1

Introduction

Suppose you have to solve an ordinary differential equation of, say, second order, for example,

$$y'' = (x - y)y'^3, \qquad y' \equiv dy/dx, \quad \text{etc.}, \tag{1.1}$$

or

$$y'' = x^n y^2, \qquad n \neq 0. \tag{1.2}$$

How can you proceed? First, you will certainly check whether the differential equation belongs to a class you know how to treat, for preference to the class of linear differential equations, or whether it can be transformed into a member of such a class by simple transformations of the dependent and independent variables. If unsuccessful, you may look up a textbook or a collection of solutions. And if that does not help and the problem is of some importance, you will try more elaborate transformation methods and ad hoc *ansätze* to find the solution. But it sometimes may and will happen that you have to leave the problem unsolved, with the uneasy feeling that there might be a method you have overlooked or you are unaware of, and the question of whether the solution to your differential equation can be given in terms of elementary, simple functions will still be open. [For the differential equations (1.1) and (1.2), the answer to that question will be given in Sections 5.2 and 7.5.]

For a partial differential equation such as

$$u_{,xx}u_{,yy} - u_{,xy}^2 - 1 = 0, \tag{1.3}$$

the situation is similar, although the specific questions are different. Since really powerful methods to deal with partial differential equations of second order are easily available only for linear equations, you may first try to transform the given differential equation into a linear one [for the example (1.3) that can be done, see Exercise 21.4.6]. If unsuccessful, you may perhaps set out to find as many special solutions as possible by using standard techniques such as separation of variables (writing u as a product or a sum of functions of different variables) or reduction of variables (assuming that u depends on less independent variables). But again the question remains open whether these methods will work and whether there are other methods that may prove useful.

In both cases, for ordinary and for partial differential equations, the unmistakable answer to all these questions is that in the majority of the cases where exact solutions of a differential equation can be found, the underlying property is a symmetry of that equation. Moreover, there are practicable methods to find those symmetries (if they exist) and to use them in solving the differential equation.

This book is devoted to the study of symmetries of differential equations, with the emphasis on how to use symmetries to find solutions. The typical way to define and find a symmetry is to first characterize a class of admitted transformations of variables and then look for special transformations in this class that leave a given differential equation invariant. The more general the admissible transformations are, the more symmetries will exist – but the more difficult it will be to find them and to exploit them to give an integration procedure. I have therefore arranged the material according to the generality of the allowed transformations, beginning with Lie point transformations and symmetries and ending with generalized (dynamical and Lie–Bäcklund) symmetries.

Some aspects of the theory could have been treated simultaneously for ordinary and for partial differential equations. To make the book easier to read, the subject has been divided. Thus some paragraphs of the second part on partial differential equations are in essence slightly generalized repetitions of what has already been said in the first part in the context of ordinary differential equations. Although that does not imply that the second part is completely self-contained, one can start reading there if interested only in partial differential equations, at the price of meeting a comparatively concise presentation from the very beginning.

The book is written in an intuitive fashion; none of the proofs is really watertight in that the necessary assumptions are not clearly stated (e.g., most of the functions that appear are tacitly assumed to be differentiable or even analytic). The emphasis is always on presenting the main ideas and giving advice on how to use them in practice. Most of the material is standard and long known. No references are given, therefore, in the text, but a small guide to the existing literature is added as an appendix.

I

Ordinary differential equations

2

Point transformations and their generators

2.1 One-parameter groups of point transformations and their infinitesimal generators

As stated in the introduction, our main goal is to use symmetries of differential equations for their integration. To do this, we need a proper definition of a symmetry, and this in turn requires some knowledge of transformations and their generators.

When dealing with differential equations, one very often tries to simplify the equation by an appropriate change of variables, that is, by a transformation of the independent variable x and the dependent variable y,

$$\tilde{x} = \tilde{x}(x, y), \qquad \tilde{y} = \tilde{y}(x, y). \tag{2.1}$$

We call this a *point transformation* (as contrasted with, e.g., contact transformations, which will be discussed in Chapter 11); it maps points (x, y) into points $(\tilde{x}, \tilde{y})$.

In the context of symmetries we have to consider point transformations that depend on (at least) one arbitrary parameter ε,

$$\tilde{x} = \tilde{x}(x, y; \varepsilon), \qquad \tilde{y} = \tilde{y}(x, y; \varepsilon), \tag{2.2}$$

and that furthermore have the properties that they are invertible, that repeated applications yield a transformation of the same family, for example,

$$\tilde{\tilde{x}} = \tilde{\tilde{x}}(\tilde{x}, \tilde{y}; \tilde{\varepsilon}) = \tilde{\tilde{x}}(x, y; \tilde{\tilde{\varepsilon}}) \tag{2.3}$$

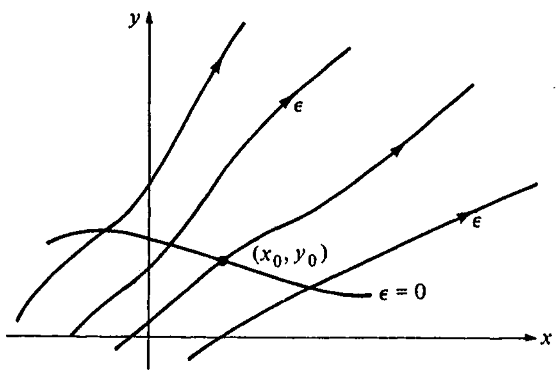

Figure 2.1. Action of a one-parameter group of transformations.

for some $\tilde{\tilde{\varepsilon}} = \tilde{\tilde{\varepsilon}}(\tilde{\varepsilon}, \varepsilon)$, and that the identity is contained for, say, $\varepsilon = 0$:

$$\tilde{x}(x, y; 0) = x, \qquad \tilde{y}(x, y; 0) = y. \tag{2.4}$$

In fact, these properties ensure that the transformations (2.2) form a *one-parameter group of point transformations.*

A simple example of a one-parameter group is given by the rotations

$$\tilde{x} = x \cos \varepsilon - y \sin \varepsilon, \qquad \tilde{y} = x \sin \varepsilon + y \cos \varepsilon. \tag{2.5}$$

On the other hand, the reflection

$$\tilde{x} = -x, \qquad \tilde{y} = -y \tag{2.6}$$

is a point transformation that, although useful, does not constitute a one-parameter group.

The one-parameter group (2.2) and its action can best be visualized as motion in an x–y plane. To do this, take (for $\varepsilon = 0$) an arbitrary point (x_0, y_0) in that plane. When the parameter ε varies, the images $(\tilde{x}_0, \tilde{y}_0)$ of (x_0, y_0) will move along some line. Repeating this for different initial points, one obtains the picture given in Figure 2.1, each curve representing points that can be transformed into each other under the action of the group. They are called the orbits of the group. This picture can also be interpreted in terms of the flow and the stream lines of some fluid.

The picture of Figure 2.1 suggests that a different representation of the transformation group given by equations (2.2) should be possible: the set of curves given in Figure 2.1 is completely characterized by the field of its tangent vectors **X**, see Figure 2.2, and vice versa!

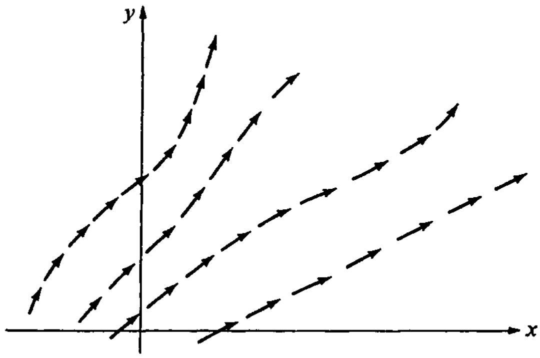

Figure 2.2. The field of tangent vectors **X** associated with orbits of a one-parameter group.

This idea can be given a concise form by considering infinitesimal transformations. We take an arbitrary point (x, y) and write

$$\begin{aligned} \tilde{x}(x, y; \varepsilon) &= x + \varepsilon\xi(x, y) + \cdots = x + \varepsilon \mathbf{X}x + \cdots, \\ \tilde{y}(x, y; \varepsilon) &= y + \varepsilon\eta(x, y) + \cdots = y + \varepsilon \mathbf{X}y + \cdots, \end{aligned} \tag{2.7}$$

where the functions ξ and η are defined by

$$\xi(x, y) = \left.\frac{\partial \tilde{x}}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad \eta(x, y) = \left.\frac{\partial \tilde{y}}{\partial \varepsilon}\right|_{\varepsilon=0}, \tag{2.8}$$

and the operator **X** is given by

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{2.9}$$

Obviously, the components of the tangent vector **X** are exactly ξ and η.

The operator **X** is called the *infinitesimal generator* of the transformation. "Generator" indicates that repeated application of the infinitesimal transformation will generate the finite transformation, which is a different way of expressing the fact that the integral curves of the vector field **X** are the group orbits: that is, by integrating

$$\frac{\partial \tilde{x}}{\partial \varepsilon} = \xi(\tilde{x}, \tilde{y}), \qquad \frac{\partial \tilde{y}}{\partial \varepsilon} = \eta(\tilde{x}, \tilde{y}) \tag{2.10}$$

with the initial values x, y at $\varepsilon = 0$, we will arrive at the finite transformation (2.2).

The infinitesimal generator uniquely determines the orbits of the group, but the orbits will give the generator only up to a constant factor: if we rescale the parameter ε by $\varepsilon = f(\hat{\varepsilon}), f(0) = 0, f'(0) \neq 0$, the definition (2.8) of ξ and η yields

$$\hat{\xi} = \frac{\partial \tilde{x}}{\partial \hat{\varepsilon}}\bigg|_{\hat{\varepsilon}=0} = \frac{\partial \tilde{x}}{\partial \varepsilon} f'(\hat{\varepsilon})\bigg|_{\varepsilon=0} = f'(0)\xi, \quad \hat{\eta} = f'(0)\eta \tag{2.11}$$

(the tangent vector $\mathbf{X}$ of the orbits has no fixed scale).

As an illustration of transformations and their generators, let us consider some examples. For the rotations (2.5) in the x–y plane we have

$$\frac{\partial \tilde{x}}{\partial \varepsilon}\bigg|_{\varepsilon=0} = -y, \qquad \frac{\partial \tilde{y}}{\partial \varepsilon}\bigg|_{\varepsilon=0} = x, \tag{2.12}$$

so that the corresponding generator is given by

$$\mathbf{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{2.13}$$

For a translation (shift of the origin of x) we obtain

$$\tilde{x} = x + \varepsilon, \qquad \tilde{y} = y, \qquad \mathbf{X} = \frac{\partial}{\partial x}. \tag{2.14}$$

The inverse problem is to find the finite transformation when the generator is given. If we have

$$\mathbf{X} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{2.15}$$

which group corresponds to it? Of course we have to integrate (2.10), that is,

$$\frac{\partial \tilde{x}}{\partial \varepsilon} = \tilde{x}, \qquad \frac{\partial \tilde{y}}{\partial \varepsilon} = \tilde{y}. \tag{2.16}$$

The solution with initial values $\tilde{x}(0) = x$, $\tilde{y}(0) = y$ is obviously

$$\tilde{x} = e^{\varepsilon}x, \qquad \tilde{y} = e^{\varepsilon}y. \tag{2.17}$$

This is a (special) scaling – or similarity – transformation: all variables are multiplied by the same constant factor.

Looking back at the last few pages, the reader may wonder why we so strongly preferred the transformations that form a (Lie) group and the generators of those transformations. The reason is that although transformations (2.1) that are not members of one-parameter groups may and will occur in the context of symmetries of differential equations, it is only with the help of the generators **X** that we will be able to *find* (and use) symmetries. This is mainly due to the fact that although the transformations themselves can be very complicated, the generators are always *linear* operators.

2.2 Transformation laws and normal forms of generators

The generators **X** as given by (2.9) explicitly refer to the variables x and y. How do the components ξ and η change if we introduce new variables $u(x, y)$ and $v(x, y)$ instead of x and y?

We slightly generalize the question by considering generators in more than two variables,

$$\mathbf{X} = b^i(x^n)\frac{\partial}{\partial x^i}, \qquad i = 1, \ldots, N \tag{2.18}$$

(summation over the repeated index i). Performing a transformation

$$x^{i'} = x^{i'}(x^i), \qquad |\partial x^{i'}/\partial x^i| \neq 0, \tag{2.19}$$

gives because of

$$\frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial}{\partial x^{i'}} \tag{2.20}$$

the transformation law

$$\mathbf{X} = b^{i'}\frac{\partial}{\partial x^{i'}}, \qquad b^{i'} = \frac{\partial x^{i'}}{\partial x^i}b^i. \tag{2.21}$$

As is to be expected, the components b^i of the generator **X** transform as the (contravariant) components of a vector.

We can make use of this transformation law to write the generator **X** in a different form. Because of

$$\mathbf{X}x^n = b^i\frac{\partial}{\partial x^i}x^n = b^n, \qquad \mathbf{X}x^{n'} = b^{n'}, \tag{2.22}$$

this form is

$$\mathbf{X} = (\mathbf{X}x^i)\frac{\partial}{\partial x^i} = (\mathbf{X}x^{i'})\frac{\partial}{\partial x^{i'}}. \tag{2.23}$$

It clearly indicates how to calculate the components of $\mathbf{X}$ in the new coordinates $x^{i'}(x^i)$ if $\mathbf{X}$ is known in coordinates x^i: one simply has to apply $\mathbf{X}$ to the new coordinates.

If, for example, we want to express the generator (2.15) in coordinates

$$u = y/x, \qquad v = xy, \tag{2.24}$$

we obtain immediately from (2.23).

$$\mathbf{X}u = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u = 0, \qquad \mathbf{X}v = 2xy = 2v, \tag{2.25}$$

that is,

$$\mathbf{X} = 2v\frac{\partial}{\partial v}. \tag{2.26}$$

A second example is provided by the generator (2.13) of a rotation if we want to express it in polar coordinates $r = (x^2 + y^2)^{1/2}$, $\varphi = \arctan y/x$. The result is $\mathbf{X}r = 0$, $\mathbf{X}\varphi = 1$, that is,

$$\mathbf{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial}{\partial \varphi}. \tag{2.27}$$

Obviously polar coordinates are better adjusted to describing rotations than Cartesian coordinates are. This (trivial) statement leads to the question: are there always coordinates that are maximally adjusted to a given one-parameter group of transformations? The answer is yes, there always exist coordinates such that – for an arbitrary number N of coordinates x^i – the generator (2.18) takes the simple form

$$\mathbf{X} = \frac{\partial}{\partial s}. \tag{2.28}$$

We call (2.28) the *normal form* of the generator $\mathbf{X}$.

To prove the above assertion, we could refer to the theory of partial differential equations, which shows that the system of equations

$$\mathbf{X}s = b^i\frac{\partial s}{\partial x^i} = 1, \qquad \mathbf{X}x^{n'} = b^i\frac{\partial x^{n'}}{\partial x^i} = 0,$$
$$i = 1, \ldots, N, \quad n' = 2, \ldots, N, \tag{2.29}$$

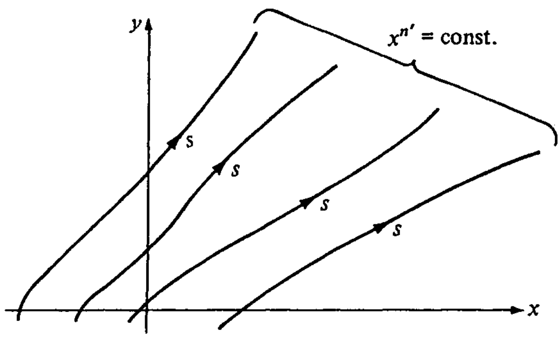

Figure 2.3. Transformation to coordinates $(s, x^{n'})$ for which $\mathbf{X} = \partial/\partial s$.

always has a nontrivial solution $\{s(x^i), x^{n'}(x^i)\}$. But it is more instructive to refer to Figure 2.3 and simply to state that if in an N-dimensional space (coordinates x^i) we have a congruence of one-dimensional group orbits covering the space smoothly and completely, then we can take these curves as coordinate lines $x^{n'} = \text{const}$ and take s as a parameter along these curves. In these coordinates the only non-zero component of the tangent vector $\mathbf{X}$ is in the s-direction, and by a suitable choice of the parameter s (different on each orbit!) we can arrange that this s-component equals unity.

To find the explicit transformation that brings a given generator into its normal form $\mathbf{X} = \partial/\partial s$ can be a difficult task. But in most applications that occur in the context of differential equations we shall be able to perform it. Comparing Figures 2.1 and 2.3, we see that it obviously amounts to finding the finite transformation and then converting this to obtain $\varepsilon = s(x, y)$, see Exercise 2.2.

2.3 Extensions of transformations and their generators

If we want to apply a point transformation (2.1) or (2.2) to a differential equation

$$H(x, y, y', y'', \dots, y^{(n)}) = 0, \qquad y' \equiv \mathrm{d}y/\mathrm{d}x, \quad \text{etc.,} \tag{2.30}$$

we have to know how to transform the derivatives $y^{(n)}$, that is, how to extend (or to prolong) the point transformation to the derivatives. Of course, this is trivially done by defining

$$\tilde{y}' = \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{x}} = \frac{\mathrm{d}\tilde{y}(x, y; \varepsilon)}{\mathrm{d}\tilde{x}(x, y; \varepsilon)} = \frac{y'(\partial\tilde{y}/\partial y) + (\partial\tilde{y}/\partial x)}{y'(\partial\tilde{x}/\partial y) + (\partial\tilde{x}/\partial x)} = \tilde{y}'(x, y, y'; \varepsilon),$$

$$\tilde{y}'' = \frac{\mathrm{d}\tilde{y}'}{\mathrm{d}\tilde{x}} = \tilde{y}''(x, y, y', y''; \varepsilon), \quad \text{etc.;} \tag{2.31}$$