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Profinite Groups

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1 Inverse and Direct Limits

1.1 Inverse or Projective Limits

In this section we define the concept of inverse (or projective) limit and establish some of its elementary properties. Rather than developing the concept and establishing those properties under the most general conditions, we restrict ourselves to inverse limits of topological spaces or topological groups. We leave the reader the task of extending and translating the concepts and results obtained here to other objects such as sets, (topological) rings, modules, graphs..., or to more general categories.

Let $I = (I, \preceq)$ denote a *directed partially ordered set* or *directed poset*, that is, I is a set with a binary relation $\preceq$ satisfying the following conditions:

(a) $i \preceq i$, for $i \in I$;
(b) $i \preceq j$ and $j \preceq k$ imply $i \preceq k$, for $i, j, k \in I$;
(c) $i \preceq j$ and $j \preceq i$ imply $i = j$, for $i, j \in I$; and
(d) if $i, j \in I$, there exists some $k \in I$ such that $i, j \preceq k$.

An *inverse* or *projective system* of topological spaces (respectively, topological groups) over I, consists of a collection $\{X_i \mid i \in I\}$ of topological spaces (respectively, topological groups) indexed by I, and a collection of continuous mappings (respectively, continuous group homomorphisms) $\varphi_{ij} : X_i \longrightarrow X_j$, defined whenever $i \succeq j$, such that the diagrams of the form

$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & X_k \\ & X_j \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

commute whenever they are defined, i.e., whenever $i, j, k \in I$ and $i \succeq j \succeq k$. In addition we assume that φ_{ii} is the identity mapping id_{X_i} on X_i. We shall denote such a system by $\{X_i, \varphi_{ij}, I\}$, or by $\{X_i, \varphi_{ij}\}$ if the index set I is clearly understood. If X is a fixed topological space (respectively, topological group), we denote by $\{X, \mathrm{id}\}$ the inverse system $\{X_i, \varphi_{ij}, I\}$, where $X_i = X$ for all $i \in I$, and φ_{ij} is the identity mapping $\mathrm{id} : X \longrightarrow X$. We say that $\{X, \mathrm{id}\}$ is the *constant inverse system* on X.

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Let Y be a topological space (respectively, topological group), $\{X_i, \varphi_{ij}, I\}$ an inverse system of topological spaces (respectively, topological groups) over a directed poset I, and let $\psi_i : Y \longrightarrow X_i$ be a continuous mapping (respectively, continuous group homomorphism) for each $i \in I$. These mappings ψ_i are said to be *compatible* if $\varphi_{ij}\psi_i = \psi_j$ whenever $j \preceq i$.

One says that a topological space (respectively, topological group) X together with compatible continuous mappings (respectively, continuous homomorphisms)

$$\varphi_i : X \longrightarrow X_i \quad (i \in I)$$

is an *inverse limit* or a *projective limit* of the inverse system $\{X_i, \varphi_{ij}, I\}$ if the following universal property is satisfied:

$$\begin{tikzcd} Y \arrow[r, dotted, "\psi"] \arrow[dr, "\psi_i"'] & X \arrow[d, "\varphi_i"] \\ & X_i \end{tikzcd}$$

whenever Y is a topological space (respectively, topological group) and $\psi_i : Y \longrightarrow X_i$ $(i \in I)$ is a set of compatible continuous mappings (respectively, continuous homomorphisms), then there is a unique continuous mapping (respectively, continuous homomorphism) $\psi : Y \longrightarrow X$ such that $\varphi_i\psi = \psi_i$ for all $i \in I$. We say that ψ is "induced" or "determined" by the compatible homomorphisms ψ_i.

The maps $\varphi_i : X \longrightarrow X_i$ are called *projections*. The projection maps φ_i are not necessarily surjections. We denote the inverse limit by (X, φ_i), or often simply by X, by abuse of notation.

If $\{X_i, I\}$ is a collection of topological spaces (respectively, topological groups) indexed by a set I, its *direct product* or *cartesian product* is the topological space (respectively, topological group) $\prod_{i\in I} X_i$, endowed with the product topology. In the case of topological groups the group operation is defined coordinatewise.

Proposition 1.1.1 *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of topological spaces (respectively, topological groups) over a directed poset I. Then*

(a) *There exists an inverse limit of the inverse system $\{X_i, \varphi_{ij}, I\}$;*
(b) *This limit is unique in the following sense. If (X, φ_i) and (Y, ψ_i) are two limits of the inverse system $\{X_i, \varphi_{ij}, I\}$, then there is a unique homeomorphism (respectively, topological isomorphism) $\varphi : X \longrightarrow Y$ such that $\psi_i\psi = \varphi_i$ for each $i \in I$.*

Proof. (a) Define X as the subspace (respectively, subgroup) of the direct product $\prod_{i\in I} X_i$ of topological spaces (respectively, topological groups) consisting of those tuples (x_i) that satisfy the condition $\varphi_{ij}(x_i) = x_j$ if $i \succeq j$.

Let

$$\varphi_i : X \longrightarrow X_i$$

denote the restriction of the canonical projection $\prod_{i\in I} X_i \longrightarrow X_i$. Then one easily checks that each φ_i is continuous (respectively, a continuous homomorphism), and that (X, φ_i) is an inverse limit.

(b) Suppose (X, φ_i) and (Y, ψ_i) are two inverse limits of the inverse system $\{X_i, \varphi_{ij}, I\}$.

$$\begin{tikzcd}
X \arrow[rr, dotted, bend left, "\varphi"] \arrow[dr, "\varphi_i"'] & & Y \arrow[ll, dotted, bend left, "\psi"] \arrow[dl, "\psi_i"] \\
& X_i &
\end{tikzcd}$$

Since the maps $\psi_i : Y \longrightarrow X_i$ are compatible, the universal property of the inverse limit (X, φ_i) shows that there exists a unique continuous mapping (respectively, continuous homomorphism) $\psi : Y \longrightarrow X$ such that $\varphi_i \psi = \psi_i$ for all $i \in I$. Similarly, since the maps $\varphi_i : X \longrightarrow X_i$ are compatible and (Y, ψ_i) is an inverse limit, there exists a unique continuous mapping (respectively, continuous homomorphism) $\varphi : X \longrightarrow Y$ such that $\psi_i \varphi = \varphi_i$ for all $i \in I$. Next observe that

$$\begin{tikzcd}
X \arrow[rr, dotted, bend left, "\psi\varphi"] \arrow[rr, dotted, bend right, "\mathrm{id}_X"'] \arrow[dr, "\varphi_i"'] & & X \arrow[dl, "\varphi_i"] \\
& X_i &
\end{tikzcd}$$

commutes for each $i \in I$. Since, by definition, there is only one map satisfying this property, one has that $\psi\varphi = \mathrm{id}_X$. Similarly, $\varphi\psi = \mathrm{id}_Y$. Thus φ is a homeomorphism (respectively, topological isomorphism). □

If $\{X_i, \varphi_{ij}, I\}$ is an inverse system, we shall denote its inverse limit by $\varprojlim_{i\in I} X_i$, or $\varprojlim_i X_i$, or $\varprojlim_I X_i$, or $\varprojlim X_i$, depending on the context.

Lemma 1.1.2 *If $\{X_i, \varphi_{ij}\}$ is an inverse system of Hausdorff topological spaces (respectively, topological groups), then $\varprojlim X_i$ is a closed subspace (respectively, closed subgroup) of $\prod_{i\in I} X_i$.*

Proof. Let $(x_i) \in \left(\prod X_i\right) - \left(\varprojlim X_i\right)$. Then there exist $r, s \in I$ with $r \succeq s$ and $\varphi_{rs}(x_r) \neq x_s$. Choose open disjoint neighborhoods U and V of $\varphi_{rs}(x_r)$ and x_s in X_s, respectively. Let U' be an open neighborhood of x_r in X_r, such that $\varphi_{rs}(U') \subseteq U$. Consider the basic open subset $W = \prod_{i\in I} V_i$ of $\prod_{i\in I} X_i$ where $V_r = U'$, $V_s = V$ and $U_i = X_i$ for $i \neq r, s$. Then W is a open neighborhood of (x_i) in $\prod_{i\in I} X_i$, disjoint from $\varprojlim X_i$. This shows that $\varprojlim X_i$ is closed. □

A topological space is *totally disconnected* if every point in the space is its own connected component. For example, a space with the discrete topology is totally disconnected, and so is the rational line. It is easily checked that the direct product of totally disconnected spaces is totally disconnected. The following result is an immediate consequence of Tychonoff's theorem, that asserts that the direct product of compact spaces is compact (cf. Bourbaki [1989], Ch. 1, Theorem 3), and the fact that a closed subset of a compact space is compact.

Proposition 1.1.3 *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of compact Hausdorff totally disconnected topological spaces* (*respectively, topological groups*) *over the directed set I. Then*

$$\varprojlim_{i \in I} X_i$$

is also a compact Hausdorff totally disconnected topological space (*respectively, topological group*).

Proposition 1.1.4 *Let $\{X_i, \varphi_{ij}\}$ be an inverse system of compact Hausdorff nonempty topological spaces X_i over the directed set I. Then*

$$\varprojlim_{i \in I} X_i$$

is nonempty. In particular, the inverse limit of an inverse system of nonempty finite sets is nonempty.

Proof. For each $j \in I$, define a subset Y_j of $\prod X_i$ to consist of those (x_i) with the property $\varphi_{jk}(x_j) = x_k$ whenever $k \preceq j$. Using the axiom of choice and an argument similar to the one used in Lemma 1.1.2, one easily checks that each Y_j is a nonempty closed subset of $\prod X_i$. Observe that if $j \preceq j'$, then $Y_j \supseteq Y_{j'}$; it follows that the collection of subsets $\{Y_j \mid j \in I\}$ has the finite intersection property (i.e., any intersection of finitely many Y_j is nonempty), since the poset I is directed. Then, one deduces from the compactness of $\prod X_i$ that $\bigcap Y_j$ is nonempty. Since

$$\varprojlim_{i \in I} X_i = \bigcap_{j \in I} Y_j,$$

the result follows. □

Let $\{X_i, \varphi_{ij}, I\}$ and $\{X'_i, \varphi'_{ij}, I\}$ be inverse systems of topological spaces (respectively, topological groups) over the same directed poset I. A *map* or a *morphism* of inverse systems

$$\Theta : \{X_i, \varphi_{ij}\} \longrightarrow \{X'_i, \varphi'_{ij}\},$$

consists of a collection of continuous mappings (respectively, continuous homomorphisms) $\theta_i : X_i \longrightarrow X'_i$ ($i \in I$) such that if $i \preceq j$, then the following diagram commutes

$$\begin{array}{ccc} X_j & \xrightarrow{\varphi_{ji}} & X_i \\ \theta_j \big\downarrow & & \big\downarrow \theta_i \\ X'_j & \xrightarrow{\varphi'_{ji}} & X'_i \end{array}$$

We say that the mappings θ_i are the *components* of Θ. A map

$$\Theta : \{X_i, \varphi_{ij}, I\} \longrightarrow \{X_i, \varphi_{ij}, I\}$$

of an inverse system to itself, whose components $\theta_i : X_i \longrightarrow X_i$ $(i \in I)$ are identity mappings, is called the identity map of the system $\{X_i, \varphi_{ij}, I\}$, and it is usually denoted by id. Composition of maps of inverse systems is defined in a natural way. That is, if

$$\Theta : \{X_i, \varphi_{ij}\} \longrightarrow \{X'_i, \varphi'_{ij}\},$$

with components θ_i, and

$$\Psi : \{X'_i, \varphi'_{ij}\} \longrightarrow \{X''_i, \varphi''_{ij}\},$$

with components ψ_i, are maps of inverse systems, then the components of the composition map

$$\Psi\Theta : \{X_i, \varphi_{ij}\} \longrightarrow \{X''_i, \varphi''_{ij}\},$$

are $\psi_i\theta_i$, $i \in I$. Thus one obtains a category of inverse systems of topological spaces (respectively, topological groups), whose objects are inverse systems of topological spaces (respectively, topological groups), and whose morphisms are maps of inverse systems.

Let $\{X_i, \varphi_{ij}\}$ and $\{X'_i, \varphi'_{ij}\}$ be inverse systems of topological spaces (respectively, topological groups) over the same directed poset I, and let $(X = \varprojlim X_i, \varphi_i)$ and $(X' = \varprojlim X'_i, \varphi'_i)$ be their corresponding inverse limits. Assume that

$$\Theta : \{X_i, \varphi_{ij}, I\} \longrightarrow \{X'_i, \varphi'_{ij}, I\}$$

is a map of inverse systems with components $\theta_i : X_i \longrightarrow X'_i$. Then the collection of compatible mappings

$$\theta_i\varphi_i : X \longrightarrow X'_i$$

induces a continuous mapping (respectively, continuous homomorphism)

$$\varprojlim \Theta = \varprojlim_{i \in I} \theta_i : \varprojlim_{i \in I} X_i \longrightarrow \varprojlim_{i \in I} X'_i.$$

Observe that $\varprojlim$ is a functor from the category of inverse systems of topological spaces (respectively,topological groups) over I to the category of topological spaces (respectively, topological groups); that is, $\varprojlim(\Psi\Theta) = \varprojlim \Psi \varprojlim \Theta$, and if id is the identity map on the inverse system $\{X_i, \varphi_{ij}, I\}$, then $\varprojlim \mathrm{id}$ is the identity map on the topological space (respectively, topological group) $\varprojlim_{i\in I} X_i$.

If the components $\theta_i : X_i \longrightarrow X_i'$ of a map $\Theta : \{X_i, \varphi_{ij}\} \longrightarrow \{X_i', \varphi_{ij}'\}$ of inverse systems are embeddings, then obviously, so is

$$\varprojlim \theta_i : \varprojlim X_i \hookrightarrow \varprojlim X_i'.$$

In contrast, if each of the components θ_i is an onto mapping, $\varprojlim \theta_i$ is not necessarily onto. For example, consider the natural numbers $I = \mathbf{N}$, with the usual partial ordering, as our indexing poset; define two inverse systems (of discrete spaces) over I as follows: the constant inverse system $\{\mathbf{Z}, \mathrm{id}\}$, and the inverse system $\{\mathbf{Z}/p^n\mathbf{Z}, \varphi_{nm}\}$, where $\varphi_{nm} : \mathbf{Z}/p^n\mathbf{Z} \longrightarrow \mathbf{Z}/p^m\mathbf{Z}$ is the natural projection for $m \leq n$. For each $n \in \mathbf{N}$, define $\theta_n : \mathbf{Z} \longrightarrow \mathbf{Z}/p^n\mathbf{Z}$ to be the canonical epimorphism; then

$$\Theta = \{\theta_n\} : \{\mathbf{Z}, \mathrm{id}\} \longrightarrow \{\mathbf{Z}/p^n\mathbf{Z}, \varphi_{nm}\}$$

is a map of inverse systems. Observe that the inverse limit of the first system is $\mathbf{Z}$, while the inverse limit of the second can be identified with

$$\varprojlim \mathbf{Z}/p^n\mathbf{Z} = \{(x_n) \mid x_n \in \mathbf{Z}, x_n \equiv x_m \pmod{p^m} \text{ if } m \leq n\}.$$

The image of $\mathbf{Z}$ in $\varprojlim \mathbf{Z}/p^n\mathbf{Z}$ under $\varprojlim \theta_n$ is the set of all constant tuples $\{(a_n) \mid a_n = t, t \in \mathbf{Z}\}$. On the other hand, the tuple (b_n), where $b_n = 1 + p + \cdots + p^{n-1}$, is in $\varprojlim \mathbf{Z}/p^n\mathbf{Z}$, but it is not constant. Thus $\varprojlim \theta_n$ is not onto.

However, for inverse systems of compact Hausdorff spaces, one has the following result.

Lemma 1.1.5 *Let $\Theta : \{X_i, \varphi_{ij}, I\} \longrightarrow \{X_i', \varphi_{ij}', I\}$ be a map of inverse systems of compact Hausdorff topological spaces (respectively, topological groups), and assume that each component $\theta_i : X_i \longrightarrow X_i'$ $(i \in I)$ is onto. Then*

$$\varprojlim \Theta = \varprojlim_{i\in I} \theta_i : \varprojlim_{i\in I} X_i \longrightarrow \varprojlim_{i\in I} X_i'$$

is onto.

Proof. Let $(x_i') \in \varprojlim X_i'$. Put $\widetilde{X}_i = \theta_i^{-1}(x_i')$ $(i \in I)$. Since $\widetilde{X}_i$ is closed in the compact space X_i, it follows that $\widetilde{X}_i$ is compact $(i \in I)$. Observe that $\varphi_{ij}(\widetilde{X}_i) \subseteq \widetilde{X}_j$ for $i \succeq j$. Therefore, $\{\widetilde{X}_i, \varphi_{ij}\}$ is an inverse system of nonempty compact topological spaces (respectively, compact topological groups). By Proposition 1.1.4, $\varprojlim \widetilde{X}_i \neq \emptyset$. Let $(x_i) \in \varprojlim \widetilde{X}_i \subseteq \varprojlim X_i$. Then one has $(\varprojlim \Theta)(x_i) = (x_i')$. □

Corollary 1.1.6 *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of compact Hausdorff spaces and X a compact Hausdorff space. Suppose that $\{\varphi_i : X \longrightarrow X_i\}_{i\in I}$ is a set of compatible continuous surjective mappings. Then the corresponding induced mapping $\theta : X \longrightarrow \varprojlim X_i$ is onto.*

Proof. Consider the constant inverse system $\{X, \mathrm{id}\}$ over I. The collection $\{\theta_i\}_{i\in I}$ can be thought of as a map from $\{X, \mathrm{id}, I\}$ to $\{X_i, \varphi_{ij}, I\}$. Then $\theta = \varprojlim \theta_i$, and the result follows from the above proposition. □

Lemma 1.1.7 *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of topological spaces over a directed set I, and let $\rho_i : X \longrightarrow X_i$ be compatible surjections from the space X onto the spaces X_i $(i \in I)$. Then either $\varprojlim X_i = \emptyset$ or the corresponding induced mapping $\rho : X \longrightarrow \varprojlim X_i$ maps X onto a dense subset of $\varprojlim X_i$.*

Proof. Suppose $\varprojlim X_i \neq \emptyset$. A general basic open subset V of $\varprojlim X_i$ can be described as follows: let $i_1, \dots, i_n$ be a finite subset of I and let U_{i_j} be an open subset of X_{i_j} $(j = 1, \dots, n)$; let

$$V = (\varprojlim X_i) \cap \left(\prod_{i\in I} V_i\right)$$

where $V_{i_j} = U_{i_j}$ $(j = 1, \dots, n)$ and $V_i = X_i$ for $i \neq i_1, \dots, i_n$. Assume such V is not empty. We have to show that $\rho(X) \cap V \neq \emptyset$. Let $i_0 \succeq i_1, \dots, i_n$, and let $y = (y_i) \in V$. Choose $x \in X$ so that $\rho_{i_0}(x) = y_{i_0}$. Then $\rho(x) \in V$. □

Corollary 1.1.8 *Let $\{X_i, \varphi_{ij}\}$ be an inverse system of compact Hausdorff spaces, $X = \varprojlim X_i$, and let $\varphi_i : X \longrightarrow X_i$ be the projections.*

(a) *If Y is a closed subspace of X, then $Y = \varprojlim \varphi_i(Y)$.*

(b) *If Y is a subspace of X, then*

$$\overline{Y} = \varprojlim \varphi_i(Y),$$

where $\overline{Y}$ is the closure of Y in X.

(c) *If Y and Y' are subspaces of X and $\varphi_i(Y) = \varphi_i(Y')$ for each i, then their closures in X coincide: $\overline{Y} = \overline{Y'}$.*

Proof. (a) Observe that there are obvious embeddings

$$Y \hookrightarrow \varprojlim \varphi_i(Y) \hookrightarrow \varprojlim X_i = X.$$

Moreover, by Corollary 1.1.6, the first of these embeddings is onto. Hence, $Y = \varprojlim \varphi_i(Y)$.

(b) According to Lemma 1.1.7, Y embeds as a dense subset of $\varprojlim \varphi_i(Y)$. Arguing as in Lemma 1.1.2 one sees that $\varprojlim \varphi_i(Y)$ is closed in X. Hence the result follows.

(c) This follows from (a) and (b). □

Let $(I, \preceq)$ be a directed poset. Assume that I' is a subset of I in such a way that $(I', \preceq)$ becomes a directed poset. We say that I' is *cofinal* in I if for every $i \in I$ there is some $i' \in I'$ such that $i \preceq i'$. If $\{X_i, \varphi_{ij}, I\}$ is an inverse system and I' is cofinal in I, then $\{X_i, \varphi_{ij}, I'\}$ becomes an inverse system in an obvious way, and we say that $\{X_i, \varphi_{ij}, I'\}$ is a *cofinal subsystem* of $\{X_i, \varphi_{ij}, I\}$.

Assume that $\{X_i, \varphi_{ij}, I'\}$ is a cofinal subsystem of $\{X_i, \varphi_{ij}, I\}$ and denote by $(\varprojlim_{i' \in I'} X_{i'}, \varphi'_{i'})$ and $(\varprojlim_{i \in I} X_i, \varphi_i)$ their corresponding inverse limits. For $j \in I$, let $j' \in I'$ be such that $j' \succeq j$. Define

$$\overline{\varphi}_j : \varprojlim_{I'} X_{i'} \longrightarrow X_j$$

as the composition of canonical mappings $\varphi_{j'j}\varphi'_{j'}$. Observe that the maps $\overline{\varphi}_j$ are well-defined (independent of the choice of j') and compatible. Hence they induce a map

$$\overline{\varphi} : \varprojlim_{I'} X_{i'} \longrightarrow \varprojlim_{I} X_i$$

such that $\varphi_j \overline{\varphi} = \overline{\varphi}_j$ $(j \in I)$. We claim that the mapping $\overline{\varphi}$ is a bijection. Note that if $(x_{i'}) \in \varprojlim_{i' \in I'} X_{i'}$ and $\overline{\varphi}(x_{i'}) = (y_i)$, then $y_{i'} = x_{i'}$ for $i' \in I'$. It follows that $\overline{\varphi}$ is an injection since I' is cofinal in I. To see that $\overline{\varphi}$ is a surjection, let $(y_i) \in \varprojlim_{i \in I} X_i$ and consider the element $(x_{i'})$, where $x_{i'} = y_{i'}$ for every $i' \in I'$. Then $(x_{i'}) \in \varprojlim_{i' \in I'} X_{i'}$ and clearly, $\overline{\varphi}(x_{i'}) = (y_i)$. This proves the claim. We record these results in the following lemma.

Lemma 1.1.9 *Let $\{X_i, \varphi_{ij}, I\}$ be a inverse system of compact topological spaces* (*respectively, compact topological groups*) *over a directed poset I and assume that I' is a cofinal subset of I. Then*

$$\varprojlim_{i \in I} X_i \cong \varprojlim_{i' \in I'} X_{i'}.$$

Proof. According to the above observations,

$$\overline{\varphi} : \varprojlim_{I'} X_{i'} \longrightarrow \varprojlim_{I} X_i$$

is a continuous bijection (respectively, group isomorphism). Since $\varprojlim_{i' \in I'} X_{i'}$ and $\varprojlim_{i \in I} X_i$ are compact spaces (respectively, compact topological groups), it follows that $\overline{\varphi}$ is a homeomorphism (respectively, topological isomorphism). We identify $\varprojlim_{i' \in I'} X_{i'}$ and $\varprojlim_{i \in I} X_i$ by means of this homeomorphism (respectively, topological isomorphism). □

An inverse system $\{X_i, \varphi_{ij}, I\}$ is called a *surjective inverse system* if each of the mappings φ_{ij} $(i \succeq j)$ is surjective. By Corollary 1.1.8(a), for any inverse system $\{X_i, \varphi_{ij}, I\}$, there is a corresponding surjective inverse system $\{\varphi_i(X), \varphi'_{ij}, I\}$ (where φ'_{ij} is just the restriction of φ_{ij} to $\varphi_i(X)$) with the same inverse limit X.

Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of topological spaces X_i over a poset I. Put $X = \varprojlim X_i$, and let $\varphi_j : X \longrightarrow X_j$ be the projection map. Assume that $X \neq \emptyset$. If φ_j is a surjection for each $i \in I$, then evidently $\varphi_{rs} : X_r \longrightarrow X_s$ is a surjection for all $r, s \in I$ with $r \succeq s$. The converse is not necessarily true. However, as the following proposition shows, the converse holds if one assumes in addition that each of the X_i is compact.

Proposition 1.1.10 *Let $\{X_i, \varphi_{ij}, I\}$ be a surjective inverse system of compact Hausdorff nonempty topological spaces X_i over a poset I. Then for each $j \in I$, the projection map $\varphi_j : \varprojlim X_i \longrightarrow X_j$ is a surjection.*

Proof. Fix $j \in I$. The set $I_j = \{i \in I \mid i \succeq j\}$ is cofinal in I; so, by Lemma 1.1.9, $\varprojlim_{i \in I_j} X_i \cong \varprojlim_{i \in I} X_i$. Therefore, we may assume that $i \succeq j$ for every $i \in I$. Let $x_j \in X_j$ and set $Y_r = \varphi_{rj}^{-1}(x_j)$ for $r \in I$. Since φ_{rj} is onto and continuous, Y_r is a nonempty compact subset of X_r $(r \in I)$. Furthermore, if $r \succeq s$ are indices in I, then $\varphi_{rs}(Y_r) \subseteq Y_s$. Hence $\{Y_r, \varphi_{rs}, I\}$ is an inverse system. According to Proposition 1.1.4, $\varprojlim Y_r \neq \emptyset$. Let $(y_r) \in \varprojlim Y_i \subseteq \varprojlim X_i$. Then $\varphi_j(y_r) = x_j$. $\square$

In what follows we shall be specially interested in topological spaces X that arise as inverse limits

$$X = \varprojlim_{i \in I} X_i$$

of finite spaces X_i endowed with the discrete topology. We call such a space a *profinite space* or a *Boolean space*. Before we give some characterizations of profinite spaces, we need the following lemma.

Lemma 1.1.11 *Let X be a compact Hausdorff topological space and let $x \in X$. Then the connected component C of x is the intersection of all clopen (i.e., closed and open) neighborhoods of x.*

Proof. Let $\{U_t \mid t \in T\}$ be the family of all clopen neighborhoods of x, and put

$$A = \bigcap_{t \in T} U_t.$$

It is clear that every clopen neighborhood of x contains the connected component C of x; and so $C \subseteq A$. Therefore, it suffices to show that A is connected. Assume that $A = U \cup V$, $U \cap V = \emptyset$ with both U and V closed in A (and so, in X). We need to prove that either U or V is empty. Since X is Hausdorff and U and V are compact and disjoint, there exist open sets U' and V' in X such that $U' \supseteq U$, $V' \supseteq V$ and $U' \cap V' = \emptyset$. So,

$$[X - (U' \cup V')] \cap A = \emptyset.$$

Now, $X - (U' \cup V')$ is closed; hence, by the compactness of X, there exists a finite subfamily T' of T such that

$$[X - (U' \cup V')] \cap \left[\bigcap_{t' \in T'} U_{t'} \right] = \emptyset.$$

Observe that $B = \bigcap_{t' \in T'} U_{t'}$ is a clopen neighborhood of x, since T' is finite. On the other hand,

$$x \in (B \cap U') \cup (B \cap V') = B.$$

Say $x \in B \cap U'$. Plainly $B \cap U'$ is open, but it is also closed because $B \cap V'$ is open and $(X - B \cap V') \cap B = B \cap U'$. Therefore, $A \subseteq B \cap U' \subseteq U'$. Hence $A \cap V \subseteq A \cap V' = \emptyset$, and thus $V = \emptyset$. □

We say that an equivalence relation R on a topological space X is *open* (respectively, *closed*) if for every $x \in X$, the equivalence class xR of x is open (respectively, closed) in X. If R is open, then it is closed (xR is the complement of a union of open sets).

Observe that R is open in the above sense if and only if R considered as a subset of $X \times X$ is open. Indeed, assume that R is open, and let $(x, y) \in R$ ($x, y \in X$); then $xR \times yR$ is an open neighborhood of (x, y) contained in R, and hence R is an open subset of $X \times X$. Conversely, assume that R is an open subset of $X \times X$; since $(x, x) \in R$, there exists an open neighborhood U of x in X such that $U \times U \subseteq R$; hence $U \subseteq xR$, proving that xR is open in X, and thus that R is an open equivalence relation.

Theorem 1.1.12 *Let X be a topological space. Then the following conditions are equivalent.*

(a) *X is a profinite space*;
(b) *X is compact Hausdorff and totally disconnected*;
(c) *X is compact Hausdorff and admits a base of clopen sets for its topology.*

Proof. (a) ⇒ (b): Let X be a profinite space. Say $X = \varprojlim_{i \in I} X_i$, where each X_i is a finite space. By Proposition 1.1.3, X is compact Hausdorff and totally disconnected.

(b) ⇒ (c): Let X be a compact Hausdorff and totally disconnected space. Let W be an open neighborhood of a point x in X. We must show that W contains a clopen neighborhood of x. Let $\{U_t \mid t \in T\}$ be the family of all clopen neighborhoods of x. According to Lemma 1.1.11,

$$\{x\} = \bigcap_{t \in T} U_t.$$

Since $X - W$ is closed and disjoint from $\bigcap_{t\in T} U_t$, we deduce from the compactness of X that there is a finite subset T' of T such that

$$(X - W) \cap \left(\bigcap_{t\in T'} U_t \right) = \emptyset.$$

Thus $\bigcap_{t\in T'} U_t$ is a clopen neighborhood of x contained in W, as desired.

(c) $\Rightarrow$ (a): Suppose that X is compact Hausdorff and admits a base of clopen sets for its topology. Denote by $\mathcal{R}$ the collection of all open equivalence relations R on X; for such R, the space X/R is finite and discrete since X is compact. The set $\mathcal{R}$ is naturally ordered as follows: if $R, R' \in \mathcal{R}$, then $R \succeq R'$ if and only if $xR \subseteq xR'$ for all $x \in X$. Then $\mathcal{R}$ is a poset. To see that this poset is directed, let R_1 and R_2 be two equivalence relations on X. Define its intersection $R_1 \cap R_2$ to be the equivalence relation corresponding to the partition of X obtained by intersecting each equivalence class of R_1 with each equivalence class of R_2. Clearly $R_1 \cap R_2 \succeq R_1, R_2$. Now, if $R, R' \in \mathcal{R}$ and $R \succeq R'$, define $\varphi_{RR'} : X/R \longrightarrow X/R'$ by $\varphi_{RR'}(xR) = xR'$. Then $\{X/R, \varphi_{RR'}\}$ is an inverse system over $\mathcal{R}$. We shall show that

$$X \cong \varprojlim_{R\in\mathcal{R}} X/R.$$

Let

$$\psi : X \longrightarrow \varprojlim_{R\in\mathcal{R}} X/R$$

be the continuous mapping induced by the canonical continuous surjections

$$\psi_R : X \longrightarrow X/R.$$

By Corollary 1.1.6, ψ is a continuous surjection. To prove that ψ is a homeomorphism, it suffices then to prove that it is an injection, since X is compact. Let $x, y \in X$. By hypothesis, there exists a clopen neighborhood U of x that excludes y. Consider the equivalence relation R' on X with two equivalence classes: U and $X - U$. Clearly, $R' \in \mathcal{R}$ and $\psi_{R'}(x) \neq \psi_{R'}(y)$. So, $\psi(x) \neq \psi(y)$. Thus, ψ is an injection. $\square$

A topological space X is said to *satisfy the second axiom of countability* if it has a countable base of open sets; such space is also called *second countable* or *countably based*. A topological space X is said to *satisfy the first axiom of countability* if each point of X has a countable fundamental system of neighborhoods; such space is also called *first countable*.

Corollary 1.1.13 *A profinite space X is second countable if and only if*

$$X \cong \varprojlim{}_{i\in I} X_i,$$

where $(I, \preceq)$ is a countable totally ordered set and each X_i is a finite discrete space.

Proof. Suppose X is profinite and second countable. Consider the set $\mathcal{R}$ of all open equivalence relations on X. For $R \in \mathcal{R}$, xR is a finite union of basic open set. Hence $\mathcal{R}$ is countable. Say $\mathcal{R} = \{R_1, R_2, \dots\}$. For each natural number i, define $R'_i = R_1 \cap \cdots \cap R_i$. Then $R'_1 \preceq R'_2 \preceq \cdots$ and $\{R'_i \mid i \in \mathbf{N}\}$ is cofinal in $\mathcal{R}$. As seen in the proof of the implication (c) $\Rightarrow$ (a) in the theorem, $X = \varprojlim_{R\in\mathcal{R}} X/R$. Thus $X = \varprojlim_{i\in\mathbf{N}} X/R'_i$.

Conversely assume that $X = \varprojlim_{i\in I} X_i$, where the poset $(I, \preceq)$ is countable and each X_i is a finite discrete space. Then obviously $\prod_{i\in I} X_i$ is second countable and profinite; thus so is X. $\square$

Exercise 1.1.14 Let $\{X_i \mid i \in I\}$ be a collection of spaces. Prove that

$$\prod_{i\in I} X_i$$

can be expressed as an inverse limit of direct products $\prod_{i\in F} X_i$, where F runs through the finite subsets of I.

Exercise 1.1.15 Let $\{X_i, \varphi_{ij}\}$ be an inverse system of topological spaces indexed by a poset I, $X = \varprojlim X_i$, and denote by $\varphi_i : X \longrightarrow X_i$ the projection map. Assume that for each $i \in I$, $\mathcal{U}_i$ is a base of open sets of X_i. Prove that $\{\varphi_i^{-1}(U) \mid U \in \mathcal{U}_i, i \in I\}$ is a base of open sets of X.

Lemma 1.1.16

(a) *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of profinite spaces. Let*

$$X = \varprojlim_{i\in I} X_i$$

and denote by $\varphi_i : X \longrightarrow X_i$ the projection map $(i \in I)$. Let $\rho : X \longrightarrow Y$ be a continuous mapping onto a discrete finite space Y. Then ρ factors through some φ_k, that is, there exists some $k \in I$ and some continuous mapping $\rho' : X_k \longrightarrow Y$ such that $\rho = \rho'\varphi_k$.

(b) *Let $\{G_i, \varphi_{ij}, I\}$ be an inverse system of topological groups with underlying profinite spaces. Let*

$$G = \varprojlim_{i\in I} G_i$$

and denote by $\varphi_i : G \longrightarrow G_i$ the projection continuous homomorphism $(i \in I)$. Let $\beta : G \longrightarrow H$ be a continuous homomorphism into a discrete finite group H. Then β factors through some φ_k, that is, there exists some $k \in I$ and some continuous homomorphism $\beta' : G_k \longrightarrow H$ such that $\beta = \beta'\varphi_k$.

Proof. (a) Assume first that each φ_i is a surjection. Let $Y = \{y_1, \dots, y_r\}$, and consider the clopen subsets $U_i = \rho^{-1}(y_i)$ $(i = 1, \dots, r)$ of X. Clearly $X = \bigcup_{i=1}^r U_i$, and $U_i \cap U_j = \emptyset$ if $i \neq j$. Fix i. For each $x \in U_i$ choose an index $k_x \in I$ and a clopen neighborhood $V_x = V_x^i$ of $\varphi_{k_x}(x)$ in X_{k_x} such that $\varphi_{k_x}^{-1}(V_x) \subseteq U_i$ (see Exercise 1.1.15). Put $W_x = \varphi_{k_x}^{-1}(V_x)$. By the compactness of U_i, there are finitely many points $x_1, \dots, x_{t_i}$ in U_i such that $U_i = W_{x_1} \cup \cdots \cup W_{x_{t_i}}$. Choose an index $k \in I$ such that $k \geq k_{x_1}, \dots, k_{x_{t_i}}$. Replacing V_{x_s} by $\varphi_{kk_{x_s}}^{-1}(V_{x_s})$ $(s = 1, \dots, t_i)$, we may assume that $k_{x_1} = \cdots = k_{x_{t_i}} = k$. Note that this k depends on i; however, since I is directed, we may assume that in fact k is valid for all $i = 1, \dots, r$. Hence we have constructed clopen subsets $V_1^i, \dots, V_{t_i}^i$ of X_k such that $U_i = \bigcup_{s=1}^{t_i} \varphi_k^{-1}(V_s^i)$ $(i = 1, \dots, r)$. Put $V^i = \bigcup_{s=1}^{t_i} V_s^i$. Then $V^i \cap V^j = \emptyset$ if $i \neq j$ $(1 \leq i, j \leq r)$; furthermore, $X_k = \bigcup_{i=1}^{r} V^i$ since φ_k is a surjection. Define $\rho' : X_k \longrightarrow Y$ by $\rho'(x) = y_i$ if $x \in V^i$. Then ρ' is a continuous mapping since the V^i are clopen and form a disjoint covering of X. Clearly $\rho = \rho' \varphi_k$.

To finish part (a), consider now the case when the projection maps φ_i are not necessarily surjective. By the construction above, there exists some $k \in I$ and a continuous surjection $\mu : \varphi_k(X) \longrightarrow Y$ such that $\rho = \mu \varphi_k$. Hence, it suffices to extend μ to a continuous map $\rho' : X_k \longrightarrow Y$. Put $Z = \varphi_k(X)$. For each $i = 1, \dots, r$, let $W_i = \mu^{-1}(y_i)$. Then $Z = W_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} W_r$ and each W_i is clopen in Z. Since X_k is a profinite space and Z is closed in X_k, there exist clopen subsets $W_1', \dots, W_r'$ of X_k such that $X_k = W_1' \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} W_r'$ and $W_i = W_i' \cap Z$ $(i = 1, \dots, r)$. Define $\rho'(x) = y_i$ for $x \in W_i'$ $(i = 1, \dots, r)$. Then ρ' is clearly continuous and extends μ. This ends the proof of part (a).

(b) Thinking of G, H and each G_i as topological spaces, we infer from part (a) that β factors through a continuous function $\beta_{i_0} : G_{i_0} \longrightarrow H$, for some $i_0 \in I$. However β_{i_0} need not be a homomorphism. Put $I_0 = \{i \in I \mid i \succeq i_0\}$. For each $i \in I_0$, define $\beta_i : G_i \longrightarrow H$, by $\beta_i = \beta_{i_0} \varphi_{ii_0}$; then clearly $\beta = \beta_i \varphi_i$. We claim that for some $k \in I_0$, the map β_k is a homomorphism. To see this consider the continuous map

$$\eta : G \times G \longrightarrow H \times H, \qquad (g_1, g_2) \mapsto (\beta(g_1)\beta(g_2), \beta(g_1 g_2)),$$

and the analogous continuous maps $\eta_i : G_i \times G_i \longrightarrow H \times H$, for each $i \in I_0$, replacing β by β_i. It is easy to check that

$$G \times G = \varprojlim_{i \succeq i_0} G_i \times G_i, \qquad \eta = \varprojlim_{i \in I_0} \eta_i,$$

and

$$\eta(G \times G) = \varprojlim_{i \in I_0} \eta_i(G_i \times G_i) = \bigcap_{i \succeq i_0} \eta_i(G_i \times G_i).$$

Since $\eta_i(G_i \times G_i)$ is contained in the finite set $H \times H$ and since I_0 is a directed poset, it follows that

$$\eta(G \times G) = \eta_k(G_k \times G_k),$$

for some $k \in I_0$. Next observe that since β is a homomorphism, $\eta(G \times G) \subseteq \Delta = \{(h, h) \mid h \in H\}$. Therefore $\eta_k(G_k \times G_k) \subseteq \Delta$; thus η_k is a homomorphism. Put $\beta' = \eta_k$. $\square$

1.2 Direct or Inductive Limits

In this section we study direct (or inductive) systems and their limits. The definitions and some of the properties obtained here are found by dualizing the corresponding ones in the case of inverse (or projective) limits developed in Section 1.1; however there some specific results for direct limits that we want to emphasize. Again, we shall not try to develop the theory under the most general conditions; we are mainly interested in direct limits of abelian groups (or modules). So, to avoid unnecessary repetitions, we shall work within the category of abelian groups and leave the reader the task of translating the results for other categories (sets, rings, modules, graphs, etc.).

Let $I = (I, \preceq)$ be a partially ordered set (see Section 1.1) A *direct* or *inductive system* of abelian groups over I consists of a collection $\{A_i\}$ of abelian groups indexed by I and a collection of homomorphisms $\varphi_{ij} : A_i \longrightarrow A_j$, defined whenever $i \preceq j$, such that the diagrams of the form

$$\begin{tikzcd} A_i \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & A_k \\ & A_j \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

commute whenever $i \preceq j \preceq k$.

In addition, we assume that φ_{ii} is the identity mapping id_{A_i} on A_i. We shall denote such a system by $\{A_i, \varphi_{ij}, I\}$, or by $\{A_i, \varphi_{ij}\}$ if the index set I is clearly understood. If A is a fixed abelian group, we denote by $\{A, \mathrm{id}\}$ the direct system $\{A_i, \varphi_{ij}\}$, where $A_i = A$ for all $i \in I$, and φ_{ij} is the identity mapping $\mathrm{id} : A \longrightarrow A$. We say that $\{A, \mathrm{id}\}$ is the *constant direct system* on A.

Let A be an abelian group, $\{A_i, \varphi_{ij}, I\}$ a direct system of abelian groups over a directed poset I and assume that $\psi_i : A_i \longrightarrow A$ is a homomorphism for each $i \in I$. These mappings ψ_i are said to be *compatible* if $\psi_j \varphi_{ij} = \psi_i$ whenever $i \preceq j$. One says that an abelian group A together with compatible homomorphisms

$$\varphi_i : A_i \longrightarrow A$$

$(i \in I)$ is a *direct limit* or an *inductive limit* of the direct system $\{A_i, \varphi_{ij}, I\}$, if the following universal property is satisfied:

$$\begin{tikzcd} A \arrow[r, dotted, "\psi"] & B \\ A_i \arrow[u, "\varphi_i"] \arrow[ur, "\psi_i"'] & \end{tikzcd}$$

whenever B is an abelian group and $\psi_i : A_i \longrightarrow B$ $(i \in I)$ is a set of compatible homomorphisms, then there exists a unique homomorphism

$$\psi : A \longrightarrow B$$

such that $\psi\varphi_i = \psi_i$ for all $i \in I$. We say that ψ is "induced" or "determined" by the compatible homomorphisms ψ_i.

Proposition 1.2.1 *Let $\{A_i, \varphi_{ij}, I\}$ be a direct system of abelian groups over a directed poset I. Then there exists a direct limit of the system. Moreover, this limit is unique in the following sense: if (A, φ_i) and (A', φ'_i) are two limits, then there is a unique isomorphism $\eta : A \longrightarrow A'$ such that $\varphi'_i = \eta\varphi_i$ for each $i \in I$.*

Proof. The uniqueness is immediate. To show the existence of the direct limit of the system $\{A_i, \varphi_{ij}, I\}$, let U be the disjoint union of the groups A_i. Define a relation $\sim$ on U as follows: we say that $x \in A_i$ is equivalent to $y \in A_j$ if there exists $k \in I$ with $k \succeq i, j$ such that $\varphi_{ik}(x) = \varphi_{jk}(y)$. This is an equivalence relation. Denote by $\tilde{x}$ the equivalence class of $x \in A_i$ under this relation. Denote by A the set of all equivalence classes of U. Given $x \in A_i$ and $y \in A_j$ consider an index $k \in I$ with $k \succeq i, j$, and define $\tilde{x} + \tilde{y}$ to be the class of $\varphi_{ik}(x) + \varphi_{jk}(y)$; this is easily seen to be well-defined. Then A becomes an abelian group under this operation (its zero element is the class represented by the zero of A_i for any $i \in I$). For each $i \in I$, let $\varphi_i : A_i \longrightarrow A$ be given by $\varphi_i(x) = \tilde{x}$; then φ_i is a homomorphism. To check that (A, φ_i) is a direct limit of the direct system $\{A_i, \varphi_{ij}, I\}$, let $\psi_i : A_i \longrightarrow B$ $(i \in I)$ be a collection of compatible homomorphisms into an abelian group B. Define the induced homomorphism $\psi : A \longrightarrow B$ as follows. Let $a \in A$; say $a = \varphi_i(x)$ for some $x \in A_i$ and $i \in I$. Then define $\psi(a) = \psi_i(x)$. Observe that ψ is a well-defined homomorphism and $\psi\varphi_i = \psi_i$ for all $i \in I$. Furthermore, ψ is the only possible homomorphism satisfying these conditions. $\square$

If $\{A_i, \varphi_{ij}, I\}$ is a direct system, we denote its direct limit by $\varinjlim_{i\in I} A_i$, or $\varinjlim_i A_i$, or $\varinjlim_I A_i$, or $\varinjlim A_i$, depending on the context.

Exercise 1.2.2 Let $\{A_i, \varphi_{ij}, I\}$ be a direct system of abelian groups over a directed poset I, and let I' be a cofinal subset of I. Show that the groups $\{A_i \mid i \in I'\}$ form in a natural way a direct system of abelian groups over I', and

$$\varinjlim_{i\in I} A_i = \varinjlim_{i\in I'} A_i.$$

The following exercise provides an alternative way of constructing direct limits; this procedure is the dual of the construction for inverse limits used in the proof of Proposition 1.1.1.

Exercise 1.2.3 Let $\{A_i, \varphi_{ij}, I\}$ be a direct system of abelian groups over a directed poset I. Define A to be the quotient group of the direct sum $\bigoplus_{i\in I} A_i$ modulo the subgroup R generated by the elements of the form $\varphi_{ij}(x) - x$ for all $x \in A_i$, $i \in I$ and $i \preceq j$. There are natural homomorphisms $\varphi_i : A_i \longrightarrow A$. Prove that A together with these homomorphisms is a direct limit of the system $\{A_i, \varphi_{ij}, I\}$.

Proposition 1.2.4 *Let* $\{A_i, \varphi_{ij}\}$ *be a direct system of abelian groups over a directed poset* I, $A = \varinjlim A_i$ *its direct limit and* $\varphi_i : A_i \longrightarrow A$ *the canonical homomorphisms. Then*

(a) $A = \bigcup_{i\in I} \varphi_i(A_i)$;
(b) *Let* $x \in A_i$ *and assume* $\varphi_i(x) = 0$; *then there exists some* $k \succeq i$ *such that* $\varphi_{ik}(x) = 0$;
(c) *If* φ_{ik} *is an injection for each* $k \succeq i$, *then* φ_i *is an injection*;
(d) *If* φ_{ik} *is onto for each* $k \succeq i$, *then* φ_i *is a surjection.*

Proof. Part (a) is obvious from our construction. To prove (b), note that $\varphi_i(x) = 0$ means that $\tilde{x} = \tilde{0}$, where $0 \in A_j$ for some $j \in I$ (we use the notation of the proof of Proposition 1.2.1). Therefore, there exists $k \succeq i, j$ such that $\varphi_{ik}(x) = \varphi_{jk}(0) = 0$. Part (c) follows from (b). To show (d), let $a \in A$; then, by construction, $a = \tilde{y}$, where $y \in A_j$ for some $j \in I$. Choose $k \succeq i, j$. Since φ_{ik} is onto, there exists $x \in A_i$ such that $\varphi_{ik}(x) = \varphi_{jk}(y)$; therefore $\varphi_i(x) = \tilde{x} = \tilde{y} = a$. □

Example 1.2.5

(1) The prototype of a direct limit is a union. If an abelian group A is a union $A = \bigcup_{i\in I} A_i$ of subgroups A_i, then A is the direct limit of the subgroup generated by the finite unions $\bigcup_{j\in J} A_j$, where J ranges over the finite subsets of I. Conversely, if

$$A = \varinjlim_{i\in I} A_i$$

is a direct limit of a direct system $\{A_i, \varphi_{ij}, I\}$, and if $\varphi_i : A_i \longrightarrow A$ are the canonical maps, then

$$A = \bigcup_{i\in I} \varphi_i(A_i).$$

(2) Every abelian group A is a direct limit of its finitely generated subgroups. In particular, if A is torsion, it is the direct limit of its finite subgroups.
(3) Let p be a prime number. We use the notation C_{p^∞} for the *p-quasicyclic* or Prüfer group, i.e., the group of p^nth complex roots of unity, with n running over all non-negative integers. Equivalently, C_{p^∞} can be defined as the direct limit

$$C_{p^\infty} = \varinjlim_{n} C_{p^n},$$

of the direct system of cyclic groups $\{C_{p^n}, \varphi_{nm}\}$, where the homomorphism $\varphi_{nm} : C_{p^n} \longrightarrow C_{p^m}$, defined for $n \le m$, is the natural injection.

A map

$$\Psi : \{A_i, \varphi_{ij}, I\} \longrightarrow \{A'_i, \varphi'_{ij}, I\}$$

of direct systems $\{A_i, \varphi_{ij}, I\}$ and $\{A'_i, \varphi'_{ij}, I\}$ over the same directed poset I consists of a collection of homomorphisms

$$\psi_i : A_i \longrightarrow A'_i \quad (i \in I)$$

that commute with the canonical maps φ_{ij} and φ'_{ij}. That is, whenever $i \preceq j$, we have a commuting square

$$\begin{tikzcd} A_i \arrow[r, "\varphi_{ij}"] \arrow[d, "\psi_i"'] & A_j \arrow[d, "\psi_j"] \\ A'_i \arrow[r, "\varphi'_{ij}"'] & A'_j \end{tikzcd}$$

We refer to the homomorphisms ψ_{ij} as the *components* of the map Ψ.

Direct systems of abelian groups over a fixed poset I together with their maps, as defined above, form in a natural way a category. (This category is in fact an abelian category; although the analogous category of direct systems of sets, say, is not abelian.)

Let

$$\{A_i, \varphi_{ij}, I\} \quad \text{and} \quad \{A'_i, \varphi'_{ij}, I\}$$

be direct systems over the same poset $(I, \preceq)$, and let

$$A = \varinjlim A_i \quad \text{and} \quad A' = \varinjlim A'_i$$

be their corresponding direct limits, with canonical maps $\varphi_i : A_i \longrightarrow A$ and $\varphi'_i : A'_i \longrightarrow A'$, respectively. Associated with each map

$$\Psi = \{\psi_i\} : \{A_i, \varphi_{ij}, I\} \longrightarrow \{A'_i, \varphi'_{ij}, I\}$$

of direct systems, there is a homomorphism

$$\varinjlim \Psi = A \longrightarrow A'$$

defined by the universal property of direct limits:

$$\varinjlim \Psi = \varinjlim{}_{i \in I} \psi_i.$$

This is the unique homomorphism induced by the compatible maps

$$\varphi'_i \psi_i : A_i \longrightarrow A' \quad (i \in I).$$

With these definitions, it is straightforward to verify that $\varinjlim(\Psi\Psi') = \varinjlim(\Psi) \varinjlim(\Psi')$ and $\varinjlim(\mathrm{id}_{\{A_i, \varphi_{ij}, I\}}) = \mathrm{id}_{\varinjlim A_i}$; in other words, $\varinjlim$ is a functor from the category of direct systems of abelian groups over the same poset, to the category of abelian groups.

We restate all this as part of the following proposition.

Proposition 1.2.6 *Let I be a fixed poset. Then the collection $\mathfrak{D}$ of all direct systems of abelian groups over I and their maps form an abelian category. Furthermore, $\varinjlim$ is an exact (covariant) functor from $\mathfrak{D}$ to the category of abelian groups.*

The proof of this proposition follows easily from repeated applications of Proposition 1.2.4; we leave the details to the reader.

1.3 Notes, Comments and Further Reading

The material in this chapter is standard. For more details on inverse and direct limits the reader may consult, e.g., Eilenberg and Steenrod [1952], Bourbaki [1989] or Fuchs [1970].