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Jordan Algebras in the Algebraic Renaissance: Finite-Dimensional Jordan Algebras over Algebraically Closed Fields

The next stage in the history of Jordan algebras was taken over by algebraists. While the physicists lost interest in the search for an exceptional setting for quantum mechanics (the philosophical objections to the theory paling in comparison to its amazing achievements), the algebraists found unsuspected relations between, on the one hand, the strange exceptional simple Albert algebra of dimension 27 and, on the other hand, the five equally strange exceptional simple Lie groups and algebras of types G_2, F_4, E_6, E_7, E_8 of dimensions 14, 52, 78, 133, 248. While these had been discovered by Wilhelm Killing and Elie Cartan in the 1890s, they were known only through their multiplication tables: there was no concrete representation for them (the way there was for the four great classes A_n, B_n, C_n, D_n discovered by Sophus Lie in the 1870s). During the 1930s Jacobson discovered that the Lie group G_2 could be realized as the automorphism group (and the Lie algebra G_2 as the derivation algebra) of a Cayley algebra, and in the early 1950s Chevalley, Schafer, Freudenthal, and others discovered that the Lie group F_4 could be realized as the automorphism group (and the Lie algebra F_4 as the derivation algebra) of the Albert algebra, that the group E_6 could be realized as the isotopy group (and the algebra E_6 as the structure algebra) of the Albert algebra, and that the algebra E_7 could be realized as the superstructure Lie algebra of the Albert algebra. [E_8 was connected to the Albert algebra in a more complicated manner.]

These unexpected connections between the physicists' orphan child and other important areas of mathematics, spurred algebraists to consider Jordan algebras over more general fields. By the late 1940s the J–vN–W structure theory had been extended by A.A. Albert, F. and N. Jacobson, and others to finite-dimensional Jordan algebras over an arbitrary algebraically closed field of characteristic not 2, with essentially the same cast of characters appearing in the title roles.

2.1 Linear Algebras over General Scalars

We begin our algebraic history by recalling the basic categorical concepts for general nonassociative algebras OVER AN ARBITRARY RING OF SCALARS Φ. When dealing with Jordan algebras we will have to assume that $\frac{1}{2} \in \Phi$, and we will point this out explicitly. An *algebra* is simultaneously a ring and a module over a ring of scalars Φ, such that the ring multiplication interacts correctly with the linear structure.

Linear Algebra Definition. *A* **ring of scalars** *is a unital commutative associative ring* Φ. *A (nonassociative)* **linear algebra** *over* Φ *(or* Φ*-algebra, for short) is a* Φ*-module* A *equipped with a* Φ*-bilinear product* $\mathrm{A} \times \mathrm{A} \longrightarrow \mathrm{A}$ *(abbreviated by juxtaposition* $(x, y) \mapsto xy$*). Bilinearity is equivalent to the condition that the product satisfies the left and right distributive laws*

$$x(y+z) = xy + xz, \quad (y+z)x = yx + zx,$$

and that scalars flit in and out of products,

$$(\alpha x)y = x(\alpha y) = \alpha(xy),$$

for all elements x, y, z *in* A. *The algebra is* **unital** *if there exists a (two-sided)* **unit element** 1 *satisfying* $1x = x1 = x$ *for all* x.

Notice that we do not require associativity of the product nor existence of a unit element in A (though we always demand a unit scalar in Φ). Lack of a unit is easy to repair: we can always enlarge a linear algebra slightly to get a unital algebra.

Unital Hull Definition. *Any linear algebra can be imbedded as an ideal in its* **unital hull**

$$\widehat{\mathrm{A}} := \Phi\hat{1} \oplus \mathrm{A}, \quad (\alpha\hat{1} \oplus x)(\beta\hat{1} \oplus y) := \alpha\beta\hat{1} \oplus (\alpha y + \beta x + xy).$$

A is always an ideal in $\widehat{\mathrm{A}}$ since multiplication by the new elements $\alpha\hat{1}$ are just scalar multiplications; this means that we can often conveniently formulate results inside A making use of its unital hull. For example, in an associative algebra the left ideal $Ax + \Phi x$ generated by an element x can be written succinctly as $\widehat{\mathrm{A}}x$ (the left ideal $\mathrm{A}x$ needn't contain x if A is not already unital).

2.2 Categorical Nonsense

We have the usual notions of morphisms, sub-objects, and quotients for linear algebras.

Morphism Definition. *A* **homomorphism** $\varphi : \mathrm{A} \to \mathrm{A}'$ *is a linear map of* Φ*-modules which preserves multiplication,*

$$\varphi(xy) = \varphi(x)\varphi(y);$$

an **anti-homomorphism** *is a linear map which reverses multiplication,*

$$\varphi(xy) = \varphi(y)\varphi(x).$$

The kernel $\mathrm{Ker}(\varphi) := \varphi^{-1}(0')$ *is the set of elements mapped into* $0' \in \mathrm{A}'$, *and the* image $\mathrm{Im}(\varphi) := \varphi(\mathrm{A})$ *is the range of the map. An* **isomorphism** *is a bijective homomorphism; we say that* A *is* **isomorphic to** A', *or* A *and* A' **are isomorphic** (*written* $\mathrm{A} \cong \mathrm{A}'$), *if there is an isomorphism of* A *onto* A'. *An* **automorphism** *is an isomorphism of an algebra with itself. We have corresponding notions of* **anti-isomorphism** *and* **anti-automorphism** *for anti-homomorphisms.*

$*$**-Algebra Definition**. *An* **involution** *is an anti-automorphism of period* 2,

$$\varphi(xy) = \varphi(y)\varphi(x) \quad \textit{and} \quad \varphi(\varphi(x)) = x.$$

We will often be concerned with involutions, since they are a rich source of Jordan algebras. The natural notion of morphism in the category of $*$**-algebras** (*algebras together with a choice of involution*) *is that of* $*$**-homomorphism** $(\mathrm{A}, *) \to (\mathrm{A}', *')$, *which is a homomorphism* $\varphi : \mathrm{A} \to \mathrm{A}'$ *of algebras which preserves the involutions,* $\varphi \circ * = *' \circ \varphi$ (*i.e.,* $\varphi(x^*) = \varphi(x)^{*'}$ *for all* $x \in \mathrm{A}$).

One important involution is the standard involution on a quaternion or octonion algebra.

Ideal Definition. *A* **subalgebra** $\mathrm{B} \leq \mathrm{A}$ *of a linear algebra* A *is a* Φ*-submodule closed under multiplication:* $\mathrm{BB} \subseteq \mathrm{B}$. *An* **ideal** $B \triangleleft \mathrm{A}$ *of* A *is a* Φ*-submodule closed under left and right multiplication by* A: $\mathrm{AB} + \mathrm{BA} \subseteq \mathrm{B}$. *If* A *has an involution, a* $*$**-ideal** *is an ideal invariant under the involution:* $\mathrm{B} \triangleleft \mathrm{A}$ *and* $\mathrm{B}^* \subseteq \mathrm{B}$. *We will always use* $\mathbf{0}$ *to denote the zero* submodule, *while ordinary* 0 *will denote the zero* scalar, vector, *or* transformation (*context will decide which is meant*).

Quotient Definition. *Any ideal* $\mathrm{B} \triangleleft \mathrm{A}$ *is the kernel of the* canonical homomorphism $\pi : x \mapsto \overline{x}$ *of* A *onto the* **quotient algebra** $\overline{\mathrm{A}} = \mathrm{A}/\mathrm{B}$ (*consisting of all cosets* $\overline{x} := [x]_{\mathrm{B}} := x + \mathrm{B}$ *with the induced operations* $\alpha\overline{x} := \overline{\alpha x}$, $\overline{x} + \overline{y} := \overline{x + y}$, $\overline{x}\ \overline{y} := \overline{xy}$). *The quotient* A/B *of a* $*$*-algebra by a* $*$*-ideal is again a* $*$*-algebra under the induced involution* $\overline{x}^* := \overline{x^*}$.

We have the usual tripartite theorem relating homomorphisms and quotients.

Fundamental Theorem of Homomorphisms. *For homomorphisms (and similarly for $*$-homomorphisms) we have:*

(I) *If $\varphi : A \to A'$ is a homomorphism, then* $\mathrm{Ker}(\varphi) \triangleleft \mathrm{A}$, $\mathrm{Im}(\varphi) \leq \mathrm{A}'$, *and* $\mathrm{A}/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi)$ *under the map* $\overline{\varphi}(\overline{x}) := \varphi(x)$.

(II) *There is a* 1*-to-*1 *correspondence between the ideals (respectively subalgebras)* $\overline{\mathrm{C}}$ *of the quotient* $\overline{\mathrm{A}} = \mathrm{A}/\mathrm{B}$ *and those* C *of* A *which contain* B, *given by* $\mathrm{C} \mapsto \pi(\mathrm{C})$ *and* $\overline{\mathrm{C}} \mapsto \pi^{-1}(\overline{\mathrm{C}})$; *for such ideals* C *we have* $\overline{\mathrm{A}}/\overline{\mathrm{C}} \cong \mathrm{A}/\mathrm{C}$.

(III) *If* $\mathrm{B} \triangleleft \mathrm{A}$, $\mathrm{C} \leq \mathrm{A}$ *then* $\mathrm{C}/(\mathrm{C} \cap \mathrm{B}) \cong (\mathrm{C} + \mathrm{B})/\mathrm{B}$ *under the map* $\varphi([x]_{\mathrm{C}\cap\mathrm{B}}) = [x]_{\mathrm{B}}$.

As usual, we have a way of gluing different algebras together as a direct sum in such a way that the individual pieces don't interfere with each other.

Direct Sum Definition. *The* **direct sum** $\mathrm{A}_1 \boxplus \cdots \boxplus \mathrm{A}_n$ *of a finite number of algebras is the Cartesian product* $\mathrm{A}_1 \times \cdots \times \mathrm{A}_n$ *under the* componentwise operations

$$\begin{aligned} \alpha(x_1, \ldots, x_n) &:= (\alpha x_1, \ldots, \alpha x_n), \\ (x_1, \ldots, x_n) + (y_1, \ldots, y_n) &:= (x_1 + y_1, \ldots, x_n + y_n), \\ (x_1, \ldots, x_n)(y_1, \ldots, y_n) &:= (x_1 y_1, \ldots, x_n y_n). \end{aligned}$$

We will consistently write an algebra *direct sum with* $\boxplus$, *and a mere* module *direct sum with* $\oplus$.

In algebras with finiteness conditions we only need to consider finite direct sums of algebras. Direct sums are the most useful (but rarest) "rigid" decompositions, and are the goal of many structure theories. In the wide-open spaces of the infinite-dimensional world, direct sums (finite or otherwise) do not suffice, and we need to deal with infinite direct products.

Direct Product Definition. *The* **direct product** $\prod_{i \in I} \mathrm{A}_i$ *of an arbitrary family of algebraic systems* A_i *indexed by a set* I *is the Cartesian product* $\times_{i \in I} \mathrm{A}_i$ *under the componentwise operations. We may think of this as all "strings"* $a = \prod a_i$ *or "I-tuples"* $a = (\ldots a_i \ldots)$ *of elements, one from each family member* A_i, *or more rigorously as all maps* $a : I \to \cup_i A_i$ *such that* $a(i) \in \mathrm{A}_i$ *for each* i, *under the pointwise operations.*

The **direct sum** $\boxplus_{i \in I} \mathrm{A}_i$ *is the subalgebra of all tuples with only a finite number of nonzero entries (so they can be represented as a finite sum* $a_{i_1} + \cdots + a_{i_n}$ *of elements* $a_j \in \mathrm{A}_j$*). For each* $i \in I$ *we have* canonical projections π_i *of both the direct product and direct sum onto the* i*th component* A_i.[1]

[1] The direct product and sum are often called the *product* $\prod \mathrm{A}_i$ and *coproduct* $\coprod \mathrm{A}_i$; especially in algebraic topology, these appear as "dual" objects. For finite index sets, the two concepts coincide.

Infinite direct products are complicated objects, more topological or combinatorial than algebraic. They play a crucial role in the "logic" of algebras through the construction of ultraproducts. Semi-direct product decompositions are fairly "loose," but are crucial in the study of radicals.

Subdirect Product Definition. *An algebra is a* **subdirect product** $\mathrm{A} \precapprox \prod_{i\in I} \mathrm{A}_i$ *of algebras (more often and more inaccurately called a* **semidirect sum**) *if there is* (1) *a monomorphism* $\varphi : \mathrm{A} \to \prod_{i\in I} \mathrm{A}_i$ *such that* (2) *for each* $i \in I$ *the canonical projection* $\pi_i(\varphi(\mathrm{A})) = \mathrm{A}_i$ *maps onto all of* A_i*. By the Fundamental Theorem,* (2) *is equivalent to* $\mathrm{A}_i \cong \mathrm{A}/\mathrm{K}_i$ *for an ideal* $\mathrm{K}_i \triangleleft \mathrm{A}$*, and* (1) *is equivalent to* $\bigcap_I \mathrm{K}_i = \mathbf{0}$*. Thus a semi-direct product decomposition of* A *is essentially the same as a "disjoint" family of ideals.*

For example, the integers $\mathbb{Z}$ are a subdirect product of fields $\mathbb{Z}_p$ for any infinite collection of primes p, and even of $\mathbb{Z}_{p^n}$ for a fixed p but infinitely many n.

In a philosophical sense, an algebra A can be recovered from an ideal B and its quotient A/B. The basic building blocks are those algebras which cannot be built up from smaller pieces, i.e., have no smaller ingredients B.

Simple Definition. *An algebra is* **simple** *if it has no proper ideals and is not trivial,* $\mathrm{AA} \neq \mathbf{0}$*. Analogously, a* $*$*-algebra is* $*$-**simple** *if it has no proper* $*$*-ideals and is not trivial. Here a submodule* B *is* proper *if it is not zero or the whole module,* $\mathrm{B} \neq \mathbf{0}, \mathrm{A}$*. An algebra is* **semisimple** *if it is a finite direct sum of simple algebras.*

2.3 Commutators and Associators

We can reformulate the algebra conditions in terms of the **left and right multiplication operators** L_x and R_x by the element x, defined by

$$L_x(y) := xy =: R_y(x).$$

Bilinearity of the product just means the map $L : x \mapsto L_x$ (or equivalently the map $R : y \mapsto R_y$) is a linear mapping from the Φ-module A into the Φ-module $\mathcal{E}nd_\Phi(\mathrm{A})$ of Φ-linear transformations on A.

The product (and the algebra) is **commutative** if $xy = yx$ for all x, y, and **skew** if $xy = -yx$ for all x, y; in terms of operators, commutativity means that $L_x = R_x$ for all x, and skewness means that $L_x = -R_x$ for all x, so in either case we can dispense with the right multiplications and work only with the L_x. In working with the commutative law, it is convenient to introduce the **commutator**

$$[x, y] := xy - yx,$$

which measures how far two elements are from commuting: x and y commute iff their commutator is zero. In these terms the commutative law is $[x, y] = 0$, so an algebra is commutative iff all commutators vanish.

The product (and the algebra) is **associative** if $(xy)z = x(yz)$ for all x, y, z, in which case we drop all parentheses and write the product as xyz. We can interpret associativity in three ways as an operator identity, depending on which of x, y, z we treat as the variable: on z it says that $L_{xy} = L_x L_y$, i.e., that L is a homomorphism of A into $\mathcal{E}nd_\Phi(\mathrm{A})$; on x it says that $R_z R_y = R_{yz}$, i.e., that R is an anti-homomorphism; on y it says that $R_z L_x = L_x R_z$, i.e., that all left multiplications L_x commute with all right multiplications R_z.[2] It is similarly convenient to introduce the **associator**

$$[x, y, z] := (xy)z - x(yz),$$

which measures how far three elements are from associating: x, y, z associate iff their associator is zero. In these terms an algebra is associative iff all its associators vanish, and the Jordan identity becomes $[x^2, y, x] = 0$.

Nonassociativity can never be repaired, it is an incurable illness. Instead, we can focus on the parts of an algebra which do behave associatively. The **nucleus** $\mathcal{N}uc(\mathrm{A})$ of a linear algebra A is the part which "associates" with all other elements, the elements n which hop blithely over parentheses:

$$\mathcal{N}uc(\mathrm{A}): \ (nx)y = n(xy), \ (xn)y = x(ny), \ (xy)n = x(yn)$$

for all x, y in A. In terms of associators, nuclear elements are those which vanish when put into an associator,

$$\mathcal{N}uc(\mathrm{A}) := \{n \in \mathrm{A} \mid [n, \mathrm{A}, \mathrm{A}] = [\mathrm{A}, n, \mathrm{A}] = [\mathrm{A}, \mathrm{A}, n] = \mathbf{0}\}.$$

Nuclear elements will play a role in several situations (such as forming nuclear isotopes, or considering involutions whose hermitian elements are all nuclear). The associative ring theorist Jerry Martindale offers this advice for proving theorems about nonassociative algebras: never multiply more than two elements together at a time. We can extend this secret for success even further: when multiplying n elements together, make sure that at least n–2 of them belong to the nucleus!

Another useful general concept is that of the **center** $\mathcal{C}ent(\mathrm{A})$, the set of elements c which both commute and associate, and therefore act like scalars:

$$\mathcal{C}ent(\mathrm{A}): \ cx = xc, \ c(xy) = (cx)y = x(cy),$$

[2] Most algebraists of yore were right-handed, i.e., they wrote their maps on the right: a linear transformation T on V had values xT, the matrix of T with respect to an ordered basis was built up row by row, and composition $S \circ T$ meant first do S and then T. For them, the natural multiplication was $R_y, xR_y = xy$. Modern algebraists are all raised as left-handers, writing maps on the left ($f(x)$ instead of xf), as learned in the calculus cradle, and building matrices column by column. Whichever hand you use, in dealing with modules over noncommutative rings of scalars it is important to *keep the scalars on the opposite side of the vectors from the operators*, so linear maps have either $T(x\alpha) = (Tx)\alpha$ or $(\alpha x)T = \alpha(xT)$. Since the dual V^* of a left (resp. right) vector space V over a noncommutative division algebra Δ is a *right* (resp. *left*) vector space over Δ, it is important to be ambidextrous, writing a linear map as $T(x)$ on V, but its adjoint as $(x^*)T^*$ on the dual.

or in terms of associators and commutators

$$\mathcal{C}ent(\mathrm{A}) := \{c \in \mathcal{N}uc(\mathrm{A}) \mid [c, \mathrm{A}] = \mathbf{0}\}.$$

Any unital algebra may be considered as an algebra over its center, which is a ring of scalars over Φ: we simply replace the original scalars by the center with scalar multiplication $c \cdot x := cx$. If A is unital then $\Phi 1 \subseteq \mathcal{C}ent(\mathrm{A})$, and the original scalar action is preserved in the form $\alpha x = (\alpha 1) \cdot x$. In most cases the center forms the "natural" scalars for the algebra; a unital Φ-algebra is **central** if its center is precisely $\Phi 1$. Central-simple algebras (those which are central and simple) are crucial building-blocks of a structure theory.

2.4 Lie and Jordan Algebras

In defining Jordan algebras over general scalars, the theory always required the existence of a scalar $\frac{1}{2}$ (ruling out characteristic 2) to make sense of its basic examples, the special algebras under the Jordan product. Outside this restriction, the structure theory worked smoothly and uniformly in all characteristics.

Jordan Algebra Definition. *If Φ is a commutative associative ring of scalars containing $\frac{1}{2}$, a* **Jordan algebra** *over Φ is a linear algebra* J *equipped with a commutative product $p(x, y) = x \bullet y$ which satisfies the Jordan identity. In terms of commutators and associators these can be expressed as*

$$\text{(JAX1)} \qquad [x, y] = 0 \qquad \text{(Commutative Law).}$$
$$\text{(JAX2)} \qquad [x^2, y, x] = 0 \qquad \text{(Jordan Identity).}$$

The product is usually denoted by $x \bullet y$ rather than by mere juxtaposition. In operator terms, the axioms can be expressed as saying that left and right multiplications coincide, and left multiplication by x^2 commutes with left multiplication by x:

$$(\text{JAX1}^{op}) \qquad L_x = R_x, \qquad (\text{JAX2}^{op}) \qquad [L_{x^2}, L_x] = 0.$$

Lie algebras can be defined over general rings, though in practice pathologies crop up as soon as you leave characteristic 0 for characteristic p (and by the time you reach characteristic 2 almost nothing remains of the structure theory).

Lie Algebra Definition. *A* **Lie algebra**[3] *over any ring of scalars Φ is a linear algebra* L *equipped with an anti-commutative product, universally denoted by brackets $p(x, y) := [x, y]$, satisfying the Jacobi identity*

$$\text{(LAX1)} \qquad [x, y] = -[y, x] \qquad \text{(Anti-commutative Law),}$$
$$\text{(LAX2)} \qquad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{(Jacobi Identity).}$$

[3] "Lee" as in Sophus or Sara or Robert E., not "Lye."

We can write these axioms too as illuminating operator identities:

$$(\mathrm{LAX1}^{op}) \quad L_x = -R_x, \qquad (\mathrm{LAX2}^{op}) \quad L_{[x,y]} = [L_x, L_y],$$

so that L is a homomorphism $\mathrm{L} \to \mathcal{E}nd_\Phi(\mathrm{L})^-$ of Lie algebras (called the *adjoint representation,* with the left multiplication map called the *adjoint map* $Ad(x) := L_x$). The use of the bracket for the product conflicts with the usual notation for the commutator, which would be $[x, y] - [y, x] = 2[x, y]$, but this shows that there is no point in using commutators in Lie algebras to measure commutativity: the bracket says it all.

2.5 The Three Basic Examples Revisited

The creation of the plus and minus algebras A^+, A^- makes sense for arbitrary linear algebras, and these produce Jordan and Lie algebras when A is associative. These are the first (and most important) examples of Jordan and Lie algebras.

Full Example. *If* A *is any linear algebra with product* xy *over a ring of scalars* Φ *containing* $\frac{1}{2}$, *the* plus algebra A^+ *denotes the linear* Φ*-algebra with commutative "Jordan product"*

$$\mathrm{A}^+ : \qquad x \bullet y := \tfrac{1}{2}(xy + yx).$$

If A *is an associative* Φ*-algebra, then* A^+ *is a Jordan* Φ*-algebra.*

Just as everyone should show, *once and only once in his or her life,* that every associative algebra A gives rise to a Lie algebra A^- by verifying directly the anti-commutativity and Jacobi identity for the commutator product, so should everyone show that A also gives rise to a Jordan algebra A^+ by verifying directly the commutativity and Jordan identity for the anti-commutator product.

The previous notions of speciality and exceptionality also make sense in general.

Special Definition. *A Jordan algebra is* **special** *if it can be imbedded in an algebra* A^+ *for* A *associative* (*i.e., if it is isomorphic to a subalgebra of some* A^+), *otherwise it is* **exceptional**. *We usually think of special algebras as* living inside *associative algebras.*

As before, the most important examples of special Jordan or Lie subalgebras are the algebras of hermitian or skew elements of an associative algebra with involution.

Hermitian Example. *If a linear algebra* A *has an involution* $*$, *then* $\mathcal{H}(\mathrm{A}, *)$ *denotes the* hermitian elements $x^* = x$. *It is easy to see that if* A *is an associative* Φ*-algebra with involution, then* $\mathcal{H}(\mathrm{A}, *)$ *is a Jordan* Φ*-subalgebra of* A^+.

The third basic example of a special Jordan algebra is a spin factor, which has no natural Lie analogue.

Spin Factor Example. *We define a linear* Φ*-algebra structure* $\mathcal{JS}pin_n(\Phi)$ *on* $\Phi 1 \oplus \Phi^n$ *over an arbitrary ring of scalars* Φ *by having* 1 *act as unit element and defining the product of vectors* $\mathbf{v}, \mathbf{w} \in \Phi^n$ *to be the scalar multiple of* 1 *given by the dot product* $\langle \mathbf{v}, \mathbf{w} \rangle$ (*for column vectors this is* $\mathbf{v}^{tr}\mathbf{w}$),

$$\mathbf{v} \bullet \mathbf{w} := \langle \mathbf{v}, \mathbf{w} \rangle 1,$$

so the global expression for the product is

$$(\alpha 1 \oplus \mathbf{v}) \bullet (\beta 1 \oplus \mathbf{w}) := \big(\alpha\beta + \langle \mathbf{v}, \mathbf{w} \rangle\big) 1 \oplus \big(\alpha \mathbf{w} + \beta \mathbf{v}\big).$$

Spin factors over general scalars are Jordan algebras just as they were over the reals, by symmetry of the dot product and the fact that L_{x^2} is a linear combination of $L_x, \mathbb{1}_J$, and again they can be imbedded in hermitian $2^n \times 2^n$ matrices over Φ.

2.6 Jordan Matrix Algebras with Associative Coordinates

An important special case of a Hermitian Jordan algebra $\mathcal{H}(\mathrm{A}, *)$ is that where the linear algebra $\mathrm{A} = \mathcal{M}_n(\mathrm{D})$ is the algebra of $n \times n$ matrices over a *coordinate algebra* $(\mathrm{D}, -)$ (a unital linear algebra with involution $d \mapsto \bar{d}$). These are especially useful since one can give an explicit "multiplication table" for hermitian matrices in terms of the coordinates of the matrices, and the properties of $\mathcal{H}$ closely reflect those of D.

Hermitian Matrix Example. *For an arbitrary linear* $*$*-algebra* D *with involution* $\overline{}$, *the conjugate transpose mapping* $X^* := \overline{X}^{tr}$ $\big(\overline{X} := (\overline{x_{ij}})\big)$ *is an involution on the linear algebra* $\mathcal{M}_n(\mathrm{D})$ *of all* $n \times n$ *matrices with entries from* D *under the usual matrix product* XY. *The* Φ*-module* $\mathcal{H}_n(\mathrm{D}, -)$ *of all hermitian matrices* $X^* = X$ *with respect to this involution is closed under the Jordan product* $X \bullet Y = \frac{1}{2}(XY + YX)$.[4]

Using the multiplication table one can show why the exceptional Jordan matrix algebras in the Jordan–von Neumann–Wigner Theorem stop at $n = 3$: in order to produce a Jordan matrix algebra, the coordinates must be alternative if $n = 3$ and associative if $n \geq 4$.[5]

[4] If we used a single symbol $\mathcal{D} = (\mathrm{D}, -)$ for a $*$-algebra, the hermitian example would take the form $\mathcal{H}_n(\mathcal{D})$. Though this notation more clearly reveals that $\mathcal{H}_n$ is a functor from the categories of associative $*$-algebras to Jordan algebras, we will almost always include the involution in the notation. The one exception is for composition algebras with their standard involution: we write $\mathcal{H}_n(\mathrm{C})$ when C is $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$, or an octonion algebra O.

[5] In fact, any "respectable" Jordan algebras of "degree" 4 or more (whether or not they have the specific form of matrix algebras) must be special.

Associative Coordinates Theorem. *If the hermitian matrix algebra $\mathcal{H}_n(\mathrm{D},-)$ for $n \geq 4$ and $\frac{1}{2} \in \Phi$ is a Jordan algebra under the product $X \bullet Y = \frac{1}{2}(XY + YX)$, then* D *must be associative and $\mathcal{H}_n(\mathrm{D},-)$ is a special Jordan algebra.*

2.7 Jordan Matrix Algebras with Alternative Coordinates

When $n = 3$ we can even allow D to be slightly nonassociative: the coordinate algebra must be alternative.

Alternative Algebra Definition. *A linear algebra* D *is* **alternative** *if it satisfies the* **Left and Right Alternative Laws**

$$\text{(AltAX1)} \qquad x^2y = x(xy) \qquad \text{(Left Alternative Law)},$$
$$\text{(AltAX2)} \qquad yx^2 = (yx)x \qquad \text{(Right Alternative Law)}$$

for all x, y in D*. An alternative algebra is automatically* **flexible***,*

$$\text{(AltAX3)} \qquad (xy)x = x(yx) \qquad \text{(Flexible Law)}.$$

In terms of associators or operators these identities may be expressed as

$$[x, x, y] = [y, x, x] = [x, y, x] = 0, \quad \textit{or}$$
$$L_{x^2} = (L_x)^2, \quad R_{x^2} = (R_x)^2, \quad L_x R_x = R_x L_x.$$

From the associator conditions we see that alternativity is equivalent to the associator $[x, y, z]$ being an *alternating* multilinear function of its arguments (in the sense that it vanishes if any two of its variables are equal). Perhaps it would be better to call the algebras *alternating* instead of *alternative*. Notice that the nuclearity conditions can be written in terms of associators as $[n, x, y] = [x, n, y] = [x, y, n] = 0$, so by alternation nuclearity reduces to $[n, x, y] = 0$ in alternative algebras.

It is not hard to see that for a matrix algebra $\mathcal{H}_3(\mathrm{D},-)$ to be a Jordan algebra it is necessary that the coordinate algebra D be alternative and that the diagonal coordinates, the hermitian elements $\mathcal{H}(\mathrm{D},-)$, lie in the nucleus. The converse is true, but painful to prove. Note that in the octonions the hermitian elements do even better: they are *scalars* lying in $\Phi 1$.

Alternative Coordinates Theorem. *The hermitian matrix algebra $\mathcal{H}_3(\mathrm{D},-)$ over Φ containing $\frac{1}{2}$ is a Jordan algebra iff the $*$-algebra* D *is alternative with nuclear involution, i.e., its hermitian elements are contained in the nucleus,*

$$[\mathcal{H}(\mathrm{D},-), \mathrm{D}, \mathrm{D}] = \mathbf{0}.$$

2.8 The n-Squares Problem

Historically, the first nonassociative algebra, the Cayley numbers (progenitor of the theory of alternative algebras), arose in the context of the number-theoretic problem of quadratic forms permitting composition. We will show how this number-theoretic question can be transformed into one concerning certain algebraic systems, the composition algebras, and then how a precise description of these algebras leads to precisely one nonassociative coordinate algebra suitable for constructing Jordan algebras, the 8-dimensional octonion algebra with scalar involution.

It was known to Diophantus that sums of two squares could be *composed,* i.e., that the product of two such terms could be written as another sum of two squares: $(x_0^2 + x_1^2)(y_0^2 + y_1^2) = (x_0y_0 - x_1y_1)^2 + (x_0y_1 + x_1y_0)^2$. Indian mathematicians were aware that this could be generalized to other "binary" (two-variable) quadratic forms, yielding a "two-square formula"

$$(x_0^2 + \lambda x_1^2)(y_0^2 + \lambda y_1^2) = (x_0y_0 - \lambda x_1y_1)^2 + \lambda(x_0y_1 + x_1y_0)^2 = z_0^2 + \lambda z_1^2.$$

In 1748 Euler used an extension of this to "quaternary" (4-variable) quadratic forms $x_0^2 + x_1^2 + x_2^2 + x_3^2$, and in 1770 Lagrange used a general "4-square formula":

$$\begin{aligned}(x_0^2 + \lambda x_1^2 + \mu x_2^2 + \lambda\mu x_3^2) \times (y_0^2 + \lambda y_1^2 + \mu y_2^2 + \lambda\mu y_3^2)&\\ = z_0^2 + \lambda z_1^2 + \mu z_2^2 + \lambda\mu z_3^2&\end{aligned}$$

for z_i defined by

$$\begin{aligned}
z_0 &:= x_0y_0 - \lambda x_1y_1 - \mu x_2y_2 - \lambda\mu x_3y_3,\\
z_1 &:= x_0y_1 + x_1y_0 + \mu x_2y_3 - \mu x_3y_2,\\
z_2 &:= x_0y_2 - \lambda x_1y_3 + x_2y_0 + \lambda x_3y_1,\\
z_3 &:= x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0.
\end{aligned}$$

In 1845 an "8-square formula" was discovered by Cayley; J.T. Graves claimed to have discovered this earlier, and in fact C.F. Degan had already noted a more general formula in 1818:

$$\begin{aligned}
&(x_0^2 + \lambda x_1^2 + \mu x_2^2 + \lambda\mu x_3^2 + \nu x_4^2 + \lambda\nu x_5^2 + \mu\nu x_6^2 + \lambda\mu\nu x_7^2)\\
&\quad\times (y_0^2 + \lambda y_1^2 + \mu y_2^2 + \lambda\mu y_3^2 + \nu y_4^2 + \lambda\nu y_5^2 + \mu\nu y_6^2 + \lambda\mu\nu y_7^2)\\
&\qquad = (z_0^2 + \lambda z_1^2 + \mu z_2^2 + \lambda\mu z_3^2 + \nu z_4^2 + \lambda\nu z_5^2 + \mu\nu z_6^2 + \lambda\mu\nu z_7^2)
\end{aligned}$$

for z_i defined by

$$\begin{aligned}
z_0 &:= x_0y_0 - \lambda x_1y_1 - \mu x_2y_2 - \lambda\mu x_3y_3 - \nu x_4y_4 - \lambda\nu x_5y_5 - \mu\nu x_6y_6 - \lambda\mu\nu x_7y_7,\\
z_1 &:= x_0y_1 + x_1y_0 + \mu x_2y_3 - \mu x_3y_2 + \nu x_4y_5 - \nu x_5y_4 - \mu\nu x_6y_7 + \mu\nu x_7y_6,\\
z_2 &:= x_0y_2 - \lambda x_1y_3 + x_2y_0 + \lambda x_3y_1 + \nu x_4y_6 + \lambda\nu x_5y_7 - \nu x_6y_4 - \lambda\nu x_7y_5,\\
z_3 &:= x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0 + \nu x_4y_7 - \nu x_5y_6 + \nu x_6y_5 - \nu x_7y_4,\\
z_4 &:= x_0y_4 - \lambda x_1y_5 - \mu x_2y_6 - \lambda\mu x_3y_7 + x_4y_0 + \lambda x_5y_1 + \mu x_6y_2 + \lambda\mu x_7y_3,\\
z_5 &:= x_0y_5 + x_1y_4 - \mu x_2y_7 + \mu x_3y_6 - x_4y_1 + x_5y_0 - \mu x_6y_3 + \mu x_7y_2,\\
z_6 &:= x_0y_6 + \lambda x_1y_7 + x_2y_4 - \lambda x_3y_5 - x_4y_2 + \lambda x_5y_3 + x_6y_0 - \lambda x_7y_1,\\
z_7 &:= x_0y_7 - x_1y_6 + x_2y_5 + x_3y_4 - x_4y_3 - x_5y_2 + x_6y_1 + x_7y_0.
\end{aligned}$$

This is clearly not the sort of formula you stumble upon during a casual mathematical stroll. Indeed, this is too cumbersome to tackle directly, with its mysterious distribution of plus and minus signs and assorted scalars.

2.9 Forms Permitting Composition

A more concise and conceptual approach is needed. If we interpret the variables as coordinates of a vector $x = (x_0, \dots, x_7)$ in an 8-dimensional vector space, then the expression $x_0^2 + \lambda x_1^2 + \mu x_2^2 + \lambda\mu x_3^2 + \nu x_4^2 + \lambda\nu x_5^2 + \mu\nu x_6^2 + \lambda\mu\nu x_7^2$ defines a quadratic norm form $N(x)$ on this space. The 8-square formula asserts that this quadratic form *permits* (or admits) *composition* in the sense that $N(x)N(y) = N(z)$, where the "composite" $z = (z_0, \dots, z_7)$ is automatically a bilinear function of x and y (i.e., each of its coordinates z_i is a bilinear function of the x_j and y_k). We may think of $z = x \cdot y$ as some sort of "product" of x and y. This product is linear in x and y, but it need not be commutative or associative. Thus the existence of an n-squares formula is equivalent to the existence of an n-dimensional algebra with product $x \cdot y$ and distinguished basis $e_0, \dots, e_{n-1}$ such that $N(x) = N(x_0 e_0 + \cdots + x_{n-1} e_{n-1}) = \sum_{i=0}^{n} \lambda_i x_i^2$ permits composition $N(x)N(y) = N(x \cdot y)$ (in the classical case all $\lambda_i = 1$, and this is a "pure" sum of squares). The element $e_0 = (1, 0, \dots, 0)$ (having $x_0 = 1$, all other $x_i = 0$) acts as unit element: $e_0 \cdot y = y$, $x \cdot e_0 = x$. When the quadratic form is anisotropic ($N(x) = 0 \Longrightarrow x = 0$) the algebra is a "division algebra": it has no divisors of zero, $x, y \neq 0 \Longrightarrow x \cdot y \neq 0$, so in the finite-dimensional case the *injectivity* of left and right multiplications makes them *bijections*.

The algebra behind the 2-square formula is just the *complex numbers* $\mathbb{C}$: $z = x_0 1 + x_1 i$ with basis $1, i$ over the reals, where 1 acts as identity and $i^2 = -1$ and $N(z) = x_0^2 + x_1^2 = z\overline{z}$ is the ordinary norm squared (where $\overline{z} = x_0 1 - x_1 i$ is the ordinary complex conjugate). This interpretation was well known to Gauss. The 4-squares formula led Hamilton to the *quaternions* $\mathbb{H}$ consisting of all $x = x_0 1 + x_1 i + x_2 j + x_3 k$, where the formula for $x \cdot y$ means that the basis elements $1, i, j, k$ satisfy the now-familiar rules

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Clearly, this algebra is no longer commutative. Again $N(x) = x\overline{x}$ is the ordinary norm squared (where $\overline{x} = x_0 1 - x_1 i - x_2 j - x_3 k$ is the ordinary quaternion conjugate).

Clifford and Hamilton invented 8-dimensional algebras (biquaternions), which were merely the direct sum $\mathbb{H} \boxplus \mathbb{H}$ of two quaternion algebras. Because of the presence of zero divisors, these algebras were of minor interest. Cayley was the first to use the 8-square formula to create an 8-dimensional division algebra $\mathbb{K}$ of *octonions* or *Cayley numbers*. By 1847 he recognized that this algebra was not commutative or associative, with basis $e_0, \ldots, e_7 = 1, i, j, k, \ell, i\ell, j\ell, k\ell$ with multiplication table

$$e_0 e_i = e_i e_0 = e_i, \quad e_i^2 = -1, \quad e_i e_j = -e_j e_i = e_k$$
$$\text{for} \quad ijk = 123, 145, 624, 653, 725, 734, 176.$$

A subsequent flood of (false!!) higher-dimensional algebras carried names such as quadrinions, quines, pluquaternions, nonions, tettarions, plutonions. Ireland especially seemed a factory for such counterfeit division algebras. In 1878 Frobenius showed that the only *associative* division algebras over the reals (permitting composition or not) are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ of dimensions 1, 2, 4. In 1898 Hurwitz proved via group representations that the only quadratic forms permitting composition over the reals are the standard ones of dimension 1, 2, 4, 8; A.A. Albert later gave an algebra-theoretic proof over a general field of scalars (with an addition by Irving Kaplansky to include characteristic 2 and non-unital algebras). Only recently was it established that the only *finite-dimensional* real nonassociative division algebras have dimensions 1, 2, 4, 8; the algebras themselves were not classified, and the proof was topological rather than algebraic.

2.10 Composition Algebras

The most important alternative algebras with nuclear involutions are the composition algebras. A **composition algebra** is a unital algebra having a nondegenerate quadratic norm form N which **permits composition**,

$$Q(1) = 1, \qquad Q(xy) = Q(x)Q(y).$$

In general, a quadratic form Q on a Φ-module V is **nondegenerate** if all nonzero elements in the module contribute to the values of the form. The slackers (the set of elements which contribute nothing) are gathered in the **radical**

$$\mathcal{R}ad(Q) := \{z \in V \mid Q(z) = Q(z, V) = 0\},$$

so nondegeneracy means that $\mathcal{R}ad(Q) = \mathbf{0}$.[6]

Even better than nondegeneracy is anisotropy. A vector x is **isotropic** if it has “zero weight” $Q(x) = 0$, and **anisotropic** if $Q(x) \neq 0$. A form is isotropic if it has nonzero isotropic vectors, and anisotropic if it has none:

$$Q \textit{ anisotropic} \quad \textit{iff} \quad Q(x) = 0 \Longleftrightarrow x = 0.$$

For example, the positive definite norm form $Q(x) = x \cdot x$ on any Euclidean space is anisotropic. Clearly, any anisotropic form is nondegenerate.

[6] Since $Q(z) = \frac{1}{2}Q(z,z)$, when $\frac{1}{2} \in \Phi$ the radical of the quadratic form reduces to the usual radical $\mathcal{R}ad(Q(\cdot,\cdot)) := \{z \in V \mid Q(z,V) = 0\}$ of the associated bilinear form $Q(\cdot,\cdot)$ (the vectors which are “orthogonal to everybody”). But in characteristic 2 there is an important difference between the radical of the quadratic form and the “bilinear radical” of its associated bilinear form.

2.11 The Cayley–Dickson Construction and Process

The famous Hurwitz Theorem of 1898 states that over the real numbers composition algebras can exist only in dimensions $1, 2, 4$, and 8. In 1958 Nathan Jacobson gave a beautiful "bootstrap" method, showing clearly how all composition algebras are generated internally, by repeated "doubling" (of the module, the multiplication, the involution, and the norm) starting from any composition subalgebra. As its name suggests, the Cayley–Dickson doubling process is due to A.A. Albert.

Cayley–Dickson Definition. *The* **Cayley–Dickson Construction** *builds a new $*$-algebra out of an old one together with a choice of scalar. If* A *is a unital linear algebra with involution $a \mapsto \bar{a}$ whose norms satisfy $a\bar{a} = n(a)1$ for scalars $n(a) \in \Phi$, and μ is an invertible scalar in Φ, then the Cayley–Dickson algebra*

$$\mathcal{KD}(\mathrm{A}, \mu) = \mathrm{A} \oplus \mathrm{A}m$$

is obtained by doubling the module A (*adjoining a formal copy* $\mathrm{A}m$) *and defining a product, scalar involution, and norm by the* **Cayley–Dickson Recipe**:

$$\begin{aligned}(a \oplus bm)(c \oplus dm) &= (ac + \mu \bar{d} b) \oplus (da + b\bar{c})m, \\ (a \oplus bm)^* &= \bar{a} \oplus -bm, \\ N(a \oplus bm) &= n(a) - \mu n(b).\end{aligned}$$

The **Cayley–Dickson Process** *consists of iterating the Cayley–Dickson Construction over and over again. Over a field Φ the Process iterates the Construction starting from the* 1*-dimensional* $\mathrm{A}_0 = \Phi$ (*the* **scalars**) *with trivial involution and nondegenerate norm $N(\alpha) = \alpha^2$ to get a 2-dimensional commutative* **binarion algebra** $\mathrm{A}_1 = \mathcal{KD}(\mathrm{A}_0, \mu_1) = \Phi \oplus \Phi i$ $(i^2 = \mu_1 1)$ *with nontrivial involution,*[7] *then a 4-dimensional noncommutative* **quaternion algebra** $\mathrm{A}_2 = \mathcal{KD}(\mathrm{A}_1, \mu_2) = \mathrm{A}_1 \oplus \mathrm{A}_1 j$ $(j^2 = \mu_2 1)$, *and finally an 8-dimensional nonassociative* **octonion algebra** $\mathrm{A}_3 = \mathcal{KD}(\mathrm{A}_2, \mu_3) = \mathrm{A}_2 \oplus \mathrm{A}_2 \ell$ $(\ell^2 = \mu_3 1)$, *all with nondegenerate norms.*

Thus octonion algebras are obtained by gluing two copies of a quaternion algebra together by the Cayley–Dickson Recipe. If the Cayley–Dickson doubling process is carried beyond dimension 8, the resulting algebras no longer permit composition and are no longer alternative (so cannot be used in constructing Jordan matrix algebras). Jacobson's Bootstrap Theorem shows that over a field Φ the algebras with involution obtained from the Cayley–Dickson Process are precisely the *composition algebras with standard involution* over Φ: every composition algebra arises by this construction. If we take $\mathrm{A}_0 = \Phi = \mathbb{R}$

[7] In characteristic 2, starting from Φ the construction produces larger and larger algebras with trivial involution and possibly degenerate norm; to get out of the rut, one must construct *by hand* the binarion algebra $\mathrm{A}_1 := \Phi 1 + \Phi v$ where v $[\approx \frac{1}{2}(1+i)]$ has $v^2 := v - \nu 1$, $v^* = 1 - v$, with nondegenerate norm $N(\alpha + \beta v) := \alpha^2 + \alpha\beta + \nu\beta^2$.

the reals and $\mu_1 = \mu_2 = \mu_3 = -1$ in the Cayley–Dickson Process, then A_1 is the complex numbers $\mathbb{C}$, A_2 is Hamilton's quaternions $\mathbb{H}$ (the Hamiltonions), and A_3 is Cayley's octonions $\mathbb{K}$ (the Caylions, Cayley numbers, or Cayley algebra), precisely as in the Jordan–von Neumann–Wigner Theorem.

Notice that we are adopting the convention that the dimension 4 composition algebras will all be called (generalized) *quaternion* algebras (as is standard in noncommutative ring theory) and denoted by Q; by analogy, the dimension 8 composition algebras will be called (generalized) *octonion* algebras, and denoted by O (even though this looks dangerously like zero), and the dimension 2 composition algebras will all be called *binarion* algebras and denoted by B. In the alternative literature the octonion algebras are called *Cayley algebras*, but we will reserve the term *Cayley* for the unique 8-dimensional real division algebra $\mathbb{K}$ (*the* Cayley algebra), just as Hamilton's quaternions are the unique 4-dimensional real division algebra $\mathbb{H}$. There is no generally accepted term for the 2-dimensional composition algebras, but that won't stop us from calling them binarions. If the 1-dimensional scalars insist on having a high-falutin' name too, we can call them *unarions*.

Notice that a composition algebra C consists of a unital algebra *plus a choice of norm form N*, and therefore always carries a *standard involution* $\bar{x} = N(x,1)1 - x$. Thus a composition algebra is always a $*$-algebra (and the $*$ determines the norm, $N(x)1 = x\bar{x}$).

2.12 Split Composition Algebras

We will often be concerned with *split unarions, binarions, quaternions,* and *octonions*. Over an algebraically closed field the composition algebras are all "split." This is an imprecise metaphysical term, meaning roughly that the system is "completely isotropic," as far removed from an "anisotropic" or "division system" as possible, as well as being defined in some simple way over the integers.[8] Each category separately must decide on its own definition of "split." For example, in the theory of finite-dimensional associative algebras we define a *split* simple algebra over Φ to be a matrix algebra $\mathcal{M}_n(\Phi)$ coordinatized by the ground field. The theory of central-simple algebras shows that every simple algebra $\mathcal{M}_n(\Delta)$ coordinatized by a division algebra Δ becomes split in some scalar extension, because of the amazing fact that finite-dimensional division algebras Δ can *be split* (turned into $\mathcal{M}_r(\Omega)$) by tensoring with a *splitting field* Ω; in particular, every division algebra has square dimension $\dim_\Phi(\Delta) = r^2$ over its center! In the theory of quadratic forms, a "split" form would have "maximal Witt index," represented relative to a suitable basis by the matrix consisting of hyperbolic planes $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ down the diagonal, with an additional 1×1 matrix (1) if the dimension is odd, $Q(\sum_{i=1}^n (\alpha_{2i-1}x_{2i-1} \oplus \alpha_{2i}x_{2i}) + \alpha x_{2n+1}) = \sum_{i=1}^n \alpha_{2i-1}\alpha_{2i} + \alpha_{2n+1}^2$.

[8] This has no relation to "split" exact sequences $0 \to A \to B \to C \to 0$, which have to do with the middle term "splitting" as a *semi-direct sum* $B \cong A \oplus C$.

The split composition algebras over an arbitrary ring of scalars (not just a field) are defined as follows.

Split Definition. *The* **split composition algebras** *over a scalar ring* Φ *are defined to be those* $*$*-algebras of dimension* 2^{n-1}, $n = 1, 2, 3, 4$, *isomorphic to the following models*:

SPLIT UNARIONS $\mathcal{U}(\Phi) := \Phi$, *the scalars* Φ *with trivial involution* $\bar{\alpha} := \alpha$ *and norm* $N(\alpha) := \alpha^2$;

SPLIT BINARIONS $\mathcal{B}(\Phi) = \Phi \boxplus \Phi$, *a direct sum of scalars with the standard* (*exchange*) *involution* $(\alpha, \beta) \mapsto (\beta, \alpha)$ *and norm* $N(\alpha, \beta) := \alpha\beta$;

SPLIT QUATERNIONS $\mathcal{Q}(\Phi)$ *with standard involution, i.e., the algebra* $\mathcal{M}_2(\Phi)$ *of* 2×2 *matrices with* symplectic involution $\overline{a} = \left(\begin{smallmatrix} \beta & -\gamma \\ -\delta & \alpha \end{smallmatrix}\right)$ *for* $a = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ *and norm* $N(a) := \det(a)$;

SPLIT OCTONIONS $\mathcal{O}(\Phi) = \mathcal{Q}(\Phi) \oplus \mathcal{Q}(\Phi)\ell$ *with standard involution* $\overline{a \oplus b\ell} = \overline{a} - b\ell$ *and norm* $N(a \oplus b\ell) := \det(a) - \det(b)$.

There is (up to isomorphism) a *unique* split composition algebra of given dimension over a given Φ, and the constructions $\Phi \mapsto \mathcal{U}(\Phi), \mathcal{B}(\Phi), \mathcal{Q}(\Phi), \mathcal{O}(\Phi)$ are functors from the category of scalar rings to the category of composition algebras.

Notice that over the reals these split composition algebras are at the opposite extreme from the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$ occurring in the J–vN–W classification. They are obtained from the Cayley–Dickson process by choosing all the ingredients to be $\mu_i = 1$ instead of $\mu_i = -1$. It is an important fact that composition algebras over a field are either division algebras or split: as soon as the quadratic norm form is the least bit isotropic (some nonzero element has norm zero) then it is split as a quadratic form, and the algebra has proper idempotents and splits entirely:

$$N \textit{ anisotropic } \Longleftrightarrow \mathcal{KD} \textit{ division}, \qquad N \textit{ isotropic } \Longleftrightarrow \mathcal{KD} \textit{ split.}$$

This *dichotomy* for composition algebras, of being entirely anisotropic (division algebra) or entirely isotropic (split), does not hold for quadratic forms in general, or for other algebraic systems. In Jordan algebras there is a *trichotomy*: an algebra can be *anisotropic* ("division algebra"), *reduced* (has nonzero idempotents but coordinate ring a division algebra), or *split* (nonzero idempotents and coordinate ring the ground field). The **split Albert algebra** $\mathcal{A}lb(\Phi)$ over Φ is the 27-dimensional Jordan algebra of 3×3 hermitian matrices over the split octonion algebra (with standard involution),

$$\mathcal{A}lb(\Phi) := \mathcal{H}_3(\mathcal{O}(\Phi)) \qquad \text{(split Albert).}$$

As we will see in the next section, over an algebraically closed field this is the only exceptional Jordan algebra. But over general fields we can have reduced Albert algebras $\mathcal{H}_3(\mathrm{O})$ for non-split octonion algebras, and (as first shown by Albert) we can even have Albert division algebras (though these can't be represented in the form of 3×3 matrices, which would always have a non-invertible idempotent E_{11}).

2.13 Classification

We now return from our long digression on general linear algebras, and consider the development of Jordan theory during the Algebraic Renaissance, whose crowning achievement was the classification of simple Jordan algebras over an arbitrary algebraically closed field Φ (of characteristic not 2, of course!). As in the J–vN–W Theorem, the classification of simple Jordan algebras proceeds according to "degree," where the *degree* is the maximal number of supplementary orthogonal idempotents (analogous to the matrix units E_{ii}). From another point of view, the degree is the degree of the *generic minimum polynomial* of the algebra, the "generic" polynomial $m_x(\lambda) = \lambda^n - m_1(x)\lambda^{n-1} + \cdots + (-1)^n m_n(x)$ ($m_i : \mathrm{J} \to \Phi$ homogeneous of degree i) of minimal degree satisfied by all x, $m_x(x) = 0$. Degree 1 algebras are just the 1-dimensional Φ^+; the degree 2 algebras are the $\mathcal{JS}pin_n$; the degree n algebras for $n \geq 3$ are all Jordan matrix algebras $\mathcal{H}_n(\mathrm{C})$ where the coordinate $*$-algebras C are precisely the split composition algebras over Φ with their standard involutions. This leads immediately to the basic classification of finite-dimensional Jordan algebras over an algebraically closed field.

Renaissance Structure Theorem. *Consider finite-dimensional Jordan algebras* J *over an algebraically closed field* Φ *of characteristic* $\neq 2$.

- *The radical of* J *is the maximal nilpotent ideal, and the quotient* $\mathrm{J}/\mathcal{R}ad(\mathrm{J})$ *is semisimple.*
- *An algebra is semisimple iff it is a finite direct sum of simple ideals. In this case, the algebra has a unit element, and its simple decomposition is unique: the simple summands are precisely the minimal ideals.*
- *Every simple algebra is automatically central-simple over* Φ.
- *An algebra is simple iff it is isomorphic to exactly one of*:

Ground Field Φ^+ *of degree* 1,

Spin Factor $\mathcal{JS}pin_n(\Phi)$ *of degree* 2, *for* $n \geq 2$,

Hermitian Matrices $\mathcal{H}_n(\mathrm{C}(\Phi))$ *of degree* $n \geq 3$ *coordinatized by a split composition algebra* $\mathrm{C}(\Phi)$ (Split Unarion, Split Binarion, Split Quaternion, or Split Octonion Matrices):

$\mathcal{H}_n(\Phi)$ *for* Φ *the ground field,*

$\mathcal{H}_n(\mathcal{B}(\Phi)) \cong \mathcal{M}_n(\Phi)^+$ *for* $\mathcal{B}(\Phi)$ *the split binarions,*

$\mathcal{H}_n(\mathcal{Q}(\Phi))$ *for* $\mathcal{Q}(\Phi)$ *the split quaternions,*

$\mathcal{A}lb(\Phi) = \mathcal{H}_3(\mathcal{O}(\Phi))$ *for* $\mathcal{O}(\Phi)$ *the split octonions.*

Once more, the only exceptional algebra in the list is the 27-dimensional split Albert algebra. Note that the 1-dimensional algebra $\mathcal{JS}pin_0$ is the same as the ground field; the 2-dimensional $\mathcal{JS}pin_1 \cong \mathcal{B}(\Phi)$ is not simple when Φ is algebraically closed, so only $\mathcal{JS}pin_n$ for $n \geq 2$ contribute new simple algebras.

We are beginning to isolate the Albert algebras conceptually; even though the split Albert algebra and the real Albert algebra discovered by Jordan, von Neumann, and Wigner appear to fit into the family of Jordan matrix algebras, we will see in the next chapter that their non-reduced forms really come via a completely different construction out of a cubic form.