Lecture Notes in Mathematics 1958

Compactifying Moduli Spaces for Abelian Varieties

Bearbeitet von
Martin C Olsson

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Preliminaries

In this chapter we review some background material used in the main part of the text. The experienced reader may wish to just skim this chapter for our notational conventions and then proceed to chapter 3.

2.1 Abelian Schemes and Torsors

2.1.1. If S is an algebraic space, an *abelian algebraic space* over S is a proper smooth group algebraic space A/S with geometrically connected fibers. An important fact due to Raynaud [13, 1.9] is that when S is a scheme any abelian algebraic space over S is in fact a scheme.

A *semi-abelian scheme* over a scheme S is a smooth commutative group scheme G/S such that every geometric fiber of G is an extension of an abelian scheme by a torus.

2.1.2. If $f : P \to S$ is a proper morphism of algebraic spaces with geometrically connected and reduced fibers, then the map $\mathscr{O}_S \to f_*\mathscr{O}_P$ is an isomorphism. From a general theorem of Artin [5, 7.3] it follows that the Picard functor $\underline{\mathrm{Pic}}(P)$ defined to be the sheaf with respect to the fppf topology of the presheaf

$$T/S \mapsto \{\text{isomorphism classes of invertible sheaves on } P_T\}$$

is an algebraic space locally of finite presentation over S (see also 1.5.7)

The case we will be interested in is when P is a torsor under an abelian algebraic space A/S. In this case define a subfunctor $\underline{\mathrm{Pic}}^0(P) \subset \underline{\mathrm{Pic}}(P)$ as follows. The points of $\underline{\mathrm{Pic}}^0(P)$ over a scheme-valued point $\bar{t} : \mathrm{Spec}(\Omega) \to S$ with Ω an algebraically closed field is the subgroup of $\underline{\mathrm{Pic}}(P)(\Omega)$ consisting of isomorphism classes of line bundles $\mathscr{L}$ such that for every $a \in A_{\bar{t}}(\Omega)$ the line bundles $t_a^*\mathscr{L}$ and $\mathscr{L}$ are isomorphic (note that this depends only on the isomorphism class of $\mathscr{L}$). The subfunctor $\underline{\mathrm{Pic}}^0(P) \subset \underline{\mathrm{Pic}}(P)$ is defined by

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associating to any scheme T/S the subset of $\underline{\mathrm{Pic}}(P)(T)$ of isomorphism classes of line bundles $\mathscr{L}$ such that for every algebraically closed field Ω and point $\bar{t} : \mathrm{Spec}(\Omega) \to S$ the image of $[\mathscr{L}]$ in $\underline{\mathrm{Pic}}(P)(\Omega)$ is in $\underline{\mathrm{Pic}}^0(P)(\Omega)$.

In the case when $P = A$ is the trivial torsor, the subfunctor $\underline{\mathrm{Pic}}^0(P) \subset \underline{\mathrm{Pic}}(P)$ is the dual abelian scheme, denoted A^t.

Proposition 2.1.3. *The subfunctor $\underline{\mathrm{Pic}}^0(P) \subset \underline{\mathrm{Pic}}(P)$ is a smooth proper algebraic space over S. Tensor product defines the structure of an abelian algebraic space on $\underline{\mathrm{Pic}}^0(P)$.*

Proof. Since $\underline{\mathrm{Pic}}^0(P) \subset \underline{\mathrm{Pic}}(P)$ is in any case a subsheaf (with respect to the étale topology) it suffices to prove the proposition after replacing S by an étale cover. We may therefore assume that P is a trivial torsor and that S is a scheme. In this case $\underline{\mathrm{Pic}}^0(P)$ is the dual abelian scheme of A as mentioned above. □

2.1.4. For any S–scheme T, the set of isomorphisms of A_T–torsors $\iota : A_T \to P_T$ is canonically in bijection with the set $P(T)$. Any such isomorphism ι defines an isomorphism $\iota_* : A^t \to \underline{\mathrm{Pic}}^0(P)$. This defines a morphism of schemes

$$A^t \times P \to \underline{\mathrm{Pic}}^0(P), \quad ([\mathscr{L}], \iota) \mapsto [\iota_* \mathscr{L}]. \tag{2.1.4.1}$$

Proposition 2.1.5. *The morphism 2.1.4.1 factors uniquely as*

$$A^t \times P \xrightarrow{\mathrm{pr}_1} A^t \xrightarrow{\sigma} \underline{\mathrm{Pic}}^0(P), \tag{2.1.5.1}$$

where σ is an isomorphism.

Proof. The uniqueness of the factorization is clear. To prove the existence we can by descent theory work étale locally on S and may therefore assume that S is a scheme and that $P \simeq A$ is the trivial torsor. In this case the map 2.1.4.1 is identified with the translation action

$$A^t \times A \to A^t, \quad ([\mathscr{L}], a) \mapsto [t_a^* \mathscr{L}].$$

It is well known that this action is trivial (see for example [3, 4.1.12]). □

2.1.6. Note in particular that an invertible sheaf $\mathscr{L}$ on P defines a homomorphism $\lambda_{\mathscr{L}} : A \to A^t$ by

$$A \to \underline{\mathrm{Pic}}^0(P) \simeq A^t, \quad a \mapsto [t_a^* \mathscr{L} \otimes \mathscr{L}^{-1}].$$

2.1.7. The dual abelian scheme A^t/S has a very useful description which does not require the sheafification involved in the definition of $\underline{\mathrm{Pic}}$ in general. Namely define a *rigidified line bundle* on A to be a pair $(\mathscr{L}, \iota)$, where $\mathscr{L}$ is a line bundle on A and $\iota : \mathscr{O}_S \to e^* \mathscr{L}$ is an isomorphism, where $e : S \to A$ is the identity section. Any isomorphism $(\mathscr{L}, \iota) \to (\mathscr{L}', \iota')$ between two such pairs is unique if it exists. Using this one shows that $\underline{\mathrm{Pic}}(A)$ can also be viewed as representing the functor

$$T/S \mapsto \{\text{isomorphism classes of rigidified line bundles on } A_T\}.$$

In particular, over $A \times_S A^t$ there is a tautological line bundle $\mathscr{B}$, called the *Poincare bundle*, with a trivialization of the restriction of $\mathscr{B}$ to $\{e\} \times A^t$.

If $\mathscr{M}$ is a line bundle on A, then for any scheme-valued point $a \in A$ the line bundle

$$t_a^*\mathscr{M} \otimes \mathscr{M}^{-1} \otimes_{\mathscr{O}_S} \mathscr{M}^{-1}(a) \otimes_{\mathscr{O}_S} \mathscr{M}(e) \qquad (2.1.7.1)$$

has a canonical rigidification. We obtain a map $\lambda_{\mathscr{M}} : A \to A^t$ by sending $a \in A$ to 2.1.7.1.

2.1.8. Let B be a scheme and A/B an abelian scheme over B. Fix a finitely generated free abelian group X with associated torus T. A semi-abelian scheme G sitting in an exact sequence

$$0 \to T \to G \to A \to 0 \qquad (2.1.8.1)$$

defines a homomorphism $c : X \to A^t$ as follows. The group X is the character group of T, so any $x \in X$ defines an extension

$$0 \to \mathbb{G}_m \to E_x \to A \to 0 \qquad (2.1.8.2)$$

by pushing 2.1.8.1 out along the homomorphism $x : T \to \mathbb{G}_m$. Let $\mathscr{L}_x$ denote the corresponding line bundle on A. The identity element of G induces a trivialization of $\mathscr{L}_x(0)$ and hence $\mathscr{L}_x$ is a rigidified line bundle. Moreover it follows from the construction that there is a canonical isomorphism of rigidied line bundles

$$\mathscr{L}_x \otimes \mathscr{L}_{x'} \simeq \mathscr{L}_{x+x'} \qquad (2.1.8.3)$$

for $x, x' \in X$. In particular, the $\mathscr{O}_A$–module

$$\oplus_{x \in X} \mathscr{L}_x \qquad (2.1.8.4)$$

has a natural algebra structure and there is a canonical isomorphism over A

$$G \to \underline{\mathrm{Spec}}_A(\oplus_{x \in X} \mathscr{L}_x). \qquad (2.1.8.5)$$

For any $a \in A(B)$ there exists étale locally on B an isomorphism $t_a^*\mathscr{L}_x \to \mathscr{L}_x$. Indeed étale locally on B there exists a lifting $\tilde{a} \in G(B)$ of a. Then the map

$$t_{\tilde{a}}^* : \mathscr{O}_G \to \mathscr{O}_G$$

induces an isomorphism $t_{\tilde{a}}^*\mathscr{L}_x \to \mathscr{L}_x$ since G is commutative.

In terms of the isomorphism 2.1.8.5, the group structure on G can be described as follows. A (scheme-valued) point $g \in G(S)$ is given by a point $a \in A(S)$ together with trivializations $\iota_x : a^*\mathscr{L}_x \to \mathscr{O}_S$ which are compatible in the sense that for any two $x, x' \in X$ the diagram

$$\begin{array}{ccc} a^*\mathscr{L}_x \otimes a^*\mathscr{L}_{x'} & \xrightarrow{\iota_x \otimes \iota_{x'}} & \mathscr{L}_x \otimes \mathscr{L}_{x'} \\ {\scriptstyle 2.1.8.3}\downarrow & & \downarrow{\scriptstyle 2.1.8.3} \\ a^*\mathscr{L}_{x+x'} & \xrightarrow{\iota_{x+x'}} & \mathscr{L}_{x+x'} \end{array} \qquad (2.1.8.6)$$

commutes. Such trivializations ι_x are equivalent to the structure of rigidified line bundles on the $t_a^*\mathscr{L}_x$. The translation action of a point $g = (a, \{\iota_x\})$ on G can be described as follows.

Namely, since the translation action of A on A^t is trivial, there exists a unique isomorphism of rigidified line bundles $t_a^*\mathscr{L}_x \to \mathscr{L}_x$ (where $t_a^*\mathscr{L}_x$ is rigidified using ι_x). This defines a morphism of algebras

$$t_a^*(\oplus_x \mathscr{L}_x) \to \oplus_{x\in X}\mathscr{L}_x \qquad (2.1.8.7)$$

which gives the translation action of g on G.

Conversely any homomorphism $c : X \to A^t$ defines a semi-abelian scheme G sitting in an extension 2.1.8.1. Indeed for every $x \in X$ let $\mathscr{L}_x$ be the rigidified line bundle corresponding to $c(x)$, and set

$$\mathscr{O}_G := \oplus_{x\in X}\mathscr{L}_x.$$

Then $\mathscr{O}_G$ is given an algebra structure using the unique isomorphism of rigidified line bundles

$$\mathscr{L}_x \otimes \mathscr{L}_{x'} \to \mathscr{L}_{x+x'}$$

corresponding to the fact that c is a homomorphism. The group structure on G is defined as in the preceding paragraph.

2.2 Biextensions

We only review the aspects of the theory we need in what follows. For a complete account see [19, VII] and [11].

2.2.1. Let S be a scheme and F and G abelian sheaves in the topos S_{Et} of all S–schemes with the big étale topology. In the applications the sheaves F and G will be either abelian schemes over S or constant sheaves associated to some abelian group.

A $\mathbb{G}_m$*–biextension of* $F \times G$ is a sheaf of sets E with a map $\pi : E \to F \times G$ and the following additional structure:

(i) A faithful action of $\mathbb{G}_m$ on E over $F \times G$ such that the quotient sheaf $[E/\mathbb{G}_m]$ is isomorphic to $F \times G$ via the map induced by π.
(ii) For a local section $q \in G$, let $E_{-,q} \subset E$ denote $\pi^{-1}(F \times q)$. Then we require for any local section $q \in G$ a structure on $E_{-,q}$ of an extension of abelian groups

$$0 \to \mathbb{G}_m \to E_{-,q} \to F \to 0 \qquad (2.2.1.1)$$

compatible with the $\mathbb{G}_m$–action from (i). This structure is determined by isomorphisms of $\mathbb{G}_m$–torsors over F

$$\varphi_{p,p';q} : E_{p,q} \wedge E_{p',q} \to E_{p+p',q} \tag{2.2.1.2}$$

satisfying certain compatibilities (see [19, VII.2.1]). Here p and p' are local sections of F and $E_{p,q} \subset E$ denotes the subsheaf of elements mapping to $(p,q) \in F \times G$.

(iii) Similarly, for a local section $p \in F$, let $E_{p,-} \subset E$ denote $\pi^{-1}(p \times G)$. Then we require for any local section $p \in F$ a structure on $E_{p,-}$ of an extension of abelian groups

$$0 \to \mathbb{G}_m \to E_{p,-} \to G \to 0 \tag{2.2.1.3}$$

compatible with the $\mathbb{G}_m$–action from (i). This structure is determined by isomorphisms of $\mathbb{G}_m$–torsors

$$\psi_{p;q,q'} : E_{p,q} \wedge E_{p,q'} \to E_{p,q+q'} \tag{2.2.1.4}$$

satisfying certain compatibilities [19, VII.2.1], where q and q' are local sections of G.

(iv) For any local sections $p, p' \in F$ and $q, q' \in G$ the diagram

$$\begin{array}{ccccc}
E_{p,q}\wedge E_{p,q'}\wedge E_{p',q}\wedge E_{p',q'} & & \xrightarrow{\text{flip}} & & E_{p,q}\wedge E_{p',q}\wedge E_{p,q'}\wedge E_{p',q'} \\
\downarrow{\scriptstyle \psi_{p;q,q'}\wedge\psi_{p';q,q'}} & & & & \downarrow{\scriptstyle \varphi_{p,p';q}\wedge\varphi_{p,p';q'}} \\
E_{p,q+q'} \wedge E_{p',q+q'} & & & & E_{p+p',q} \wedge E_{p+p';q'} \\
& \searrow{\scriptstyle \varphi_{p,p';q+q'}} & & \swarrow{\scriptstyle \psi_{p+p';q,q'}} & \\
& & E_{p+p',q+q'} & &
\end{array} \tag{2.2.1.5}$$

commutes.

If $\pi : E \to F \times G$ and $\pi' : E' \to F \times G$ are two $\mathbb{G}_m$–biextensions of $F \times G$, then a morphism of $\mathbb{G}_m$–biextensions $f : E \to E'$ is a map of sheaves over $F \times G$ compatible with the $\mathbb{G}_m$–actions and the maps $\varphi_{p,p';q}$ and $\psi_{p;q,q'}$. Note that (i) implies that any morphism of $\mathbb{G}_m$–biextensions of $F \times G$ is automatically an isomorphism. The collection of $\mathbb{G}_m$–biextensions of $F \times G$ therefore form a groupoid denoted $\underline{\text{Biext}}(F, G; \mathbb{G}_m)$.

The *trivial* $\mathbb{G}_m$–biextension of $F \times G$ is defined to be the product $\mathbb{G}_m \times F \times G$ with $\mathbb{G}_m$–action on the first factor, and the map to $F \times G$ given by projection to the last two factors. The maps $\varphi_{p,p';q}$ and $\psi_{p;q,q'}$ are induced by the group law on $\mathbb{G}_m \times F \times G$. If E is a $\mathbb{G}_m$–biextension of $F \times G$, then a *trivialization* of E is an isomorphism from the trivial $\mathbb{G}_m$–biextension of $F \times G$ to E.

If $f : F' \to F$ and $g : G' \to G$ are morphisms of abelian sheaves and $\pi : E \to F \times G$ is a $\mathbb{G}_m$–biextension of $F \times G$, then the pullback $\pi' : E \times_{F\times G} (F' \times G') \to F' \times G'$ has a natural structure of a $\mathbb{G}_m$-biextension of $F' \times G'$ induced by the structure on E. We therefore have a pullback functor

$$(f \times g)^* : \underline{\mathrm{Biext}}(F, G; \mathbb{G}_m) \to \underline{\mathrm{Biext}}(F', G'; \mathbb{G}_m). \tag{2.2.1.6}$$

For later use, let us discuss some examples of pullbacks.

2.2.2. Let $E \in \underline{\mathrm{Biext}}(F, G; \mathbb{G}_m)$ be a biextension, and let $f : F \to F$ be the zero map and $g : G \to G$ the identity map. Denote by E' the pullback

$$E' := (f \times g)^* E. \tag{2.2.2.1}$$

For any sections $(p, q) \in F \times G$ we have $E'_{p,q} = E_{0,q}$. Let $e_q \in E_{0,q}$ denote the section corresponding to the identity element in the group $E_{-,q}$.

Lemma 2.2.3 *The sections e_q define a trivialization of E'.*

Proof. Let $s_{p,q} \in E'_{p,q}$ denote the section $e_q \in E'_{p,\,q} = E_{0,q}$.

Let $\psi'_{p\,;\,q,\,q'}$ and $\varphi'_{p,\,p';\,q}$ be the maps giving E' the biextension structure. Then we need to show the following:

(i) For every $p, p' \in F$ and $q \in G$ the map

$$\varphi'_{p,p';q} : E'_{p,q} \wedge E'_{p',q} \to E'_{p+p',q} \tag{2.2.3.1}$$

sends $s_{p,q} \wedge s_{p',q}$ to $s_{p+p',q}$;

(ii) For every $p \in F$ and $q, q' \in G$ the map

$$\psi'_{p;q,q'} : E'_{p,q} \wedge E'_{p,q'} \to E_{p,q+q'} \tag{2.2.3.2}$$

sends $s_{p,q} \wedge s_{p,q'}$ to $s_{p,q+q'}$.

Statement (i) is immediate as $\varphi'_{p,p';q}$ is simply the map

$$E_{0,q} \wedge E_{0,q} \to E_{0,q} \tag{2.2.3.3}$$

induced by the group law on $E_{-,q}$.

For statement (ii), note that the map $\psi'_{p;q,q'}$ is given by the map

$$\psi_{0;q,q'} : E_{0,q} \wedge E_{0,q'} \to E_{0,q+q'}. \tag{2.2.3.4}$$

Now the identity element $e_{q+q'} \in E_{0,q+q'}$ is characterized by the condition that

$$\varphi_{0,0;q+q'}(e_{q+q'} \wedge e_{q+q'}) = e_{q+q'}. \tag{2.2.3.5}$$

We show that $\psi_{0;q,q'}(e_q \wedge e_{q'})$ also has this property.

By the commutativity of 2.2.1.5, the diagram

$$\begin{array}{ccc} E_{0,q}\wedge E_{0,q'}\wedge E_{0,q}\wedge E_{0,q'} & \xrightarrow{\text{flip}} & E_{0,q}\wedge E_{0,q}\wedge E_{0,q'}\wedge E_{0,q'} \\ \downarrow{\scriptstyle \psi_{0;q,q'}\wedge\psi_{0;q,q'}} & & \downarrow{\scriptstyle \varphi_{0,0;q}\wedge\varphi_{0,0;q'}} \\ E_{0,q+q'}\wedge E_{0,q+q'} & & E_{0,q}\wedge E_{0;q'} \\ & {\scriptstyle\varphi_{0,0;q+q'}}\searrow \qquad \swarrow{\scriptstyle\psi_{0;q,q'}} & \\ & E_{0,q+q'} & \end{array} \tag{2.2.3.6}$$

commutes. Chasing the section

$$e_q \wedge e_{q'} \wedge e_q \wedge e_{q'} \in E_{0,q} \wedge E_{0,q'} \wedge E_{0,q} \wedge E_{0,q'} \tag{2.2.3.7}$$

along the two paths to $E_{0,q+q'}$ we obtain

$$\varphi_{0,0;q+q'}((\psi_{0;q,q'}(e_q \wedge e_{q'})) \wedge (\psi_{0;q,q'}(e_q \wedge e_{q'}))) = \psi_{0;q,q'}(e_q \wedge e_{q'}) \tag{2.2.3.8}$$

as desired. □

2.2.4. Similarly, if we take f to be the identity map and g to be the zero morphism, then the sections $f_p \in E_{p,0}$ corresponding to the identity elements of the groups $E_{p,-}$ define a trivialization of the biextension

$$E'' := (\mathrm{id} \times 0)^* E. \tag{2.2.4.1}$$

2.2.5. We can also pullback a biextension E of $F \times G$ by $\mathbb{G}_m$ along the map

$$(-1) : F \times G \to F \times G, \quad (a, b) \mapsto (-a, -b). \tag{2.2.5.1}$$

Concretely for local sections $(p, q) \in F \times G$ we have

$$((-1)^* E)_{p,q} = E_{-p,-q}. \tag{2.2.5.2}$$

Let

$$\sigma_{p,q} : E_{p,q} \to ((-1)^* E)_{p,q} = E_{-p,-q} \tag{2.2.5.3}$$

by the isomorphism of $\mathbb{G}_m$-torsors characterized by the condition that the diagram

$$\begin{array}{ccc} E_{p,q} \wedge E_{p,-q} & \xrightarrow{\sigma_{p,q}\times \mathrm{id}} & E_{-p,-q} \wedge E_{p,-q} \\ \downarrow{\scriptstyle \psi_{p;q,-q}} & & \downarrow{\scriptstyle \varphi_{-p,p;-q}} \\ E_{p,0} & \xrightarrow{\text{can}} & E_{0,-q} \end{array} \tag{2.2.5.4}$$

commutes, where can denotes the unique isomorphism of $\mathbb{G}_m$-torsors sending the identity element of $E_{p,-}$ (an element of $E_{p,0}$) to the identity element of $E_{-,q}$ (an element of $E_{0,-q}$).

Lemma 2.2.6 *The maps $\sigma_{p,q}$ define an isomorphism of $\mathbb{G}_m$-biextensions of $F \times G$*

$$\sigma : E \to (-1)^*E. \tag{2.2.6.1}$$

Proof. We can rewrite the diagram 2.2.5.4 as the diagram

$$\begin{array}{ccc}
(E \wedge (1\times(-1))^*E)_{p,q} & \xrightarrow{\sigma_{p,q}\times \mathrm{id}} & (((-1)\times(-1))^*E \wedge (1\times(-1))^*E)_{p,q} \\
\downarrow \psi_{p;q,-q} & & \downarrow \varphi_{-p,p;-q} \\
((1\times 0)^*E)_{p,q} & & ((0\times(-1))^*E)_{p,q} \\
& \searrow^{e^{-1}} \quad \nearrow^{f} & \\
& \mathbb{G}_m, &
\end{array} \tag{2.2.6.2}$$

where e and f are the maps of biextensions defined in 2.2.2 and 2.2.4. Since the maps $\psi_{p;q,-q}$ and $\varphi_{-p,p;-q}$ induce morphisms of biextensions

$$E \wedge (1\times(-1))^*E \to (1\times 0)^*E \tag{2.2.6.3}$$

and

$$((-1)\times(-1))^*E \wedge (1\times(-1))^*E \to (0\times -1)^*E \tag{2.2.6.4}$$

by [11, 1.2] this proves that the maps $\sigma_{p,q}$ are obtained from a composition of morphisms of biextensions (which is also a morphism of biextensions). □

2.2.7. If $F = G$, then the "flip" map $\iota : F\times F \to F \times F$ sending (a,b) to (b,a) induces a functor

$$\iota^* : \underline{\mathrm{Biext}}(F,F;\mathbb{G}_m) \to \underline{\mathrm{Biext}}(F,F;\mathbb{G}_m). \tag{2.2.7.1}$$

A *symmetric $\mathbb{G}_m$–biextension of F* is defined to be a $\mathbb{G}_m$–biextension E of $F\times F$ together with an isomorphism $\lambda : \iota^*E \to E$ such that $\lambda \circ i^*(\lambda) = \mathrm{id}$ (note that $(\iota^*)^2 = \mathrm{id}$). If (E', λ') is a second symmetric $\mathbb{G}_m$–biextension of F, then a morphism $(E',\lambda') \to (E,\lambda)$ is a morphism $h : E' \to E$ of biextensions such that the diagram

$$\begin{array}{ccc}
\iota^*E' & \xrightarrow{\lambda'} & E' \\
{\scriptstyle \iota^*(h)}\downarrow & & \downarrow{\scriptstyle h} \\
\iota^*E & \xrightarrow{\lambda} & E
\end{array} \tag{2.2.7.2}$$

commutes. We denote by $\underline{\mathrm{Biext}}^{\mathrm{sym}}(F,\mathbb{G}_m)$ the category of symmetric $\mathbb{G}_m$–biextensions of F. The "flip" map $\mathbb{G}_m \times F \times F \to \mathbb{G}_m \times F \times F$ sending (u,a,b) to (u,b,a) induces the structure of a symmetric $\mathbb{G}_m$–biextension on the trivial $\mathbb{G}_m$–biextension of $F\times F$. As above, we therefore have a notion of a trivialization of a symmetric $\mathbb{G}_m$–biextension of F.

2.2.8. Let X be a free abelian group of finite rank, and view X as a constant sheaf. In this case the group of automorphisms of a symmetric $\mathbb{G}_m$–biextension $\pi : E \to X \times X$ of X is canonically isomorphic to $\mathrm{Hom}(S^2X, \mathbb{G}_m)$, where S^2X denotes the second symmetric power of X. To see this let $h : E \to E$ be such an automorphism. Since h is a morphism over $X \times X$ and E is a $\mathbb{G}_m$–torsor over $X \times X$, for any local section $e \in E$ there exists a unique element $u \in \mathbb{G}_m$ such that $h(e) = u(e)$. Furthermore, since h is compatible with the $\mathbb{G}_m$–action the element u depends only on $\pi(e)$. We therefore obtain a set map $b : X \times X \to \mathbb{G}_m$ by associating to any pair (x, y) the section of $\mathbb{G}_m$ obtained by locally choosing a lifting $e \in E$ of (x, y) and sending (x, y) to the corresponding unit $u \in \mathbb{G}_m$. Compatibility with (2.2.1 (ii) and (iii)) implies that this map b in fact is bilinear. Furthermore, the commutativity of 2.2.7.2 amounts to the condition $b(x, y) = b(y, x)$. Thus any automorphism h of E is determined by a map $b : S^2X \to \mathbb{G}_m$. Conversely any such map b induces an automorphism by the formula

$$e \mapsto b(\pi(e)) \cdot e. \tag{2.2.8.1}$$

2.2.9. Let A/S be an abelian scheme, A^t/S the dual abelian scheme, and $\lambda : A \to A^t$ a principal polarization defined by an invertible sheaf $\mathscr{M}$ on A. Via the isomorphism λ, the Poincare bundle $\mathscr{B} \to A \times A^t$ defines a $\mathbb{G}_m$–torsor (denoted by the same letter) $\pi : \mathscr{B} \to A \times A$. This torsor can be described as follows. For an integer n and subset $I \subset \{1, \ldots, n\}$, let $m_I : A^{\times n} \to A$ be the map

$$(a_1, \ldots, a_n) \mapsto \sum_{i \in I} a_i, \tag{2.2.9.1}$$

where $A^{\times n}$ denotes the n–fold fiber product over S of A with itself. If I is the empty set then m_I sends all points of A^n to the identity element of A. Then $\mathscr{B} \to A \times A$ is canonically isomorphic to

$$\Lambda(\mathscr{M}) := \bigotimes_{I \subset \{1,2\}} m_I^* \mathscr{M}^{(-1)^{\mathrm{card}(I)}}. \tag{2.2.9.2}$$

In other words, for any two scheme–valued points $a, b \in A$, the fiber of $\mathscr{B}$ over $(a, b) \in A \times A$ is equal to

$$\mathscr{M}(a + b) \otimes \mathscr{M}(a)^{-1} \otimes \mathscr{M}(b)^{-1} \otimes \mathscr{M}(0). \tag{2.2.9.3}$$

Note also that the definition of $\Lambda(\mathscr{M})$ is symmetric in the two factors of $A \times A$ so there is a canonical isomorphism $\iota : \mathscr{B} \to \mathscr{B}$ over the flip map $A \times A \to A \times A$.

The theorem of the cube [11, 2.4] provides a canonical isomorphism

$$\rho : \mathscr{O}_{A^3} \to \Theta(\mathscr{M}) := \bigotimes_{I \subset \{1,2,3\}} m_I^* \mathscr{M}^{(-1)^{\mathrm{card}(I)}}. \tag{2.2.9.4}$$

For any three scheme–valued points $a, b, c \in A$ this gives a canonical isomorphism

$$\mathscr{M}(a+b) \otimes \mathscr{M}^{-1}(a) \otimes \mathscr{M}^{-1}(b) \otimes \mathscr{M}(0) \otimes \mathscr{M}(a+c) \otimes \mathscr{M}^{-1}(a) \otimes \mathscr{M}^{-1}(c)$$
$$\downarrow$$
$$\mathscr{M}(a+b+c) \otimes \mathscr{M}^{-1}(b+c) \otimes \mathscr{M}^{-1}(a). \tag{2.2.9.5}$$

For points $p, p', q \in A$ this induces an isomorphism

$$\psi_{p;q,q'} : \mathscr{B}_{p,q} \otimes \mathscr{B}_{p,q'} \to \mathscr{B}_{p,q+q'}, \tag{2.2.9.6}$$

and also by symmetry for $p, q, q' \in A$ an isomorphism

$$\varphi_{p,p';q} : \mathscr{B}_{p,q} \otimes \mathscr{B}_{p',q} \to \mathscr{B}_{p+p',q}. \tag{2.2.9.7}$$

It is shown in [11, 2.4] that these maps together with the above defined map ι give $\mathscr{B}$ the structure of a symmetric $\mathbb{G}_m$–biextension of A.

2.2.10. Let X be a free abelian group of finite rank, let $(A, \mathscr{M})$ be an abelian scheme with an invertible sheaf defining a principal polarization over some base S with $\mathrm{Pic}(S) = 0$, and let $c : X \to A(S)$ be a homomorphism. We fix a rigidification of $\mathscr{M}$. Pulling back $\mathscr{M}$ along c we obtain a $\mathbb{G}_m$–torsor W over X (viewed as a constant sheaf on the category of S–schemes). Let

$$\lambda_{\mathscr{M}} : A \to A^t, \quad a \in A \mapsto t_a^*\mathscr{M} \otimes \mathscr{M}^{-1} \otimes \mathscr{M}^{-1}(a) \otimes \mathscr{M}(0) \tag{2.2.10.1}$$

be the isomorphism defined by $\mathscr{M}$ (where we view A^t as classifying invertible sheaves rigidified along 0), and let $\mathscr{B} \to A \times A$ denote the symmetric $\mathbb{G}_m$–biextension defined by the Poincare bundle. Pulling back along $c \times c : X \times X \to A \times A$ we also obtain a symmetric $\mathbb{G}_m$–biextension E of X. From above we know that for $(x, y) \in X \times X$ the fiber $E_{x,y}$ is equal to

$$\mathscr{M}(c(x) + c(y)) \otimes \mathscr{M}(c(x))^{-1} \otimes \mathscr{M}(c(y))^{-1} \otimes \mathscr{M}(0). \tag{2.2.10.2}$$

Now let $\psi : X \to c^*\mathscr{M}^{-1}$ be a trivialization. This trivialization ψ defines a trivialization τ of the $\mathbb{G}_m$–torsor over $X \times X$ underlying E by sending $(x, y) \in X \times X$ to

$$\psi(x+y) \otimes \psi(x)^{-1} \otimes \psi(y)^{-1} \otimes \psi(0).$$

In what follows it will be important to make explicit the additional conditions on ψ needed for τ to be compatible with the symmetric biextension structure.

For a point $a \in A$, let L_a denote the rigidified invertible sheaf corresponding to $\lambda_{\mathscr{M}}(a)$. The sheaf L_a is equal to the restriction of the Poincare bundle to $A \simeq A \times \{a\} \subset A \times A$. In particular, for a point $b \in A$ we have $L_a(b) \simeq \mathscr{B}_{(b,a)}$. For $x \in X$, we also sometimes write L_x for $L_{c(x)}$ if no confusion seems likely to arise.

Lemma 2.2.11 *For any integer* $d \geq 0$ *and* $x, y \in X$ *the sheaves* $t_{c(y)}^*(\mathscr{M}^d \otimes L_x)$ *and* $\mathscr{M}^d \otimes L_{x+dy}$ *on* A *are non–canonically isomorphic, where* $t_{c(y)} : A \to A$ *denotes translation by* $c(y)$.

Proof. View A^t as the connected component of the space classifying isomorphism classes of line bundles on A. Then the isomorphism $\lambda : A \to A^t$ sends a scheme–valued point $a \in A$ to the isomorphism class of $L_a := t_a^*\mathscr{M} \otimes \mathscr{M}^{-1}$. Since λ is a homomorphism, there exists an isomorphism

$$(t_x^*\mathscr{M}) \otimes (t_y^*\mathscr{M}) \simeq \mathscr{M} \otimes t_{x+y}^*\mathscr{M}. \tag{2.2.11.1}$$

It follows that

$$t_y^*(\mathscr{M}^d \otimes L_x) \simeq t_y^*(\mathscr{M}^{d-1} \otimes t_x^*\mathscr{M}) \simeq t_y^*\mathscr{M}^{d-1} \otimes t_{x+y}^*\mathscr{M} \simeq \mathscr{M}^{d-1} \otimes t_{x+dy}^*\mathscr{M}, \tag{2.2.11.2}$$

and also

$$\mathscr{M}^d \otimes L_{x+dy} \simeq \mathscr{M}^{d-1} \otimes t_{x+dy}^*\mathscr{M}. \tag{2.2.11.3}$$

It follows that étale locally on S the two line bundles in the lemma are isomorphic. It follows that the functor on S–schemes

$$T/S \mapsto \{\text{isomorphisms } t_{c(y)}^*(\mathscr{M}^d \otimes L_x) \to \mathscr{M}^d \otimes L_{x+dy} \text{ over } A_T\} \tag{2.2.11.4}$$

is a $\mathbb{G}_m$–torsor. Since $\mathrm{Pic}(S) = 0$ this torsor is trivial so there exists an isomorphism over S. $\square$

2.2.12. It follows that to give an isomorphism

$$t_{c(y)}^*(\mathscr{M}^d \otimes L_x) \to \mathscr{M}^d \otimes L_{x+dy} \tag{2.2.12.1}$$

is equivalent to giving an isomorphism of $\mathscr{O}_S$–modules

$$(\mathscr{M}^d \otimes L_x)(c(y)) \to \mathscr{M}^d(0) \otimes L_{x+dy}(0). \tag{2.2.12.2}$$

Since $\mathscr{M}^d \otimes L_{x+dy}$ is rigidified at 0 this in turn is equivalent to a trivialization of $(\mathscr{M}^d \otimes L_x)(c(y))$. On the other hand, there is a canonical isomorphism

$$(\mathscr{M}^d \otimes L_x)(c(y)) \simeq \mathscr{M}(c(y))^d \otimes \mathscr{B}_{(c(y),c(x))}. \tag{2.2.12.3}$$

It follows that ψ and τ define an isomorphism

$$\psi(y)^d\tau(y,x) : t_{c(y)}^*(\mathscr{M}^d \otimes L_x) \to \mathscr{M}^d \otimes L_{x+dy}. \tag{2.2.12.4}$$

Proposition 2.2.13. *The trivialization* τ *is compatible with the symmetric biextension structure on* $(c \times c)^*\mathscr{B}$ *if and only if the following two conditions hold (see the proof for explanation of the numbering):*
(iii)' *For any* $x, x', y \in X$ *and* $d \geq 0$ *the diagram*

$$\begin{array}{ccc} t^*_{c(y)}(\mathscr{M}^d \otimes L_x) \otimes t^*_{c(y)}(\mathscr{M}^{d'} \otimes L_{x'}) & \xrightarrow{\text{can}} & t^*_{c(y)}(\mathscr{M}^{d+d'} \otimes L_{x+x'}) \\ \downarrow{\scriptstyle \psi(y)^d\tau(y,x)\otimes\psi(y)^{d'}\tau(y,x')} & & \downarrow{\scriptstyle \psi(y)^{d+d'}\tau(y,x+x')} \\ (\mathscr{M}^d \otimes L_{x+dy}) \otimes (\mathscr{M}^{d'} \otimes L_{x'+d'y}) & \xrightarrow{\text{can}} & \mathscr{M}^{d+d'} \otimes L_{x+x'+(d+d')y} \end{array} \tag{2.2.13.1}$$

commutes, where "can" *denotes the canonical isomorphisms described in* 4.1.10.2.
(ii)' *For any $x, y, y' \in X$, the diagram*

$$\begin{array}{ccc} t^*_{c(y+y')}(\mathscr{M}^d \otimes L_x) & \xrightarrow{\psi(y)^d\tau(y,x)} & t^*_{c(y')}(\mathscr{M}^d \otimes L_{x+dy}) \\ & {\scriptstyle \psi(y+y')^d\tau(y+y',x)} \searrow \quad \swarrow {\scriptstyle \psi(y')^d\tau(y',x+dy)} & \\ & \mathscr{M}^d \otimes L_{x+d(y+y')} & \end{array} \tag{2.2.13.2}$$

commutes.

Proof. First we claim that (iii)' is equivalent to compatibility with the structure in (2.2.1 (iii)). To see this note first that 2.2.13.1 clearly commutes when $x = x' = 0$. From this it follows that it suffices to consider the case when $d = 0$. In this case the commutativity of 2.2.13.1 amounts to the statement that the image of $\tau(y, x) \otimes \tau(y, x')$ under the canonical map

$$\mathscr{B}_{(y,x)} \otimes \mathscr{B}_{(y,x')} = L_x(y) \otimes L_{x'}(y) \to L_{x+x'}(y) = \mathscr{B}_{(y,x+x')} \tag{2.2.13.3}$$

is equal to $\tau(y, x + x')$. This is precisely compatibility with the structure in (2.2.1 (iii)).

Next we claim that condition (ii)' in the case when $d = 0$ is equivalent to compatibility of τ with (2.2.1 (ii)). Indeed in this case the composite map

$$(\psi(y')^d\tau(y', x + dy)) \circ (\psi(y)^d\tau(y, x)) \tag{2.2.13.4}$$

is equal to the map induced by the image of $\tau(y, x) \otimes \tau(y', x)$ under the map

$$L_x(y) \otimes L_x(y') = \mathscr{B}_{(y,x)} \otimes \mathscr{B}_{(y',x)} \to \mathscr{B}_{(y+y',x)} \simeq L_x(y + y'). \tag{2.2.13.5}$$

Note also that (ii)' holds in the case when $x = 0$ by the definition of $\tau(y, y')$.

Condition (ii)' in general follows from these two special cases and (iii)'. To see this note that there is a commutative diagram

$$\begin{array}{ccc} t^*_{c(y+y')}(\mathscr{M}^d) \otimes t^*_{c(y+y')}L_x & \xrightarrow{\text{can}} & t^*_{c(y+y')}(\mathscr{M}^d \otimes L_x) \\ \downarrow & & \downarrow{\scriptstyle \psi(y)^d\tau(y,x)} \\ t^*_{c(y')}(\mathscr{M}^d \otimes L_{dy}) \otimes t^*_{c(y')}(L_x) & \xrightarrow{\text{can}} & t^*_{c(y')}(\mathscr{M}^d \otimes L_{x+dy}) \\ \downarrow & & \downarrow{\scriptstyle \psi(y')^d\tau(y',x+dy)} \\ \mathscr{M}^d \otimes L_{d(y+y')} \otimes L_x & \xrightarrow{\text{can}} & \mathscr{M}^d \otimes L_{x+d(y+y')} \end{array} , \tag{2.2.13.6}$$

where the left column is obtained by taking the tensor product of the maps 2.2.13.4 in the cases $x = 0$ and $d = 0$. Then (iii)' implies that it suffices to show that the composite of the left column is equal to the tensor product of $\psi(y + y')^d$ and $\tau(y + y', x)$ which follows from the above special cases.

Finally compatibility with (2.2.1 (iv)) is automatic by the definition of τ, as is the compatibility with the isomorphism ι giving the symmetric structure. $\square$

2.3 Logarithmic Geometry

In this section we review the necessary parts of the theory of logarithmic geometry developed by Fontaine, Illusie, and Kato. For complete treatments of the theory we recommend [24] and [41].

2.3.1. Let us start by reviewing some terminology from the theory of monoids.

We only consider commutative monoids with unit, and morphisms of monoids are required to preserve the unit element. We usually write the monoid law additively (the main exception being a ring R viewed as a monoid under multiplication).

The inclusion functor

$$\text{(abelian groups)} \subset \text{(monoids)}$$

has a left adjoint sending a monoid M to the group M^{gp} which is the quotient of the set of pairs $\{(a, b) \in M \times M\}$ by the equivalence relation

$$(a, b) \sim (c, d) \ \text{ if there exists } s \in M \text{ such that } s + a + d = s + b + c.$$

The group structure on M^{gp} is induced by the addition

$$(a, b) + (c, d) = (a + c, b + d).$$

The adjunction map $\pi : M \to M^{gp}$ sends $a \in M$ to $(a, 0)$.

A monoid is called *integral* if for every $a \in M$ the translation map

$$M \to M, \quad b \mapsto a + b$$

is injective. The monoid M is called *coherent* if M is a finitely generated monoid, and *fine* if M is coherent and integral.

Let X be a scheme.

Definition 2.3.2. (i) A *pre-log structure* on X is a pair (M, α), where M is a sheaf of monoids on the étale site of X and $\alpha : M \to \mathscr{O}_X$ is a morphism of sheaves of monoids (where $\mathscr{O}_X$ is viewed as a monoid under multiplication).

(ii) A pre-log structure (M, α) is called a *log structure* if the map α induces a bijection $\alpha^{-1}(\mathscr{O}_X^*) \to \mathscr{O}_X^*$.

(iii) A *log scheme* is a pair (X, M_X) consisting of a scheme X and a log structure M_X on X.

Remark 2.3.3. As in (iii) above, when dealing with (pre-)log structures we usually omit the map α from the notation and write simply M for the pair (M, α).

Remark 2.3.4. If (X, M) is a log scheme then the units $M^* \subset M$ are by the definition of log structure identified with $\mathscr{O}_X^*$ via the map $\alpha : M \to \mathscr{O}_X$. We let $\lambda : \mathscr{O}_X^* \hookrightarrow M$ be the resulting inclusion. The monoid law on M defines an action of $\mathscr{O}_X^*$ on M by translation. The quotient $\overline{M} := M/\mathscr{O}_X^*$ has a natural monoid structure induced by the monoid structure on M.

Remark 2.3.5. The notion of log structure makes sense in any ringed topos. Using the étale topology on algebraic spaces and Deligne-Mumford stacks and the lisse-étale topology on Artin stacks (see for example [31, §12]) we can therefore also talk about log algebraic spaces and log algebraic stacks.

2.3.6. The natural inclusion functor

$$(\text{log structures on } X) \hookrightarrow (\text{pre-log structures on } X)$$

has a left adjoint $M \mapsto M^a$. The log structure M^a is obtained from M by setting M^a equal to the pushout $M \oplus_{\alpha^{-1}\mathscr{O}_X^*} \mathscr{O}_X^*$ in the category of sheaves of monoids of the diagram

$$\begin{array}{ccc} \alpha^{-1}\mathscr{O}_X^* & \xrightarrow{\lambda} & M \\ \downarrow & & \\ \mathscr{O}_X^*, & & \end{array} \qquad (2.3.6.1)$$

with the map to $\mathscr{O}_X$ induced by the map $M \to \mathscr{O}_X$ and the inclusion $\mathscr{O}_X^* \hookrightarrow \mathscr{O}_X$. We refer to M^a as the *log structure associated to M*.

The basic example of this construction is the following. If P is a finitely generated integral monoid and $\beta : P \to \Gamma(X, \mathscr{O}_X)$ is a morphism of monoids, we obtain a pre-log structure by viewing P as a constant sheaf with the map to $\mathscr{O}_X$ defined by β . By passing to the associated log structure we obtain a log structure on X.

Definition 2.3.7. A log structure M on X is *fine* if there exists an étale cover $\{U_i \to X\}_{i\in I}$ and finitely generated integral monoids $\{P_i\}_{i\in I}$ with maps $\beta_i : P_i \to \Gamma(U_i, \mathscr{O}_{U_i})$ such that the restriction $M|_{U_i}$ is isomorphic to the log structure defined by the pair (P_i, β_i).

Definition 2.3.8. A *chart* for a fine log structure M on a scheme X is a map $\beta : P \to \Gamma(X, M)$ from a fine monoid P such that the induced map

$$P^a \to M$$

is an isomorphism, where P^a is the log structure associated to the prelog structure defined by the composite

$$P \xrightarrow{\beta} \Gamma(X, M) \xrightarrow{\alpha} \Gamma(X, \mathscr{O}_X).$$

Example 2.3.9. Let R be a ring. If P is a finitely generated integral monoid, we write $\mathrm{Spec}(P \to R[P])$ for the log scheme whose underlying scheme is $\mathrm{Spec}(R[P])$ (where $R[P]$ is the monoid algebra on P) and whose log structure is associated to the prelog structure given by the natural map of monoids $P \to R[P]$.

2.3.10. If $f : X \to Y$ is a morphism of schemes and M_Y is a log structure on Y, then the pullback f^*M_Y of M_Y to X is defined to be the log structure associated to the prelog structure

$$f^{-1}M_Y \to f^{-1}\mathscr{O}_Y \to \mathscr{O}_X. \tag{2.3.10.1}$$

One checks immediately that if M_Y is fine then f^*M_Y is also fine.

This construction enables one to define a category of log schemes: A morphism $(X, M_X) \to (Y, M_Y)$ is a pair (f, f^b), where $f : X \to Y$ is a morphism of schemes and $f^b : f^*M_Y \to M_X$ is a morphism of log structures on X.

Example 2.3.11. Let $\theta : Q \to P$ be a morphism of fine monoids, and let R be a ring. Then θ induces a natural morphism of fine log schemes

$$\mathrm{Spec}(P \to R[P]) \to \mathrm{Spec}(Q \to R[Q]).$$

Many of the classical notions (e.g. smooth, flat, local complete intersection...) have logarithmic analogues. The key notions we need in this paper are the following:

Definition 2.3.12. (i) A morphism $(f, f^b) : (X, M_X) \to (Y, M_Y)$ of log schemes is *strict* if the map $f^b : f^*M_Y \to M_X$ is an isomorphism.

(ii) A morphism $(f, f^b) : (X, M_X) \to (Y, M_Y)$ is a *closed immersion* (resp. *strict closed immersion*) if $f : X \to Y$ is a closed immersion and $f^b : f^*M_X \to M_Y$ is surjective (resp. an isomorphism).

(iii) A morphism $(f, f^b) : (X, M_X) \to (Y, M_Y)$ is *log smooth* (resp. *log étale*) if $f : X \to Y$ is locally of finite presentation and for every commutative diagram

$$\begin{array}{ccc} (T_0, M_{T_0}) & \xrightarrow{a} & (X, M_X) \\ {\scriptstyle j}\downarrow & & \downarrow \\ (T, M_T) & \longrightarrow & (Y, M_Y) \end{array} \tag{2.3.12.1}$$

with j a strict closed immersion defined by a nilpotent ideal, there exists (resp. there exists a unique) morphism $(T, M_T) \to (X, M_X)$ filling in the diagram.

Remark 2.3.13. If (X, M) is a fine log scheme (i.e. a scheme with fine log structure) then giving a chart $P \to \Gamma(X, M)$ is equivalent to giving a strict morphism of log schemes

$$(X, M) \to \mathrm{Spec}(P \to \mathbb{Z}[P]). \tag{2.3.13.1}$$

Lemma 2.3.14 *Let $(f, f^b) : (X, M_X) \to (Y, M_Y)$ be a morphism of fine log schemes, and let $h : U \to X$ be a smooth surjection. Denote by M_U the pullback of M_X to U so we have a commutative diagram of log schemes*

$$(U, M_U) \xrightarrow{(h,h^b)} (X, M_X) \xrightarrow{(f,f^b)} (Y, M_Y), \qquad (g,g^b) : (U, M_U) \to (Y, M_Y). \tag{2.3.14.1}$$

Then (f, f^b) is log smooth if and only if (g, g^b) is log smooth.

Proof. This follows immediately from the definition of a log smooth morphism. □

2.3.15. One of the most remarkable aspects of the logarithmic theory is that the notion of log smoothness behaves so much like the usual notion of smoothness for schemes (a stack–theoretic "explanation" for this phenomenon is given in [42]). In particular, as we now explain the étale local structure of log smooth morphisms is very simple and there is a good deformation theory of log smooth morphisms.

Theorem 2.3.16 ([24, 3.5]) *Let $f : (X, M_X) \to (Y, M_Y)$ be a log smooth morphism of fine log schemes, let $\bar{x} \to X$ be a geometric point and let $\bar{y} \to Y$ be the composite $\bar{x} \to X \to Y$. Then after replacing X and Y by étale neighborhoods of $\bar{x}$ and $\bar{y}$ respectively, there exists charts $\beta_X : P \to M_X$, $\beta_Y : Q \to M_Y$, and a morphism $\theta : Q \to P$ such that the following hold:*
(i) *The diagram of fine log schemes*

$$\begin{array}{ccc} (X, M_X) & \xrightarrow{\beta_X} & \operatorname{Spec}(P \to \mathbb{Z}[P]) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \theta} \\ (Y, M_Y) & \xrightarrow{\beta_Y} & \operatorname{Spec}(Q \to \mathbb{Z}[Q]) \end{array} \tag{2.3.16.1}$$

commutes.
(ii) *The induced map*

$$X \to Y \times_{\operatorname{Spec}(\mathbb{Z}[Q])} \operatorname{Spec}(\mathbb{Z}[P]) \tag{2.3.16.2}$$

is étale.
(iii) *The kernel of $\theta^{gp} : Q^{gp} \to P^{gp}$ is a finite group and the orders of* $\operatorname{Ker}(\theta^{gp})$ *and the torsion part of* $\operatorname{Coker}(\theta^{gp})$ *are invertible in $k(x)$.*

Conversely if étale locally there exists charts satisfying the above conditions then the morphism $(X, M_X) \to (Y, M_Y)$ is log smooth.

Example 2.3.17. If S is a scheme which we view as a log scheme with the trivial log structure $\mathscr{O}_S^* \hookrightarrow \mathscr{O}_S$, then a morphism of fine log schemes

$(X, M_X) \to (S, \mathscr{O}_S^*)$ is log smooth if and only if étale locally on S and X the log scheme (X, M_X) is isomorphic to $\mathrm{Spec}(P \to \mathscr{O}_S[P])$ for some finitely generated integral monoid P. Thus in the case of trivial log structure on the base log smoothness essentially amounts to "toric singularities".

Example 2.3.18. Probably the most important example from the point of view of degenerations is the following. Let k be a field and M_k the log structure on k associated to the map $\mathbb{N} \to k$ sending all nonzero elements to 0. Let $\Delta : \mathbb{N} \to \mathbb{N}^2$ be the diagonal map and set $X = \mathrm{Spec}(k \otimes_{k[\mathbb{N}]} k[\mathbb{N}^2]) = \mathrm{Spec}(k[x, y]/(xy))$ with log structure M_X induced by the natural map $\mathbb{N}^2 \to k \otimes_{k[\mathbb{N}]} k[\mathbb{N}^2]$. Then the morphism

$$(X, M_X) \to (\mathrm{Spec}(k), M_k) \tag{2.3.18.1}$$

is log smooth.

2.3.19. One technical difficulty that arises when dealing with log smoothness is that in general the underlying morphism of schemes of a log smooth morphism need not be flat. All the examples considered in this text will satisfy an additional property that ensures that the underlying morphism of schemes is flat. A morphism of integral monoids $\theta : P \to Q$ is called *integral* if the map of algebras $\mathbb{Z}[P] \to \mathbb{Z}[Q]$ induced by θ is flat (see [24, 4.1] for several other characterizations of this property). A morphism of log schemes $f : (X, M_X) \to (Y, M_Y)$ is called *integral* if for every geometric point $\bar{x} \to X$ the map $f^{-1}\overline{M}_{Y,f(\bar{x})} \to \overline{M}_{X,\bar{x}}$ is an integral morphism of monoids. By [24, 4.5], if $f : (X, M_X) \to (Y, M_Y)$ is a log smooth and integral morphism of fine log schemes then the underlying morphism $X \to Y$ is flat.

2.3.20. As in the case of schemes without log structures, the notion of log smoothness is intimately tied to differentials.

Let $f : (X, M_X) \to (Y, M_Y)$ be a morphism of fine log schemes locally of finite presentation. For a scheme T and a quasi–coherent sheaf I on T, let $T[I]$ denote the scheme with same underlying topological space as that of T, but with structure sheaf the $\mathscr{O}_T$–algebra $\mathscr{O}_T \oplus I$ with algebra structure given by $(a + i)(c + j) = ac + (aj + ci)$. The ideal I defines a closed immersion $j : T \hookrightarrow T[I]$ for which the natural map $T[I] \to T$ induced by $\mathscr{O}_T \to \mathscr{O}_T[I]$ sending a to a is a retraction. If M_T is a fine log structure on T, let $M_{T[I]}$ denote the log structure on $T[I]$ obtained by pullback along $\mathscr{O}_T \to \mathscr{O}_T[I]$ so that we have a diagram of fine log schemes

$$(T, M_T) \xrightarrow{\quad j \quad} (T[I], M_{T[I]}) \xrightarrow{\quad \pi \quad} (T, M_T). \tag{2.3.20.1}$$

The above construction is functorial in the pair (T, I).

Consider now the functor F on the category of quasi–coherent sheaves on X associating to any quasi-coherent sheaf I the set of morphisms of fine log schemes $(X[I], M_{X[I]}) \to (X, M_X)$ filling in the commutative diagram

$$\begin{array}{ccc} (X, M_X) & \xrightarrow{\text{id}} & (X, M_X) \\ j \downarrow & & \downarrow f \\ (X[I], M_{X[I]}) & \xrightarrow{f \circ \pi} & (Y, M_Y). \end{array} \tag{2.3.20.2}$$

Theorem 2.3.21 ([24, 3.9]) *There exists a (necessarily unique) quasi–coherent sheaf $\Omega^1_{(X,M_X)/(Y,M_Y)}$ on X and an isomorphism of functors*

$$F \simeq \mathrm{Hom}(\Omega^1_{(X,M_X)/(Y,M_Y)}, -). \tag{2.3.21.1}$$

2.3.22. The sheaf $\Omega^1_{(X,M_X)/(Y,M_Y)}$ is called the sheaf of *logarithmic differentials* of (X, M_X) over (Y, M_Y). Note that the identity map

$$\Omega^1_{(X,M_X)/(Y,M_Y)} \to \Omega^1_{(X,M_X)/(Y,M_Y)} \tag{2.3.22.1}$$

defines a morphism

$$\rho : (X[\Omega^1_{(X,M_X)/(Y,M_Y)}], M_{X[\Omega^1_{(X,M_X)/(Y,M_Y)}]}) \to (X, M_X). \tag{2.3.22.2}$$

This defines in particular a morphism $\rho^* : \mathscr{O}_X \to \mathscr{O}_X[\Omega^1_{(X,M_X)/(Y,M_Y)}]$. Taking the difference of this morphism and the morphism $\pi^* : \mathscr{O}_X \to \mathscr{O}_X[\Omega^1_{(X,M_X)/(Y,M_Y)}]$ we obtain a derivation $d : \mathscr{O}_X \to \Omega^1_{(X,M_X)/(Y,M_Y)}$. This defines in particular a morphism of quasi–coherent sheaves

$$\Omega^1_{X/Y} \to \Omega^1_{(X,M_X)/(Y,M_Y)}. \tag{2.3.22.3}$$

Remark 2.3.23. Note that in the case when the log structures M_X and M_Y are trivial, we recover the usual Kahler differentials.

Remark 2.3.24. The sheaf of logarithmic differentials $\Omega^1_{(X,M_X)/(Y,M_Y)}$ has the following more concrete description (see [24, 1.7]). It is the quotient of the $\mathscr{O}_X$-module

$$\Omega^1_{X/Y} \oplus (\mathscr{O}_X \otimes_{\mathbb{Z}} M_X^{\mathrm{gp}})$$

by the $\mathscr{O}_X$-submodule generated locally by sections of the following form:

(i) $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$, where $a \in M_X$.
(ii) $(0, 1 \otimes a)$ for $a \in M_X$ in the image of $f^{-1}M_Y$.

The following summarizes the basic properties of logarithmic differentials:

Theorem 2.3.25 ([41, 3.2.1 and 3.2.3]) (i) *If $f : (X, M_X) \to (Y, M_Y)$ is a log smooth morphism of fine log schemes, then $\Omega^1_{(X,M_X)/(Y,M_Y)}$ is a locally free $\mathscr{O}_X$–module of finite type.*
(ii) *For any composite*

$$(X, M_X) \xrightarrow{f} (Y, M_Y) \xrightarrow{g} (S, M_S) \tag{2.3.25.1}$$

there is an associated exact sequence

$$f^*\Omega^1_{(Y,M_Y)/(S,M_S)} \xrightarrow{s} \Omega^1_{(X,M_X)/(S,M_S)} \longrightarrow \Omega^1_{(X,M_X)/(Y,M_Y)} \longrightarrow 0. \tag{2.3.25.2}$$

If gf is log smooth, then f is log smooth if and only if s is injective and the image is locally a direct summand.

Example 2.3.26. Let P be a finitely generated integral monoid, and let (X, M_X) denote the log scheme $\mathrm{Spec}(P \to \mathbb{Z}[P])$. Then one can show that $\Omega^1_{(X,M_X)/\mathbb{Z}} \simeq \mathscr{O}_X \otimes_{\mathbb{Z}} P^{gp}$ [24, 1.8]. The differential $d : \mathscr{O}_X \to \Omega^1_{(X,M_X)/\mathbb{Z}}$ sends a section $e_p \in \mathbb{Z}[P]$ which is the image of an element $p \in P$ to $e_p \otimes p$. In this case when $P = \mathbb{N}^r$ with standard generators e_i $(1 \leq i \leq r)$ the module $\Omega^1_{(X,M_X)/\mathbb{Z}}$ is isomorphic to the classically defined module of logarithmic differentials on $\mathbb{A}^r \simeq \mathrm{Spec}(\mathbb{Z}[X_1, \ldots, X_r])$ with the section $1 \otimes e_i \in \mathscr{O}_X \otimes_{\mathbb{Z}} \mathbb{Z}^r$ playing the role of $d\log(X_i)$.

2.3.27. If $f : (X, M_X) \to (Y, M_Y)$ is log smooth the dual of the vector bundle $\Omega^1_{(X,M_X)/(Y,M_Y)}$, denoted $T_{(X,M_X)/(Y,M_Y)}$, is called the *log tangent bundle*.

As in the classical case of schemes, the cohomology of the log tangent bundle controls the deformation theory of log smooth morphisms. Let $i : (Y_0, M_{Y_0}) \hookrightarrow (Y, M_Y)$ be a strict closed immersion defined by a square–zero quasi–coherent ideal $I \subset \mathscr{O}_Y$, and let $f_0 : (X_0, M_{X_0}) \to (Y_0, M_{Y_0})$ be a log smooth and integral morphism. A *log smooth deformation of* f_0 is a commutative diagram of log schemes

$$\begin{array}{ccc} (X_0, M_{X_0}) & \xrightarrow{j} & (X, M_X) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ (Y_0, M_{Y_0}) & \xrightarrow{i} & (Y, M_Y), \end{array} \tag{2.3.27.1}$$

where j is a strict closed immersion, and the underlying diagram of schemes is cartesian. Note that since f_0 is assumed integral any deformation f is also integral from which it follows that $X \to Y$ is a flat deformation of $X_0 \to Y_0$.

Theorem 2.3.28 ([24, 3.14]) (i) *There is a canonical obstruction*

$$o \in H^2(X_0, I \otimes_{\mathscr{O}_{Y_0}} T_{(X_0,M_{X_0})/(Y_0,M_{Y_0})}) \tag{2.3.28.1}$$

whose vanishing is necessary and sufficient for there to exists a log smooth deformation of f_0.
(ii) *If $o = 0$ then the set of isomorphism classes of log smooth deformations of f_0 is canonically a torsor under $H^1(X_0, I \otimes_{\mathscr{O}_{Y_0}} T_{(X_0,M_{X_0})/(Y_0,M_{Y_0})})$.*
(iii) *For any log smooth deformation f of f_0, the group of automorphisms of f is canonically isomorphic to $H^0(X_0, I \otimes_{\mathscr{O}_{Y_0}} T_{(X_0,M_{X_0})/(Y_0,M_{Y_0})})$.*

2.3.29. For technical reasons we will also need the notion of saturated log scheme.

A fine monoid P is called *saturated* if for any $p \in P^{\mathrm{gp}}$ for which there exists $n \geq 1$ such that $np \in P$ we have $p \in P$.

A fine log structure M on a scheme X is called *saturated* if for every geometric point $\bar{x} \to X$ the monoid $\overline{M}_{\bar{x}}$ is saturated. We also say that the log scheme (X, M) is *saturated* if M is a saturated log structure.

Proposition 2.3.30 ([41, Chapter II, 2.4.5]). *The inclusion functor*

$$(\text{saturated fine log schemes}) \hookrightarrow (\text{fine log schemes}) \tag{2.3.30.1}$$

has a right adjoint $(X, M) \mapsto (X^{\mathrm{sat}}, M^{\mathrm{sat}})$.

Remark 2.3.31. The log scheme $(X^{\mathrm{sat}}, M^{\mathrm{sat}})$ is called the *saturation* of (X, M).

Remark 2.3.32. Locally the saturation of a fine log scheme (X, M) can be described as follows. Let P be a finitely generated integral monoid, and assume given a strict morphism

$$(X, M) \to \mathrm{Spec}(P \to \mathbb{Z}[P]). \tag{2.3.32.1}$$

Let $P' \subset P^{gp}$ be the submonoid of elements $p \in P^{gp}$ for which there exists an integer $n > 0$ such that $np \in P$. Then P' is a saturated monoid. The saturation $(X^{\mathrm{sat}}, M^{\mathrm{sat}})$ is then equal to the scheme

$$X^{\mathrm{sat}} := X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P']) \tag{2.3.32.2}$$

with the log structure M^{sat} equal to the pullback of the log structure on $\mathrm{Spec}(P' \to \mathbb{Z}[P'])$.

Remark 2.3.33. Following standard practice, we usually write ‘fs’ for ‘fine and saturated’.

2.3.34. The category of log schemes (resp. fine log schemes, fs log schemes) has fiber products. Given a diagram of log schemes

$$\begin{array}{ccc} & & (X, M_X) \\ & & \downarrow{\scriptstyle a} \\ (Y, M_Y) & \xrightarrow{b} & (Z, M_Z) \end{array} \tag{2.3.34.1}$$

the fiber product in the category of log schemes is equal to the scheme $X \times_Z Y$ with log structure the pushout

$$\mathrm{pr}_1^* M_X \oplus_{p^* M_Z} \mathrm{pr}_2^* M_Y \tag{2.3.34.2}$$

with the natural map to $\mathscr{O}_{X\times_Z Y}$. Here we write $p : X \times_Z Y \to Z$ for the projection.

For fiber products in the categories of fine and fs log schemes more care has to be taken, as the inclusion functors

$$\text{(fs log schemes)} \hookrightarrow \text{(fine log schemes)} \hookrightarrow \text{(log schemes)} \tag{2.3.34.3}$$

do not preserve fiber products. To understand this, the key case to consider is the following. Let

$$\begin{array}{ccc} & & P \\ & & \uparrow \\ Q & \longleftarrow & R \end{array} \tag{2.3.34.4}$$

by a diagram of fine (resp. fs) monoids, and consider the resulting diagram of log schemes

$$\begin{array}{ccc} & & \mathrm{Spec}(P \to \mathbb{Z}[P])\ . \\ & & \downarrow \\ \mathrm{Spec}(Q \to \mathbb{Z}[Q]) & \longrightarrow & \mathrm{Spec}(R \to \mathbb{Z}[R]) \end{array} \tag{2.3.34.5}$$

Let $P \oplus_R Q$ be the pushout in the category of monoids. Then this pushout need not be fine (resp. fs). Let $(P \oplus_R Q)^{\mathrm{int}}$ be the image of $P \oplus_R Q$ in $(P \oplus_R Q)^{\mathrm{gp}}$. Then $(P \oplus_R Q)^{\mathrm{int}}$ is the pushout of the diagram 2.3.34.4 in the category of integral monoids, and in the fs case the pushout of 2.3.34.4 in the category of fs monoids is given by the saturation $(P \oplus_R Q)^{\mathrm{sat}}$ of $(P \oplus_R Q)^{\mathrm{int}}$. The fiber product of 2.3.34.5 in the category of fine (resp. fs) log schemes is then equal to

$$\mathrm{Spec}((P \oplus_R Q)^{\mathrm{int}} \to \mathrm{Spec}(\mathbb{Z}[(P \oplus_R Q)^{\mathrm{int}}]) \tag{2.3.34.6}$$

$$(\text{resp.}\ \ \mathrm{Spec}((P \oplus_R Q)^{\mathrm{sat}} \to \mathrm{Spec}(\mathbb{Z}[(P \oplus_R Q)^{\mathrm{sat}}])). \tag{2.3.34.7}$$

2.4 Summary of Alexeev's Results

For the convenience of the reader we summarize in this section the main results of Alexeev [3]. At various points in the work that follows we have found it convenient to reduce certain proofs to earlier results of Alexeev instead of proving everything "from scratch".

2.4.1. Recall [3, 1.1.6] that a reduced scheme P is called *seminormal* if for every reduced scheme P' and proper bijective morphism $f : P' \to P$ such that for every $p' \in P'$ mapping to $p \in P$ the map $k(p) \to k(p')$ is an isomorphism, the morphism f is an isomorphism.

In his paper, Alexeev considers two kinds of moduli problems. The first concerns polarized toric varieties and the second abelian varities. We summarize here the main results from both problems. The reader only interested in abelian varieties can skip to 2.4.7.

Broken Toric Varieties.

2.4.2. Let X be a finitely generated free abelian group, and let $T = \mathrm{Spec}(\mathbb{Z}[X])$ be the corresponding torus. If S is a scheme, we write T_S for the base change of T to S.

2.4.3. Let $B = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field k, and let P/B be an affine integral scheme with action of the torus T_B such that the action has only finitely many orbits. Write $P = \mathrm{Spec}(R)$, and assume that for every $p \in P(k)$ the stabilizer group scheme $H_p \subset T_B$ of p is connected and reduced.

The T_B–action on P defines (and is defined by) an X–grading $R = \oplus_{\chi \in X} R_\chi$. Since R is an integral domain the set

$$S := \{\chi \in X | R_\chi \neq 0\} \subset X \qquad (2.4.3.1)$$

is a submonoid of X. Since the T_B–action has only finitely many orbits, there exists an orbit which is dense in P. Pick a point $p \in P(k)$ in this orbit so that the map $T_B \to P$ sending a scheme-valued point $g \in T_B$ to $g(p) \in P$ is dominant. This map defines a T_B–invariant inclusion $R \hookrightarrow k[X]$. This shows in particular that each R_χ is a k–vector space of dimension 1 or 0. Furthermore, if $X' \subset X$ denotes the subgroup generated by S and $T \to T'$ the associated quotient torus of T, then the map $T_B \to P$ defines by $p \in P(k)$ factors through a dense open immersion $T'_B \hookrightarrow P$. Furthermore we obtain a morphism $\mathrm{Spec}(k[S]) \to P$ which by [3, 2.3.13] is an isomorphism if P is seminormal.

More generally, if P is reduced, affine, but not necessarily irreducible, we obtain a collection of cones $S_i \subset X$ as follows. Let $\{P_i \subset P\}$ be the closures of the orbits of the T–action. Since each P_i is an irreducible scheme we get a collection of cones $S_i \subset X$. If $P_i \subset P_j$ then one sees from the construction that S_i is a face of S_j.

2.4.4. We can also use this to study the projective situation. Let $(P, \mathscr{L})$ be a projective scheme with T–action over k with a linearized ample line bundle $\mathscr{L}$. Assume that P is seminormal and connected, the T–action has only finitely many orbits, and that for every point $p \in P$ the stabilizer group scheme is reduced.

Set $R = \oplus_{i \geq 0} H^0(P, \mathscr{L}^{\otimes i})$. Let $\mathbb{T}$ denote the torus $T \times \mathbb{G}_m$. The $\mathbb{Z}$–grading on R defines an action of $\mathbb{G}_m$ on R which one can show commutes with the T–action thereby defining an action of $\mathbb{T}$ on R. Applying the above discussion to R with this $\mathbb{T}$–action we obtain a collection of cones $S_i \subset \mathbb{Z} \oplus X$. Let $\Delta \subset X_{\mathbb{R}}$ be the union of the intersections of the cones $S_{i,\mathbb{R}}$ with the set

$(1, X_{\mathbb{R}}) \subset \mathbb{Z} \oplus X_{\mathbb{R}}$. Then one can show that Δ is a polytope in $X_{\mathbb{R}}$ and that the $S_{i,\mathbb{R}} \cap \Delta$ define a paving of $X_{\mathbb{R}}$ in the sense of 3.1.1 below.

2.4.5. The association of the polytope Δ to $(P, \mathscr{L})$ in the preceding paragraph behaves well in families. If S is a scheme and $f : P \to S$ is a projective flat morphism with $T = \mathrm{Spec}(\mathbb{Z}[X])$ a torus acting on P over S, and $\mathscr{L}$ is a T–linearized ample invertible sheaf on P then for any geometric point $\bar{s} \to S$ the fiber $P_{\bar{s}}$ defines a polytope $\Delta_{\bar{s}} \subset X_{\mathbb{R}}$. As explained in [3, 2.10.1], if S is connected then the polytopes $\Delta_{\bar{s}}$ are all equal.

For a fixed polytope $Q \subset X_{\mathbb{R}}$, this enables one to define a stack $\mathcal{TP}^{\mathrm{fr}}[Q]$ which to any scheme S associates the groupoid of triples $(f : P \to S, \mathscr{L}, \theta \in f_*\mathscr{L})$ as follows:

(i) $f : P \to S$ is a proper flat morphism of schemes, and T acts on P over S.
(ii) For every geometric point $\bar{s} \to S$ the fiber $P_{\bar{s}}$ is seminormal and connected with associated polytope $Q \subset X_{\mathbb{R}}$.
(iii) $\mathscr{L}$ is a relatively ample invertible sheaf on P.
(iv) $\theta \in f_*\mathscr{L}$ is a section such that for every geometric point $\bar{s} \to S$ the zero locus of the section $\theta_{\bar{s}} \in H^0(P_{\bar{s}}, \mathscr{L}_{\bar{s}})$ does not contain any T-orbit.

Theorem 2.4.6 ([3, 2.10.10]) *The stack $\mathcal{TP}^{\mathrm{fr}}[Q]$ is a proper algebraic stack over $\mathbb{Z}$ with finite diagonal.*

Abelian Varieties

2.4.7. Following Alexeev [3, 1.1.3.2], a *stable semiabelic variety* over an algebraically closed field k is a scheme P/k together with an action of a semiabelian scheme G/k such that the following condition holds:

1. The dimension of G is equal to the dimension of each irreducible component of P.
2. P is seminormal.
3. There are only finitely many orbits for the G–action.
4. The stabilizer group scheme of every point of P is connected, reduced, and lies in the toric part T of G.

A *stable semiabelic pair* is a projective stable semiabelic variety P (with action of the semiabelian group scheme G) together with an ample line bundle $\mathscr{L}$ on P and a section $\theta \in H^0(P, \mathscr{L})$ whose zero locus does not contain any G–orbits.

2.4.8. If S is a scheme, then a *stable semiabelic pair* over S is a collection of data $(G, f : P \to S, \mathscr{L}, \theta \in f_*\mathscr{L})$ where:

1. G is a semiabelian scheme over S.
2. $f : P \to S$ is a projective flat morphism, and G acts on P over S.
3. $\mathscr{L}$ is a relatively ample invertible sheaf on P.

4. $\theta \in f_*\mathscr{L}$ is a section.
5. For every geometric point $\bar{s} \to S$ the fiber $(G_{\bar{s}}, P_{\bar{s}}, \mathscr{L}_{\bar{s}}, \theta_{\bar{s}} \in H^0(P_{\bar{s}}, \mathscr{L}_{\bar{s}}))$ is a stable semiabelic pair.

One can show that the sheaf $f_*\mathscr{L}$ is locally free and that its formation commutes with arbitrary base change on S. We define the *degree* of a stable semiabelic pair to be the rank of $f_*\mathscr{L}$.

2.4.9. Fix integers g and d, and let $\mathscr{M}_{g,d}$ denote the stack over $\mathbb{Z}$ which to any scheme S associates the groupoid of stable semiabelic pairs $(G, P, \mathscr{L}, \theta)$ with G of dimension g and $\mathscr{L}$ of degree d. Let $\mathscr{A}_{g,d} \subset \mathscr{M}_{g,d}$ denote the substack classifying pairs $(G, P, \mathscr{L}, \theta)$, where G is an abelian scheme.

By a standard Hilbert scheme argument one sees that the diagonal of $\mathscr{M}_{g,d}$ is representable. It therefore makes sense to talk about the connected components of $\mathscr{M}_{g,d}$ (we ignore the question of whether the stack $\mathscr{M}_{g,d}$ is algebraic; presumably this is the case). Furthermore the inclusion $\mathscr{A}_{g,d} \subset \mathscr{M}_{g,d}$ is representable by open immersions. Let $\mathscr{A}^{\mathrm{Alex}}_{g,d}$ denote the union of the connected components of $\mathscr{M}_{g,d}$ meeting $\mathscr{A}_{g,d}$.

Theorem 2.4.10 ([3, 5.10.1]) *The stack $\mathscr{A}^{\mathrm{Alex}}_{g,d}$ is a proper Artin stack over $\mathbb{Z}$ with finite diagonal.*

2.4.11. When $d = 1$, the stack $\mathscr{A}_{g,1}$ is canonically isomorphic to the moduli stack of principally polarized abelian varieties of dimension g. To see this denote temporarily by $\mathscr{A}'_g$ the moduli stack of principally polarized abelian varieties. Using 2.1.6 we get a functor

$$F : \mathscr{A}_{g,1} \to \mathscr{A}'_g, \quad (A, P, \mathscr{L}, \theta) \mapsto (A, \lambda_{\mathscr{L}} : A \to A^t), \qquad (2.4.11.1)$$

which we claim is an isomorphism. Evidently every object of $\mathscr{A}'_g$ is étale locally in the essential image of F, so it suffices to show that F is fully faithful. This is the content of the following lemma:

Lemma 2.4.12 *Let S be a scheme and A/S an abelian scheme. Let $(P_i, \mathscr{L}_i, \theta_i)$ $(i = 1, 2)$ be two collections of data as follows:*

(i) $f_i : P_i \to S$ is an A-torsor;
(ii) $\mathscr{L}_i$ is a relatively ample invertible sheaf on P_i such that $f_{i*}\mathscr{L}_i$ is locally free of rank 1 on S;
(iii) $\theta_i \in f_{i*}\mathscr{L}_i$ is a section which is nonzero in every fiber.

Assume further that the two isomorphisms

$$\lambda_{\mathscr{L}_i} : A \to A^t \qquad (2.4.12.1)$$

are equal. Then there exists a unique pair (g, g^b) of isomorphisms

$$g : P_1 \to P_2, \quad g^b : g^*\mathscr{L}_2 \to \mathscr{L}_1, \qquad (2.4.12.2)$$

where g is an isomorphism of A-torsors and g^b is an isomorphism of line bundle on P_1 such that $g^{b*}(\theta_2) = \theta_1$).

Proof. Using the uniqueness we may by descent theory work locally on the fppf topology on S. We may therefore assume that the torsors P_i are trivial. Choose some trivializations $\iota_i : A \to P_i$.

Any isomorphism of torsors $g : A \to A$ is necessarily translation by a point. Indeed any isomorphism g can be written as $t_a \circ h$, where $h : A \to A$ is a homomorphism. Since g is also supposed to commute with the A-action we have for every $\alpha \in A$

$$g(\alpha) = g(e) + \alpha \tag{2.4.12.3}$$

which gives

$$h(\alpha) + a = a + \alpha. \tag{2.4.12.4}$$

Therefore h is the identity and g must be translation by a point.

Next note that since $\lambda_{\mathscr{L}_1} = \lambda_{\mathscr{L}_2}$ we have étale locally on S

$$t_a^*(\mathscr{L}_1 \otimes \mathscr{L}_2^{-1}) \simeq \mathscr{L}_1 \otimes \mathscr{L}_2^{-1} \tag{2.4.12.5}$$

for all $a \in A$. It follows that $\mathscr{L}_1 \otimes \mathscr{L}_2^{-1} \in \underline{\mathrm{Pic}}^0_{A/S}(S)$. In particular, there exists a unique point $a \in A$ such that $t_a^*\mathscr{L}_2 \simeq \mathscr{L}_1$. We take translation by this element a to be the map g. The choice of g^b is then uniquely determined by the condition $g^{b*}(\theta_2) = \theta_1$. $\square$

Remark 2.4.13. The space $\mathscr{A}_{g,1}^{\mathrm{Alex}}$ therefore provides a compactification of $\mathscr{A}_{g,1}$. However, in general the stack $\mathscr{A}_{g,1}^{\mathrm{Alex}}$ has many irreducible componets. One of the main problems dealt with in this paper is how to "carve out" the component of $\mathscr{A}_{g,1}^{\mathrm{Alex}}$ and also to generalize the theory to higher degree polarizations.