exponents are $1.497, 0$, and -22.46, which imply that the Lorenz system has sensitive dependence on initial conditions.

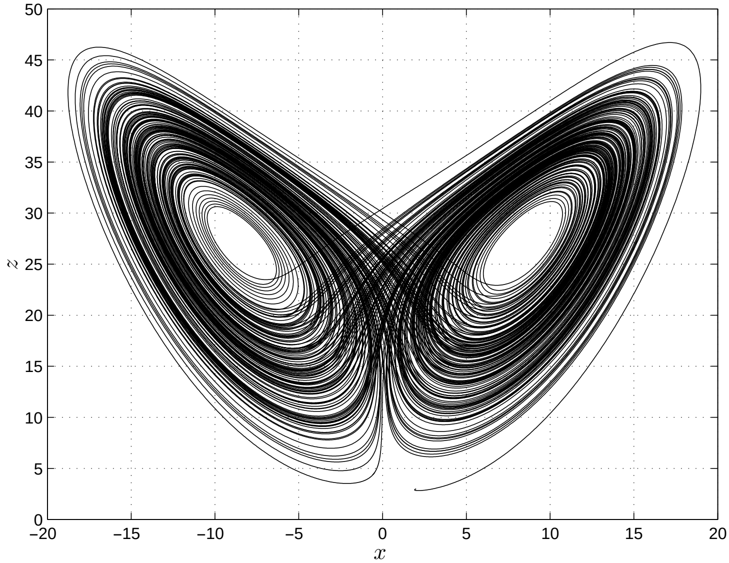

Fig. 2.28 The Lorenz attractor

Remark 2.13. When we refer to a computer to visualize the numerical solution of a chaotic system, an important issue arises. If we take into account the combined influence of round-off errors in numerical computations and the property of divergence of nearby trajectories for chaotic behavior, how can we trust numerical computations of trajectories to give us reliable results? (We note that the same problem arises in experimental measurements in which ‘noise’ plays the role of round-off errors.) If the system’s behavior is chaotic, then even small numerical errors are amplified exponentially in time. Perhaps all of our results for chaotic systems are artifacts of the numerical computation procedure. Even if they are not artifacts, perhaps the numerical values of the properties depend critically on the computational procedures. If that is true, how general are our results? Although it is difficult to answer these questions once and for all, it is comforting to know that while it is true that the details of a particular trajectory do depend on the round-off errors in the numerical computation, the trajectory actually calculated does follow very closely some trajectory of the system. That is, the trajectory one calculates might not be exactly the one he thinks, but it is very close to one of the possible trajectories of the system. In more technical terms, we say that the computed trajectory shadows some possible trajectories of the system. (A proof of this shadowing property for chaotic systems is given in [6] and [14].) In general, we are most often interested in properties that are averaged over a trajectory; in many cases those average values are independent of the particular trajectory we follow. So, as long as we follow some possible trajectory for the system, we can have confidence that our results are a good characterization of the system's behavior. Recently, W. Tucker's work [15] strengthened the above discussion, in which he has shown, using a computer-assisted proof, that the Lorenz system not only has sensitive dependence on initial conditions, but also has a chaotic attractor. □

Although a full mathematical analysis of the observed attractor is still lacking, some of the attractor's properties have been established through a combination of numerical evidence and theoretical arguments. Before presenting the analysis we first consider the following three facts about system (2.33).

(i) The trace of the Jacobian matrix

$$\mathrm{tr}[Df(x,y,z)] = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(b+\sigma+1) < 0$$

is constant and negative along orbits. Thus, any three-dimensional volume of initial conditions is contracted along orbits at a rate equal to

$$\gamma = -(b+\sigma+1) < 0;$$

that is to say, the system is *dissipative*.

(ii) It is possible to define a trapping region such that all orbits outside of it tend to it, and no orbits ever leave it. To see this, consider the function

$$V(x,y,z) = x^2 + y^2 + (z-r-\sigma)^2 = K^2(r+\sigma)^2 \tag{2.34}$$

defining a sphere with center at $(x = y = 0; z = \sigma + r)$ and radius $K(\sigma + r)$. The time derivative of (2.34) along the solution of (2.33) is

$$\dot{V}(x,y,z) = -2\sigma x^2 - 2y^2 - 2b\left(z - \frac{r+\sigma}{2}\right)^2 + b\frac{(r+\sigma)^2}{2}.$$

$\dot{V} = 0$ defines an ellipsoid outside of which $\dot{V} < 0$. For a sufficiently large value of the radius (for sufficiently large K, given r and σ), the sphere (2.34) will contain all three fixed points and all orbits on the boundary of the sphere will point inward. Consequently, system (2.33) is 'trapped' inside the sphere.

(iii) Numerical evidence indicates that for $r \in (13.8, 14)$, there exist two symmetric *homoclinic orbits* (that is, orbits that connect a fixed point to itself) asymptotically approaching the origin for $t \to \pm\infty$, tangentially to the z axis; see the sketch in Fig. 2.29.

Keeping the three facts in mind, we can investigate the Lorenz system by constructing a geometric model which, under a certain hypothesis, provides a reasonable approximation of the dynamics of the original model with 'canonical' parameter values $\sigma = 10$, $b = 8/3$, and $r > r_H$.

We first consider a system of differential equations in $\mathbb{R}^3$ depending on a parameter μ with the following properties:

(i) for a certain value μ_h of the parameter, there exists a pair of symmetrical homoclinic orbits, asymptotically converging to the origin, and tangential to the positive z axis;

(ii) the origin is a saddle-point equilibrium and the dynamics in a neighborhood N of the equilibrium, for μ in a neighborhood of μ_h, is approximated by the system

$$\begin{cases} \dot{x} = \lambda_1 x, \\ \dot{y} = \lambda_2 y, \\ \dot{z} = \lambda_3 z, \end{cases}$$

where $\lambda_2 < \lambda_3 < 0 < \lambda_1$ and $-\lambda_3/\lambda_1 < 1$;

(iii) the system is invariant under the change of coordinates $(x,y,z) \to (-x,-y,z)$.

Under these conditions, for $(x,y,z) \in N$ and μ near μ_h, it is possible to construct a two-dimensional cross section Σ, such that the transversal intersections of orbits with Σ define a two-dimensional Poincaré map $P\colon \Sigma \to \Sigma$. For values of $|x|$ and $|\mu - \mu_h|$ sufficiently small, the dynamics of P can further be approximated by a one-dimensional, noninvertible map $G_\mu[-a,a] - \{0\} \to \mathbb{R}$ defined on an interval of the x axis, but not at $x = 0$.

A typical formulation of the map G_μ is

$$G_\mu(x) = \begin{cases} a\mu + cx^\delta, & \text{if } x > 0; \\ -a\mu - c|x|^\delta, & \text{if } x < 0; \end{cases}$$

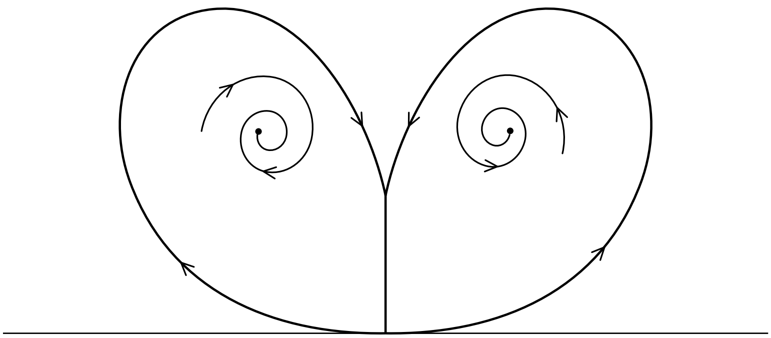

Fig. 2.29 Homoclinic orbits in the Lorenz model

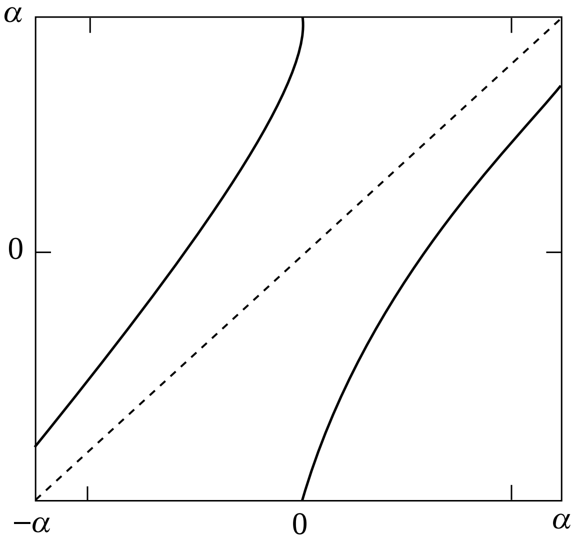

Fig. 2.30 One-dimensional map for the Lorenz model

where $a < 0$, $c > 0$, $\delta = -\lambda_3/\lambda_1$, and $0 < \delta < 1$.

Assuming that the one-dimensional approximation remains valid for values of the parameter μ outside the neighborhood of $\mu = \mu_h$ (that is, outside the neighborhood of the homoclinic orbit), values of $\mu > \mu_h$ can be chosen so that there exists a closed interval $[-\alpha, \alpha]$ with $\alpha > 0$ such that $G_\mu[-\alpha, \alpha]\backslash\{0\} \to [-\alpha, \alpha]$ and

$$\lim_{x \to 0^-} G_\mu(x) = \alpha > 0, \quad \lim_{x \to 0^+} G_\mu(x) = -\alpha < 0.$$

Then, $G'_\mu(x) > 1$ for $x \in [-\alpha, \alpha]$ $(x \neq 0)$, and $\lim_{x \to 0^\pm} G'_\mu(x) = +\infty$. The map G_μ on the interval is depicted in Fig. 2.30.

Because $G'_\mu(x) > 1$ for all $x \in [-\alpha, \alpha]$ $(x \neq 0)$, G_μ is a piecewise-expanding map and has therefore sensitive dependence on initial conditions. There are no fixed points or stable periodic points and most orbits on the interval $[-\alpha, 0) \cup (0, \alpha]$ are attracted to a chaotic invariant set. Although increasing μ beyond the homoclinic value μ_h leads to stable chaotic motion, if we take μ very large, the system reverts to simpler dynamical behavior and stable periodic orbits reappear.

Remark 2.14. The idea that some essential aspects of the dynamics of the original system (2.33) could be described by a one-dimensional map was first put forward by Lorenz himself. In order to ascertain whether the numerically observed attractor could be periodic rather than chaotic, he plotted successive maxima of the variable z along an orbit on the numerically observed attractor. In doing so, he discovered that the plot of z_{n+1} against z_n has a simple shape, as illustrated in Fig. 2.31. The points of the plot lie almost exactly on a curve whose form changes as the parameter r varies. Setting $\sigma = 10$, $b = 8/3$, and $r = 28$ (the traditional 'chaotic values'), we

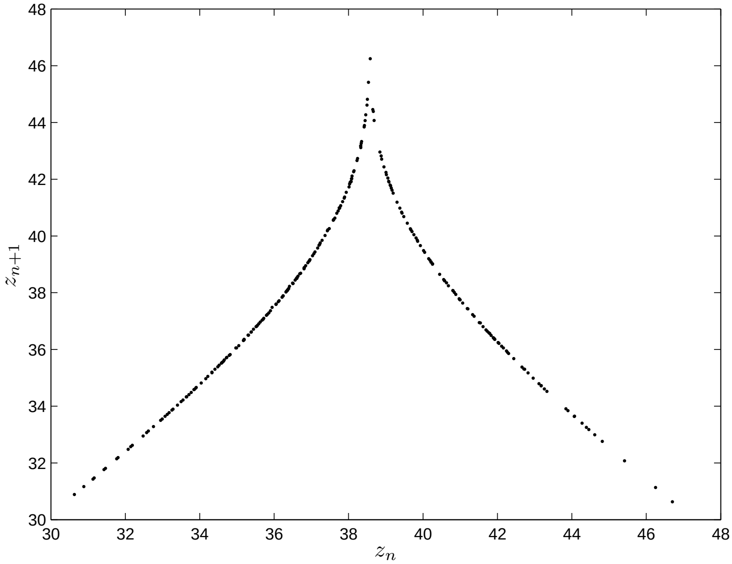

Fig. 2.31 Successive maxima of z for the Lorenz attractor

obtain a curve resembling a distorted tent map. It has slope everywhere greater than 1 in absolute value so, again, it approximates a piecewise-expanding map. For such a map there cannot be stable fixed points or stable periodic orbits and, for randomly chosen initial values, orbits converge to a chaotic attractor. □

2.15 Basics of Functional Differential Equations Theory

In many applications, the system under consideration is not governed by a principle of causality; that is, the future state of the system is dependent not only on the present state but also on the past states. The theory about these systems has been extensively developed, which formed the framework of the subject – *functional differential equations* (FDEs). In this section, we only provide some necessary notions and theorems about FDEs so that this book can be read smoothly.

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. Let $C = C([-\tau, 0], \mathbb{R}^n)$ denote the space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$. We designate the norm of an element ϕ in C by $\|\phi\|_\tau = \sup_{-\tau \le \theta \le 0} \|\phi(\theta)\|$. Each $x \in \mathbb{R}^n$ can also be looked as an element in C: $x(\theta) = x$, $\theta \in [-\tau, 0]$. If $\sigma \in \mathbb{R}$, $\alpha \ge 0$, and $x \in C[\sigma - \tau, \sigma + \alpha]$, then, for any $t \in [\sigma, \sigma + \alpha]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$, $-\tau \le \theta \le 0$. If D is a subset of $\mathbb{R} \times C$, $f: D \to \mathbb{R}^n$ is a given function, and '$\cdot$' represents the right-hand derivative, we say that the relation

$$\dot{x} = f(t, x_t) \tag{2.35}$$

is a retarded functional differential equation on D and will denote this equation by RFDE. We write RFDE(f) if we wish to emphasize that the equation is defined by f.

Definition 2.26. A function x is said to be a solution of (2.35) on $[\sigma - \tau, \sigma + \alpha]$ if there are $\sigma \in \mathbb{R}$ and $\alpha > 0$ such that $x \in C([\sigma - \tau, \sigma + \alpha], \mathbb{R}^n)$, $(t, x_t) \in D$, and $x(t)$ satisfies (2.35) for $t \in [\sigma, \sigma + \alpha)$. For given $\sigma \in \mathbb{R}$, $\phi \in C$, we say that $x(\sigma, \phi, f)$ is a solution of (2.35) with initial value ϕ at σ or simply a solution through (σ, ϕ) if there is an $\alpha > 0$ such that $x(\sigma, \phi, f)$ is a solution of (2.35) on $[\sigma - \tau, \sigma + \alpha]$ and $x_\sigma(\sigma, \phi, f) = \phi$. □

In the following we only state the main results; the proofs can be found in [8].

Theorem 2.18 (Existence). Suppose that Ω is an open subset in $\mathbb{R} \times C$ and $f^0 \in C(\Omega, \mathbb{R}^n)$. If $(\sigma, \phi) \in \Omega$, then there is a solution of the RFDE(f^0) passing through (σ, ϕ). More generally, if $W \subseteq \Omega$ is compact and $f^0 \in C(\Omega, \mathbb{R}^n)$ is given, then there is a neighborhood $V \subseteq \Omega$ of W such that $f^0 \in C^0(V, \mathbb{R}^n)$, there is a neighborhood $U \subseteq C^0(V, \mathbb{R}^n)$ of f^0, and an $\alpha > 0$ such that, for any $(\sigma, \phi) \in W$, $f \in U$, there is a solution $x(\sigma, \phi, f)$ of the RFDE(f) through (σ, ϕ) which exists on $[\sigma - \tau, \sigma + \alpha]$. □

Theorem 2.19 (Continuous Dependence). Suppose that $\Omega \subseteq \mathbb{R} \times C$ is open, $(\sigma^0, \phi^0) \in \sigma$, $f^0 \in C(\Omega, \mathbb{R}^n)$, and x_0 is a solution of the RFDE(f^0) through (σ^0, ϕ^0) which exists and is unique on $[\sigma^0 - \tau, b]$. Let $W^0 \subseteq \Omega$ be the compact set defined by

$$W^0 = \{(t, x_t^0) | t \in [\sigma^0, b]\}$$

and let V^0 be a neighborhood of W^0 on which f^0 is bounded. If (σ^k, ϕ^k, f^k), $k = 1, 2, \ldots$, satisfies $\sigma^k \to \sigma^0$, $\phi^k \to \phi^0$, and $\|f^k - f^0\|_{V^0} \to 0$ as $k \to \infty$, then there is a k^0 such that the RFDE(f^k) for $k \geq k^0$ is such that each solution $x^k = x^k(\sigma^k, \phi^k, f^k)$ through (σ^k, ϕ^k) exists on $[\sigma^k - \tau, b]$ and $x^k \to x^0$ uniformly on $[\sigma^0 - \tau, b]$. Since all x^k may not be defined on $[\sigma^0 - \tau, b]$, by $x^k \to x^0$ uniformly on $[\sigma^0 - r, b]$, we mean that, for any $\varepsilon > 0$, there is a $k_1(\varepsilon)$ such that $x^k(t)$, $k \geq k_1(\varepsilon)$, is defined on $[\sigma^0 - \tau + \varepsilon, b]$, and $x^k \to x^0$ uniformly on $[\sigma^0 - \tau + \varepsilon, b]$. □

Theorem 2.20. Suppose that Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in Ω, i.e.,

$$\|f(t, \phi) - f(t, \psi)\| \leq K \|\phi - \psi\|,$$

for arbitrary $\phi, \psi \in C$, where K is a constant. If $(\sigma, \phi) \in \Omega$, then there is a unique solution of (2.35) through (σ, ϕ). □

Definition 2.27. If, for all $(\sigma, \phi) \in \mathbb{R} \times C$, the solution of (2.35) through (σ, ϕ), $x(\sigma, \phi)(t)$, exists on $[\sigma - \tau, \infty)$, then we say that (2.35) has global solutions. □

Theorem 2.21. If there exist continuous functions $M, N\colon \mathbb{R} \to \mathbb{R}^+$ such that

$$\|f(t,\phi)\| \le M(t) + N(t)\|\phi\|, \quad (t,\phi) \in \mathbb{R} \times C,$$

then (2.35) has global solutions. □

In the following, we introduce the stability theory for (2.35). Analogous to what we have done with ordinary differential equations, the stability of a general solution $x(t)$ of (2.35) is equivalent to the stability of the zero solution of a new RFDE. Therefore, without loss of generality, we assume that $f(t,0) = 0$ for all $t \in \mathbb{R}$.

Definition 2.28.

(i) The solution $x = 0$ of (2.35) is said to be stable if, for any $\sigma \in \mathbb{R}$ and $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon,\sigma) > 0$ such that $\|\phi\|_\tau < \delta$ implies that $\|x(t,\sigma,\phi)\| < \varepsilon$ for $t \ge \sigma$.
(ii) The solution $x = 0$ of (2.35) is said to be uniformly stable if the δ in (i) is independent of σ.
(iii) The solution $x = 0$ of (2.35) is said to be attractive if, for any $\sigma \in \mathbb{R}$, there exists a $b = b(\sigma)$ such that $\|\phi\|_\tau \le b$ implies that $x(t,\sigma,\phi) \to 0$ $(t \to \infty)$. That is, for any $\varepsilon > 0$ and $\|\phi\|_\tau \le b$, there exists a $T(\sigma,\varepsilon,\phi)$ such that $\|x(t,\sigma,\phi)\| < \varepsilon$ whenever $t \ge \sigma + T(\sigma,\varepsilon,\phi)$. If $b = +\infty$, the solution $x = 0$ of (2.35) is said to be globally attractive.
(iv) The solution $x = 0$ of (2.35) is said to be asymptotically stable if it is both stable and attractive.
(v) The solution $x = 0$ of (2.35) is said to be uniformly attractive if in (iii) b is independent of σ and T only depends on ε.
(vi) The solution $x = 0$ of (2.35) is said to be uniformly asymptotically stable if it is both uniformly stable and uniformly attractive.
(vii) The solution $x = 0$ of (2.35) is said to be globally asymptotically stable if it is stable and globally attractive. □

Definition 2.29. The solution $x = 0$ of (2.35) is said to be exponentially stable if there exists a $\beta > 0$ and, for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that, for any $\sigma \in \mathbb{R}$, $\|\phi\|_\tau < \delta$ implies that

$$\|x(t,\sigma,\phi)\| \le \varepsilon \exp[-\beta(t-\sigma)]$$

for $t \ge \sigma$. The solution $x = 0$ of (2.35) is said to be globally exponentially stable if there exist a $\beta > 0$ and an $\eta > 0$ such that, for any $(\sigma,\phi) \in \mathbb{R} \times C$, the following inequality holds:

$$\|x(t,\sigma,\phi)\| \le \eta\|\phi\|_\tau \exp[-\beta(t-\sigma)]$$

for $t \ge \sigma$. □

Definition 2.30. A solution $x(t,\sigma,\phi)$ of (2.35) is bounded if there is a $\beta(\sigma,\phi)$ such that $\|x(t,\sigma,\phi)\| < \beta(\sigma,\phi)$ for $t \ge \sigma - \tau$. The solution is uniformly bounded if, for any $\alpha > 0$, there is a $\beta = \beta(\alpha) > 0$ such that, for all $\sigma \in \mathbb{R}$, $\phi \in C$, and $\|\phi\| \le \alpha$, we have $\|x(t,\sigma,\phi)\| \le \beta(\alpha)$ for all $t \ge \sigma$. □

In the following, we introduce some sufficient conditions for the stability of the solution $x = 0$ of (2.35) which generalize the second method of Lyapunov for ordinary differential equations.

If the functional $V\colon \mathbb{R} \times C \to \mathbb{R}^+$ is continuous and $x(t, \sigma, \phi)$ is the solution of (2.35) through (σ, ϕ), we define

$$\dot{V}(t, \phi) = \overline{\lim_{h \to 0^+}} \frac{1}{h}[V(t+h, x_{t+h}(t, \phi)) - V(t, \phi)].$$

The function $\dot{V}(t, \phi)$ is the upper right-hand derivative of $V(t, \phi)$ along the solution of (2.35).

If the function $V\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is continuous, we define

$$\dot{V}(t, \phi(0)) = \overline{\lim_{h \to 0^+}} \frac{1}{h}[V(t+h, x(t, \phi)(t+h)) - V(t, \phi(0))].$$

The function $\dot{V}(t, \phi(0))$ is the upper right-hand derivative of $V(t, x)$ along the solution of (2.35).

Sometimes we write $\dot{V}_{(2.35)}(t, \phi)$ and $\dot{V}_{(2.35)}(t, \phi(0))$ to emphasize the dependence on (2.35), respectively.

Theorem 2.22. Suppose that $f\colon \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded set of C) into bounded sets of $\mathbb{R}^n$, and $u, v, w\colon \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous functional $V\colon \mathbb{R} \times C \to \mathbb{R}^+$ such that

$$u(\|\phi(0)\|) \le V(t, \phi) \le v(\|\phi\|_\tau),$$

$$\dot{V}_{(2.35)}(t, \phi) \le -w(\|\phi(0)\|),$$

then the solution $x = 0$ of (2.35) is uniformly stable. If $u(s) \to \infty$ as $s \to \infty$, the solutions of (2.35) are uniformly bounded. If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable. □

Corollary 2.1. Suppose that $f\colon \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded set of C) into bounded sets of $\mathbb{R}^n$, and $u, w\colon \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $w(s)$ are positive for $s > 0$, $u(0) = w(0) = 0$, and $u(s) \to \infty$ $(s \to \infty)$. If there is a continuous functional $V\colon \mathbb{R} \times C \to \mathbb{R}^+$ such that

$$u(\|\phi(0)\|) \le V(t, \phi),\ V(t, 0) = 0,$$

$$\dot{V}_{(2.35)}(t, \phi) \le -w(\|\phi(0)\|),$$

then the solution $x = 0$ of (2.35) is globally asymptotically stable. □

Theorem 2.23. Suppose that $f\colon \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded set of C) into bounded sets of $\mathbb{R}^n$, and $u, v, w\colon \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s), v(s)$, and $w(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. Suppose that $P(s)$ is a continuous and nondecreasing function satisfying $P(s) > s$ for $s > 0$. If there is a continuous function $V\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^+$ such that

$$u(\|x\|) \le V(t,\phi) \le v(\|x\|)$$

and

$$\dot{V}_{(2.35)}(t,\phi(0)) \le -w(\|\phi(0)\|) \ \text{ when } V(t+\theta,\phi(\theta)) < P(V(t,\phi(0))), \theta \in [-\tau,0],$$

then, the solution $x = 0$ of (2.35) is uniformly asymptotically stable. Moreover, if $u(s) \to \infty$ as $s \to \infty$, then the solution $x = 0$ of (2.35) is globally asymptotically stable. □

Theorem 2.24. Suppose that α, β, p, and μ are positive constants. If there is a continuous function $V\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^+$ such that

$$\alpha\|x\|^p \le V(t,x) \le \beta\|x\|^p$$

and

$$\dot{V}_{(2.35)}(t,\phi(0)) \le -\mu V(t,\phi(0)) \text{ when } \sup_{-\tau\le\theta\le 0} [e^{\mu\theta}V(t+\theta,x(t+\theta))] = V(t,x(t)),$$

then, the solution $x = 0$ of (2.35) is globally exponentially stable. □

2.16 Summary

One basic goal in studying dynamical systems is to explore how the trajectories of a system evolve as time proceeds. So, in this chapter, we began with the theorems on existence and uniqueness of solutions of ordinary differential equations. In the subsequent sections we focused on a special class of solutions: equilibrium solutions. Besides that notion, we introduced other definitions such as fixed point, periodic orbit, quasiperiodic orbit, ω-limit set, invariant set, etc., and, furthermore, the key concepts of the book, chaos and chaotic attractors, were introduced. Two powerful tools in studying chaotic systems, Lyapunov exponents and symbolic dynamics, were discussed briefly. The stability issue was discussed and a detailed category of the types of fixed point of planar systems for both continuous time and discrete time was provided. Three famous examples on chaos were carefully presented through which we wanted to give a concrete understanding about chaos. Finally, we provided some necessary preliminaries on retarded functional differential equations for the purpose of self-containment of the book.

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