1
Introduction

1.0. This chapter is a preliminary discussion of *finite-dimensional smooth* (*infinitely differentiable*) *real manifolds*, the main protagonists of this book.

Why are smooth manifolds important?

Well, we live *in* a manifold (a four-dimensional one, according to Einstein) and *on* a manifold (the Earth's surface, whose model is the sphere S^2). We are surrounded by manifolds: The surface of a coffee cup is a manifold (namely, the torus $S^1 \times S^1$, more often described as the surface of a doughnut or an anchor ring, or as the tube of an automobile tire); a shirt is a two-dimensional manifold with boundary.

Processes taking place in nature are often adequately modeled by points moving on a manifold, especially if they involve no discontinuities or catastrophes. (Incidentally, catastrophes — in nature or on the stock market — as studied in "catastrophe theory" may not be manifolds, but then they are smooth maps of manifolds.)

What is more important from the point of view of this book, is that *manifolds arise quite naturally in various branches of mathematics* (in *algebra* and *analysis* as well as in *geometry*) *and its applications* (especially *mechanics*). Before trying to explain what smooth manifolds are, we give some examples.

1.1. The configuration space Rot(3) **of a rotating solid in space.** Consider a solid body in space fixed by a hinge O that allows it to rotate in any direction (Figure 1.1). We want to describe the set of positions of the body, or, as it is called in classical mechanics, its *configuration space*. One way of going about it is to choose a coordinate system $Oxyz$ and

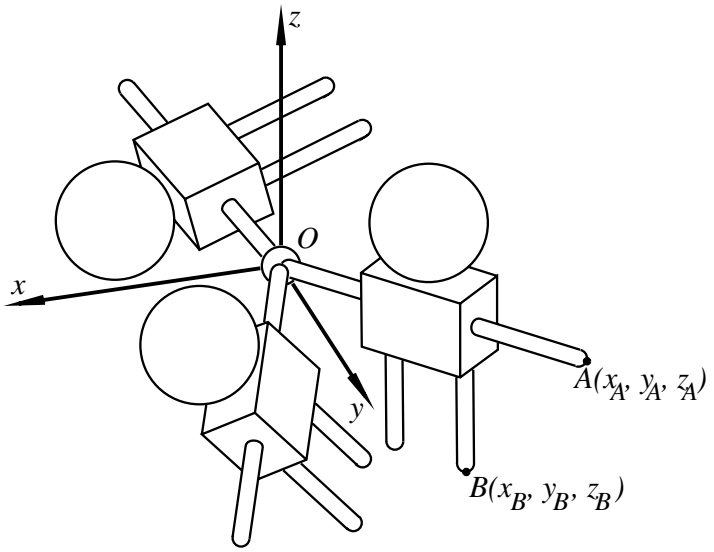

Figure 1.1. Rotating solid.

determine the body's position by the coordinates (x_A, y_A, z_A), (x_B, y_B, z_B) of two of its points A, B. But this is obviously not an economical choice of parameters: It is intuitively clear that only *three* real parameters are required, at least when the solid is not displaced too greatly from its initial position OA_0B_0. Indeed, two parameters determine the direction of OA (e.g., x_A, y_A; see Figure 1.1), and one more is needed to show how the solid is turned about the OA axis (e.g., the angle $\varphi_B = B_0'OB$, where AB_0' is parallel to A_0B_0).

It should be noted that these are not ordinary Euclidean coordinates; the positions of the solid do not correspond bijectively in any natural way to ordinary three-dimensional space $\mathbb{R}^3$. Indeed, if we rotate AB through the angle $\varphi = 2\pi$, the solid does not acquire a new position; it returns to the position OAB; besides, *two* positions of OA correspond to the coordinates (x_A, y_A): For the second one, A is *below* the Oxy plane. However, *locally,* say near the initial position OA_0B_0, there is a bijective correspondence between the position of the solid and a neighborhood of the origin in 3-space $\mathbb{R}^3$, given by the map $OAB \mapsto (x_A, y_A, \varphi_B)$. Thus the configuration space $\mathrm{Rot}(3)$ of a rotating solid is an object that can be described locally by three Euclidean coordinates, but *globally* has a more complicated structure.

1.2. An algebraic surface V. In nine-dimensional Euclidean space $\mathbb{R}^9$ consider the set of points satisfying the following system of six algebraic

equations:

$$\begin{cases} x_1^2+x_2^2+x_3^2=1; & x_1x_4+x_2x_5+x_3x_6=0; \\ x_4^2+x_5^2+x_6^2=1; & x_1x_7+x_2x_8+x_3x_9=0; \\ x_7^2+x_8^2+x_9^2=1; & x_4x_7+x_5x_8+x_6x_9=0. \end{cases}$$

This happens to be a nice three-dimensional surface in $\mathbb{R}^9$ ($3=9-6$). It is not difficult (try!) to describe a bijective map of a neighborhood of any point (say $(1,0,0,0,1,0,0,0,1)$) of the surface onto a neighborhood of the origin of Euclidean 3-space. But this map cannot be extended to cover the entire surface, which is compact (why?). Thus again we have an example of an object V locally like 3-space, but with a different global structure.

It should perhaps be pointed out that the solution set of six algebraic equations with nine unknowns chosen at random will not always have such a simple local structure; it may have self-intersections and other *singularities.* (This is one of the reasons why algebraic geometry, which studies such *algebraic varieties,* as they are called, is not a part of smooth manifold theory.)

1.3. Three-dimensional projective space $\mathbb{R}P^3$. In four-dimensional Euclidean space $\mathbb{R}^4$ consider the set of all straight lines passing through the origin. We want to view this set as a "space" whose "points" are the lines. Each "point" of this space — called *projective space* $\mathbb{R}P^3$ by nineteenth century geometers — is determined by the line's directing vector (a_1,a_2,a_3,a_4), $\sum a_i^2 \neq 0$, i.e., a quadruple of real numbers. Since proportional quadruples define the same line, each point of $\mathbb{R}P^3$ is an equivalence class of proportional quadruples of numbers, denoted by $P=(a_1:a_2:a_3:a_4)$, where (a_1,a_2,a_3,a_4) is any representative of the class. In the vicinity of each point, $\mathbb{R}P^3$ is like $\mathbb{R}^3$. Indeed, if we are given a point $P_0=\left(a_1^0:a_2^0:a_3^0:a_4^0\right)$ for which $a_4^0\neq 0$, it can be written in the form $P_0=\left(a_1^0/a_4^0:a_2^0/a_4^0:a_3^0/a_4^0:1\right)$ and the three ratios viewed as its three coordinates. If we consider all the points P for which $a_4\neq 0$ and take $x_1=a_1/a_4$; $x_2=a_2/a_4$; $x_3=a_3/a_4$ to be their coordinates, we obtain a bijection of a neighborhood of P_0 onto $\mathbb{R}^3$. This neighborhood, together with three similar neighborhoods (for $a_1\neq 0$, $a_2\neq 0$, $a_3\neq 0$), covers all the points of $\mathbb{R}P^3$. But points belonging to more than one neighborhood are assigned to different triples of coordinates (e.g., the point $(6:12:2:3)$ will have the coordinates $\left(2,4,\frac{2}{3}\right)$ in one system of coordinates and $\left(3,6,\frac{3}{2}\right)$ in another). Thus the overall structure of $\mathbb{R}P^3$ is not that of $\mathbb{R}^3$.

1.4. The special orthogonal group $\mathrm{SO}(3)$. Consider the group $\mathrm{SO}(3)$ of orientation-preserving isometries of $\mathbb{R}^3$. In a fixed orthonormal basis, each element $A\in\mathrm{SO}(3)$ is defined by an orthogonal positive definite matrix, thus by nine real numbers ($9=3\times 3$). But of course, fewer than 9 numbers are needed to determine A. In canonical form, the matrix of A

will be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix},$$

and A is defined if we know φ and are given the eigenvector corresponding to the eigenvalue $\lambda = 1$ (two real coordinates a, b are needed for that, since eigenvectors are defined up to a scalar multiplier). Thus again three coordinates (φ, a, b) determine elements of SO(3), and they are Euclidean coordinates only locally.

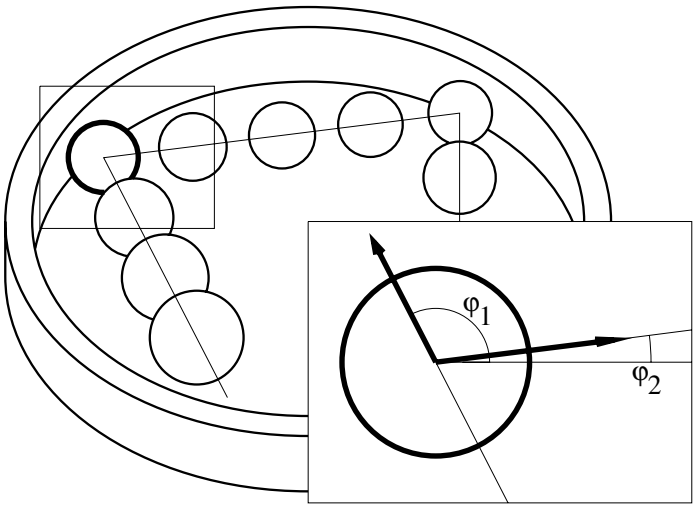

Figure 1.2.

1.5. The phase space of billiards on a disk $\mathcal{B}(D^2)$. A tiny billiard ball P moves with unit velocity in a closed disk D^2, bouncing off its circular boundary C in the natural way (angle of incidence = angle of reflection). We want to describe the *phase space* $\mathcal{B}(D^2)$ of this mechanical system, whose "points" are all the possible *states of the system* (each state being defined by the position of P and the direction of its velocity vector). Since each state is determined by three coordinates $(x, y; \varphi)$ (Figure 1.2), it would seem that as a set, $\mathcal{B}(D^2)$ is $D^2 \times S^1$, where S^1 is the unit circle ($S^1 = \mathbb{R} \bmod 2\pi$). But this is not the case, because at the moment of collision with the boundary, say at (x_0, y_0), the direction of the velocity vector jumps from φ_1 to φ_2 (see Figure 1.2), so that we must identify the states

$$(x_0, y_0, \varphi_1) \equiv (x_0, y_0, \varphi_2). \tag{1.1}$$

Thus $\mathcal{B}(D^2) = (D^2 \times S^1)/\sim$, where $/\sim$ denotes the factorization defined by the equivalence relation of all the identifications (1.1) due to all possible collisions with the boundary C.

Since the identifications take place only on C, all the points of

$$\mathcal{B}^0(D^2) = \operatorname{Int} D^2 \times S^1 = (\operatorname{Int} D^2 \times S^1)/\sim ,$$

where $\operatorname{Int} D^2 = D^2 \smallsetminus C$ is the interior of D^2, have neighborhoods with a structure like that of open sets in $\mathbb{R}^3$ (with coordinates $(x, y; \varphi)$). It is a rather nice fact (not obvious to the beginner) that after identifications the "boundary states" $(x, y; \varphi)$, $(x, y) \in C$, also have such neighborhoods, so that again $\mathcal{B}(D^2)$ is locally like $\mathbb{R}^3$, but not like $\mathbb{R}^3$ globally (as we shall later show).

As a more sophisticated example, the advanced reader might try to describe the phase space of billiards in a right triangle with an acute angle of (a) $\pi/6$; (b) $\sqrt{2}\pi/4$.

1.6. The five examples of three-dimensional manifolds described above all come from different sources: classical mechanics 1.1, algebraic geometry 1.2, classical geometry 1.3, linear algebra 1.4, and mechanics 1.5. The advanced reader has not failed to notice that 1.1–1.4 are actually examples of *one and the same manifold* (appearing in different garb):

$$\operatorname{Rot}(3) = V = \mathbb{R}P^3 = \operatorname{SO}(3).$$

To be more precise, the first four manifolds are all "diffeomorphic," i.e., equivalent as smooth manifolds (the definition is given in Section 6.7). As for Example 1.5, $\mathcal{B}(D^2)$ differs from (i.e., is not diffeomorphic to) the other manifolds, because it happens to be diffeomorphic to the three-dimensional sphere S^3 (the beginner should not be discouraged if he fails to see this; it is not obvious).

What is the moral of the story? The history of mathematics teaches us that if the same object appears in different guises in various branches of mathematics and its applications, and plays an important role there, then it should be studied intrinsically, as a separate concept. That was what happened to such fundamental concepts as *group* and *linear space,* and is true of the no less important concept of *smooth manifold.*

1.7. The examples show us that a manifold M is a point set locally like Euclidean space $\mathbb{R}^n$ with global structure not necessarily that of $\mathbb{R}^n$. How does one go about studying such an object? Since there are Euclidean coordinates near each point, we can try to cover M with *coordinate neighborhoods* (or *charts,* or *local coordinate systems*, as they are also called). A family of charts covering M is called an *atlas.* The term is evocative; indeed, a geographical atlas is a set of charts or maps of the manifold S^2 (the Earth's surface) in that sense.

In order to use the separate charts to study the overall structure of M, we must know how to move from one chart to the next, thus "gluing together"

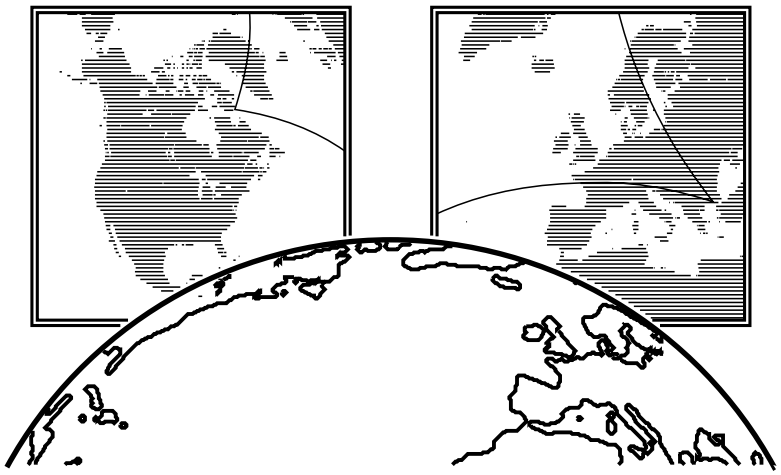

Figure 1.3.

the charts along their common parts, so as to recover M (see Figure 1.3). In less intuitive language, we must be in possession of *coordinate transformations,* expressing the coordinates of points of any chart in terms of those of a neighboring chart. (The industrious reader might profit by actually writing out these transformations for the case of the four-charts atlas of $\mathbb{R}P^3$ described in 1.3.)

If we wish to obtain a *smooth* manifold in this way, we must require that the coordinate transformations be "nice" functions (in a certain sense). We then arrive at the *coordinate* or *classical approach* to smooth manifolds. It is developed in detail in Chapter 5.

1.8. Perhaps more important is the *algebraic approach* to the study of manifolds. In it we forget about charts and coordinate transformations and work only with the $\mathbb{R}$-*algebra* $\mathcal{F}_M$ of *smooth functions* $f\colon\ M \to \mathbb{R}$ on the manifold M. It turns out that $\mathcal{F}_M$ entirely determines M and is a convenient object to work with.

An attempt to give the reader an intuitive understanding of the natural philosophy underlying the algebraic approach is undertaken in the next sections.

1.9. In the description of a classical *physical system* or process, the key notion is the *state* of the system. Thus, in classical mechanics, the state of a moving point is described by its position and velocity at the given moment of time. The state of a given gas from the point of view of thermodynamics is described by its temperature, volume, and pressure, etc. In order to actually assess the state of a given system, the experimentalist must use various *measuring devices* whose *readings* describe the state.

Suppose M is the set of all states of the classical physical system S. Then to each measuring device D there corresponds a function f_D on the

set M, assigning to each state $s \in M$ the reading $f_D(s)$ (a real number) that the device D yields in that state. From the physical point of view, we are interested only in those characteristics of each state that can be measured in principle, so that the set M of all states is described by the collection Φ_S of all functions f_D, where the D's are measuring devices (possibly imaginary ones, since it is not necessary — nor indeed practically possible — to construct all possible measuring devices). Thus, theoretically, *a physical system S is nothing more that the collection Φ_S of all functions determined by adequate measuring devices* (*real or imagined*) *on S.*

1.10. Now, if the functions $f_1, \dots, f_k$ correspond to the measuring devices $D_1, \dots, D_k$ of the physical system S, and $\varphi(x_1, \dots, x_k)$ is any "nice" real-valued function in k real variables, then in principle it is possible to construct a device D such that the corresponding function f_D is the composite function $\varphi(f_1, \dots, f_k)$. Indeed, such a device may be obtained by constructing an auxiliary device, synthesizing the value $\varphi(x_1, \dots, x_k)$ from input entries $x_1, \dots, x_k$ (this can always be done if φ is nice enough), and then "plugging in" the outputs $(f_1, \dots, f_k)$ of the devices $D_1, \dots, D_k$ into the inputs $(x_1, \dots, x_k)$ of the auxiliary device. Let us denote this device D by $\varphi(D_1, \dots, D_k)$.

In particular, if we take $\varphi(x_1, x_2) = x_1 + x_2$ (or $\varphi(x) = \lambda x$, $\lambda \in \mathbb{R}$, or $\varphi(x_1, x_2) = x_1 x_2$), we can construct the devices $D_1 + D_2$ (or λD_i, or $D_1 D_2$) from any given devices D_1, D_2. In other words, if $f_i = f_{D_i} \in \Phi_S$, then the functions $f_1 + f_2$, λf_i, $f_1 f_2$ also belong to Φ_S.

Thus the set Φ_S of all functions $f = f_D$ describing the system S has the structure of an *algebra over* $\mathbb{R}$ (or $\mathbb{R}$-*algebra*).

1.11. Actually, the set Φ_S of *all* functions $f_D : M_S \to \mathbb{R}$ is much too large and cumbersome for most classical problems. Systems (and processes) described in classical physics are usually continuous or smooth in some sense. Discontinuous functions f_D are irrelevant to their description; only "smoothly working" measuring devices D are needed. Moreover, the problems of classical physics are usually set in terms of differential equations, so that we must be able to take derivatives of the relevant functions from Φ_S as many times as we wish. Thus we are led to consider, rather than Φ_S, the smaller set $\mathcal{F}_S$ of *smooth functions* $f_D \colon M_S \to \mathbb{R}$.

The set $\mathcal{F}_S$ inherits an $\mathbb{R}$-algebra structure from the inclusion $\mathcal{F}_S \subset \Phi_S$, but from now on we shall forget about Φ, since the *smooth $\mathbb{R}$-algebra $\mathcal{F}_S$* will be our main object of study.

1.12. Let us describe in more detail what the algebra $\mathcal{F}_S$ might be like in classical situations. For example, from the point of view of classical mechanics, a system S of N points in space is adequately described by the positions and velocities of the points, so that we need $6N$ measuring devices D_i to record them. Then the algebra $\mathcal{F}_S$ consists of all elements of the form $\varphi(f_1, \dots, f_{6N})$, where the f_i are the "basic functions" determined by the devices D_i, while $\varphi \colon \mathbb{R}^{6N} \to \mathbb{R}$ is any nice (smooth) function.

In more complicated situations, certain *relations* among the basis functions f_i may arise. For example, if we are studying a system of two mass points joined by a rigid rod of negligible mass, we have the relation

$$\sum_{i=1}^{3}(f_i - f_{i+3})^2 = r^2,$$

where r is the length of the rod and the functions f_i (respectively f_{i+3}) measure the ith coordinate of the first (respectively second) mass point. (There is another relation for the velocity components, which the reader might want to write out explicitly.)

Generalizing, we can say that there usually exists a basis system of devices $D_1, \dots, D_k$ adequately describing the system S (from the chosen point of view). Then the $\mathbb{R}$-algebra $\mathcal{F}_S$ consists of all elements of the form $\varphi(f_1, \dots, f_k)$, where $\varphi\colon \mathbb{R}^k \to \mathbb{R}$ is a nice function and the $f_i = f_{D_i}$ are the relevant measurements (given by the devices D_i) that may be involved in relations of the form $F(f_1, \dots, f_k) \equiv 0$.

Then $\mathcal{F}_S$ may be described as follows. Let $\mathbb{R}^k$ be Euclidean space with coordinates $f_1, \dots, f_k$ and $U = \{(f_1, \dots, f_k) \mid a_i < f_i < b_i\}$, where the open intervals $]a_i, b_i[$ contain all the possible readings given by the devices D_i. The relations $F_j(f_1, \dots, f_k) = 0$ between the basis variables $f_1, \dots, f_k$ determine a surface M in U. Then $\mathcal{F}_S$ *is the* $\mathbb{R}$*-algebra of all smooth functions on the surface* M.

1.13. Example (thermodynamics of an ideal gas). Consider a certain volume of ideal gas. From the point of view of thermodynamics, we are interested in the following measurements: the volume V, the pressure p, and the absolute temperature T of the gas. These parameters, as is well known, satisfy the relation $pV = cT$, where c is a certain constant. Since $0 < p < \infty$, $0 < V < \infty$, and $0 < T < \infty$, the domain U is the first octant in the space $\mathbb{R}^3 \ni (V, p, T)$, and the hypersurface M in this domain is given by the equation $pV = cT$. The relevant $\mathbb{R}$-algebra $\mathcal{F}$ consists of all smooth functions on M.

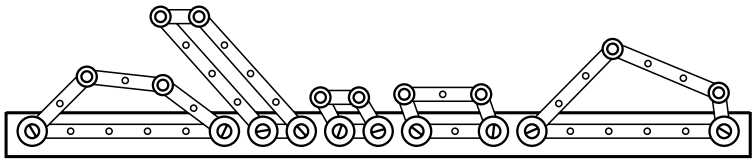

Figure 1.4. Hinge mechanisms $(5; 2, 2, 2)$, $(1; 4, 1, 4)$, $(1; 1, 1, 1)$, $(2; 1, 2, 1)$, $(5; 3, 3, 1)$.

1.14. Example (plane hinge mechanisms). Such a mechanism (see Figure 1.4) consists of $n > 3$ ideal rods in the plane of lengths, say, $(l_1; l_2, \dots, l_n)$; the rods are joined in cyclic order to each other by ideal hinges at their

endpoints; the hinges of the first rod (and hence the rod itself) are fixed to the plane; the other hinges and rods move freely (insofar as the configuration allows them to); the rods can sweep freely over ("through") each other. Obviously, the configuration space of a hinge mechanism is determined completely by the sequence of lengths of its rods. So, one can refer to a concrete mechanism just by indicating the corresponding sequence, for instance, $(5; 2, 3, 2)$. The reader is invited to solve the following problems in the process of reading the book. The first of them she/he can attack even now.

Exercise. Describe the configuration spaces of the following hinge mechanisms:

1. *Quadrilaterals*: $(5; 2, 2, 2)$; $(1; 4, 1, 4)$; $(1; 1, 1, 1)$; $(2; 1, 2, 1)$; $(5; 3, 3, 1)$.
2. *Pentagons*: $(3.9; 1, 1, 1, 1)$; $(1; 4, 1, 1, 4)$; $(6; 6, 2, 2, 6)$; $(1; 1, 1, 1, 1)$.

The reader will enjoy discovering that the configuration space of $(1; 1, 1, 1, 1)$ is the sphere with four handles.

Exercise. Show that the configuration space of a pentagon depends only on the set of lengths of the rods and not on the order in which the rods are joined to each other.

Exercise. Show that the configuration space of the hinge mechanism $(n - \alpha; 1, \dots, 1)$ consisting of $n + 1$ rods is:

1. The sphere S^{n-2} if $\alpha = \frac{1}{2}$.
2. The $(n-2)$-dimensional torus $T^{n-2} = S^1 \times \cdots \times S^1$ if $\alpha = \frac{3}{2}$.

1.15. So far we have not said anything to explain what a state $s \in M_S$ of our physical system S really is, relying on the reader's physical intuition. But once the set of relevant functions $\mathcal{F}_S$ has been specified, this can easily be done in a mathematically rigorous and physically meaningful way.

The methodological basis of physical considerations is measurement. Therefore, two states of our system must be considered identical if and only if all the relevant measuring devices yield the same readings. Hence each state $s \in M_S$ is entirely determined by the readings in this state on all the relevant measuring devices, i.e., by the correspondence $\mathcal{F}_S \to \mathbb{R}$ assigning to each $f_D \in \mathcal{F}_S$ its reading (in the state s) $f_D(s) \in \mathbb{R}$. This assignment will clearly be an $\mathbb{R}$-algebra homomorphism. Thus we can say, by definition, that *any state s of our system is simply an $\mathbb{R}$-algebra homomorphism $s\colon \mathcal{F}_S \to \mathbb{R}$*. The set of all $\mathbb{R}$-algebra homomorphisms $\mathcal{F}_S \to \mathbb{R}$ will be denoted by $|\mathcal{F}_S|$; it should coincide with the set M_S of all states of the system.

1.16. Summarizing Sections 1.9–1.15, we can say that *any classical physical system is described by an appropriate collection of measuring devices*,

each state of the system being the collection of readings that this state determines on the measuring devices.

The sentence in italics may be translated into mathematical language by means of the following dictionary:

- physical system = manifold, M;
- state of the system = point of the manifold, $x \in M$;
- measuring device = function on M, $f \in \mathcal{F}$;
- adequate collection of measuring devices = smooth $\mathbb{R}$-algebra, $\mathcal{F}$;
- reading on a device = value of the function, $f(x)$;
- collection of readings in the given state = $\mathbb{R}$-algebra homomorphism

$$x\colon \mathcal{F} \to \mathbb{R}, \quad f \mapsto f(x).$$

The resulting translation reads: *Any manifold M is determined by the smooth $\mathbb{R}$-algebra $\mathcal{F}$ of functions on it, each point x on M being the $\mathbb{R}$-algebra homomorphism $\mathcal{F} \to \mathbb{R}$ that assigns to every function $f \in \mathcal{F}$ its value $f(x)$ at the point x.*

1.17. Mathematically, the crucial idea in the previous sentence is the identification of points $x \in M$ of a manifold and $\mathbb{R}$-algebra homomorphisms $x\colon \mathcal{F} \to \mathbb{R}$ of its $\mathbb{R}$-algebra of functions $\mathcal{F}$, governed by the formula

$$x(f) = f(x). \tag{1.2}$$

This formula, read from left to right, defines the homomorphism $x\colon \mathcal{F} \to \mathbb{R}$ when the functions $f \in \mathcal{F}$ are given. Read from right to left, it defines the functions $f\colon M \to \mathbb{R}$, when the homomorphisms $x \in M$ are known.

Thus formula (1.2) is right in the middle of the important duality relationship existing between points of a manifold and functions on it, a duality similar to, but much more delicate than, the one between vectors and covectors in linear algebra.

1.18. In the general mathematical situation, the identification $M \leftrightarrow |\mathcal{F}|$ between the set M on which the functions $f \in \mathcal{F}$ are defined and the family of all $\mathbb{R}$-algebra homomorphisms $\mathcal{F} \to \mathbb{R}$ cannot be correctly carried out. This is because, first of all, $|\mathcal{F}|$ may turn out to be "much smaller" than M (an example is given in Section 3.6) or "bigger" than M, as we can see from the following example:

Example. Suppose M is the set $\mathbb{N}$ of natural numbers and $\mathcal{F}$ is the set of all functions on $\mathbb{N}$ (i.e., sequences $\{a(k)\}$) such that the limit $\lim_{k\to\infty} a(k)$ exists and is finite. Then the homomorphism

$$\alpha\colon \mathcal{F} \to \mathbb{R}, \quad \{a(k)\} \mapsto \lim_{k\to\infty} a(k),$$

does not correspond to any point of $M = \mathbb{N}$.

◀ Indeed, if α did correspond to some point $n \in \mathbb{N}$, we would have by (1.2)

$$n(a(\cdot)) = a(n),$$

so that

$$\lim_{k\to\infty} a(k) = \alpha(a(\cdot)) = n(a(\cdot)) = a(n)$$

for any sequence $\{a(k)\}$. But this is not the case, say, for the sequence $a_i = 0$, $i \leqslant n$, $a_i = 1$, $i > n$. Thus $|\mathcal{F}|$ is bigger than M, at least by the homomorphism α. ▶

However, we can always add to $\mathbb{N}$ the "point at infinity" ∞ and extend the sequences (elements of $\mathcal{F}$) by putting $a(\infty) = \lim_{k\to\infty} a(k)$, thus viewing the sequences in $\mathcal{F}$ as functions on $\mathbb{N} \cup \{\infty\}$. Then obviously the homomorphism above corresponds to the "point" ∞.

This trick of adding *points at infinity* (or imaginary points, improper points, points of the absolute, etc.) is extremely useful and will be exploited to great advantage in Chapter 8.

1.19. In our mathematical development of the algebraic approach (Chapter 3) we shall start from an $\mathbb{R}$-algebra $\mathcal{F}$ of abstract elements called "functions." Of course, $\mathcal{F}$ will not be just any algebra; it must meet certain "smoothness" requirements. Roughly speaking, the algebra $\mathcal{F}$ must be smooth in the sense that locally (the meaning of that word must be defined in abstract algebraic terms!) it is like the $\mathbb{R}$-algebra $C^\infty(\mathbb{R}^n)$ of infinitely differentiable functions in $\mathbb{R}^n$. This will be the algebraic way of saying that the manifold M is locally like $\mathbb{R}^n$; it will be explained rigorously and in detail in Chapter 3. When the smoothness requirements are met, it will turn out that $\mathcal{F}$ entirely determines the manifold M as the set $|\mathcal{F}|$ of all $\mathbb{R}$-algebra homomorphisms of $\mathcal{F}$ into $\mathbb{R}$, and $\mathcal{F}$ can be identified with the $\mathbb{R}$-algebra of smooth functions on M. The algebraic definition of smooth manifold appears in the first section of Chapter 4.

1.20. Smoothness requirements are also needed in the classical coordinate approach, developed in detail below (see Chapter 5). In particular, coordinate transformations must be infinitely differentiable. The rigorous coordinate definition of a smooth manifold appears in Section 5.8.

1.21. The two definitions of smooth manifold (in which the algebraic approach and the coordinate approach result) are of course equivalent. This is proved in Chapter 7 below. Essentially, this book is a detailed exposition of these two approaches to the notion of smooth manifold and their equivalence, involving many examples, including a more rigorous treatment of the examples given in Sections 1.1–1.5 above.