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Volumes on Normed and Finsler Spaces

J. C. ÁLVAREZ PAIVA AND A. C. THOMPSON

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1. Introduction

The study of volumes and areas on normed and Finsler spaces is a relatively new field that comprises and unifies large domains of convexity, geometric tomography, and integral geometry. It opens many classical unsolved problems in these fields to powerful techniques in global differential geometry, and suggests new challenging problems that are delightfully geometric and simple to state.

Keywords: Minkowski geometry, Hausdorff measure, Holmes–Thompson volume, Finsler manifold, isoperimetric inequality.

The theory starts with a simple question: How does one measure volume on a finite-dimensional normed space? At first sight, this question may seem either unmotivated or trivial: normed spaces are metric spaces and we can measure volume using the Hausdorff measure, period. However, if one starts asking simple, naive questions one discovers the depth of the problem. Even if one is unwilling to consider that definitions of volume other than the Hausdorff measure are not only possible but may even be better, one is faced with questions such as these: What is the $(n-1)$-dimensional Hausdorff measure of the unit sphere of an n-dimensional normed space? Do flat regions minimize area? For what normed spaces are metric balls also the solutions of the isoperimetric problem? These questions, first posed in convex-geometric language by Busemann and Petty [1956], are still open, at least in their full generality. However, one should not assume too quickly that the subject is impossible. Some beautiful results and striking connections have been found. For example, the fact that the $(n-1)$-Hausdorff measure in a normed space determines the norm is equivalent to the fact that the areas of the central sections determine a convex body that is symmetric with respect to the origin. This, in turn, follows from the study of the spherical Radon transform. The fact that regions in hyperplanes are area-minimizing is equivalent to the fact that the intersection body of a convex body that is symmetric with respect to the origin is also convex.

But the true interest of the theory can only be appreciated if one is willing to challenge Busemann's dictum that the natural volume in a normed or Finsler space is the Hausdorff measure. Indeed, thinking of a normed or Finsler space as an anisotropic medium where the speed of a light ray depends on its direction, we are led to consider a completely different notion of volume, which has become known as the *Holmes–Thompson volume.* This notion of volume, introduced in [Holmes and Thompson 1979], uncovers striking connections between integral geometry, convexity, and Hamiltonian systems. For example, in a recent series of papers, [Schneider and Wieacker 1997], [Alvarez and Fernandes 1998], [Alvarez and Fernandes 1999], [Schneider 2001], and [Schneider 2002], it was shown that the classical integral geometric formulas in Euclidean spaces can be extended to normed and even to projective Finsler spaces (the solutions of Hilbert's fourth problem) if the areas of submanifolds are measured with the Holmes–Thompson definition. That extensions of this kind are not possible with the Busemann definition was shown by Schneider [Schneider 2001].

Using Finsler techniques, Burago and Ivanov [2001] have proved that a flat two-dimensional disc in a finite-dimensional normed space minimizes area among all other immersed discs with the same boundary. Ivanov [2001] has shown, among other things, that Pu's isosystolic inequality for Riemannian metrics in the projective plane extends to Finsler metrics, and the Finslerian extension of Berger's infinitesimal isosystolic inequality for Riemannian metrics on real projective spaces of arbitrary dimension has been proved by Álvarez [2002].

Despite these and other recent interdisciplinary results, we believe that the most surprising feature of the Holmes–Thompson definition is the way in which it organizes a large portion of convexity into a coherent theory. For example, the sharp upper bound for the volume of the unit ball of a normed space is given by the Blaschke–Santaló inequality; the conjectured sharp lower bound is Mahler's conjecture; and the reconstruction of the norm from the area functional is equivalent to the famous Minkowski's problem of reconstructing a convex body from the knowledge of its curvature as a function of its unit normals.

In this paper, we have attempted to provide students and researchers in Finsler and global differential geometry with a clear and concise introduction to the theory of volumes on normed and Finsler spaces. To do this, we have avoided the temptation to use auxiliary Euclidean structures to describe the various concepts and constructions. While these auxiliary structures may render some of the proofs simpler, they hinder the understanding of the subject and make the application of the ideas and techniques to Finsler spaces much more cumbersome. We also believe that by presenting the results and techniques in intrinsic terms we can make some of the beautiful results of convexity more accessible and enticing to differential geometers.

In the course of our writing we had to make some tough choices as to what material should be left out as either too advanced or too specialized. At the end we decided that we would concentrate on the basic questions and techniques of the theory, while doing our best to exhibit the general abstract framework that makes the theory of volumes on normed spaces into a sort of Grand Unified Theory for many problems in convexity and Finsler geometry. As a result there is little Finsler geometry *per se* in the pages that follow. However, just as tensors, forms, spinors, and Clifford algebras developed in invariant form have immediate use in Riemannian geometry, the more geometric constructions with norms, convex bodies, and k-volume densities that make up the heart of this paper have immediate applications to Finsler geometry.

2. A Short Review of the Geometry of Normed Spaces

This section is a quick review of the geometry of finite-dimensional normed spaces adapted to the needs and language of Finsler geometry. Unless stated otherwise, *all vector spaces in this article are real and finite-dimensional.* We suggest that the reader merely browse through this section and come back to it if and when it becomes necessary.

Definition 2.1. A *norm* on a vector space X is a function

$$\|\cdot\| : X \to [0, \infty)$$

satisfying the following properties of positivity, homogeneity, and convexity:

(1) If $\|\boldsymbol{x}\| = 0$, then $\boldsymbol{x} = \boldsymbol{0}$;

(2) If t is a real number, then $\|t\boldsymbol{x}\| = |t|\|\boldsymbol{x}\|$;
(3) For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in X, $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

If $(X, \|\cdot\|)$ is a finite-dimensional normed space, the set

$$B_X := \{\boldsymbol{x} \in X : \|\boldsymbol{x}\| \leq 1\}$$

is the *unit ball* of X and the boundary of B_X, S_X, is its *unit sphere.* Notice that B_X is a compact, convex set with nonempty interior. In short, it is a *convex body* in X. Moreover, it is symmetric with respect to the origin. Conversely, if $B \subset X$ is a *centered convex body* (*i.e.,* a convex body symmetric with respect to the origin), it is the unit ball of the norm

$$\|\boldsymbol{x}\| := \inf\{t \geq 0 : t\boldsymbol{y} = \boldsymbol{x} \text{ for some } \boldsymbol{y} \in B\}.$$

We shall now describe various classes of normed spaces that will appear repeatedly throughout the paper.

Euclidean spaces. A Euclidean structure on a finite-dimensional vector space X is a choice of a symmetric, positive-definite quadratic form $\mathcal{Q} : X \to \mathbb{R}$. The normed space $(X, \mathcal{Q}^{1/2})$ will be called a Euclidean space. Note that a normed space is Euclidean if and only if its unit sphere is an ellipsoid, which is if and only if the norm satisfies the parallelogram identity:

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2.$$

EXERCISE 2.2. Let $B \subset \mathbb{R}^n$ be a convex body symmetric with respect to the origin. Show that if the intersection of B with every 2-dimensional plane passing through the origin is an ellipse, then B is an ellipsoid.

The ℓ_p spaces. If $p \geq 1$ is a real number, the function

$$\|\boldsymbol{x}\|_p := \big(|x_1|^p + \cdots + |x_n|^p\big)^{1/p}$$

is a norm on $\mathbb{R}^n$. When p tends to infinity, $\|\boldsymbol{x}\|_p$ converges to

$$\|\boldsymbol{x}\|_\infty := \max\{|x_1|, \ldots, |x_n|\}.$$

The normed space $(\mathbb{R}^n, \|\cdot\|_p)$, $1 \leq p \leq \infty$, is denoted by ℓ_p^n.

The unit ball of ℓ_∞^n is the n-dimensional cube of side length two, while the unit ball of ℓ_1^n is the n-dimensional *cross-polytope.* In general, norms whose unit balls are polytopes are called *crystalline norms.*

Subspaces of $L_1([0,1], dx)$. The space of measurable functions $f : [0,1] \to \mathbb{R}$ satisfying

$$\|f\| := \int_0^1 |f(x)|\, dx < \infty$$

is a normed space denoted by $L_1([0,1], dx)$. The geometry of finite-dimensional subspaces of $L_1([0,1], dx)$ is closely related to problems of volume, area, and integral geometry on normed and Finsler spaces. In the next few paragraphs, we will summarize the properties of these subspaces that will be used in the rest of the paper. For proofs, references, and to learn more about hypermetric spaces, we recommend the landmark paper [Bolker 1969], as well as the surveys [Schneider and Weil 1983] and [Goodey and Weil 1993].

First we begin with a beautiful metric characterization of the subspaces of $L_1([0,1],dx)$.

Definition 2.3. A metric space (M,d) is said to be *hypermetric* if it satisfies the following stronger version of the triangle inequality: If $m_1,\dots,m_k$ are elements of M and $b_1,\dots,b_k$ are integers with $\sum_i b_i = 1$, then

$$\sum_{i,j=1}^{k} d(m_i,m_j)b_ib_j \le 0.$$

Theorem 2.4. *A finite-dimensional normed space is hypermetric if and only if it is isometric to a subspace of* $L_1([0,1],dx)$.

An important analytic characterization of a hypermetric normed space can be given through the Fourier transform of its norm:

Theorem 2.5. *A norm on* $\mathbb{R}^n$ *is hypermetric if and only if its distributional Fourier transform is a nonnegative measure.*

The characterizations above, important as they are, are hard to grasp at first sight. A much more visual approach will be given after we review the duality of normed spaces.

Minkowski spaces. Minkowski spaces are normed spaces with strict smoothness and convexity properties. In precise terms, a norm $\|\cdot\|$ on a vector space X is said to be a *Minkowski norm* if it is smooth outside the origin and the Hessian of the function $\|\cdot\|^2$ at every nonzero point is a positive-definite quadratic form.

The unit sphere of a Minkowski space X is a smooth convex hypersurface S_X such that for any Euclidean structure on X the principal curvatures of S_X are positive.

2.1. Maps between normed spaces. An important feature of the geometry of normed spaces is that the space of linear maps between two normed spaces carries a natural norm.

Definition 2.6. If $T : X \to Y$ is a linear map between normed spaces $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$, we define the *norm of* T as the supremum of $\|T\boldsymbol{x}\|_Y$ taken over all vectors $\boldsymbol{x} \in X$ with $\|\boldsymbol{x}\|_X \le 1$.

A linear map $T : X \to Y$ is said to be *short* if its norm is less than or equal to one. In other words, a short linear map does not increase distances. Two important types of short linear maps between normed spaces are isometric embeddings and isometric submersions:

DEFINITION 2.7. An injective linear map $T : X \to Y$ between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is said to be an *isometric embedding* if $\|T\boldsymbol{x}\|_Y = \|\boldsymbol{x}\|_X$ for all vectors $\boldsymbol{x} \in X$.

DEFINITION 2.8. A surjective linear map $T : X \to Y$ between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is said to be an *isometric submersion* if

$$\|T\boldsymbol{x}\|_Y = \inf\{\|\boldsymbol{v}\|_X : \boldsymbol{v} \in X \text{ and } T\boldsymbol{v} = T\boldsymbol{x}\}$$

for all vectors $x \in X$.

In terms of the unit balls, $T : X \to Y$ is an isometric embedding if and only if $T(B_X) = T(X) \cap B_Y$, and T is an isometric submersion if and only if $T(B_X) = B_Y$.

2.2. Dual spaces and polar bodies. From the previous paragraph, we see that if $(X, \|\cdot\|)$ is a normed space, then the set of all linear maps onto the one-dimensional normed space $(\mathbb{R}, |\cdot|)$ carries a natural norm. The resulting normed space is called the *dual* of $(X, \|\cdot\|)$ and is denoted by $(X^*, \|\cdot\|^*)$. It is easy to see that the double dual (*i.e.,* the dual of the dual) of a finite-dimensional normed space can be naturally identified with the space itself. The unit ball of $(X^*, \|\cdot\|^*)$ is said to be the *polar* of the unit ball of $(X, \|\cdot\|)$.

Example. Hölder's inequality implies that if $p > 1$, the dual of ℓ_p^n is ℓ_q^n, where $1/p + 1/q = 1$. Likewise, it is easy to see that the dual of ℓ_1^n is ℓ_∞^n.

If $T : X \mapsto Y$ is a linear map then the *dual map* $T^* : Y^* \mapsto X^*$ is defined by

$$(T^*\boldsymbol{\xi})(\boldsymbol{x}) = \boldsymbol{\xi}(T\boldsymbol{x}).$$

Note that $\|T^*\| = \|T\|$.

EXERCISE 2.9. Show that if $T : X \to Y$ is an isometric embedding between normed spaces X and Y, the dual map $T^* : Y^* \to X^*$ is an isometric submersion.

Many of the geometric constructions in convex geometry and the geometry of normed spaces are functorial. More precisely, if we denote by $\mathcal{N}$ the category whose objects are finite-dimensional normed spaces and whose morphisms are short linear maps, many classical constructions define functors from $\mathcal{N}$ to itself.

EXERCISE 2.10. Show that the assignment $(X, \|\cdot\|) \mapsto (X^*, \|\cdot\|^*)$ is a contravariant functor from $\mathcal{N}$ to $\mathcal{N}$.

Duals of hypermetric normed spaces. As advertised earlier in this section, the notion of duality allows us to give a more geometric characterization of hypermetric spaces.

DEFINITION 2.11. A polytope in a vector space X is said to be a *zonotope* if all of its faces are symmetric. A convex body is said to be a *zonoid* if it is the limit (in the Hausdorff topology) of zonotopes.

Notice that an n-dimensional cube, as well as all its linear projections, are zonotopes. In fact, it can be shown that any zonotope is the linear projection of a cube (see, for example, Theorem 3.3 in [Bolker 1969]).

THEOREM 2.12. *Let X be a finite-dimensional normed space with unit ball B_X. The dual of X is hypermetric if and only if B_X is a zonoid.*

Notice that this immediately implies that the space ℓ_1^n, $n \geq 1$, is hypermetric.

Duality in Minkowski spaces. If $(X, \|\cdot\|)$ is a Minkowski space, the differential of the function $L := \|\cdot\|^2/2$,

$$dL(\boldsymbol{x})(\boldsymbol{y}) := \frac{1}{2}\frac{d}{dt}\|\boldsymbol{x} + t\boldsymbol{y}\|^2_{t=0},$$

is a continuous linear map from X to X^* that is smooth outside the origin and homogeneous of degree one. This map is usually called the *Legendre transform,* although that term is also used to describe some related concepts (see, for example, § 2.2 in [Hörmander 1994]). The following exercise describes the most important properties of the Legendre transform.

EXERCISE 2.13. Let $(X, \|\cdot\|)$ be a Minkowski space and let

$$\mathfrak{L} : X \setminus \mathbf{0} \to X^* \setminus \mathbf{0}$$

be its Legendre transform.

(1) Show that if $\boldsymbol{x} \in X$ is a unit vector, then $\mathfrak{L}(\boldsymbol{x})$ is the unique covector $\boldsymbol{\xi} \in X^*$ such that the equation $\boldsymbol{\xi} \cdot \boldsymbol{y} = 1$ describes the tangent plane to the unit sphere S_X at the point $\boldsymbol{x}$.
(2) Show that the Legendre transform defines a diffeomorphism between the unit sphere and its polar.
(3) Show that the inverse of the Legendre transform from $X \setminus \mathbf{0}$ to $X^* \setminus \mathbf{0}$ is just the Legendre transform from $X^* \setminus \mathbf{0}$ to $X \setminus \mathbf{0}$.
(4) Show that the Legendre transform is linear if and only if X is a Euclidean space.

EXERCISE 2.14. Show that a normed space is a Minkowski space if its unit sphere and the unit sphere of its dual are smooth.

2.3. Sociology of normed spaces. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a finite-dimensional vector space X, it is easy to see that there are positive numbers m and M such that

$$m\|\cdot\|_2 \leq \|\cdot\|_1 \leq M\|\cdot\|_2.$$

If we take the numbers m and M such that the inequalities are sharp, then $\log(M/m)$ is a good measure of how far away one norm is from the other.

For example, the following well-known result states that we can always approximate a norm by one whose unit sphere is a polytope or by one such that its unit sphere and the unit sphere of its dual are smooth.

PROPOSITION 2.15. *Let $\|\cdot\|$ be a norm on the finite-dimensional vector space X. Given $\varepsilon > 0$, there exist a crystalline norm $\|\cdot\|_1$ and a Minkowski norm $\|\cdot\|_2$ on X such that*

$$\|\cdot\|_1 \leq \|\cdot\| \leq (1+\varepsilon)\|\cdot\|_1,$$
$$\|\cdot\|_2 \leq \|\cdot\| \leq (1+\varepsilon)\|\cdot\|_2.$$

For a short proof see Lemma 2.3.2 in [Hörmander 1994] .

In many circumstances, one wants to measure how far is one normed space from being isometric to another. The straightforward adaptation of the previous idea leads us to the following notion:

DEFINITION 2.16. The *Banach–Mazur distance* between n-dimensional normed spaces X and Y, is the infimum of the numbers $\log(\|T\|\|T^{-1}\|)$, where T ranges over all invertible linear maps from X to Y.

Notice that the Banach–Mazur distance is a distance on the set of isometry classes of n-dimensional normed spaces: two such spaces are at distance zero if and only if they are isometric.

An important question is to determine how far a general n-dimensional normed space is from being Euclidean. The answer rests on two results of independent interest:

THEOREM 2.17 (LOEWNER). *If B is a convex body in an n-dimensional vector space X, there exists a unique n-dimensional ellipsoid $E \subset B$ such that for any Lebesgue measure on X, the ratio $\mathrm{vol}(B)/\mathrm{vol}(E)$ is minimal.*

THEOREM 2.18 [John 1948]. *Let X be an n-dimensional normed space with unit ball B. If $E \subset B$ is the Loewner ellipsoid of B, then*

$$E \subset B \subset \sqrt{n}E.$$

EXERCISE 2.19. Show that the Banach–Mazur distance from an n-dimensional normed space to a Euclidean space is at most $\log(n)/2$.

The structure of the set of isometry classes of n-dimensional normed spaces is given by the following theorem (see [Thompson 1996, page 73] for references and some of the history on the subject):

THEOREM 2.20. *The set of isometry classes of n-dimensional normed spaces, $\mathcal{M}_n$, provided with the Banach–Mazur distance is a compact, connected metric space.*

The *Banach–Mazur compactum,* $\mathcal{M}_n$, enters naturally into Finsler geometry by the following construction: Let $\pi : \zeta \to M$ be a vector bundle with n-dimensional fibers such that every fiber $\zeta_m = \pi^{-1}(m)$ carries a norm that varies continuously with the base point (a *Finsler bundle*). The (continuous) map

$$\mathcal{I} : M \longrightarrow \mathcal{M}_n$$

that assigns to each point $m \in M$ the isometry class of ζ_m measures how the norms vary from point to point.

Currently, there are not many results that describe the map $\mathcal{I}$ under different geometric and/or topological hypotheses on the bundle. However the following exercise (and its extension in [Gromov 1967]) shows that such results are possible.

Exercise 2.21. Let $\pi : \zeta \to S^2$ be a Finsler bundle whose fibers are 2-dimensional. Show that if the bundle is nontrivial and the map $\mathcal{I}$ is constant, then the image of S^2 under $\mathcal{I}$ is the isometry class of 2-dimensional Euclidean spaces.

A corollary of this exercise is that if X is a three-dimensional normed space such that all its two-dimensional subspaces are isometric, then X is Euclidean. Another interesting corollary is that a Berwald (Finsler) metric on S^2 must be Riemannian.

3. Volumes on Normed Spaces

In defining the notion of volume on normed spaces, it is best to adopt an axiomatic approach. We shall impose some minimal set of conditions that are reasonable and then try to find out to what extent they can be satisfied, and to what point they determine our choices.

In a normed space, all translations are isometries. This suggests that we require the volume of a set to be invariant under translations. Since any finite-dimensional normed space is a locally compact, commutative group, we can apply the following theorem of Haar:

Theorem 3.1. *If μ is a translation-invariant measure on $\mathbb{R}^n$ for which all compact sets have finite measure and all open sets have positive measure, then μ is a constant multiple of the Lebesgue measure.*

Proofs of this theorem can be found in many places. A full account is given in [Cohn 1980] and an abbreviated version in [Thompson 1996].

In the light of Haar's theorem, in order to give a definition of volume in *every* normed space, we must assign to every normed space X a multiple of the Lebesgue measure. Since the Lebesgue measure is not intrinsically defined (it depends on a choice of basis for X), it is best to describe this assignment as a choice of a norm μ in the 1-dimensional vector space $\Lambda^n X$, where n is the dimension of X; if $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in X$, we define $\mu(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n)$ as the volume of the parallelotope formed by these vectors.

Another natural requirement is *monotonicity:* if X and Y are n-dimensional normed spaces and $T : X \to Y$ is a short linear map (*i.e.,* a linear map that does not increase distances), we require that T does not increase volumes. Notice that this implies that isometries between normed spaces are volume-preserving.

The monotonicity requirement makes a definition of volume on normed spaces into a functor from $\mathcal{N}$ to itself that takes the n-dimensional normed space $(X, \|\cdot\|)$ to the 1-dimensional normed space $(\Lambda^n X, \mu)$. While we shall often abandon this viewpoint, it is a guiding principle throughout the paper with which we would like to acquaint the reader early on.

Definition 3.2. A *definition of volume* on normed spaces assigns to every n-dimensional, $n \geq 1$, normed space X a normed space $(\Lambda^n X, \mu_X)$ with the following properties:

(1) If X and Y are n-dimensional normed spaces and $T : X \to Y$ is a short linear map , then the induced linear map $T_* : \Lambda^n X \to \Lambda^n Y$ is also short.
(2) The map $X \mapsto (\Lambda^n X, \mu_X)$ is continuous with respect to the topology induced by the Banach–Mazur distance.
(3) If X is Euclidean, then μ_X is the standard Euclidean volume on X.

Before presenting the principal definitions of volume in normed spaces, let us make the first link between these concepts and the affine geometry of convex bodies.

Exercise 3.3. Assume we have a definition of volume in normed spaces and use it to assign a number to any centrally symmetric convex body $B \subset \mathbb{R}^n$ by the following procedure: Consider $\mathbb{R}^n$ as the normed space X whose unit ball is B and compute

$$\mathcal{V}(B) := \mu_X(B) = \int_B \mu_X .$$

Show that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map, then $\mathcal{V}(B) = \mathcal{V}(T(B))$, and write the monotonicity condition in terms of the affine invariant $\mathcal{V}$.

Notice that we can turn the tables and start by considering a suitable affine invariant $\mathcal{V}$ of centered convex bodies and give a definition of volume in normed spaces by prescribing that the volume of the unit ball B of a normed space X be given by $\mathcal{V}(B)$.

Exercise 3.4. Let μ be a definition of volume for 2-dimensional normed spaces. Use John's theorem to show that if B is the unit disc of a two-dimensional normed space X, then $\pi/2 \leq \mu_X(B) \leq 2\pi$.

3.1. Examples of definitions of volume in normed spaces. The study of the four definitions of volume we shall describe below makes up the most important part of the theory of volumes on normed and Finsler spaces.

The Busemann definition. The Busemann volume of an n-dimensional normed space is that multiple of the Lebesgue measure for which the volume of the unit ball equals the volume of the Euclidean unit ball in dimension n, ε_n, . In other words, we have chosen as our affine invariant the constant ε_n, where n is the dimension of the space.