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Excerpt
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1. MOTION UNDER GRAVITY ALONE

"Fair pledges of a fruitful tree,
Why do ye fall so fast?"

Robert Herrick (1591 - 1674)

1.1 Gravity

When a small body is projected near a much larger body its trajectory is not straight but curves back towards the larger body. Newton's law of universal gravitation reveals that, when both bodies are spherically symmetric and the small projectile is outside the larger body, the force acting by the larger mass (m_1) on the smaller mass (m_2) is given by

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \tag{1.1}$$

where G is the universal gravitation constant, $\mathbf{r}(= r\hat{\mathbf{r}})$ is the vector from the centre of mass of the larger body to the centre of mass of the smaller body and $\hat{\mathbf{r}}$ is the corresponding unit vector. When the larger body is the Earth ($m_1 = m_e$) and the small projectile ($m_2 = m$) is close enough to a point fixed on the Earth's surface, the Earth may be considered to have spherical symmetry with $r \approx r_e$ (the radius of the Earth). Then equation (1.1) is replaced by

$$\mathbf{F} = -\frac{Gm_em}{r_e^2}\hat{\mathbf{j}}$$

where $\hat{\mathbf{j}}$ is a unit vector in the upward vertical direction, which is considered to be constant in both direction and magnitude.

The assumption that $\mathbf{F}$ has a constant direction is called the "flat Earth" assumption and then

$$\mathbf{F} = -mg\hat{\mathbf{j}}$$

$$= m\mathbf{g}$$

where $g = Gm_e/r_e^2$. Since $G = 6.67\times10^{-11}$ (M.K.S. units), $m_e = 5.98\times10^{24}$(kg) and $r_e = 6.38\times10^6$(m) then $g = 9.80$(ms^{-2}), and g is called the acceleration due to gravity. Its magnitude varies by less than 1% for projectiles within 30 km of the Earth's surface, and there are similarly small variations for changes in latitude. The force $m\mathbf{g}$ is referred to as the weight of any body of mass m.

1.2 Velocity and Position Vectors

The simplest approximation for a projectile's motion is to consider that the only force acting on it, after it is launched, is its weight. Then, for motion in free space, Newton's second law of motion yields

$$m\frac{d^2\mathbf{r}}{dt^2} = m\mathbf{g}$$

that is

$$\frac{d^2\mathbf{r}}{dt^2} = -g\hat{\mathbf{j}} \tag{1.2}$$

where $\mathbf{r}$ is the position vector with respect to a fixed origin on the Earth's surface.

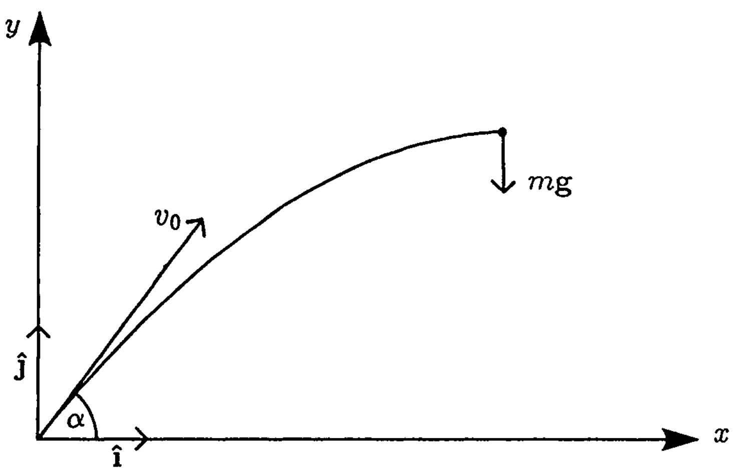

Figure 1.1 The co-ordinate and vector system

If initially ($t = 0$) the projectile is travelling at speed v_0 at an angle α to the horizontal the initial velocity vector is

$$\mathbf{v}_0 = v_0 \cos\alpha\, \hat{\mathbf{i}} + v_0 \sin\alpha\, \hat{\mathbf{j}}$$

where $\hat{\mathbf{i}}$ is a unit vector in the horizontal direction forming a right-hand system with $\hat{\mathbf{j}}$ (see Figure 1.1).

When equation (1.2) is integrated with respect to t it yields

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_0 - gt\hat{\mathbf{j}} \\ &= v_0 \cos\alpha\ \hat{\mathbf{i}} + (v_0 \sin\alpha - gt)\hat{\mathbf{j}} \qquad (1.3)\end{aligned}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the velocity vector at any time t. Note that the horizontal component of the velocity of a projectile in free space is a constant.

When equation (1.3) is integrated with respect to time, and the assumption $\mathbf{r} = \mathbf{0}$ when $t = 0$ is made, the position vector is

$$\begin{aligned}\mathbf{r} &= \mathbf{v}_0 t - \frac{1}{2}gt^2\hat{\mathbf{j}} \\ &= v_0 t \cos\alpha\ \hat{\mathbf{i}} + \left(v_0 t \sin\alpha - \frac{1}{2}gt^2\right)\hat{\mathbf{j}} \qquad (1.4)\end{aligned}$$

If the projectile is at the point (x, y), then $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and the horizontal component of the projectile's displacement is

$$x = \mathbf{r} \cdot \hat{\mathbf{i}} = v_0 t \cos\alpha$$

while the vertical component is

$$y = \mathbf{r} \cdot \hat{\mathbf{j}} = v_0 t \sin\alpha - \frac{1}{2}gt^2$$

These two equations for x and y can be thought of as defining the trajectory in parametric form. When t is eliminated from the two equations they yield

$$y = x \tan\alpha - \frac{gx^2}{2v_0^2} \sec^2\alpha \qquad (1.5)$$

which can be rearranged to give

$$\left[x - \frac{v_0^2 \sin 2\alpha}{2g}\right]^2 = \frac{2v_0^2 \cos^2\alpha}{g}\left[\frac{v_0^2 \sin^2\alpha}{2g} - y\right]$$

This shows that the trajectory is a parabola with its vertex uppermost (see Figure 1.2).

The speed (v) in any position is given by

$$\begin{aligned} v^2 &= \left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 \\ &= v_0^2 \cos^2\alpha + (v_0 \sin\alpha - gt)^2 \\ &= v_0^2 \cos^2\alpha + v_0^2 \sin^2\alpha - 2v_0 gt \sin\alpha + g^2t^2 \\ &= v_0^2 - 2gy \end{aligned}$$

which could also be obtained by noting that the mechanical energy of the projectile is conserved when only gravity acts.

The angle ψ made by the tangent to the projectile's path at any time t with the horizontal is given by

$$\begin{aligned} \tan\psi &= \frac{dy}{dt} \Big/ \frac{dx}{dt} \\ &= \frac{v_0 \sin\alpha - gt}{v_0 \cos\alpha} \\ &= \tan\alpha - \left[\frac{gt}{v_0}\right] \sec\alpha \end{aligned} \tag{1.6}$$

An alternative form is obtained from equation (1.5) by noting that the slope at any point is

$$\tan\psi = \frac{dy}{dx} = \tan\alpha - \frac{gx\sec^2\alpha}{v_0^2}$$

1.3 Point of Impact

Impact on the horizontal plane through the projection point occurs when

$$\mathbf{r} \cdot \hat{\mathbf{j}} = 0$$

Thus from equation (1.4),

$$v_0 t \sin\alpha - \frac{1}{2} g t^2 = 0$$

and so

$$t = 0 \quad \text{or} \quad \frac{2v_0 \sin\alpha}{g}$$

Now $t = 0$ corresponds to the projection point, while

$$t_f = \frac{2v_0 \sin\alpha}{g}$$

gives the time of flight to impact.

At the impact point $y = 0$ and so $v^2 = v_0^2$, giving $v = v_0$ since the speed cannot be negative. Also from equation (1.6), the direction at the impact point is given by

$$\begin{aligned} \tan\psi &= \tan\alpha - \frac{g \sec\alpha}{v_0}\left(\frac{2v_0 \sin\alpha}{g}\right) \\ &= -\tan\alpha \end{aligned}$$

Therefore

$$\psi = -\alpha \quad \text{or} \quad (\pi - \alpha)$$

but $(\pi - \alpha)$ is irrelevant on physical grounds, since $\psi = \pi - \alpha$ means that the projectile is travelling back along its trajectory. Therefore at impact on the horizontal plane through the projection point the projectile has an angle of depression α, as is obvious from the symmetry of the parabola about its axis.

Now the range (x_f) on the horizontal plane is the x-value when $t = t_f$. Therefore

$$\begin{aligned} x_f &= v_0 \cos\alpha \left(\frac{2v_0 \sin\alpha}{g}\right) \\ &= \frac{v_0^2 \sin 2\alpha}{g} \end{aligned}$$

This result can also be obtained by putting $y = 0$ in equation (1.5). For a given initial speed (v_0) the range is proportional to $\sin 2\alpha$. The maximum range occurs when $\alpha = \pi/4$ radians and has the value v_0^2/g.

1.4 The Vertex of the Trajectory

At the vertex

$$\mathbf{v} \cdot \hat{\mathbf{j}} = 0$$

and so the vertex is reached when

$$t = \frac{v_0 \sin\alpha}{g}$$

$$= \frac{1}{2} t_f$$

Thus at the vertex

$$x = \frac{v_0^2 \sin 2\alpha}{2g}$$

$$y = \frac{v_0^2 \sin^2 \alpha}{2g}$$

$$v = v_0 \cos\alpha$$

$$\psi = 0$$

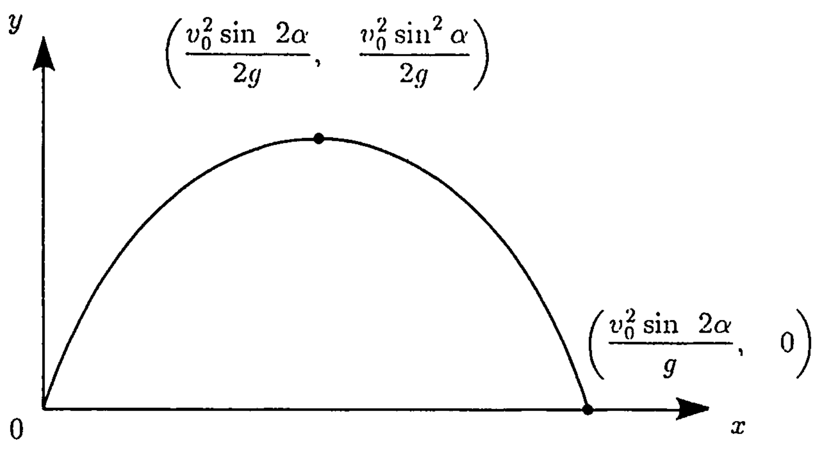

Figure 1.2 The gravity-only trajectory

Example 1.1

A player hits a baseball into the outfield against a wall 100 metres away. If it leaves the bat at an angle of 45° to the horizontal and strikes the wall 10 metres above the bat-ball contact position, what is the initial speed of the ball?

Solution

Consider equation (1.5)

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2} \sec^2 \alpha$$

On rearranging

$$v_0^2 = \frac{gx^2 \sec^2 \alpha}{2(x \tan \alpha - y)}$$

$$= \frac{9.8 \times 10000 \times 2}{2(100 - 10)}$$

and so

$$v_0 = 33$$

Thus the initial speed is 33 $\mathrm{ms}^{-1} \approx 119$ km h^{-1}.

Example 1.2

A cannon mounted on a cliff overlooking the sea can fire a shot at an angle of 30° to the horizontal with a muzzle speed of 800 ms^{-1}. If the mouth of the cannon is 100 metres vertically above the base of the cliff, find how far out from the cliff a shot will hit the water.

Solution

(Method 1)

Consider equation (1.4)

$$y = v_0 t \sin \alpha - \frac{1}{2} g t^2$$

Substitution leads to

$$-100 = 400t - 4.9t^2 \qquad (\text{since } \alpha = 30^\circ)$$

which has solutions

$$t = \frac{400 \pm \sqrt{161960}}{9.8}$$

$$\approx 81.88 (\text{ the negative answer is irrelevant})$$

Therefore

$$\begin{aligned} x &= v_0 t \cos\alpha \\ &= 81.88 \times 800 \times \cos\frac{\pi}{6} \\ &\approx 56728 \end{aligned}$$

The shot will hit the water approximately 56700 m from the base of the cliff.

(Method 2)

Rewrite equation (1.5) so that the coefficient of x^2 is unity; then

$$x^2 - \frac{v_0^2 \sin 2\alpha}{g} x + \frac{2v_0^2 y \cos^2\alpha}{g} = 0$$

Substituting the appropriate value yields

$$x^2 - 56556.76x - 9795318.4 = 0$$

The positive root is 56729, and the difference between the results from the two methods is due to the approximate time value used in Method 1.

Example 1.3

On the 1983 tour of Australia by the West Indies' cricket team Joel Garner hit a massive six during one game. The radio commentator said that it was such a big six that the ball went up as high as it went forward. If this was true, at what angle was it hit initially?

Solution

$$\text{Maximum height} = \frac{v_0^2 \sin^2\alpha}{2g}$$

$$\text{Range} = \frac{v_0^2 \sin 2\alpha}{g}$$

When maximum height equals range it is seen that

$$2\sin 2\alpha = \sin^2\alpha$$

which leads to

$$\sin\alpha = 0 \quad \text{or} \quad \tan\alpha = 4$$

Thus $\alpha = 76°$. The answer $0°$ is neglected, since the hit would not be a six.

1.5 Projection from a Different Level

For many projectile problems the projection point is at a different level from the impact point. Suppose that the impact point P is a vertical distance h above the projection point 0 which is selected as the origin. (When P is below 0, h is of course negative). The aim first of all is to determine the horizontal range and time of flight, and then to consider the maximum horizontal range.

When $h > 0$ there will be two points at this height through which the projectile passes. When $h < 0$ however, if the time of flight is restricted to positive values, there is only one point (see Figure 1.3). Now from equation (1.4) since $\mathbf{r}\cdot\hat{\mathbf{j}} = h$ then

$$h = v_0 t\sin\alpha - \frac{1}{2}gt^2$$

The solutions of this quadratic equation in t are

$$t = \frac{v_0\sin\alpha \pm \sqrt{v_0^2\sin^2\alpha - 2gh}}{g}$$

When $v_0\sin\alpha < \sqrt{2gh}$ the solution is a complex number, indicating that heights $h > v_0^2\sin^2\alpha/(2g)$ cannot be reached with these initial conditions. When $v_0\sin\alpha > \sqrt{2gh}$ there are two cases to consider. For $h > 0$ there are two possible times of flight, with the negative sign associated with the earlier time. For $h < 0$ the expression containing the negative option is neglected.

The horizontal range $(\mathbf{r}\cdot\hat{\mathbf{i}})$ is therefore

$$x = \frac{v_0\cos\alpha\left\{v_0\sin\alpha \pm \sqrt{v_0^2\sin^2\alpha - 2gh}\right\}}{g} \tag{1.7}$$

which can also be obtained from equation (1.5) directly.

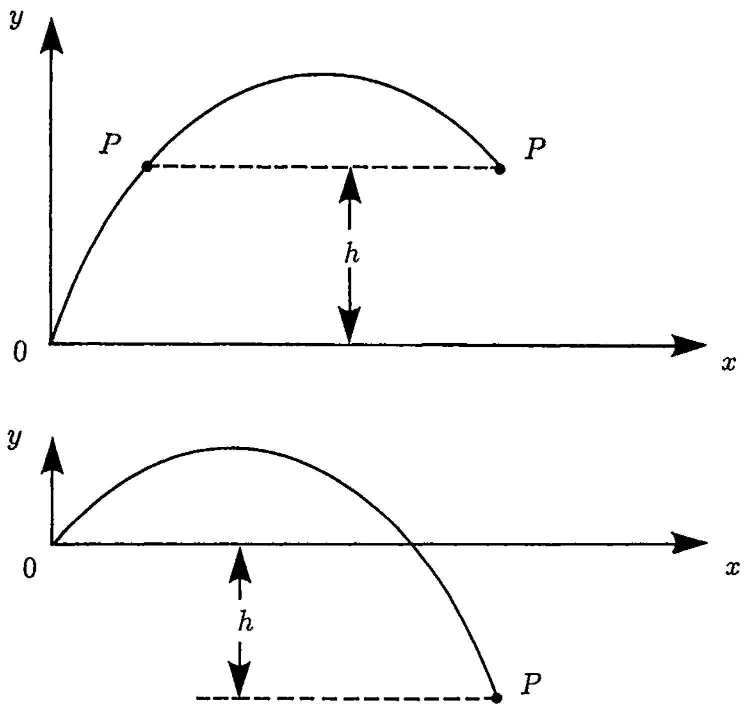

Figure 1.3 Projection through points at height (i) $h > 0$ (ii) $h < 0$

For a given initial speed v_0 the maximum horizontal range is obtained by considering $dx/d\alpha = 0$. It will be left as an exercise for the reader to determine the maximum value of x from equation (1.7). Instead an alternative derivation of the result is given by considering equation (1.5) with $y = h$. Then

$$h = x \tan\alpha - \frac{gx^2}{2v_0^2}\sec^2\alpha$$