# 1

# THE GAUSS FUNCTION

## 1.1 Historical introduction

The series

$$1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!}+\frac{a(a+1)\,(a+2)\,b(b+1)\,(b+2)}{c(c+1)\,(c+2)}\frac{z^3}{3!}+\dots, \tag{1.1.1}$$

is called the Gauss series or the ordinary hypergeometric series. It is usually represented by the symbol

$$_2F_1[a,b;\ c;\ z].$$

The variable is $z$, and $a$, $b$ and $c$ are called the parameters of the function. If either of the quantities $a$ or $b$ is a negative integer $-n$, the series has only a finite number of terms and becomes in fact a polynomial

$$_2F_1[-n,b;\ c;\ z].$$

For example, suppose that $a=-2$, then the series becomes

$$_2F_1[-2,b;\ c;\ z]=1+\frac{(-2)\,b}{c}\frac{z}{1!}+\frac{(-2)\,(-1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!}+0,$$

that is
$$_2F_1[-2,b;\ c;\ z]=1-\frac{2bz}{c}+\frac{b(b+1)\,z^2}{c(c+1)}, \tag{1.1.2}$$

since all the later terms are zero.

In his work *Arithmetica Infinitorum* (1655), the Oxford professor John Wallis (1616–1703) first used the term 'hypergeometric' (from the Greek ὑπερ, above or beyond) to denote any series which was beyond the ordinary geometric series

$$1+x+x^2+x^3+\dots.$$

In particular, he studied the series

$$1+a+a(a+1)+a(a+1)\,(a+2)+\dots.$$

During the next one hundred and fifty years many other mathematicians studied similar series, notably the Swiss L. Euler (1707–1783)† who gave amongst many other results, the famous relation

$$_2F_1[-n,b;\ c;\ z]=(1-z)^{c+n-b}\,{}_2F_1[c+n,c-b;\ c;\ z], \tag{1.1.3}$$

† Euler (1748). Full details of all references are to be found in the bibliography.

In 1770, the Frenchman, A. T. Vandermonde (1735–1796) stated his theorem, an extension of the binomial theorem, in the form

$$ {}_2F_1[-n, b; c; 1] = \frac{(c-b)(c-b+1)(c-b+2)\dots(c-b+n-1)}{c(c+1)(c+2)(c+3)\dots(c+n-1)}, \tag{1.1.4} $$

but during the next forty years the Göttingen school under C. F. Hindenberg (1741–1808) wasted much effort on various complicated extensions of the binomial and multinomial theorems. All this was changed dramatically, when on 20th January, 1812, C. F. Gauss (1777–1855) delivered his famous thesis 'Disquisitiones generales circa seriem infinitam'† before the Royal Society in Göttingen. In it, this brilliant mathematician defined the modern infinite series of (1.1.1) above and introduced the notation $F[a, b; c; z]$ for it. He also proved his famous summation theorem

$$ {}_2F_1[a, b; c; 1] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \tag{1.1.5} $$

and he gave many relations between two or more of these series. He showed clearly that he was already regarding ${}_2F_1[a, b; c; z]$ as a function in four variables, rather than as a series in $z$, and in a note added 10 February, 1812, he gave a remarkably full discussion of the convergence of such series.

The next major advance was made in 1836 by E. E. Kummer (1810–93), who first used the term 'hypergeometric' for series of the type (1.1.1) only. He showed that the differential equation

$$ z(1-z)\frac{d^2y}{dz^2} + \{c-(1+a+b)z\}\frac{dy}{dz} - aby = 0, \tag{1.1.6} $$

is satisfied by the function

$$ {}_2F_1[a, b; c; z], $$

and has in all twenty-four solutions in terms of similar Gauss functions.‡ In 1857§, G. F. B. Riemann (1826–66) extended this theory by the introduction of his $P$ functions, which in a way, are generalizations of the Gaussian

$$ {}_2F_1[a, b; c; z]. $$

Riemann also discussed the general theory of the transformation of the variable in a differential equation and this theory was applied to Kummer's work by J. Thomae who, in 1879, worked out in detail the relationships between Kummer's twenty-four solutions.‖

† Gauss (1812).
‡ Kummer (1836).
§ Riemann (1857).
‖ Thomae (1879).

The first integral representation of the Gauss function goes back to Euler† who showed that

$$ {}_2F_1[-n, b; c; z] = \frac{n!}{c(c+1)(c+2)\dots(c+n-1)} \times \int_0^1 t^{-n-1}(1-t)^{c+n-1}(1-tz)^{-b}\, dt. \quad (1.1.7)$$

The basic idea of representing a function by a contour integral with gamma functions in the integrand seems to be due to S. Pincherle (1853–1936) who used contours of a type which stems from Riemann's work. This side of the subject was developed extensively by R. Mellin and E. W. Barnes.‡ In 1907, Barnes published his contour integral representations of Kummer's twenty-four functions, and later, in 1910,§ he proved the integral analogue of Gauss's theorem

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \Gamma(a+s)\,\Gamma(b+s)\,\Gamma(c-s)\,\Gamma(d-s)\,ds = \frac{\Gamma(a+c)\,\Gamma(a+d)\,\Gamma(b+c)\,\Gamma(b+d)}{\Gamma(a+b+c+d)}. \quad (1.1.8)$$

**1.1.1 The Gauss series and its convergence.** Let us write

$$(a)_n \equiv a(a+1)(a+2)(a+3)\dots(a+n-1), \quad (1.1.1.1)$$

and in particular, $(a)_0 \equiv 1$, so that, for example $(3)_5 = 3.4.5.6.7$, $= 2520$, and $(1)_n = n!$. Then

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (1.1.1.2)$$

and

$$\lim_{n\to\infty} (a)_n = \frac{1}{\Gamma(a)}. \quad (1.1.1.3)$$

If $a$ is a negative integer $-m$, then

$$(a)_n = (-m)_n \quad \text{if} \quad m \geqslant n,$$

and

$$(a)_n = 0 \quad \text{if} \quad m < n,$$

so that $(-3)_3 = (-3)(-2)(-1) = -6$, but $(-3)_4 = 0$.

In this notation, the Gauss function becomes

$$ {}_2F_1[a, b; c; z] = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}, \quad (1.1.1.4)$$

where $a$, $b$, $c$ and $z$ may be real or complex. From this, we see that if either of the numbers $a$ or $b$ is zero or a negative integer, the function

† Euler (1748). ‡ Barnes (1907*a*).
§ Barnes (1910).

reduces to a polynomial, but if $c$ is zero or a negative integer, the function is not defined, since all but a finite number of the terms of the series become infinite. Also we have immediately

$$\frac{\mathrm{d}}{\mathrm{d}z}({}_2F_1[a,b;\,c;\,z]) = \frac{ab}{c}\,{}_2F_1[a+1,b+1;\,c+1;\,z]. \tag{1.1.1.5}$$

Some alternative notations for the Gauss function, which are in common use, are:

Appell (1926) and Bailey (1935$a$),

$${}_2F_1\left[\begin{matrix}a,b; & z\\ c; & \end{matrix}\right] = {}_2F_1[a,b;\,c;\,z], \tag{1.1.1.6}$$

$$F(a,b;\,c;\,z) = {}_2F_1[a,b;\,c;\,z], \tag{1.1.1.7}$$

Meijer (1953$c$),

$$\Phi[a,b;\,c;\,z] = {}_2F_1[a,b;\,c;\,z]/\Gamma(c), \tag{1.1.1.8}$$

MacRobert (1947), p. 352,

$$E(2;\,a,\,b;\,1;\,c;\,-1/z) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\,{}_2F_1[a,b;\,c;\,z], \tag{1.1.1.9}$$

Meijer (1941$a$),

$$G_{22}^{12}\left(-z\,\middle|\,\begin{matrix}-a, & -b\\ -1, & -c\end{matrix}\right) = -\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)\,z}\,{}_2F_1[a,b;\,c;\,z], \tag{1.1.1.10}$$

Riemann (1857),

$$P\left\{\begin{matrix}0 & \infty & 1 & \\ 0 & a & 0 & z\\ 1-c & b & c-a-b & \end{matrix}\right\} = {}_2F_1[a,b;\,c;\,z]. \tag{1.1.1.11}$$

Let $u_n = \dfrac{(a)_n\,(b)_n}{(c)_n\,(1)_n}$, then we have

$$(1+n)\,(c+n)\,u_{n+1} = (a+n)\,(b+n)\,u_n. \tag{1.1.1.12}$$

The ratio of the two successive terms $u_n$ and $u_{n+1}$ of the Gaussian series is

$$\frac{(a+n)\,(b+n)}{(c+n)\,(1+n)}z = \frac{(1+a/n)\,(1+b/n)}{(1+c/n)\,(1+1/n)}z, \tag{1.1.1.13}$$

so that as $n \to \infty$, the ratio

$$|u_{n+1}/u_n| \to |z|.$$

Hence, by D'Alembert's test†, the series is convergent for all values of $z$, real or complex such that $|z| < 1$, and divergent for all values of $z$ real or complex, such that $|z| > 1$.

† Bromwich, *Infinite Series*, (1947), p. 39.

When $|z| = 1$,

$$|u_{n+1}/u_n| = \left|\left\{1+\frac{a+b}{n}+O(1/n^2)\right\}\left\{1-\frac{1+c}{n}+O(1/n^2)\right\}\right|,$$

$$= \left|1+\frac{a+b-c-1}{n}+O(1/n^2)\right|,$$

$$\leqslant 1+\{\mathrm{Rl}\,(a+b-c-1)/n\}+O(1/n^2). \qquad (1.1.1.14)$$

Thus, when $z = 1$, by Raabe's test†, the series is convergent if $\mathrm{Rl}\,(c-a-b) > 0$, and divergent if $\mathrm{Rl}\,(c-a-b) < 0$.

It is also divergent when $\mathrm{Rl}\,(c-a-b) = 0$, for in this case

$$|u_{n+1}/u_n| > 1-\frac{1}{n}-\frac{C}{n^2},$$

where $C$ is a constant.

When $|z| = 1$, but $z \neq 1$, the series is absolutely convergent when $\mathrm{Rl}\,(c-a-b) > 0$, convergent but not absolutely so when

$$-1 < \mathrm{Rl}\,(c-a-b) \leqslant 0,$$

and divergent when $\mathrm{Rl}\,(c-a-b) < -1$. If $\mathrm{Rl}\,(c-a-b) = -1$, more delicate tests are needed. In this case, we find that

$$|u_{n+1}/u_n| = 1-\frac{\mathrm{Rl}\,(a+b-ab+1)}{n^2}+O(1/n^3). \qquad (1.1.1.15)$$

Hence the series is convergent if $\mathrm{Rl}\,(a+b) > \mathrm{Rl}\,ab$, and divergent if $\mathrm{Rl}\,(a+b) \leqslant \mathrm{Rl}\,ab$.

For example, the series

$$1-\tfrac{2}{3}+\tfrac{3}{4}-\tfrac{4}{5}+\tfrac{5}{6}-\tfrac{6}{7}+\ldots = \tfrac{1}{2}\{1+{}_2F_1[2,2;\ 3;\ -1]\}, \qquad (1.1.1.16)$$

is divergent.

We note also that

$$\frac{(a)_n\,(b)_n}{(c)_n\,n!} \to 0 \text{ as } n \to \infty, \text{ if } 0 < \mathrm{Rl}\,(1+c-a-b) < 1. \quad (1.1.1.17)$$

## 1.2 The Gauss equation

The differential equation

$$z(1-z)\frac{d^2y}{dz^2}+\{c-(1+a+b)\,z\}\frac{dy}{dz}-aby = 0, \qquad (1.2.1)$$

is called the Gauss equation or the hypergeometric equation. In the region $|z| < 1$, one solution is

$$y_1 = {}_2F_1[a,b;\ c;\ z]. \qquad (1.2.2)$$

† Bromwich (1947), p. 40.

This can be verified by direct differentiation of the series (1.1.1), and substitution in the above differential equation. But an alternative form of writing this equation is

$$\frac{d}{dz}\left(z\frac{d}{dz}+c-1\right)y = \left(z\frac{d}{dz}+a\right)\left(z\frac{d}{dz}+b\right)y, \qquad (1.2.3)$$

and this leads to an elegant proof, for

$$\left(z\frac{d}{dz}+a\right)y = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}(n+a)z^n.$$

Hence $$\left(z\frac{d}{dz}+a\right)\left(z\frac{d}{dz}+b\right)y = \sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_n n!}z^n.$$

Similarly, $$\left(z\frac{d}{dz}+c-1\right)y = \sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_{n-1} n!}z^n.$$

Hence $$\frac{d}{dz}\left(z\frac{d}{dz}+c-1\right)y = \sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_{n-1} n!}n z^{n-1},$$

$$= \sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_n n!}z^n.$$

The Gauss equation can be rewritten

$$\frac{d^2y}{dz^2}+\left\{\frac{c}{z(1-z)}-\frac{1+a+b}{1-z}\right\}\frac{dy}{dz}-\frac{ab}{z(1-z)}y = 0, \qquad (1.2.4)$$

from which 0 and 1 are seen to be regular singularities. If we write $1/z$ for $z$, we find that infinity is also a regular singularity of the Gauss equation.†

In the notation of operators, where $\Delta \equiv z\dfrac{d}{dz}$, the Gauss equation can also be written

$$\Delta(\Delta+c-1)y = z(\Delta+a)(\Delta+b)y. \qquad (1.2.5)$$

**1.2.1 The connexion with Riemann's equation.** We shall now show that any equation of the general form

$$\frac{d^2y}{dz^2}+\left(\sum_{\nu=1}^{3}\frac{A_\nu}{z-z_\nu}\right)\frac{dy}{dz}+\left(\sum_{\nu=1}^{3}\frac{B_\nu}{z-z_\nu}\right)\frac{y}{(z-z_1)(z-z_2)(z-z_3)} = 0, \qquad (1.2.1.1)$$

where $A_\nu$ and $B_\nu$ are constants, can be reduced to a Gauss equation, provided that $A_1+A_2+A_3 = 2$, to ensure that the 'point at infinity'

† Whittaker & Watson (1947), § 10.3.

is an ordinary point of the equation. We shall also exhibit the interconnexions between several well-known differential equations, as incidental to the proof given here.

First we note that in the equation (1.2.1.1) every point, including infinity, is an ordinary point of the equation, except the points $z = z_1$, $z = z_2$ and $z = z_3$. So let us write $\theta = z_2 - z_3$, $\phi = z_3 - z_1$ and $\psi = z_1 - z_2$, where $\theta + \phi + \psi = 0$. The indicial equation, formed for expansion about $z = z_1$, is

$$\rho(\rho-1) + A_1\rho + \frac{B_1}{-\phi\psi} = 0,$$

with roots $\alpha$ and $\alpha'$ say. Then we can write

$$A_1 = 1 - \alpha - \alpha' \quad \text{and} \quad B_1 = -\phi\psi\alpha\alpha'.$$

Similarly, by considering the indicial equations formed for expansions about $z = z_2$ and $z = z_3$, respectively, we can write

$$A_2 = 1 - \beta - \beta', \quad B_2 = -\psi\theta\beta\beta',$$

and

$$A_3 = 1 - \gamma - \gamma', \quad B_3 = -\theta\phi\gamma\gamma',$$

where, since $A_1 + A_2 + A_3 = 2$, we must have

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1.$$

The given equation then becomes Riemann's equation

$$\frac{d^2y}{dz^2} + \left(\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3}\right)\frac{dy}{dz}$$
$$= \left\{\frac{\alpha\alpha'}{(z-z_1)\,\theta} + \frac{\beta\beta'}{(z-z_2)\,\phi} + \frac{\gamma\gamma'}{(z-z_3)\,\psi}\right\}\frac{\theta\phi\psi y}{(z-z_1)(z-z_2)(z-z_3)}. \tag{1.2.1.2}$$

This equation is also known as Papperitz's equation.† Its solution is usually written in terms of Riemann's $P$ function as

$$u = P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}, \tag{1.2.1.3}$$

or, in terms of the Gauss function, as

$$u = (z-z_1)^{\alpha}(z-z_2)^{-\alpha-\gamma}(z-z_3)^{\gamma}$$
$$\times {}_2F_1\left[\alpha+\beta+\gamma, \alpha+\beta'+\gamma;\ 1+\alpha-\alpha';\ \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}\right]. \tag{1.2.1.4}$$

† Papperitz (1885).

Twenty-four solutions of Riemann's equation can be written down immediately, simply by interchanging the triads $(z_1, \alpha, \alpha')$, $(z_2, \beta, \beta')$ and $(z_3, \gamma, \gamma')$, in a cyclic order.

If, in (1.2.1.2) we write $t = \{(z-z_1)\,\theta\}/\{(z-z_2)\,(-\phi)\}$, and divide by $(t\phi+\theta)^4/(\theta^2\phi^2\psi^2)$, the equation (1.2.1.2) becomes

$$\frac{d^2y}{dt^2}+\left(\frac{A_1}{t}+\frac{A_3}{t-1}\right)\frac{dy}{dt}-\left(\frac{\alpha\alpha'}{t}-\beta\beta'-\frac{\gamma\gamma'}{t-1}\right)\frac{y}{t(t-1)}=0. \tag{1.2.1.5}$$

If further we write $y = t^{\alpha}(t-1)^{\gamma}\,Y$, (1.2.1.5) reduces to

$$t(t-1)\frac{d^2Y}{dt^2}+\{t(2+\alpha-\alpha'+\gamma-\gamma')-(1+\alpha-\alpha')\}\frac{dY}{dt} + \{(\alpha+\gamma)\,(1-\alpha'-\gamma')+\beta\beta'\}\,Y=0. \tag{1.2.1.6}$$

Finally, if we write $a+b$ for $1+\alpha-\alpha'+\gamma-\gamma'$, $ab$ for

$$(\alpha+\gamma)\,(1-\alpha'-\gamma')+\beta\beta'$$

and $c$ for $1+\alpha-\alpha'$, (1.2.1.6) reduces to the ordinary Gauss equation (1.2.1). Thus we see that, in general, for any equation with three ordinary singularities at $z_1$, $z_2$ and $z_3$, these singularities can be transformed into the three singularities of the Gauss equation simply by writing $z$ for $\{(z-z_1)\,(z_3-z_2)\}/\{(z-z_2)\,(z_3-z_1)\}$.

## 1.3 Kummer's twenty-four solutions

Let us assume that
$$y = z^g \sum_{n=0}^{\infty} u_n z^n, \tag{1.3.1}$$

(where $u_0 \neq 0$) is any solution of the Gauss equation (1.2.1). Then, by direct differentiation of this series, we find that

$$u_0\,g(g+c-1)\,z^{g-1}+\sum_{n=0}^{\infty}\{u_{n+1}(g+n+1) \times(g+n+c)-u_n(g+n+a)\,(g+n+b)\}\,z^{g+n}=0. \tag{1.3.2}$$

Hence we must have as the indicial equation

$$g(g+c-1)=0, \tag{1.3.3}$$

and in general

$$(g+n+c)\,(g+n+1)\,u_{n+1}=(g+n+a)\,(g+n+b)\,u_n. \tag{1.3.4}$$

The root $g = 0$ of the indicial equation (1.3.3) leads to the solution

$$y_1 = {}_2F_1[a, b;\ c;\ z],$$

provided that $c$ is not zero nor a negative integer, and the root $g = 1-c$, gives a second solution in which

$$(1+n)(2-c+n)\,u_{n+1} = (a+1-c+n)(b+1-c+n)\,u_n. \tag{1.3.5}$$

This solution is

$$y_2 = z^{1-c}\,{}_2F_1[1+a-c, 1+b-c;\ 2-c;\ z], \tag{1.3.6}$$

provided that $c$ is not a positive integer $\geqslant 2$. Hence one complete solution of the Gauss equation (1.2.1) is

$$y = A\,{}_2F_1[a, b;\ c;\ z] + B\,z^{1-c}\,{}_2F_1[1+a-c, 1+b-c;\ 2-c;\ z], \tag{1.3.7}$$

for $|z| < 1$, and for $c$ not an integer, where $A$ and $B$ are constants.

When $c = 1$, the two solutions are equivalent, and we have to follow the usual Frobenius process† in order to find that a second solution is now

$$y_2 = {}_2F_1[a, b;\ 1;\ z]\log z + \sum_{n=1}^{\infty}\left[\frac{\partial}{\partial g}\left\{\frac{(a+g)_n\,(b+g)_n}{(1+g)_n\,(1+g)_n}\right\}\right]_{g=0} z^n. \tag{1.3.8}$$

When $c = 0$, or a negative integer, the second solution (1.3.6) is still valid but the first solution has to be replaced by

$$y_1 = z^{1-c}\,{}_2F_1[1+a-c, 1+b-c;\ 2-c;\ z]\log z + \sum_{n=0}^{\infty}\left[\frac{\partial}{\partial g}\left\{\frac{(a+g)_n\,(b+g)_n}{(c+g)_n\,(1+g)_n}\right\}\right]_{g=1-c} z^n. \tag{1.3.9}$$

When $c$ is a positive integer $\geqslant 2$, the first solution is still valid but the second solution has to be replaced by

$$y_2 = {}_2F_1[a, b;\ c;\ z]\log z + \sum_{n=0}^{\infty}\left[\frac{\partial}{\partial g}\left\{\frac{(a+g)_n\,(b+g)_n}{(c+g)_n\,(1+g)_n}\right\}\right]_{g=0} z^n. \tag{1.3.10}$$

If $a$ or $b$ is a negative integer, as we have already seen, our first solution reduces to a polynomial in $z$, and if $1+a-c$ or $1+b-c$ is a negative integer, the second solution reduces to a polynomial in $z$.

When we are dealing with solutions of this type, it is useful to remember that if

$$u_n(g) = \frac{(a+g)_n\,(b+g)_n}{(c+g)_n\,(1+g)_n},$$

† Whittaker & Watson (1947), § 10.3.

then

$$\frac{1}{u_n(g)}\frac{\partial}{\partial g}\{u_n(g)\}+\frac{1}{g+n}+\frac{1}{g+c+n-1}-\frac{1}{g+a+n-1}-\frac{1}{g+b+n-1}$$

$$=\frac{1}{u_{n-1}(g)}\frac{\partial}{\partial g}\{u_{n-1}(g)\}, \qquad (1.3.11)$$

(see Copson, *Functions of a complex variable* (1950), p. 248).

Next let us substitute $(1-z)^k w$ for $y$ in the Gauss equation. It becomes

$$z(1-z)\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\{c-(a+b+1+2k)\,z\}\frac{\mathrm{d}w}{\mathrm{d}z}$$

$$+\left[\frac{k(k-1)\,z-k\{c-(a+b+1)\,z\}}{1-z}-ab\right]w=0. \qquad (1.3.12)$$

This equation is also of hypergeometric type if $1-z$ divides exactly into $k(k-1)\,z-k\{c-(a+b+1)\,z\}$, that is, if either $k=0$, or $k=c-a-b$. When $k=0$, the two solutions (1.2.2) and (1.3.6) are given, but when $k=c-a-b$, then two new solutions are given, valid in the region $|z|<1$. These are

$$y_3=(1-z)^{c-a-b}\,{}_2F_1[c-a,c-b;\;c;\;z] \qquad (1.3.13)$$

and

$$y_4=z^{1-c}(1-z)^{c-a-b}\,{}_2F_1[1-a,1-b;\;2-c;\;z]. \qquad (1.3.14)$$

Since the Gauss equation is of order two, it can have only two linearly independent solutions. Hence there must exist constants $A$ and $B$ such that

$$(1-z)^{c-a-b}\,{}_2F_1[c-a,c-b;\;c;\;z]$$

$$=A\,{}_2F_1[a,b;\;c;\;z]+Bz^{1-c}\,{}_2F_1[1+a-c,1+b-c;\;2-c;\;z].$$

Now the left-hand side of this equation can be expanded in integral powers of $z$, but $z^{1-c}$ cannot, since $c$ is not an integer, by hypothesis. Hence $B=0$. If however we put $z=0$, we find that we must have $A=1$. Hence $y_1=y_3$, that is

$${}_2F_1[a,b;\;c;\;z]=(1-z)^{c-a-b}\,{}_2F_1[c-a,c-b;\;c;\;z]. \qquad (1.3.15)$$

This is the result usually known as Euler's identity. In a similar way we can show that $y_2=y_4$, that is

$${}_2F_1[1+a-c,1+b-c;\;2-c;\;z]=(1-z)^{c-a-b}\,{}_2F_1[1-a,1-b;\;2-c;\;z]. \qquad (1.3.16)$$