Chapter 2

INDUSTRY DYNAMICS *Á LA* STACKELBERG WITH STOCHASTIC CAPITAL ACCUMULATION

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1. INTRODUCTION

Firms' entry and growth in an industry have attracted a great deal of attention within both industrial and applied economics for several decades. Ever since Gibrat's seminal contribution (Gibrat, 1931), the established wisdom has maintained that expected firm growth rates are independent of firm size, a property known as *Gibrat's Law*. Both theoretical and empirical research have been extensively carried out along this line.[1] So far, the existing literature provides heterogeneous answers to the question as the way we shall expect market dynamics to unravel, given some degree of initial asymmetry among firms.

Two relevant contributions by Lucas and Prescott (1971) and Lucas (1978) investigate entry and exit decisions in long-run competitive equilibrium models where prices, outputs and investments are driven by stochastic processes. In a pioneering paper, Jovanovic (1982) proposes a theory of noisy selection where firms enter over time and learn about their productive efficiency as they operate in the market. Those that are relatively more efficient grow and survive, while those who relatively less efficient decline and ultimately exit the industry. Hopenhayn (1992) analyzes the case of individual productivity shocks and their effects on entry, exit and market dynamics in the long-run. He finds that the steady state equilibrium implies a size distribution of firms by age cohorts, and proves that the size distribution is stochastically increasing in the age of the cohorts. Jovanovic's model is extended by Ericson and Pakes (1995) who consider two models of firm behaviour, allowing for heterogeneity among firms, idiosyncratic (or firm-specific) sources of uncertainty, and discrete outcomes (exit and/or entry).[2]

Broadly speaking, an overview of this literature leads one to think that 'older firms are bigger than younger firms'. An important question to this regard is the following: is moving first a prerequisite (i.e., a necessary condition) for a firm to become larger than its rivals, or is it a sufficient condition?

Here I propose a dynamic oligopoly model under uncertainty generalising some of the aspects treated in Lambertini (2005). Firms enter simultaneously and then compete hierarchically à la Stackelberg, at each instant over an infinite horizon. They accumulate capacity through costly investment, as in Solow's (1956) and Swan's (1956) growth model. At every instant, first the investment levels are chosen, then shocks realize and finally productive capacities are determined as a function of the shocks. Due to the formal properties of the model, the game possesses a unique and time consistent open-loop equilibrium.

The main results are as follows. The relative performance of firms depends on several factors, including the relative size of shocks as well as the relative number of leaders and followers. In particular, if investment costs are negligible, or the variance of the shock affecting the leaders is low, or again firms are subject to a common shock, then the expected profits of the representative leader exceed those of the representative follower. These results tend to confirm the acquired wisdom according to which leading entails a 'first mover advantage', when choice variables directly pertain to the size of the firm, as it also happens in static Cournot games (see Dowrick, 1986; Hamilton and Slutsky, 1990, *inter alia*). More interestingly, the opposite result is also admissible, namely, that the followers may ultimately overtake the leaders in the steady state (this can happen, for instance, when shocks are idiosyncratic, or when the variance of the shock affecting the leader is sufficiently large).

The model investigated in the present paper allows me to propose a few considerations as to the debate concerning the intertemporal growth of firms. First, the Stackelberg model described here shows that an industry equilibrium that is characterized by an uneven size distribution of firms may not necessarily be the outcome of the entry process, but may be rather the consequence of (*i*) a strategic advantage of some firms over the others, or (*ii*) an asymmetric distribution of shocks across firms that are otherwise fully symmetric and have played simultaneously along the entire history of the industry. Second, whether leaders grows more or less than followers is independent of initial conditions, which may or may not differ across firms. Consequently, in general, the present analysis does not confirm *Gibrat's Law*, since sequential play induces an asymmetry in growth rates for any admissible distribution of initial capacities across firms. Likewise, the indications provided by the Nash game are in contrast with *Gibrat's Law*, as the expected equilibrium size and performance of firms are symmetric irrespective of initial conditions, so that any asymmetric vector of initial capacities involves asymmetric growth rates in order to reach a symmetric steady state allocation (in expected value).

The remainder of the paper is structured as follows. Section 2 presents the general features of the model. The open-loop Stackelberg equilibrium is derived in section 3, while section 4 contains some comparative statics. Concluding remarks are in section 4.

2. THE SETUP

N firms operate over continuous time $t \in [0,\infty)$ in a market for differentiated goods, the demand function for variety v at any t being:

$$p_v(t) = a - q_v(t) - s\sum_{z\neq v} q_z(t)\,, \tag{1}$$

where $s \in [0,1]$ is the constant degree of substitutability between any two varieties. If $s = 0$ then each firm is a monopolist in a separate market, while on the contrary if $s = 1$ then firms supply homogeneous goods.

The game unravels following a sequential play framework. Out of the population of N firms, $f \equiv \{1,2,3,...F\}$ of them are followers while $l \equiv \{F+1, F+2, F+3,...N\}$ of them are leaders, with $F \geq 1$ and $N \geq F+1$. Each firm keeps playing the same role over the whole horizon of the game.

In order to supply the final good, firms must build up capacity (i.e., physical capital) $k_v(t)$ through intertemporal investment:

$$\frac{dk_v(t)}{dt} \equiv \dot{k}_v = I_v(t) - \delta k_v(t) + \varepsilon_v(t), \quad v = 1,2,3,...,F,F+1,...,N\,, \tag{2}$$

where $\delta \in [0,1]$ is the depreciation rate, constant and equal across firms; $\varepsilon_v(t)$ is the shock affecting firm v, and it is assumed to be i.i.d. across periods. Furthermore, for the sake of simplicity I will assume throughout the paper that

$$\varepsilon_v(t) = \varepsilon_F(t) \qquad \forall\ v = 1,2,3,...,F \tag{3}$$

$$\varepsilon_v(t) = \varepsilon_L(t) \qquad \forall\ v = F+1, F+2, F+3,...,N \tag{4}$$

$$E(\varepsilon_i) = 0; \quad E(\varepsilon_i^2) = \sigma_i^2, \quad i = F,L$$

and $E(\varepsilon_F \varepsilon_L) = E(\varepsilon_L \varepsilon_F)\sigma_{FL}^2$.

At any instant t, the sequence of events is taken to be as follows:

- firms sequentially choose investment efforts $I_v(t)$, then
- shocks realize, and finally
- the interaction between investments, depreciation and the firm-specific shock determines capacity through (2) and the dynamics of control, $dI_v(t)/dt$, for each firm.[3]

For the sake of simplicity, in the remainder I assume that $q_v(t)=k_v(t)$, i.e., all firms operate at full capacity at any instant. At any t, firm v bears the following total costs:

$$C_v(t)=b[I_v(t)]^2, \quad b\geq 0, \tag{5}$$

where marginal production cost is constant and normalised to zero in order to shrink to a minimum the set of parameters. The instantaneous profit of firm v is:

$$\pi_v(t)=p(t)-b[I_v(t)]^2. \tag{6}$$

For each firm v, the instantaneous investment effort $I_v(t)$ is the control variable, while capacity $k_v(t)$ is the state variable. The value of the state variables at $t=0$ is given by the vector $\mathrm{k}(0)=\mathrm{k}_0$. The aim of firm v consists in:

$$\max_{I_v(t)} J_v \equiv \int_0^\infty \pi_{vv}(t)e^{-\rho t}dt \tag{7}$$

subject to the relevant dynamic constraints. The factor $e^{-\rho t}$ discounts future gains, and the discount rate $\rho>0$ is assumed to be constant and common to all players. In order to solve the optimization problem, each firm defines a strategy $I_v(t)$ at each t, for any admissible $I_j(t)$, $j\neq v$. If, when choosing $I_v(t)$, firm v explicitly takes into account the stock of state variables $\mathrm{k}(t)$ at time t (or their evolution up to that time), the game is solved in closed-loop strategies. Otherwise, if controls are chosen only upon calendar time, the game is solved in open-loop strategies. On the one hand, a closed-loop solution is clearly preferable in that it accounts for feedback effects at all times during the game; however, on the other hand, it is worth stressing that the choice of the solution concept may be taken depending upon the nature of the problem at hand. Indeed, the main difference between open-loop and closed-loop approaches is that in the former, players decide by looking at the clock (i.e., calendar time), while in the latter, they decide by looking at the stock (i.e., the past history of the game). Whether the second perspective is more realistic than the first has to be evaluated within the specific framework being used, in relation with the kind of story the model itself tries to account for (Clemhout and Wan, 1994, p. 812). If controls describe something like investment plans, these can in fact be sticky enough to justify the adoption of an open-loop solution. The next question is whether open-loop rules can produce subgame perfect equilibria or not. Briefly, an equilibrium is (at least weakly) time consistent if, at any intermediate time $\tau \in [0,\infty)$, no player has an incentive to deviate from the plan initially designed at time zero (at least in view of the stocks of state variables at time τ).[4]

3. THE GAME

The Stackelberg game is taken to be solved by firms in open-loop strategies. Consider first the optimum problem for the followers, i.e., firms belonging to the set $f \equiv \{1,2,3,...,F\}$. Given that all of them are a priori symmetric and face the same problem, I will confine my attention to a single representative follower, say, firm F. Its expected value Hamiltonian is:

$$E[\mathrm{H}_F(\mathrm{k}(t),\mathrm{I}(t))] = E\left\{\left[a - k_F(t) - s\left(\sum_{i=1}^{F-1} k_i(t) + \sum_{j=F+1}^{N} k_i(t)\right)\right] k_F(t) - b[I_F(t)]^2 + \right.$$
$$+\mu_{FF}(t)[I_F(t) - \delta k_F(t) + \varepsilon_F(t)] + \sum_{i=1}^{F-1} \mu_{Fi}(t)[I_i(t) - \delta k_i(t) + \varepsilon_F(t)] +$$
$$\left. + \sum_{j=F+1}^{N} \mu_{Fj}(t)[I_j(t) - \delta k_j(t) + \varepsilon_L(t)]\right\}, \tag{8}$$

where $\mu_{Fj}(t)$ is the co-state variable associated with state variable $k_j(t)$. The first order conditions are (exponential discounting is omitted for brevity):

$$\frac{\partial E[\mathrm{H}_F(\mathrm{k}(t),\mathrm{I}(t))]}{\partial I_F(t)} = -2bI_F(t) + \mu_{FF}(t) = 0; \tag{9}$$

$$-\frac{\partial[\mathrm{H}_F(\mathrm{k}(t),\mathrm{I}(t))]}{\partial k_F(t)}=\frac{\partial\mu_{FF}(t)}{\partial t}\quad\Rightarrow \tag{10}$$

$$\frac{\partial\mu_{FF}(t)}{\partial t}=\mu_{FF}(t)\delta-a+s\left(\sum_{i=1}^{F-1}k_i(t)+\sum_{j=F+1}^{N}k_j(t)\right)+2k_F(t); \tag{11}$$

$$-\frac{\partial[\mathrm{H}_F(\mathrm{k}(t),\mathrm{I}(t))]}{\partial k_h(t)}=\frac{\partial\mu_{Fh}(t)}{\partial t}\Rightarrow\frac{\partial\mu_{Fh}(t)}{\partial t}=\mu_{Fh}(t)\delta+sk_h(t),\quad\forall\, h\neq F. \tag{12}$$

Equations (9-12) must be considered together with the initial conditions $\mathrm{k}(0)=\mathrm{k}_0$ and the transversality condition:

$$\lim_{t\to\infty}\mu_{Fv}(t)\cdot k_v(t)=0,\quad v=F+1,F+2,F+3,\ldots,N. \tag{13}$$

From (9), one obtains:

$$\mu_{FF}(t)=2bI_F(t);\quad \frac{\partial I_F(t)}{\partial t}=\frac{1}{2b}\frac{\partial\mu_{FF}(t)}{\partial t}. \tag{14}$$

Moreover, from co-state equations(11-12), one can check that the expressions of co-state variables:

$$\mu_{FF}(t)=\int\frac{\partial\mu_{FF}(t)}{\partial t};\quad \mu_{Fh}(t)=\int\frac{\partial\mu_{Fh}(t)}{\partial t},\quad\forall\, h\neq F \tag{15}$$

are independent of any rivals' controls, in particular the followers' controls. This fact proves the following result (see Xie, 1997):[5]

Lemma 1 *The Stackelberg game is uncontrollable by the leaders. Therefore, the open-loop Stackelberg equilibrium is time consistent.*

Before approaching the leader's problem, it is worth observing, again from (14), that the evolution of firm *F*'s investment does not depend on any $\mu_{Fh}(t)$. This redundancy of the dynamics of the other firms' co-state variables as to the follower's decisions is going to become useful in order to characterise the equilibrium.

Now I can characterize the leader's problem. As with the follower, again in view of the ex ante symmetry characterising the population of leaders, I may focus upon a single firm that will be taken as a representative leader, say, firm *N*. Its Hamiltonian function (in expected value) is:

$$E[\mathrm{H}_N(\mathbf{k}(t),I_N(t))]=E\left\{\left[a-k_N(t)-s\left(\sum_{i=1}^{F}k_i(t)+\sum_{j=F+1}^{N-1}k_j(t)\right)\right]k_N(t)+\right.$$
$$-b[I_N(t)]^2+\mu_{NN}(t)[I_N(t)-\delta k_N(t)+\varepsilon_L(t)]+$$
$$+\sum_{j=F+1}^{N-1}\mu_{Nj}(t)[I_j(t)-\delta k_j(t)+\varepsilon_L(t)]+\sum_{i=1}^{F}\mu_{Ni}(t)\left[\frac{\mu_{ii}(t)}{2b}-\delta k_i(t)+\varepsilon_F(t)\right]+$$
$$\left.+\sum_{i=1}^{F}\theta_{Ni}(t)\left[\frac{\partial\mu_{ii}(t)}{\partial t}\right]+\sum_{i=1}^{F}\sum_{j=1}^{N-1}\phi_{Nj}(t)\left[\frac{\partial\mu_{ij}(t)}{\partial t}\right]\right\} \tag{16}$$

where $\theta_{Ni}(t)$ and $\phi_{Nj}(t)$ are the additional co-state variables attached by the leader to the followers' co-state equations, and the expressions $\partial\mu_{ii}(t)/\partial t$ and $\partial\mu_{ij}(t)/\partial t$ are given by (11-12). Solving the leader's problem, one obtains (superscripts l and f stand for *leader* and *follower*, respectively):

Lemma 2 At *the steady state of the Stackelberg open-loop game, optimal capacities are:*

$$k^{l}=\frac{2(1+b\delta^2)(a+2b\delta\varepsilon_L)-s\left[a+2b\delta\left(\varepsilon_L+F\left(\varepsilon_F-\varepsilon_L\right)\right)\right]}{4(1+b\delta^2)^2+2s\left(1+b\delta^2\right)(N-2)-s^2(N+F-1)},$$

$$k^{f}=\frac{4\left(1+b\delta^2\right)^2\left(a+2b\delta\varepsilon_F\right)+2s(1+b\delta^2)\Gamma-s^2\Psi}{8\left(1+g\delta^2\right)^3+4s\left(1+b\delta^2\right)^2\left(N+F-3\right)+s^2\Omega-s^3\Lambda},$$

$$\Gamma\equiv a(F-2)+2b\delta\left[\varepsilon_F(N-2)-\varepsilon_L(N-F)\right];$$
$$\Psi\equiv a(2F-1)+2b\delta\left[\varepsilon_F(N+F-1-F(N-F))-\varepsilon_L(N-F)(F-1)\right];$$
$$\Omega\equiv 2\left(1+b\delta^2\right)\left[3-2(N-F)+F(N-5)\right];$$
$$\Lambda\equiv(F-1)(N+F-1).$$

Proof. The first order conditions for the representative leader are:

$$\frac{\partial E[H_N(\cdot)]}{\partial I_N(t)}=-2bI_N(t)+\mu_{NN}(t)=0; \tag{17}$$

$$-\frac{\partial E[H_N(\cdot)]}{\partial k_N(t)}=\frac{\partial\mu_{NN}(t)}{\partial t}\quad\Rightarrow \tag{18}$$

$$\frac{\partial \mu_{NN}(t)}{\partial t} = \mu_{NN}(t)\delta - a + 2k_N(t) + s\left(\sum_{i=1}^{F} k_i(t) + \sum_{j=F+1}^{N-1} k_j(t)\right) - s\sum_{i=1}^{F}\theta_{Ni}(t); \tag{19}$$

$$-\frac{\partial E[H_N(\cdot)]}{\partial k_j(t)} = \frac{\partial \mu_{Nj}(t)}{\partial t} \quad \Rightarrow \tag{20}$$

$$\frac{\partial \mu_{Nj}(t)}{\partial t} = \mu_{Nj}(t)\delta + sk_N(t) - s\sum_{i=1}^{F}\theta_{Ni}(t); \tag{21}$$

$$-\frac{\partial E[H_N(\cdot)]}{\partial k_i(t)} = \frac{\partial \mu_{Ni}(t)}{\partial t} \quad \Rightarrow \tag{22}$$

$$\frac{\partial \mu_{Ni}(t)}{\partial t} = \mu_{Ni}(t)\delta + sk_N(t) - 2\theta_{Ni}(t) - s\left(\sum_{h\neq i}\theta_{Nh}(t) + \sum_{j=1}^{N-1}\phi_{Nj}(t)\right); \tag{23}$$

$$-\frac{\partial E[H_N(\cdot)]}{\partial h_{ih}(t)} = \frac{\partial \phi_{hj}(t)}{\partial t} \quad \Rightarrow \quad \frac{\partial \phi_{hj}(t)}{\partial t} = -\delta\phi_{Nh}(t); \tag{24}$$

$$-\frac{\partial E[H_1(\cdot)]}{\partial \mu_{ii}(t)} = \frac{\partial \theta_{Ni}(t)}{\partial t} \quad \Rightarrow \quad \frac{\partial \theta_{Ni}(t)}{\partial t} = -\frac{\mu_{Ni}(t)}{2b} - \delta\theta_{Li}(t). \tag{25}$$

The above conditions are accompanied by the initial conditions $\mathrm{k}(0) = \mathrm{k}_0$ as well as the transversality condition:

$$\begin{aligned} &\lim_{t\to\infty}\mu_{Nj}(t)\cdot kj(t) = 0 \\ &\lim_{t\to\infty}\theta_{Ni}(t)\cdot \mu_{Ni}(t) = 0 \\ &\lim_{t\to\infty}\phi_{Nh}(t)\cdot \mu_{Nh}(t) = 0 \end{aligned} \tag{26}$$

for all i, j, h.

From (17) one immediately gets:

$$\mu_{NN}(t) = b + 2bI_N(t); \quad \frac{\partial I_N(t)}{\partial t} = \frac{1}{2b}\frac{\partial \mu_{NN}(t)}{\partial t} \quad \Rightarrow \tag{27}$$

$$\frac{\partial I_N(t)}{\partial t}=\frac{1}{2g}\left[\mu_{NN}(t)\delta-a+2k_N(t)+s\left(\sum_{i=1}^{F}k_i(t)+\sum_{j=F+1}^{N-1}k_j(t)\right)-s\sum_{i=1}^{F}\theta_{Ni}(t)\right]. \tag{28}$$

Additionally, from (24), we observe that $\frac{\partial \phi_{Nh}}{\partial t}=0$ if and only if $\phi_{Nh}=0$. Proceeding likewise, note that from (25), we have:

$$\frac{\partial \theta_{Ni}(t)}{\partial t}=0 \quad \Leftrightarrow \quad \theta_{Ni}(t)=-\frac{\mu_{Ni}(t)}{2b\delta}. \tag{29}$$

Now, having taken all the relevant first order conditions, I may impose symmetry across (i) leaders and (ii) follower, by setting

- $k_i(t)=k_F(t)$ for all i, $\quad k_j(t)=k_N(t)$ for all j,
- $I_i(t)=I_F(t)$ for all i, $\quad I_j(t)=I_N(t)$ for all j,

Accordingly, (19), (21) and (23) rewrite, respectively, as follows:

$$\frac{\partial \mu_{NN}(t)}{\partial t}=\mu_{NN}(t)\delta-a+2k_N(t)++s[k_F(t)F+k_N(t)(N-F-11)]-sF\theta_{Ni}(t); \tag{30}$$

$$\frac{\partial \mu_{Nj}(t)}{\partial t}=\mu_{Nj}(t)\delta+sk_N(t)-sF\theta_{Ni}(t); \tag{31}$$

$$\frac{\partial \mu_{Ni}(t)}{\partial t}=\mu_{Ni}(t)\delta+sk_N(t)-2\theta_{Ni}(t)-s[(F-1)\theta_{Nh}(t)+(N-1)\phi_{Nj}(t)]. \tag{32}$$

Since I'm looking for the characterization of the steady state equilibrium, I may use (29) and impose stationarity upon equations (32), to obtain:

$$\mu_{Ni}(t)=-\frac{2bs\delta k_N(t)}{2(1+b\delta^2)+s(F-1)}. \tag{33}$$

Then, plugging (29) and (33) into (28), one obtains the following dynamic equation for the representative leader's investment (henceforth, I omit the indication of time for the sake of brevity):

$$\frac{\partial I_N}{\partial t} \propto (a - 2b\delta I_N - sFk_F)[2(1+b\delta^2)+s(F-1)] + \\ + k_N[s^2F - (2(1+b\delta^2)+s(F-1))(s(N-F-1)+2)] \tag{34}$$

which is nil at

$$I_N^* = \frac{(a-sFk_F)[2(1+b\delta^2)+s(F-1)]}{2b\delta[2(1+b\delta^2)+s(F-1)]} + \\ = + \frac{k_N[s^2F-(2(1+b\delta^2)+s(F-1))(s(N-F-1)+2)]}{2b\delta[2(1+b\delta^2)+s(F-1)]}. \tag{35}$$

The representative follower's optimal investment is:

$$I_F^* = \frac{\mu_{FF}^*}{2b} = \frac{a-2k_F-s[k_F(F-1)+k_N(N-F)]}{2b\delta}. \tag{36}$$

Obviously the sign of $I_N^* - I_F^*$ depends, amongst other things, upon the relative number of leaders and followers, for any given $\{k_F, k_N\}$:

$$I_N^* - I_F^* \quad \propto \quad k_F(2-s)[2(1+b\delta^2)+s(F-1)] + \\ + k_N[s(4-s-2F(1-s))-2(2+b\delta^2(2-s))]. \tag{37}$$

Of course the sign of the above expression can be determined on the basis of the relative size of capacities, the steady state levels of which can be determined imposing stationarity on the kinematic equations of state variables (2):

$$k^l = \frac{2(1+b\delta^2)(a+2b\delta\varepsilon_L) - s[a+2b\delta(\varepsilon_L + F(\varepsilon_F-\varepsilon_L))]}{4(1+b\delta^2)^2 + 2s(1+b\delta^2)(N-2) - s^2(N+F-1)}; \tag{38}$$

$$k^f = \frac{4(1+b\delta^2)^2(a+2b\delta\varepsilon_F) + 2s(1+b\delta^2)\Gamma - s^2\Psi}{8(1+g\delta^2)^3 + 4s(1+b\delta^2)^2(N+F-3) + s^2\Omega - s^3\Lambda}, \tag{39}$$

where

$$\Gamma \equiv a(F-2) + 2b\delta[\varepsilon_F(N-2) - \varepsilon_L(N-F)]; \\ \Psi \equiv a(2F-1) + 2b\delta[\varepsilon_F(N+F-1-F(N-F)) - \varepsilon_L(N-F)(F-1)]; \\ \Omega \equiv 2(1+b\delta^2)[3-2(N-F)+F(N-5)]; \tag{40}$$

$\Lambda \equiv (F-1)(N+F-1)$.

Expressions (38-39) can be used to write the corresponding equilibrium expressions of $\{I^{\iota} = \delta \mathcal{I}^{\iota} + \varepsilon_L, I^{f} = \delta \mathcal{I}^{f} + \varepsilon_F\}$.

Before assessing the properties of steady state capacities, let me go briefly back to (37). The sign of this expression is difficult to determine, however there is a special case where it can be easily done. Assume $s = 1$, i.e., goods are homogeneous. If so, then

$$I_N^* - I_F^* \quad \propto \quad (1+2b\delta^2+F)k_F - (1+2b\delta^2)k_N > 0$$
$$\forall\, F > \frac{(1+2b\delta^2)(k_F - k_N)}{k_F} \equiv \hat{F} \tag{41}$$

This allows me to prove the following result:

Lemma 3 *Suppose firms supplies perfect substitute goods. In such a case:*

- *if* $k_N > k_F$, *then* $I_N^* > I_F^*$ *for all* $F > \hat{F}$;
- *if* $k_N > k_F$, *then* $I_N^* < I_F^*$ *for all* $F \in [1, \hat{F})$;
- *if* $k_N < k_F$, *then* $I_N^* > I_F^*$ *for all* $F \geq 1$.

Proof. The first two claims in the above Lemma can be shown to hold by quickly observing that, provided $k_N > k_F$, then $\hat{F} > 1$ if $k_N / k_F > (1+b\delta^2)/(1+2b\delta^2)$, which is smaller than one for all admissible values of b and δ. Therefore $k_N > k_F$ suffices to ensure that $\hat{F} > 1$. If instead $k_N < k_F$, then $\hat{F} < 0$. Hence, $F > \hat{F}$ holds trivially.

The interesting feature of Lemma 3 lies in the fact that it highlights the existence of an admissible case where the representative leader is *bigger* than the representative follower in terms of installed capacity, but nonetheless the follower invests more than the leader. This happens if the number of the followers is low enough, and seems to suggest that decreasing the intensity of competition among followers (by shrinking their number) ultimately produces an incentive for them to outperform the leaders as far as the instantaneous optimal investment effort is concerned.

Assessing the difference between steady state capital endowments and investment levels, one finds that the sign of both $k^{\iota} - k^{f}$ and $I^{\iota} - I^{f}$ may change depending upon the relative size of shocks, ε_F and ε_L. However, if $\varepsilon_F = \varepsilon_L = \varepsilon$, we have:

$$k^{\iota}-k^{f}\propto I^{\iota}-I^{f}\propto(a+2b\delta\varepsilon)/\left[8(1+b\delta^{2})^{3}+4(1+b\delta^{2})^{2}(N+F-3)s+\right.$$
$$\left.+2(1+b\delta^{2})(3-2(N-F)+F(N-5))s^{2}-(F-1)(N+F-1)\right], \qquad (42)$$

where the sign of the numerator depends on the size of the shock ε, while the sign of the denominator depends on F. In particular, the following holds:

Lemma 4 *Suppose* $\varepsilon_F=\varepsilon_L=\varepsilon$. *In such a case*:

- *if* $\varepsilon>-a/(2b\delta)$, *then* $k^{\iota}-k^{f}$ *and* $I^{\iota}-I^{f}$ *are*:
  (i) *positive for all* $F\in\left[1,\dfrac{[2(1+b\delta^{2})-s][2(1+b\delta^{2})+s(N-1)]}{s^{2}}\right)$;
  (ii) *negative for all* $F>\dfrac{[2(1+b\delta^{2})-s][2(1+b\delta^{2})+s(N-1)]}{s^{2}}$.
- *if* $\varepsilon<-a/(2b\delta)$, *then* $k^{\iota}-k^{f}$ *and* $I^{\iota}-I^{f}$ *are*:
  (i) *positive for all* $F>\dfrac{[2(1+b\delta^{2})-s][2(1+b\delta^{2})+s(N-1)]}{s^{2}}$;
  (ii) *negative for all* $F\in\left[1,\dfrac{[2(1+b\delta^{2})-s][2(1+b\delta^{2})+s(N-1)]}{s^{2}}\right)$.

Proof. The numerator of (42) is positive (resp., negative) for all ε larger (resp., smaller) than $-a/(2b\delta)$. The sign of the denominator changes in correspondence of

$$F=\frac{s-2(1+b\delta^{2})}{s}<0; \qquad (43)$$
$$F=\frac{[2(1+b\delta^{2})-s][2(1+b\delta^{2})+s(N-1)]}{s^{2}}>1.$$

Moreover, the polynomial at the denominator is positive inside the interval defined by the above roots. This suffices to prove the claim.

Using (35-36) and (38-39) together with (4), one can write the steady state Stackelberg expected equilibrium profits for a generic pair of leader and follower, $E(\pi^{\iota})$ and $E(\pi^{f})$, respectively.[6] The relative performance of the two representative firms in steady state is summarized by:

Proposition 5 *At the Stackelberg open-loop equilibrium*:

(*i*) *for all* $\sigma_L^2 \bullet \left(0, \overline{\sigma}_L^2\right)$, *then any leader's expected profits are larger than any follower's;*

(*ii*) *if* $\sigma_F^2 = \sigma_L^2 = \sigma_{FL}^2$, *then any leader's expected profits are larger than any follower's, for all admissible values of F and N;*

(*iii*) *if* $b \to 0$, *then any leader's expected profits are larger than any follower's for all admissible values of* $\sigma_F^2, \sigma_L^2, \sigma_{FL}^2, F$ *and* N *.*

Proof. Claim (*i*) of the Proposition requires simple albeit tedious algebra, the resulting threshold level $\overline{\sigma}_L^2$ being a cumbersome expression containing all the relevant parameters of the model. However, $\overline{\sigma}_L^2$ can be explicitly written in the duopoly case with F=1 and N=2,[7] where $E(\pi^l) > E(\pi^f)$ if

$$\sigma_L^2 < \frac{\{a^2(1+4b\delta^2)+16g^2\delta^2\sigma_{FL}^2 + 4b[4+b\sigma^2(29+4b\delta^2(18+b\delta^2(15+4b\delta^2)))]\sigma_F^2\}}{\varpi} \qquad (44)$$

where

$$\varpi \equiv 16b(1+b\delta^2)(1+b\delta^2(1+b\delta^2)(7+4b\delta^2)). \qquad (45)$$

As to claim (*ii*), observe that if $\sigma_F^2 = \sigma_L^2 = \sigma_{FL}^2$, then

$$E(\pi_1^l) - E(\pi_2^f) \propto [2(1+b\delta^2)-s]^2 + sF(1+b\delta^2 - s) \qquad (46)$$

To prove claim (*iii*), it suffices to observe that

$$\lim_{b\to 0}[E(\pi^l) - E(\pi^{fi})] = (4+sF)(1-s)+s^2 > 0. \qquad (47)$$

This concludes the proof.

Claim (*i*) in the above Proposition simply states the intuitive result that, if the degree of uncertainty borne by the representative leader is high enough, then, all else equal, following may be preferable to leading in terms of expected profits. Claim (*ii*) illustrates the special case in which all firms face the same shock. If so, then all that matters is having the first mover advantage at any point in time. Finally, claim (*iii*) deals with the limit case where investment costs are negligible. In this circumstance the leader is better off irrespective of the values of all other relevant parameters, the intuitive reason being that under this condition the role of uncertainty becomes immaterial.

The stability analysis of the Stackelberg open-loop game is rather cumbersome, yet it can be carried out (without resorting to numerical calculations) to verify that the steady state open-loop Stackelberg equilibrium $(k^f, k^\ell, I^f, I^\ell)$ is a saddle point.[8]

I am now in a position to make a crucial remark concerning the growth rates exhibited by leaders and followers, respectively. Evaluating the sign of $\dot{k}_L - \dot{k}_F$ is a difficult task even in special cases. However, as a quick inspection of the proof of Lemma 2 reveals, growth rates surely differ because, in general, $\dot{I}_L$ and $\dot{I}_F$ differ at all $t \in [0, \infty)$. In addition to this, firms' saddle paths to the steady state are independent of initial conditions k_0; hence, there follows that growth rates are determined by the distribution of roles across firms in the games (i.e., the timing of moves) but not their initial respective sizes (or installed capacities), and the cases in which $k_L(0) < k_F(0)$ while $k^\ell > k^f$, are both admissible. This brief discussion ultimately entails that the present setup does not yield theoretical support to *Gibrat's Law*.

4. COMPARATIVE STATICS

First, I evaluate firms' profits under perfect certainty, i.e., at $\sigma_1 = \sigma_2 = \sigma_{12} = 0$:

$$\pi^\ell = \frac{a^2[2(1+b\delta^2)+s]^2[(1+b\delta^2)(2(1+b\delta^2)+s(F-1))-s^{2F}]}{(2(1+b\delta^2)+s(F-1))[2(1+b\delta^2)(s(N-2)+2(1+b\delta^2))-s^2(N+F-1)]^2} \tag{48}$$

$$\begin{aligned}\pi^f = {} & a^2(1+b\delta^2)[2(1+b\delta^2)(s(F-2)+2(1+b\delta^2))-s^2(2F-1)]^2 / \\ & \{2(1+b\delta^2)[(3-2(N-F)+(N-5)s^2)+ \\ & +2(1+b\delta^2)(s(N+F-3)+2(1+b\delta^2))]-(F-1)(N+F-1)s^2\}^2,\end{aligned} \tag{49}$$

with

$$\pi^\ell - \pi^f \propto s^2 + [4(1+b\delta^2)+sF](1+b\delta^2-s) > 0 \tag{50}$$

for all admissible values of parameters. The deterministic case yields the well known profit ranking usually associated with games in which controls are strategic substitutes (i.e., best replies are downward sloping), as is well known from previous literature (Dowrick, 1986; and Hamilton and Slutsky, 1990, *inter alia*). However, the ranking of firms' profits may drastically change due to uncertainty. For the sake of simplicity, relabel

$\sigma_L^2 = \zeta_L, \sigma_F^2 = \zeta_F$ and $\sigma_{FL}^2 = \zeta_{FL}$. The following properties can be ascertained:

$$\frac{\partial E(\pi^l)}{\partial \zeta_L} < 0; \quad \frac{\partial E(\pi^l)}{\partial \zeta_F} > 0; \quad \frac{\partial E(\pi^l)}{\partial \zeta_{FL}} < 0 \quad \forall\, b, s, \delta, N, F\,; \tag{51}$$

$$\frac{\partial E(\pi^f)}{\partial \zeta_L} > 0; \quad \frac{\partial E(\pi^f)}{\partial \zeta_F} < 0; \quad \frac{\partial E(\pi^f)}{\partial \zeta_{FL}} < 0 \quad \forall\, b, s, \delta, N, F\,. \tag{52}$$

Moreover:

$$\left|\frac{\partial E(\pi^l)}{\partial \zeta_L}\right| > \frac{\partial E(\pi^f)}{\partial \zeta_L}; \quad \left|\frac{\partial E(\pi^f)}{\partial \zeta_F}\right| > \frac{\partial E(\pi^l)}{\partial \zeta_F}; \quad \forall\, b, s, \delta, N, F\,. \tag{53}$$

The above list of partial derivatives reveals a few facts:

- increasing the variance of the shock affecting the leaders (resp., follower) generates a positive spillover for the followers (leaders), while obviously damaging the leaders (followers) themselves;
- moreover, the former effect is smaller than the latter in absolute value;
- increasing the correlation between shocks negatively affects the performance of both leaders and followers alike.[9]

5. CONCLUSIONS

I have described a stochastic differential game in which firms invest to increase productive capacity, following time-consistent open-loop Stackelberg strategies. The equilibrium of the model highlights different growth rates along the saddle path. Accordingly, the analysis carried out in this paper is clearly in contrast with *Gibrat's Law*. Moreover, it appears that there are admissible cases where the followers's growth rates are larger than the leaders', e.g. when the representative leader is indeed bigger than the representative follower in terms of installed capacity. This may ultimately lead to situations where equilibrium profits are larger for followers than for leaders.

NOTES

[1] For early empirical studies confirming *Gibrat's Law*, see Hart and Prais (1956), Simon and Bonini (1958) and Hymer and Pashigian (1962). An exhaustive overview of empirical findings is in Audretsch, Santarelli and Vivarelli (1999). For a thorough appraisal of Gibrat's contribution, see Sutton (1997).
[2] In a subsequent paper, Pakes and Ericson (1998) evaluate the empirical implications of Jovanovic's model and their model of industry dynamics.
[3] The deterministic version of the present model has been investigated, under simultaneous play only, in the previous literature (Fudenberg and Tirole, 1983; Fershtman and Muller, 1984; Reynolds, 1987) and can be ultimately traced back to Solow (1956) and Swan (1956).
[4] A more detailed illustration of these issues can be found in Dockner, Jørgensen, Van Long and Sorger (2000, section 4.3, pp. 98-107; and Ch. 5).
[5] See also Dockner, Jørgensen, Van Long and Sorger (2000, Ch. 5) and Cellini, Lambertini and Leitmann (2005).
[6] These expressions are omitted for brevity.
[7] See Proposition 10 in Lambertini (2005, p. 455).
[8] The detailed stability properties of the Jacobian matrix are fully illustrated in Lambertini (2005, Proposition 11, p. 455) for the duopoly case.
[9] It is worth noting that this result has interesting macroeconomic implications, suggesting that integration or globalization may favour the diffusion of shocks across markets/ countries, so that firms located in markets previously separated by significantly high trade barriers are no longer protected from shocks taking place abroad.

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