Finite Rank Toeplitz Operators in the Bergman Space

Grigori Rozenblum

To Volodya Maz'ya,
an outstanding mathematician and person

Abstract We discuss resent developments in the problem of description of finite rank Toeplitz operators in different Bergman spaces and give some applications.

1 Introduction

Toeplitz operators arise in many fields of Analysis and have been an object of active study for many years. Quite a lot of questions can be asked about these operators, and these questions depend on the field where Toeplitz operators are applied.

The classical Toeplitz operator T_f in the Hardy space $H^2(S^1)$ is defined as

$$T_f u = Pfu, \tag{1.1}$$

for $u \in H^2(S^1)$, where f is a bounded function on S^1 (the weight function) and P is the Riesz projection, the orthogonal projection $P : L_2(S^1) \to H^2(S^1)$. Such operators are often called *Riesz–Toeplitz* or *Hardy–Toeplitz* operators (cf. [15], for more details). More generally, for a Hilbert space $\mathcal{H}$ of functions and a closed subspace $\mathcal{L} \subset \mathcal{H}$, the Toeplitz operator T_f in $\mathcal{L}$ acts as in (1.1), where P is the projection $P : \mathcal{H} \to \mathcal{L}$. In particular, in the case where $\mathcal{H}$ is the space $L_2(\Omega, \rho)$ for some domain $\Omega \subset \mathbb{C}^d$ and some measure ρ

Grigori Rozenblum
Department of Mathematics, Chalmers University of Technology and Department of Mathematics, University of Gothenburg, S-412 96, Gothenburg, Sweden
e-mail: grigori@math.chalmers.se

and $\mathcal{L}$ is the Bergman space $\mathcal{B}^2 = \mathcal{B}^2(\Omega, \rho)$ of analytical functions in $\mathcal{H}$, such an operator is called *Bergman–Toeplitz*; we denote it by $\mathcal{T}_f$.

Among many interesting properties of Riesz–Toeplitz operators, we mention the following *cut-off* one. If f is a bounded function and the operator T_f is compact, then f should be zero. For many other classes of operators a similar cut-off on some level is also observed. The natural question arises, whether there is a kind of cut-off property for Bergman–Toeplitz operators. Quite long ago it became a common knowledge that at least direct analogy does not take place. In [13], the conditions were found on the function f in the unit disk $\Omega = D$ guaranteeing that the operator $\mathcal{T}_f$ in $\mathcal{B}^2(D, \lambda)$ with Lebesgue measure λ belongs to the Schatten class $\mathfrak{S}_p$. So, the natural question came up: probably, it is on the finite rank level that the cut-off takes place. In other words, if a Bergman–Toeplitz operator has finite rank it should be zero.

It was known long ago that the Schatten class behavior of $\mathcal{T}_f$ is determined by the rate of convergence to zero at the boundary of the function f. Therefore, the finite rank (FR) hypothesis deals with functions f with compact support not touching the boundary of Ω. In this setting, the FR hypothesis is equivalent to the one for Toeplitz operators on the Bargmann (Fock, Segal) space consisting of analytical functions in $\mathbb{C}$, square summable with a Gaussian weight. A proof of the FR hypothesis appeared in the same paper [13], about twenty lines long. Unfortunately, there was an unrepairable fault in the proof, so the FR remained unsettled.

It was only in 2007 that the proof of the FR hypothesis was finally found, even in a more general form. The Bergman projection $\mathbf{P} : L_2 \to \mathcal{B}$ can be extended to an operator from the space of distributions $\mathcal{D}'(\Omega)$ to $\mathcal{B}^2(\Omega, \lambda)$. Let μ be a regular complex Borel measure with compact support in Ω. With μ we associate the Toeplitz operator $\mathcal{T}_\mu : \; u \mapsto \mathbf{P} u\mu$ in $\mathcal{B}^2(\Omega, \lambda)$.

In [14], the following result was established.

Theorem 1.1. *Suppose that the Toeplitz operator $\mathcal{T}_\mu$ in $\mathcal{B}^2(\Omega, \lambda)$, $\Omega \subset \mathbb{C}$ has finite rank $\mathbf{r}$. Then the measure μ is the sum of $\mathbf{r}$ point masses,*

$$\mu = \sum_{1}^{\mathbf{r}} C_k \delta_{z_j}, \; z_j \in \Omega. \tag{1.2}$$

The publication of the proof of Theorem 1.1 induced an activity around it. In two years to follow several papers appeared, where the FR theorem was generalized in different directions, and interesting applications were found in Analysis and Mathematical Physics.

In this paper, we aim for collecting and systematizing the existing results on the finite rank problem and their applications. We also present several new theorems generalizing and extending these results.

In a more vague setting, the problem discussed in the paper can be understood in the following way: is it possible that the contribution of the positive part of a real measure μ and the contribution of the negative part of μ "eat up" each other, so that the resulting Toeplitz operator becomes "trivial." In this form, the relation arises with the results by Maz'ya and Verbitsky (cf., in particular, [16, 17], where the phenomenon of the mutual compensation of positive and negative parts of the weight for embedding of Sobolev spaces was studied in detail).

2 Problem Setting

Let Ω be a domain in $\mathbb{R}^d$ or $\mathbb{C}^d$. We suppose that a measure ρ is defined on Ω, jointly absolutely continuous with Lebesgue measure. Suppose that $\mathcal{L}$ is a closed subspace in $\mathcal{H} = L_2(\Omega, \rho)$, consisting of smooth functions, $\mathcal{L} \subset C^\infty(\Omega)$. In this case, the orthogonal projection $\mathbf{P} : \mathcal{L} \to \mathcal{H}$ is an integral operator with smooth kernel,

$$\mathbf{P}u(x) = \int P(x, y)u(y)d\rho(x). \tag{2.1}$$

We call $\mathbf{P}$ the *Bergman projection* and $P(x, y)$ the *Bergman kernel* (corresponding to the subspace $\mathcal{L}$).

Let F be a distribution, compactly supported in Ω, $F \in \mathcal{E}'(\Omega)$. We denote by $\langle F, \phi\rangle$ the action of the distribution F on the function $\phi \in \mathcal{E}$. Then one can define the Toeplitz operator in $\mathcal{L}$ with *weight* F:

$$(\mathcal{T}_F u)(x) = \langle F, P(x, \cdot)u(\cdot)\rangle. \tag{2.2}$$

Formula (2.2) can be also understood in the following way. The operator $\mathbf{P}$ considered as an operator $P : \mathcal{H} \to \mathcal{L}$ has an adjoint, $P' : \mathcal{L}' \to \mathcal{H}$, so PP' is the extension of $\mathbf{P}$ to the operator $\mathcal{L}' \to \mathcal{L}$, in particular, $\mathbf{P}$ extends as an operator from $\mathcal{E}'(\Omega)$ to $\mathcal{L}$. In this setting, $Fu \in \mathcal{E}'(\Omega)$ for $u \in \mathcal{E}(\Omega)$ and the Toeplitz operator has the form

$$\mathcal{T}_F u = \mathbf{P}Fu, \tag{2.3}$$

consistently with the traditional definition of Toeplitz operators.

It is more convenient to use the description of the Toeplitz operator by means of the sesquilinear form. For $u, v \in \mathcal{L}$, we have

$$(\mathcal{T}_F u, v) = (\mathbf{P}Fu, v) = \langle \sigma Fu, \overline{\mathbf{P}v}\rangle = \langle \sigma F, u\overline{v}\rangle, \tag{2.4}$$

where σ is the Radon–Nikodym derivative of ρ with respect to the Lebesgue measure. In particular, if F is a regular Borel complex measure $F = \mu$, the corresponding Toeplitz operator acts as

$$\mathcal{T}_\mu u(x) = \int_\Omega P(x,y)u(x)d\mu(x), \tag{2.5}$$

and the quadratic form is

$$(\mathcal{T}_F u, v) = \int_\Omega u\overline{v}\sigma d\mu(x).$$

Finally, when F is a bounded function, formula (2.4) takes the form

$$(\mathcal{T}_F u, v) = \int_\Omega u\overline{v}F(x)d\rho(x). \tag{2.6}$$

Classical examples of Bergman spaces and corresponding Toeplitz operators are produced by solutions of elliptic equations and systems.

Example 2.1. Let Ω be a bounded domain in $\mathbb{C}$, $\rho = \lambda$ the Lebesgue measure, and $\mathcal{L} = \mathcal{B}^2(\Omega)$ the space of L_2 functions analytical in Ω. This is the classical Bergman space.

Example 2.2. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^d$, $d > 1$, with Lebesgue measure ρ, and let the space $\mathcal{L}$ consist of L_2 functions analytical in Ω. This is also a classical Bergman space. Here, and in Example 2.1, measures different from the Lebesgue one are also considered, especially when Ω is a ball or a (poly)disk.

Example 2.3. For a bounded domain $\Omega \subset R^d$, we set $\mathcal{L}$ to be the space of L_2 solutions of the equation $Lu = 0$, where L is an elliptic differential operator with constant coefficients. In particular, if L is the Laplacian, the space $\mathcal{L}$ is called the *harmonic Bergman space.*

Example 2.4. If Ω is a bounded domain in $\mathbb{R}^d$ with *even* $d = 2\mathbf{m}$, and $\mathbb{R}^d$ is identified with $\mathbb{C}^{\mathbf{m}}$ with variables $z_j = (x_j, y_j)$, $j = 1, \dots, \mathbf{m}$, the Bergman space of functions which are harmonic with respect to each pair (x_j, y_j) is called $\mathbf{m}$-harmonic Bergman space; if on the other hand, the space of functions $u(z)$ such that $u_\zeta(\xi_1, \xi_2) = u(\zeta(\xi_1 + i\xi_2))$ is harmonic as a function of variables ξ_1, ξ_2 for any $\zeta \in \mathbb{C}^m \setminus \{0\}$, is called pluriharmonic Bergman space.

Example 2.5. Let Ω be the whole of $\mathbb{C}^{\mathbf{m}} = \mathbb{R}^d$, with the Gaussian measure $d\rho = \exp(-|z|^2/2)d\lambda$. The subspace $\mathcal{L} \subset L_2(\mathbb{C}^{\mathbf{m}}, \rho)$ of entire analytical functions in $\mathbb{C}^{\mathbf{m}}$ is called *Fock* or *Segal–Bargmann* space.

The study of Toeplitz operators in many cases is based upon the consideration of associated infinite matrices.

Let $\Sigma_1 = \{f_j(x), x \in \Omega\}$, $\Sigma_2 = \{g_j(x), x \in \Omega\}$ be two infinite systems of functions in $\mathcal{L}$. With these systems and a distribution $F \in \mathcal{E}'(\Omega)$ we associate the matrix

$$\mathcal{A} = \mathcal{A}(F) = \mathcal{A}(F, \Sigma_1, \Sigma_2, \Omega, \rho) = (\mathcal{T}_F f_j, g_k)_{j,k=1,\dots} = (\langle \sigma F, f_j \overline{g_k} \rangle). \quad (2.7)$$

So, the matrix $\mathcal{A}$ is the matrix of the sesquilinear form of the operator $\mathcal{T}_F$ on the systems Σ_1, Σ_2. We formulate the obvious but important statement.

Proposition 2.6. *Suppose that the Toeplitz operator $\mathcal{T}_F$ has finite rank $\mathbf{r}$. Then the matrix $\mathcal{A}$ also has finite rank; moreover,* $\operatorname{rank}(\mathcal{A}) \leqslant \mathbf{r}$.

The use of matrices of the form (2.7) enables one to perform important reductions. In particular, since the domain Ω does not enter explicitly into the matrix, the rank of this matrix does not depend on the domain Ω, as long as one can chose the systems Σ_1, Σ_2 dense simultaneously in the Bergman spaces in different domains. Thus, in particular, the FR problems for the analytical Bergman spaces in bounded domains and for the Fock space are equivalent (cf. the discussion in [21].)

3 Theorem of Luecking. Extensions in Dimension 1

In this section, we present the original proof given by Luecking in [14], and give extensions in several directions.

Theorem 3.1. *Let $\Omega \subset \mathbb{C}$ be a bounded domain, with Lebesgue measure. Suppose that for some regular complex Borel measure μ, absolutely continuous with respect to the Lebesgue measure, with compact support in Ω, the Toeplitz operator $\mathcal{T}_\mu$ in the Bergman space of analytical functions has finite rank $\mathbf{r}$. Then $\mu = 0$.*

We formulate and prove here Luecking's theorem only in the case of an absolutely continuous measure; the case of more singular measures will be taken care of later, as a part of the general distributional setting. In the proof, which follows [14], we separate a lemma that will be used further on.

Lemma 3.2. *Let ϕ be a linear functional on polynomials in $z, \overline{z}$. Denote by $\mathcal{A}(\phi)$ the matrix with elements $\phi(z^j \overline{z}^k)$. Then the following are equivalent:*

1) *the matrix $\mathcal{A}(\phi)$ has finite rank not greater than $\mathbf{r}$,*

2) *for any collections of nonnegative integers $J = \{j_0, \dots, j_{\mathbf{r}}\}$ and $K = \{k_0, \dots, k_{\mathbf{r}}\}$*

$$\phi^{\otimes N} \left(\prod_{i \in (0,\mathbf{r})} z_i^{j_i} \det \overline{z_i}^{k_l} \right) = 0, \quad (3.1)$$

where $N = \mathbf{r} + 1$.

Proof. Since passing to linear combinations of rows and columns does not increase the rank of the matrix, it follows that for any polynomials $f_j(z), g_k(z)$, with $j, k = 0, \dots, \mathbf{r}$, the determinant $\operatorname{Det}(\phi(f_j \overline{g}_k))$ vanishes.

The determinant is linear in each column and ϕ is a linear functional, so we can write

$$\phi\left(f_0(z)\times\begin{vmatrix}\overline{g_0(z)} & \mu(f_1\overline{g}_0) & \dots & \phi(f_{\mathbf{r}}\overline{g}_0)\\ \overline{g_1(z)} & \phi(f_1\overline{g}_1) & \dots & \phi(f_{\mathbf{r}}\overline{g}_1)\\ \vdots & \vdots & \ddots & \vdots\\ \overline{g_{\mathbf{r}}(z)} & \phi(f_1\overline{g}_{\mathbf{r}}) & \dots & \phi(f_{\mathbf{r}}\overline{g}_{\mathbf{r}})\end{vmatrix}\right)=0.$$

We introduce the variable z_0 in place of z above and use ϕ_0 for ϕ acting in the variable z_0. We repeat this process in each column (using the variable z_j in column j and the notation ϕ_j for ϕ acting in z_j) to obtain

$$\phi_0\left(\phi_1\left(\dots\phi_{\mathbf{r}}\left(\prod_{k=0}^{\mathbf{r}} f_k(z_k)\det\big(g_j(z_k)\big)\right)\dots\right)\right)=0. \tag{3.2}$$

We now specialize to the case where each $f_i=z^{j_i}$, $g_i=z^{k_i}$ and arrive at (3.1), thus proving the implication $1\Rightarrow 2$. The converse implication follows by going along the above reasoning in the opposite direction. □

Proof of Theorem 3.1. We identify $\mathbb{C}$ and $\mathbb{R}^2$ with co-ordinates $z=x+iy$. Consider the functional $\phi(f)=\phi_\mu(f)=\int f(z)d\mu(z)$. Write Z for the N-tuple $(z_0,z_1,\dots,z_{\mathbf{r}})$ and $V_J(Z)$ for the determinant $\det\left(z_i^{k_j}\right)$. By Lemma 3.2,

$$\phi^{\otimes N}\left(Z^K\overline{V_J(Z)}\right)=0. \tag{3.3}$$

Taking finite sums of equations (3.3), we get for any polynomial $P(Z)$ in N variables:

$$\phi^{\otimes N}\left(P(Z)\overline{V_J(Z)}\right)=0. \tag{3.4}$$

By taking linear combinations of antisymmetric polynomials $V_J(Z)$ one can obtain any antisymmetric polynomial $Q(Z)$ (cf. [14]) for details). Thus,

$$\phi^{\otimes N}\left(P(Z)\overline{Q(Z)}\right)=0 \tag{3.5}$$

for any polynomial $P(Z)$ and any antisymmetric polynomial $Q(Z)$. In its turn, the polynomial $Q(Z)$ is divisible by the lowest degree antisymmetric polynomial, the Vandermonde polynomial $V(Z)=\prod_{0\leqslant j\leqslant k\leqslant\mathbf{r}}(z_j-z_k)$, $Q(Z)=Q_1(Z)V(Z)$ with a symmetric polynomial $Q_1(Z)$. We write (3.5) for Q of this form and P having the form $P(Z)=P_1(Z)V(Z)$. So we arrive at

$$\phi^{\otimes N}\left(P_1(Z)\overline{Q_1(Z)}|V(Z)|^2\right)=0\quad\text{for all symmetric } P_1 \text{ and } Q_1. \tag{3.6}$$

It is clear that finite sums of products of the form $P_1(Z)\overline{Q_1(Z)}$ (with P_1 and Q_1 symmetric) form an algebra A of functions on $\mathbb{C}$ which contains the constants and is closed under conjugation. It does not separate points because each element is constant on sets of points that are permutations of one another. Therefore, we define an equivalence relation $\sim$ on $\mathbb{C}^N : Z_1 \sim Z_2$ if and only if $Z_2 = \pi(Z_1)$ for some permutation π. Let $Z = (z_0, \ldots, z_{\mathbf{r}})$, and let $W = (w_0, \ldots, w_{\mathbf{r}})$. If $Z \not\sim W$ then the polynomials $p(t) = \prod(t - z_j)$ and $q(t) = \prod(t - w_j)$ have different zeros (or the same zeros with different orders). This implies that the coefficient of some power of t in $p(t)$ differs from the corresponding coefficient in $q(t)$. Thus, there is an elementary symmetric function that differs at Z and W. Consequently, A separates equivalence classes.

We give the quotient space $\mathbb{C}^N/\sim$ the standard quotient space topology. If K is any compact set in $\mathbb{C}^N$ that is invariant with respect to $\sim$, then $K/\sim$ is compact and Hausdorff. Also, any symmetric continuous function on $\mathbb{C}^N$ induces a continuous function on $\mathbb{C}^N/\sim$ (and conversely). Thus, we can apply the Stone–Weierstrass theorem (on $K/\sim$) to conclude that A is dense in the space of continuous symmetric functions, in the topology of uniform convergence on any compact set. Therefore, for any continuous symmetric function $f(Z)$

$$\int_{\mathbb{C}^N} f(Z)|V(Z)|^2 \, d\mu^{\otimes N}(Z) = 0. \tag{3.7}$$

If f is an arbitrary continuous function, the above integral will be the same as the corresponding integral with the symmetrization of f replacing f. This is because the function $|V(Z)|^2$ and the product measure $\mu^{\otimes N}$ are both invariant under permutations of the coordinates. We conclude that this integral vanishes for *any continuous* f and so the measure $|V(Z)|^2 \, d\mu^{\otimes N}(Z)$ must be zero. Thus, $\mu^{\otimes N}$ is supported on the set where V vanishes, i.e., on the set of Lebesgue measure zero. Since $\mu^{\otimes N}$ is absolutely continuous, it must be zero. □

The initial setting of Theorem 3.1 dealt with arbitrary measures, as it is explained in the Introduction. A more advanced result was obtained in [2], where Luecking's theorem was carried over to distributions.

Theorem 3.3. *Suppose that $F \in \mathcal{E}'(\Omega)$ is a distribution with compact support in $\Omega \subset \mathbb{C}$ and the Toeplitz operator $\mathcal{T}_F$ has finite rank $\mathbf{r}$. Then the distribution F is a finite combination of δ-distributions at some points in Ω and their derivatives,*

$$F = \sum_{j \leqslant \mathbf{r}} L_j \delta(z - z_j), \tag{3.8}$$

L_j being differential operators.

We start with some observations about distributions in $\mathcal{E}'(\mathbb{C})$. For such a distribution we denote by $\operatorname{psupp} F$ the complement of the unbounded component of the complement of $\operatorname{supp} F$.

Lemma 3.4. *Let $F \in \mathcal{E}'(\mathbb{C})$. Then the following two statements are equivalent*:

a) *there exists a distribution* $G \in \mathcal{E}'(\mathbb{C})$ *such that* $\frac{\partial G}{\partial \overline{z}} = F$; *moreover,* $\operatorname{supp} G \subset \operatorname{psupp} F$,

b) F *is orthogonal to all polynomials of* z *variable, i.e.,* $\langle F, z^k \rangle = 0$ *for all* $k \in \mathbb{Z}_+$.

Proof. The implication *a)* $\Longrightarrow$ *b)* follows from the relation

$$\langle F, z^k \rangle = \Big\langle \frac{\partial G}{\partial \overline{z}}, z^k \Big\rangle = \Big\langle G, \frac{\partial z^k}{\partial \overline{z}} \Big\rangle = 0. \tag{3.9}$$

We prove that *b)* $\Longrightarrow$ *a)*. Put $G := F * \frac{1}{\pi z} \in \mathcal{S}'(\mathbb{C})$, the convolution being well-defined because F has compact support. Since $\frac{1}{\pi z}$ is the fundamental solution of the Cauchy–Riemann operator $\frac{\partial}{\partial \overline{z}}$, we have $\frac{\partial G}{\partial \overline{z}} = F$ (cf., for example, [10, Theorem 1.2.2]). By the ellipticity of the Cauchy–Riemann operator, $\operatorname{singsupp} G \subset \operatorname{singsupp} F \subset \operatorname{supp} F$, in particular, this means that G is a smooth function outside $\operatorname{psupp} F$; moreover, G is analytic outside $\operatorname{psupp} F$ (by $\operatorname{singsupp} F$ we denote the singular support of the distribution F (cf., for example, [10], the largest open set where the distribution coincides with a smooth function). Additionally,

$$G(z) = \langle F, \frac{1}{\pi(z-w)} \rangle = \pi^{-1} \sum_{k=0}^{\infty} z^{-k-1} \langle F, w^k \rangle = 0$$

if $|z| > R$ and R is sufficiently large. By analyticity this implies $G(z) = 0$ for all z outside $\operatorname{psupp} F$. □

Proof of Theorem 3.3. The distribution F, as any distribution with compact support, is of finite order, therefore it belongs to some Sobolev space, $F \in H^s$ for certain $s \in \mathbb{R}^1$. If $s \geqslant 0$, F is a function and must be zero by Luecking's theorem. So, suppose that $s < 0$.

Consider the first $\mathbf{r} + 1$ columns in the matrix $\mathcal{A}(F)$, i.e.,

$$a_{kl} = (\mathcal{T}_F z^k, z^l) = \langle \sigma F, z^k \overline{z}^l \rangle, \quad l = 0, \dots \mathbf{r}; \quad k = 0, \dots . \tag{3.10}$$

Since the rank of the matrix $\mathcal{A}(F)$ is not greater than $\mathbf{r}$, the columns are linearly dependent, in other words, there exist coefficients $c_0, \dots, c_{\mathbf{r}}$ such that $\sum_{l=0}^{\mathbf{r}} a_{kl} c_l = 0$ for any $k \geqslant 0$. This relation can be written as

$$\langle F, z^k h_1(\overline{z}) \rangle = \langle h_1(\overline{z}) F, z^k \rangle = 0, \quad h_1(\overline{z}) = \sum_{k=0}^{\mathbf{r}} c_l \overline{z}^l. \tag{3.11}$$

Therefore, the distribution $h_1(\overline{z}) F \in H^s$ satisfies the conditions of Lemma 3.4 and hence there exists a compactly supported distribution $F^{(1)}$ such that $\frac{\partial F^{(1)}}{\partial \overline{z}} = h_1 F$. By the ellipticity of the Cauchy–Riemann operator, the distribution $F^{(1)}$ is less singular than F, $F^{(1)} \in H^{s+1}$. At the same time, Springer

http://www.springer.com/978-1-4419-1344-9

Around the Research of Vladimir Maz'ya III
Analysis and Applications
(Ed.)A. Laptev
2010, XXI, 386 p. 3 illus., Hardcover
ISBN: 978-1-4419-1344-9