1 Matching

This chapter is about matching the elements of one set with elements of another. When you have completed it you should

- be able to represent a matching problem by means of a bipartite graph
- be able to find a maximal matching
- be able to apply the Matching Augmentation algorithm
- be able to interpret allocation problems as matching problems where cost must be minimised
- know how to use the Hungarian algorithm to solve allocation problems.

1.1 Introduction

Matching the elements of two different sets is a common task which you often perform without being aware of it. When you distribute copies of a newsletter to your friends you are matching the set of newsletters with the set of friends. In this case the task is easy because any copy can be assigned to any friend.

Matching becomes more difficult when the two sets to be matched have characteristics that make certain assignments undesirable or impossible. For example, in matching people with jobs, the skills of the people and the requirements of the jobs mean that some assignments should not be made.

Bipartite graphs provide a useful way to represent such **matching problems**. You met the idea of a bipartite graph in D1 Section 2.1.

> A **bipartite graph** is a graph with two sets of nodes such that arcs only connect nodes from one set to the other and do not connect nodes within a set.

It is convenient to always consider the two sets of nodes of a bipartite graph to be the **left-nodes** and the **right-nodes**. For example, $K_{3,3}$ would be drawn as shown in Fig. 1.1.

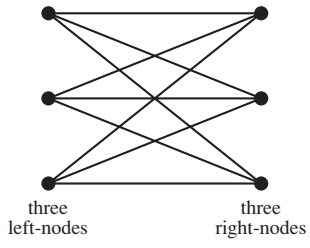


Fig. 1.1

The two sets of nodes correspond to the two sets to be matched and the arcs correspond to the assignments that can be made.

For any bipartite graph, a **matching** is a set of arcs which have no nodes in common.

A **maximal matching** is any matching which contains the largest possible number of arcs.

You have already met a matching problem in D1 Exercise 2A Question 8. Example 1.1.1 shows the problem again.

Example 1.1.1
Four Members of Parliament, Ann, Brian, Clare and David, are being considered for four cabinet posts. Ann could be Foreign Secretary or Home Secretary, Brian could be Home Secretary or the Chancellor of the Exchequer, Clare could be Foreign Secretary or the Minister for Education and David could be the Chancellor or the Home Secretary.

(a) Draw a bipartite graph to represent this situation.

(b) How many options does the Prime Minister have?

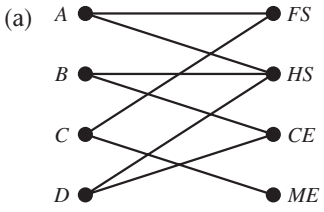


Fig. 1.2

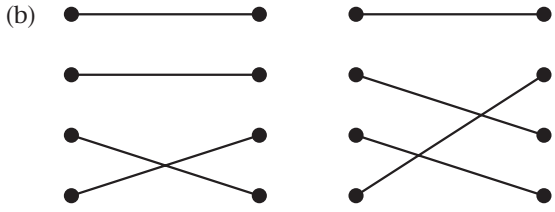


Fig. 1.3

Figure 1.2 shows the graph, and Fig. 1.3 shows the two options.

The maximal matching of Example 1.1.1 covered all of the nodes and each diagram in Fig. 1.3 is called a **complete matching**. A complete matching is not always possible.

Example 1.1.2
Four couples are booked into a small hotel. The Smiths have requested a double room and the Joneses have asked for a double room on the ground floor. The Browns require a twin-bedded room and the Greens will be happy with any room on the ground floor. The hotel manager has just four rooms to assign, three double and one twin-bedded. The twin-bedded room and one of the doubles is on the ground floor. Can the manager satisfy the requirements of all the couples?

The relevant bipartite graph is shown in Fig. 1.4.

The requirements of the Joneses, Browns and Greens cannot all be satisfied, because the Smiths are the only couple who are content with the double rooms which are not on the ground floor, and they cannot use both of them. The maximal matching has three pairings. One example is $\{S, D1\}$, $\{J, DG\}$, $\{B, TG\}$.

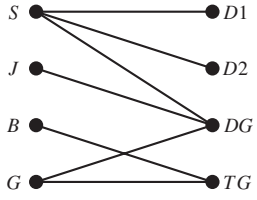


Fig. 1.4

As well as using bipartite graphs to represent the information in assignment problems, you can also use **adjacency matrices**.

The left-nodes are put down one side, and the right-nodes along the top. An entry of 1 means that the corresponding nodes are linked by an arc.

For example, the matrix in Fig. 1.5 contains the same information as the graph of Fig. 1.4.

$$\begin{array}{c} \\ S \\ J \\ B \\ G \end{array} \begin{array}{c} \begin{array}{cccc} D1 & D2 & DG & TG \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Fig. 1.5

For a problem formulated as an adjacency matrix, the task is to find the maximum number of 1s such that no two of these 1s are in the same row or in the same column. In Fig. 1.6 the solution is shown by the 1s in bold-faced type. If you were solving this problem you would probably wish to circle the 1s in the solution.

$$\begin{pmatrix} \mathbf{1} & 1 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Fig. 1.6

1.2 The Matching Augmentation algorithm

In the previous section you solved simple matching problems by inspection. As the numbers of nodes and arcs increase, inspection becomes a hit or miss affair and the chance of overlooking a maximal matching increases.

Fortunately, there is a simple algorithm which can be used to improve upon (or augment) an initial matching or to show that you have already obtained a maximal matching.

Matching Augmentation algorithm

Step 1 Consider all arcs of the matching to be directed from right to left. Consider all other arcs to be directed from left to right.

Step 2 Identify all nodes which do not belong to the matching. Create a new node, X say, joined with directed arcs to all left-nodes which do not belong to the matching.

Step 3 Give each arc a weighting of 1.

Step 4 Apply Dijkstra's algorithm from X until one of the following happens:

- a right-node which does not belong to the matching is reached
- no further labelling is possible. In this case, the initial matching cannot be improved and the algorithm stops.

Step 5 Retrace any path from an unmatched left-node to the unmatched right-node. This is called an alternating path.

Step 6 Remove from the original matching any arcs in the path of Step 5. Add to the matching the other steps in the path. This increases, by 1, the number of arcs in the matching.

Although the following example has only 10 nodes, it is nevertheless difficult to spot a maximal matching. This example will be used to illustrate the Matching Augmentation algorithm.

Example 1.2.1
A builder employs five workers: Alan, who does labouring, plastering and joinery; Betty, who does labouring and wiring; Colin, who does bricklaying and wiring; Di, who does plastering and bricklaying; and Ed, who does plastering, joinery and bricklaying. Can each of the five workers be assigned to a task so that all five tasks are covered?

First, draw a bipartite graph for this problem. In Fig. 1.7, the heavy lines represent a first try at a matching, which assigns just four workers to tasks.

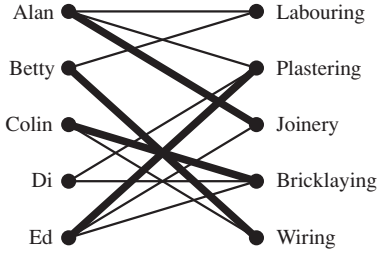


Fig. 1.7

The Matching Augmentation algorithm will show how to improve this matching, if it is possible.

The result of applying the first four steps of the Matching Augmentation algorithm to Example 1.2.1 is shown in Fig. 1.8.

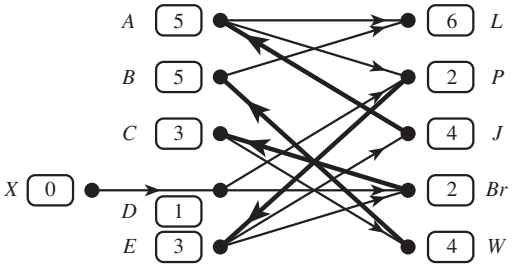


Fig. 1.8

You can see that all the paths have been given directions, as required in Step 1.

The new node X has been created and joined to D, the left-node which does not belong to the initial matching.

Dijkstra's algorithm is then applied from X, and the right-node L, which doesn't belong to the initial matching, is reached. At this stage Step 4 has been completed.

To carry out Step 5, you need to trace a path from D to L: there are two possibilities, D-P-E-J-A-L and D-Br-C-W-B-L, and you could choose either one.

Looking at the first of these, as you trace the path you cover *P-E* and *J-A*, both of which were in the initial matching. Remove them from the initial matching, and add the other arcs from the path, *D-P, E-J* and *A-L*, to the initial matching. This increases by 1 the number of arcs in the matching. In this case, the matching is now maximal. It is

A-L, *B-W*, *C-Br*, *D-P*, *E-J*.

If you had chosen the other path, *D-Br-C-W-B-L*, you would have covered *Br-C* and *W-B* from the initial matching. If you removed them, and added *D-Br, C-W* and *B-L* to the original matching, the new matching would be

A-J, *B-L*, *C-W*, *D-Br*, *E-P*,

an alternative maximal matching.

The next example illustrates how the Matching Augmentation algorithm recognises that a maximal matching has already been reached.

Example 1.2.2
A school timetabling team is trying to timetable four teachers, Andy, Barbara, Chris and Dave to four classes, denoted by *R, S, T* and *U*.

Andy can teach *R* or *S*	Barbara can teach *R, S, T* or *U*
Chris can teach *R* or *S*	Dave can teach *S*.

Draw a bipartite graph and apply the Matching Augmentation algorithm to find a maximal matching.

An initial attempt at a matching is illustrated by the heavy lines in Fig. 1.9.

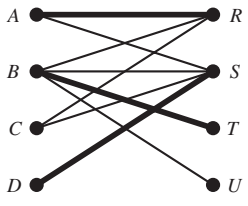


Fig. 1.9

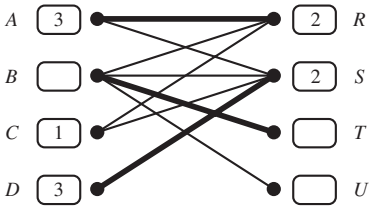


Fig. 1.10

The result of applying the Matching Augmentation algorithm is then shown in Fig. 1.10. The label *X*, that was initially created and then joined to *C* is not shown.

No further labelling is possible, so the initial matching is maximal. Notice that although it is maximal it is not complete.

For an existing non-maximal matching, *M*, of a bipartite graph, *G*, the Matching Augmentation algorithm produces what is called an alternating path.

An **alternating path** for *M* in *G* is a path which consists alternately of arcs in *M* and arcs not in *M*.

This definition makes it easier to see why the Matching Augmentation algorithm works. The following argument is not a proof, however.

Consider the two situations in which Step 4 of the algorithm stops. If Step 4 stops because no further labelling is possible, the algorithm has shown the initial matching to be maximal. If Step 4 stops because a right-node which isn't in the matching has been reached, Step 6 will improve the matching. In the second case the algorithm has constructed a path from an unmatched left-node to an unmatched right-node. The path is alternating because the only way to go from left to right is via an arc which is not in M, and the only way to go from right to left is via an arc which is in M. Because the path starts at the left and ends at the right it must contain one more arc not in M than in M, and so Step 6 of the algorithm produces a matching that contains one more arc than the initial matching.

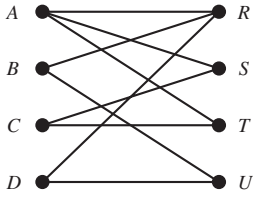
Exercise 1A

1 Apply the Matching Augmentation algorithm to the following bipartite graphs where the heavy lines represent matchings. In each case state what you can conclude.

(a)

(b)

2 Find the adjacency matrix corresponding to the bipartite graph in the figure.



Select four 1s in the matrix in such a way that no two 1s are in the same row or column. Hence find a maximal matching for the bipartite graph.

3 A large department store employs five students, Anita, Bruce, Chloe, Darminder and Errol, for the Christmas period.

The Hardware manager would be happy to use Anita, Chloe or Darminder.
The Bookshop manager would be happy to use Bruce or Errol.
The Sports manager would be happy to use Anita or Darminder.
The Electrical manager would be happy to use Darminder or Errol.
The Food hall manager would be happy to use Anita or Chloe.

(a) Draw a bipartite graph, G, to show which students are suitable for each department.

The managing director initially decides to place Anita in the Food hall, Chloe in Hardware, Darminder in Electrical and Errol in the Bookshop. However, he is then unable to place Bruce appropriately.

(b) Show the incomplete matching, M, that describes the managing director's attempted allocation.

(c) Use the Matching Augmentation algorithm to construct an alternating path for M in G, and hence find a complete matching.

4 A dance team consists of four men, Ahmed, Benny, Chris and Derek, paired with four women, Ann, Bala, Celine and Di. Ann is only prepared to dance with Ahmed, Bala will dance with Benny or Derek, Celine will dance with Chris, and Di will dance with any of the men. In their first competition, Di dances with Derek.

(a) Draw a bipartite graph, with the women on the left and the men on the right, to represent the only possible matching with Di and Derek paired.

Unfortunately, Di and Derek fall out and a different pairing has to be arranged for the second competition.

(b) Draw a bipartite graph to show the possible pairings and the incomplete matching, M, from the first competition.

(c) Apply the Matching Augmentation algorithm to obtain a complete matching for the second competition.

1.3* Maximal matching–minimum cover

This section contains extension material and may be omitted.

There is an important connection between the matching problem and the problem of finding sets of nodes which cover every arc, that is sets of nodes which contain at least one end-node of each arc of the bipartite graph. These are called **cover sets** of the graph.

Consider, for example, the graph drawn for Example 1.2.2, redrawn as Fig. 1.11.

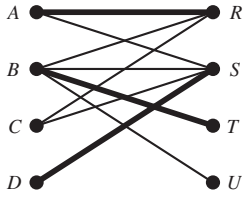


Fig. 1.11

Since A-R, B-T and D-S is a matching, a cover set must contain at least one of A or R, at least one of B or T and at least one of D or S.

In general it is clear that:

The number of arcs in any matching	$\leqslant$	the number of nodes in any cover set.

The smallest possible number of nodes in a cover set cannot, therefore, be less than the number of arcs in a maximal matching. In Example 1.2.2, this minimum is actually achieved for the cover set $\{B, R, S\}$. The interesting aspect of the connection between the matching problem and the cover set problem is that this result is always true. That is:

The number of arcs in a maximal matching	$=$	the minimum number of nodes in a cover set.

This is called the maximal matching–minimum cover result.

You can prove this result by considering the effect of applying the Matching Augmentation algorithm to bipartite graphs for which the maximal matching has already been achieved.

First, consider the following examples:

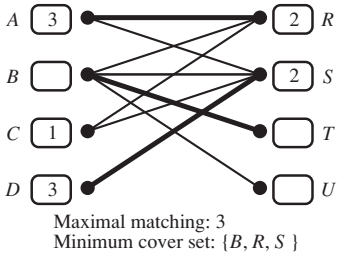


Fig. 1.12

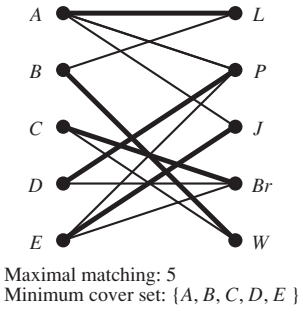


Fig. 1.13

The key to proving the maximal matching–minimum cover result is to note how the examples of cover sets in Figs. 1.12 to 1.14 are formed. In each case, they consist of the unlabelled left-nodes and labelled right-nodes.

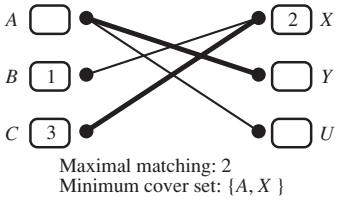


Fig. 1.14

To prove the maximal matching–minimum cover result it is therefore necessary to prove:

- the unlabelled left-nodes and labelled right-nodes form a cover set;
- the number of these nodes equals the number of arcs in a maximal matching.

These two facts follow from the following features of maximal matchings to which the Matching Augmentation algorithm has been applied. In each case, you should try to explain the reason for these features (see Miscellaneous exercise 1 Question 12).

- Each arc of the bipartite graph either has an unlabelled left-node or a labelled right-node.
- All unlabelled left-nodes are in the matching.
- All labelled right-nodes are in the matching.
- Each arc of the matching joins nodes which are either both labelled or both unlabelled.

Then:

Number of arcs of maximal matching

= number of right-nodes of matching

= number of unlabelled right-nodes of matching
+ number of labelled right-nodes of matching

= number of unlabelled left-nodes of matching
+ number of labelled right-nodes of matching

= number of unlabelled left-nodes + number of labelled right-nodes.

Each arc of the bipartite graph has either an unlabelled left-node or a labelled right-node, but not both. So the set of unlabelled left-nodes and labelled right-nodes is a cover set. It must be a minimum cover set since it has the same number of nodes as the number of arcs in a matching.

The maximal matching–minimum cover result can be useful for seeing quickly that you have found a maximal matching and need not apply the Matching Augmentation algorithm.

Example 1.3.1
For the bipartite graph shown in Fig. 1.15, find

(a) a cover set containing four nodes,

(b) a matching containing four arcs.

What can you conclude about your answers to (a) and (b)?

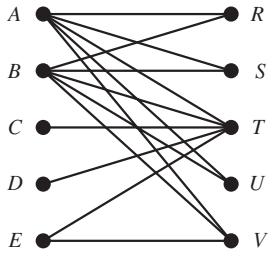


Fig. 1.15

(a) A, B, T, V.

(b) $\{A, R\}$, $\{B, S\}$, $\{C, T\}$, $\{E, V\}$.

A cover set and a matching of the same sizes have been found. The solution to part (a) is therefore a minimum cover set and the solution to part (b) is a maximal matching.

1.4 Allocation problems

Suppose that a building company has four contracts which must be completed at the same time. The work is to be done by subcontractors, each of whom can carry out only one contract. The subcontractors' quotes for each of the four jobs are shown in Table 1.16.

		Contract			
		A	B	C	D
Subcontractor	1	10	5	9	–
	2	9	6	9	6
	3	10	–	10	7
	4	9	5	9	8

Quotes are in £1000s.
'–' indicates no quote.

Table 1.16

Here there is no difficulty in finding a matching. An example is 1-A, 2-B, 3-C, 4-D. The problem is to find a matching which minimises the total cost; such a problem is called an **allocation problem**.

A good method of tackling allocation problems is to reduce the array of costs by subtracting from each element of a row (or column) the least element in that row (or column). For example, Table 1.18 is obtained from Table 1.17 by subtracting 5, 6, 7 and 5 from the first, second, third and fourth rows respectively. The smallest number in each new row is now equal to zero.

10	5	9	–
9	6	9	6
10	–	10	7
9	5	9	8

Table 1.17

	5	0	4	–
−6 →	3	0	3	0
−7 →	3	–	3	0
−5 →	4	0	4	8

Table 1.18

The solution to the original problem will then simply be the solution to the new problem with an extra cost of £23,000 because $5 + 6 + 7 + 5 = 23$. Reducing the columns of Table 1.18 in a similar way, in this case by subtracting 3 from the first and third columns, leads to the array in Table 1.19.

2	0	1	–
0	0	0	0
0	–	0	0
1	0	1	3

Table 1.19

For this matrix, it is easy to see that there are no matchings of zero cost but there are several of cost only 1. For example, Table 1.20 shows a matching of cost 1 in bold type.

2	**0**	1	–
0	0	**0**	0
0	–	0	**0**
1	0	1	3

Table 1.20

So the original problem has an optimal allocation, of cost £30,000, shown in Table 1.21.

10	**5**	9	–
9	6	**9**	6
10	–	10	**7**
9	5	9	8

Table 1.21

To solve an allocation problem, first reduce the array of costs by subtracting the least number in a row (or column) from each element of that row (or column).

Example 1.4.1
Choose five numbers from the array given below so that the sum of the five numbers is the least possible. No two numbers can be in the same row or column.

10	10	9	8	10
10	12	12	9	13
16	16	14	12	15
14	15	12	12	16
15	16	14	13	14

Reducing the array by rows leads to the array on the left; then reducing by columns leads to the array on the right, in which the minimum allocation is shown in bold type.