Introduction

Chapter 1 is an introductory chapter. Nets and assembly instructions are given for a simple hexaflexagon, the trihexaflexagon, and for a simple square flexagon. The pinch flex used to manipulate them is described. Nets for other types of flexagon are given later in the book to illustrate various points made. General assembly instructions are given for these nets.

Flexagons are a twentieth century discovery. Their early history is given in Chapter 2. In 1940 two members of a Flexagon Committee at Princeton University worked out a mathematical theory of flexagons but this was never published. The subject can be said to have reached maturity with the issue in 1962 of a comprehensive report on flexagons, but it was not published in a form which reached a wide audience.

In general the main characteristic feature of a flexagon is that it has the appearance of a polygon which may be flexed in order to display pairs of faces, around a cycle, in cyclic order. Another characteristic feature is that faces of individual polygons, known as leaves, which make up a face of a flexagon, rotate in the sense that different vertices move to the centre of a main position as a flexagon is flexed from one main position to another. The visible leaves are actually folded piles of leaves, called pats. Sometimes pats are single leaves. Alternate pats have the same structure. A pair of adjacent pats is a sector. A convenient mathematical framework for the analysis of flexagons is presented in Chapter 3, together with explanations of special technical terms. A straightforward geometric approach, without equations, is used. Geometric descriptions are used for three main purposes: firstly, to map the dynamic behaviour of flexagons, secondly to analyse their structure, and thirdly as the basis of recipes for the construction of flexagons of any desired type.

Hexaflexagons, described in Chapter 4, were the first variety of flexagon to be discovered and they have been analysed in the most detail. The leaves of a hexaflexagon are equilateral triangles. In appearance a main position of a hexaflexagon is flat and consists of six leaves, each with a vertex at the centre so there are six pats and three sectors. The outline is a regular hexagon. In some ways hexaflexagons are the simplest type of flexagon. There are only one possible type of cycle and one possible type of link between cycles. Multicycle hexaflexagons have been extensively analysed and their dynamic behaviour is well understood. There has been much interest in the design of nets for specific types of hexaflexagon whose dynamic behaviour is known. Design methods are described in detail. Extensive analysis has resulted in variations on the theme of hexaflexagons. Three are described in Chapter 5. These are a different variety of flexagon, triangle flexagons, a different way of flexing hexaflexagons, the V-flex, and origami like recreations with hexaflexagons.

Square flexagons are described in Chapter 6. They were the second variety of flexagon to be discovered. They are less well understood than hexaflexagons, partly because their dynamic behaviour is much more complex. Square flexagons have three different types of cycle and two types of link between cycles are possible. The leaves of a square flexagon are squares. In appearance a main position of a square flexagon is flat and consists of four leaves each with a vertex at the centre so there are four pats and two sectors. The outline is a square. The design of nets for specific types of square flexagons whose dynamic behaviour is known is more difficult than for hexaflexagons.

The remaining chapters are more advanced. Chapter 7 is an introduction to convex polygon flexagons. Convex polygon flexagons are generalisations of the square flexagons and triangle flexagons described in earlier chapters. Understanding of convex polygon flexagons in general is incomplete. There is an infinite family of convex polygon flexagons. Varieties are named after the constituent polygons. A feature of some varieties of convex polygon flexagon is that there may be more than one type of main position and more than one type of complete cycle. It then becomes necessary to refer to principal and subsidiary main positions and cycles. In a principal main position a convex polygon flexagon has the appearance of four leaves each with a vertex at the centre so there are four pats and two sectors. If a flexagon is regarded as a linkage then bending the leaves during flexing is not permissible. However, allowing bending during flexing does makes it easier to rationalise dynamic behaviours of the convex polygon flexagon family, and does make the manipulation of some types of convex polygon flexagon more interesting.

The first variety of the convex polygon flexagon family, the digon flexagon, can only be flexed in truncated form and then only by bending the leaves of a paper model using a push through flex. The second and third varieties are triangle flexagons and square flexagons. None of these first three varieties is typical of convex polygon flexagons. The fourth variety, pentagon flexagons, and higher varieties have characteristics in common and all can be regarded as typical members of the family. In a typical convex polygon flexagon the sum of the leaf vertex angles at the centre of a principal main position is greater than 360° so the principal main position is skew and its outline is a skew polygon. It is always possible to traverse the principal cycle of a typical convex polygon flexagon without bending the leaves of a paper model. There is always at least one subsidiary cycle. In general subsidiary cycles cannot be traversed without bending leaves. The appearance of the subsidiary main positions is different from that of the principal main positions. Various features of typical convex polygon flexagons are illustrated in Chapter 8 through descriptions of pentagon flexagons, hexagon flexagons and octagon flexagons. With octagon flexagons an additional type of flex, the twist flex, appears.

In a systematic treatment flexagons can be classified into two main infinite families. The first is the convex polygon flexagon family and the second is the star flexagon family. A principal main position of a star flexagon is flat, and has the appearance of an even number of regular polygons arranged about its centre, each with a vertex at the centre. The first two varieties of star flexagons are square flexagons and hexaflexagons. These are not typical of star flexagons. Typical star flexagons have at least eight polygons arranged about the centre of a principal main position, and the constituent polygons are regular star polygons. Interpenetration of the stellations during flexing makes the construction of paper models impossible.

Typical star flexagons are precursors to ring flexagons, which are described in Chapter 9. If all the stellations are removed from the constituent polygons of a star polygon flexagon then it becomes a ring flexagon. A principal main position of a ring flexagon has the appearance of a flat ring of an even number of regular convex polygons. The rings are regular in that each polygon is the same distance from the centre of the ring. Paper models of ring flexagons are awkward and tedious to handle. A compound flexagon is a ring flexagon in which alternate pats lie closer to the centre of a main position than do the others. There is an infinite number of compound flexagon varieties. Principal main positions are flat and have the appearance of compound rings of regular convex polygons in which alternate polygons lie closer to the centre of the ring. The leaves are regular convex polygons, and compound flexagons are named after the constituent polygons. The lines of hinges between pats do not intersect at the centres of the rings. Because of this, compound flexagons can only be flexed by bending the leaves. Flexing paper models is difficult.

Some distorted polygon flexagons are described in Chapter 10 in order to illustrate the enormous range of possibilities. A distorted polygon is a convex polygon derived from a regular convex polygon by changing the shape without changing the number of sides. The leaves of flexagons can be made from any convex polygon, but only a limited range of distorted convex polygons result in flexagons whose paper models are reasonably easy to handle. Distorted polygon flexagons are usually named after the polygons from which they are made. There are several ways in which the leaves of a flexagon made from regular polygons can be modified to produce a distorted polygon flexagon. A distorted polygon can sometimes be regarded as a partially stellated version of a regular convex polygon with a different number of sides. Alternatively, a distorted polygon can sometimes be regarded as a star polygon from which some of the stellations have been removed. If the proportions of the leaves are changed without changing the angular relationships between their sides then, in general, the dynamic behaviour of the flexagon is not affected. Changing the angular relationships between leaf sides does change the dynamic behaviour. Most distorted polygon flexagons are best regarded as variants of either convex polygon flexagons or star flexagons.

Four dimensional space is a purely theoretical idea but is nevertheless fascinating. Chapter 11 is a brief introduction to the remarkably rich and largely unexplored topic of flexahedra, which are the four dimensional analogues of flexagons. It is of course not possible to make physical models of flexahedra. It is possible to generate a flexahedron analogue of any flexagon and examples are given. There are some flexahedra which are not analogues of flexagons, and one is described. The nets of flexahedra are three dimensional so can be visualised in ordinary space. Sometimes main and intermediate positions of flexahedra, including those of the examples, are also three dimensional and hence may be visualised.

1 Making and flexing flexagons

As an introduction, nets and assembly instructions are given for two simple flexagons. The nets are laid out full size in a form suitable for photocopying. Nets for other types of flexagon are given later in the book to illustrate various points made. General assembly instructions are given for these nets. The appearance of paper models of flexagons can be improved by colouring and decorating the faces. Some decorative schemes exploit symmetries of flexagons both to create an attractive appearance and to create puzzles.

The 'pinch flex' used to manipulate flexagons in order to display different pairs of faces is described. A flexagon is flexed from a main position first to an 'intermediate position', and then to another main position. Other types of flex are sometimes used and are described later in the book.

1.1 The trihexaflexagon

The net for a paper model of a simple hexaflexagon is shown in Fig. 1.1. This is the trihexaflexagon, which was the first type of flexagon to be discovered. The trihexaflexagon is the simplest possible type of hexaflexagon (Conrad 1960, Conrad and Hartline 1962, Cundy and Rollett 1981, Gardner 1965, Gardner 1988, Hilton and Pedersen 1994, Hilton et al. 1997, Johnson 1974, Kenneway 1987, Laithwaite 1980, Liebeck 1964, McIntosh 2000h, McLean 1979, Madachy 1968, Maunsell 1954, Mitchell 1999, Oakley and Wisner 1957, Pedersen and Pedersen 1973, Wheeler 1958).

To make the trihexaflexagon photocopy the net onto 80 g/m^2 paper and cut it out. Crease the lines between triangles to form hinges. Transfer each number in brackets to the reverse face of the triangle, and delete it from the upper face. Fold the faces of triangles numbered 3 together. Join the ends of the net, shown by dashed lines on the figure, with transparent adhesive tape. The assembled trihexaflexagon is a continuous band of hinged triangles. The outline of the assembled trihexaflexagon is a hexagon, and it is in a 'main position'. One of the visible faces consists of six triangles numbered 1, each with a vertex at the centre of the hexagon. On the other visible face the triangles are numbered 2. This particular flexagon is a twisted band, so it exists in two mirror image (enantiomorphic) forms. To make the enantiomorph transfer each unbracketed

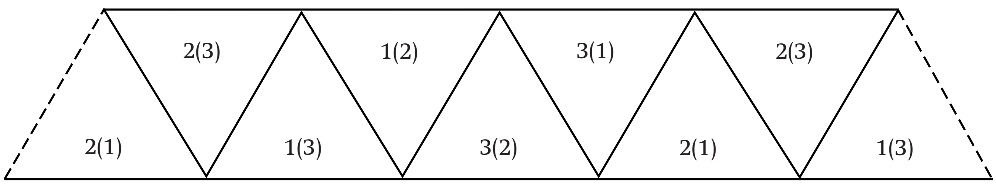

Fig. 1.1 Net for the trihexaflexagon.

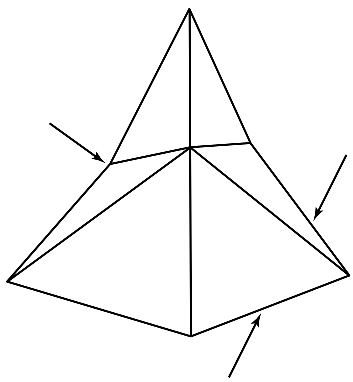

Fig. 1.2 Flexing a hexaflexagon using the pinch flex. Threefold rotational symmetry is maintained during flexing.

number to the reverse face of the triangle, and delete it from the upper face.

The individual polygons used to make flexagons, in this case triangles, are called 'leaves'. Examination of the assembled trihexaflexagon shows that it consists of alternate single leaves and folded piles of two leaves. Both the single leaves and the folded piles are called 'pats'.

1.2 The pinch flex

To 'flex' the trihexaflexagon start with the face numbered 1 uppermost. Pinch together two pats with the thumb and index finger of one hand, as shown in Fig. 1.2. Ensure that there isn't a continuous fold connecting the top of the two pats being pinched together. If there is, rotate the trihexaflexagon through 60° and start again. At the same time push the two opposite pats inwards with the index finger of the other hand. Continue until the trihexaflexagon has the appearance of three triangles connected at a common edge, and it is in an 'intermediate position'. The intermediate position has threefold rotational symmetry with an angle of 120° between each pair of triangles. For a description of various types of symmetry see Holden (1991). At the intermediate position it is possible to open the trihexaflexagon at the top of the common edge to reveal leaves numbered 3. Do this, and then flatten the trihexaflexagon. It will then have leaves numbered 3 on top and those numbered 1 underneath, and is in another main position. This manoeuvre is a ‘pinch flex’. A pinch flex has two stages. In the first stage a main position is transformed into an intermediate position, and in the second stage this intermediate is transformed into another main position. Threefold rotational symmetry is maintained while pinch flexing the trihexaflexagon.

Repeating the pinch flex, after rotating the trihexaflexagon through 60°, results in leaves numbered 2 on top and those numbered 3 underneath. A third pinch flex returns the trihexaflexagon to its initial main position with 1 on top and 2 underneath, so completing a cycle. This cycle can be repeated indefinitely. The effect is that the band of triangles is continually turned inside out. The cycle can be traversed in the reverse direction by turning the trihexaflexagon over.

The pinch flex is the basic flex used to manipulate flexagons. It differs in detail for flexagons made from other types of polygon. Other types of flex are described later.

1.3 A simple square flexagon

In a main position a square flexagon has the appearance of four squares, each with a vertex at the centre. The outline is a larger square. Fig. 1.3 shows the net for a simple square flexagon (Chapman 1961, Conrad 1960, Conrad and Hartline 1962, Gardner 1966, Johnson 1974, McIntosh 2000c, McIntosh 2000g, Mitchell 1999, Neale 1999).

This square flexagon is assembled as for the trihexaflexagon (Section 1.1), except that only the horizontal lines between leaves need to be creased. The assembled square flexagon is a twisted band of hinged squares and it exists in two enantiomorphic forms. It is flexed using the pinch flex. This is simpler than for the trihexaflexagon because there is only twofold rotational symmetry. Start with the leaves numbered 2 uppermost. Fold the square flexagon in two with the fold uppermost. Ensure that there isn’t a continuous fold connecting the tops of the pats being folded together. The square flexagon is now in an intermediate position. This has the appearance of two squares with a common side. Open the square flexagon at the top to reveal the squares numbered 3 and flatten it to a second main position. This particular square flexagon cannot be made to traverse a cycle. It can be returned to its initial main position by turning it over and flexing it. The version of the pinch flex used to manipulate square flexagons is sometimes called the ‘book flex’.

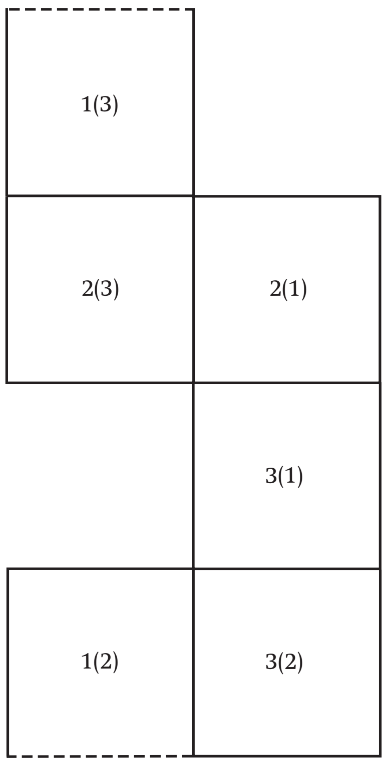

Fig. 1.3 Net for a simple square flexagon.

1.4 General assembly instructions

To assemble nets given later for other types of flexagons use the following general scheme. Copy a net onto 80 g/m^2 paper and cut it out. Crease the lines between leaves to form hinges. Ensure that adjacent leaves superimpose neatly when folded together. Transfer each number in brackets to the reverse face of the leaf, and delete it from the upper face. Copy any hinge or vertex letters on to the reverse of the net. Leaves with the same numbers are folded together. Start with the highest number and work downwards until only leaves numbered 1 and 2 are visible. Then join the ends of the net using transparent adhesive tape. Alternative or additional instructions are included in the captions for some nets.

Sometimes paper models of flexagons don't flex smoothly. If this is a problem try trimming a small amount, say 1 mm, off the edges of the net. Where a paper model of a flexagon is inherently difficult to make or to flex this is noted in the caption. Most types of flexagon are twisted bands and hence exist as an enantiomorphic pair. The net for one enantiomorph can be converted into the net for the other by reversing the

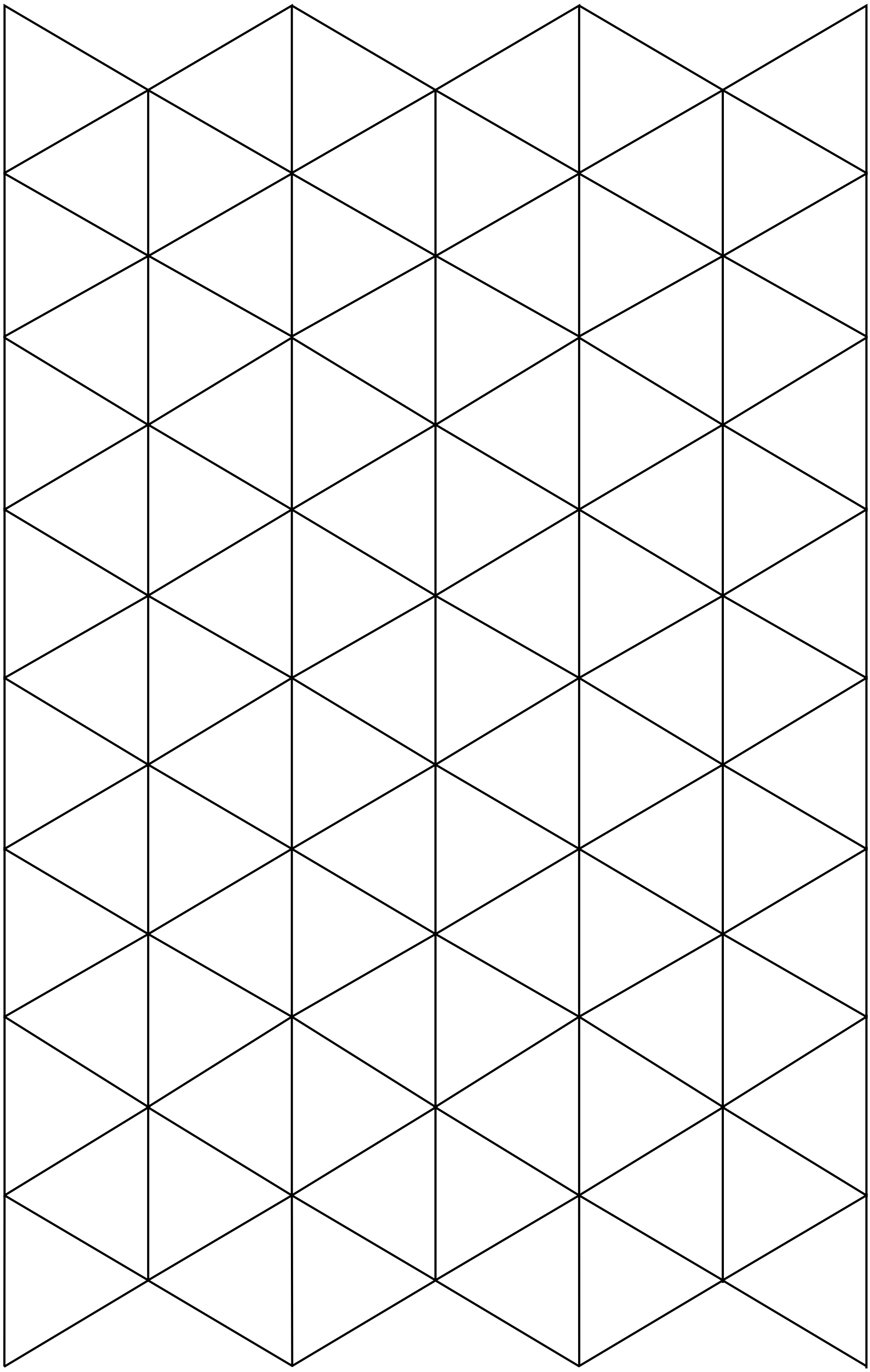

Fig. 1.4 Paper for flexagons made from equilateral triangles.

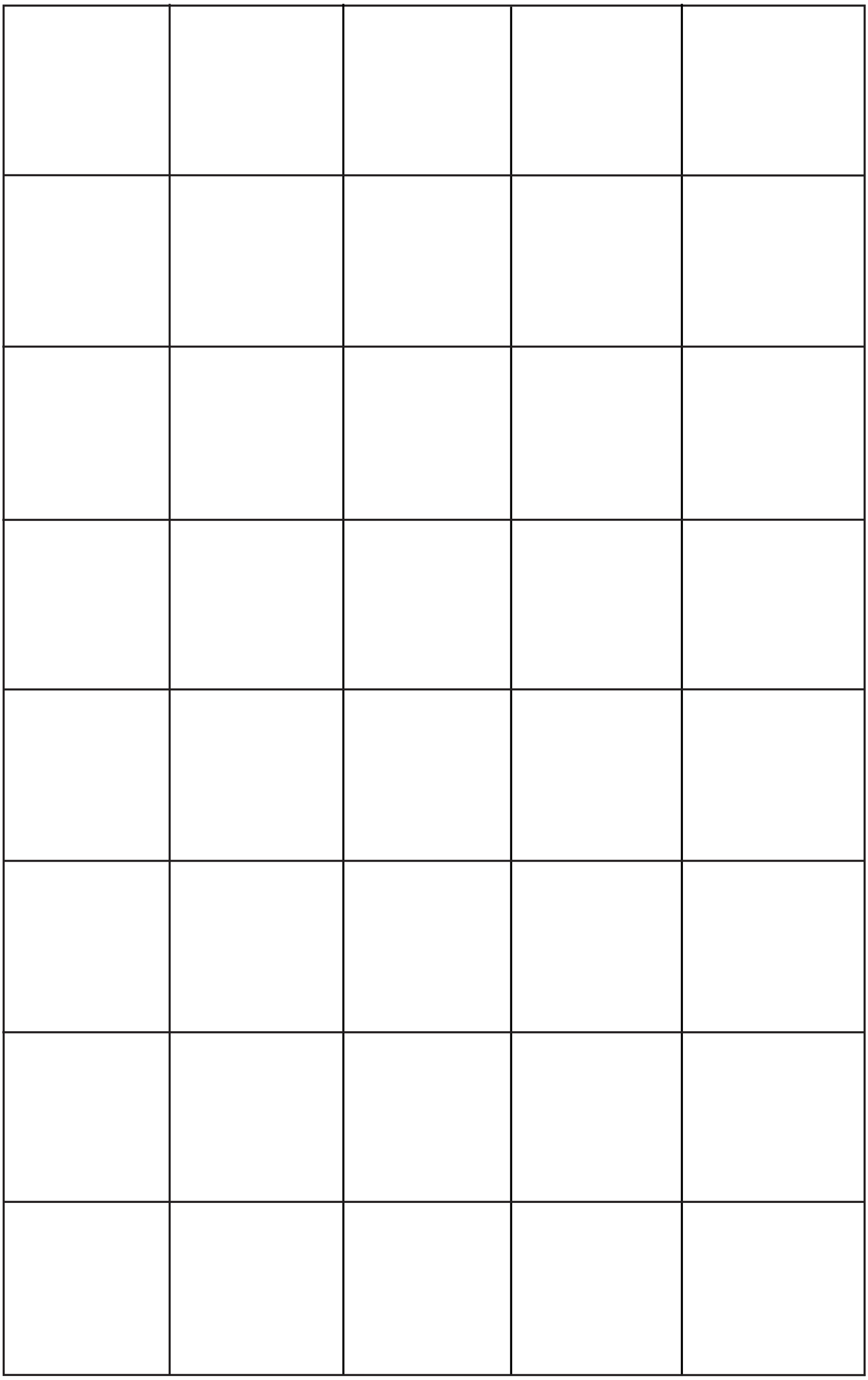

Fig. 1.5 Paper for flexagons made from squares.