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An Overview of Infinite Ergodic Theory

Jon Aaronson
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, 69978 Tel Aviv, Israel

Abstract. We review the basic ergodic theory of non-singular transformations placing special emphasis on those transformations admitting σ-finite, infinite invariant measures. The topics to be discussed include invariant measures, recurrence, ergodic theorems, pointwise dual ergodicity, distributional limits, structure and intrinsic normalizing constants.

0 Introduction

Infinite ergodic theory is the study of measure preserving transformations of infinite measure spaces. It is part of the more general study of non-singular transformations (since a measure preserving transformation is also a non-singular transformation).

This paper is an attempt at an introductory overview of the subject, and is necessarily incomplete. More information on most topics discussed here can be found in [1]. Other references are also given in the text.

Before discussing the special properties of infinite measure preserving transformations, we need to review some basic non-singular ergodic theory first.

Let $(X, \mathcal{B}, m)$ be a standard σ-finite measure space. A *non-singular transformation* of X is only defined modulo nullsets, and is a map $T : X_0 \to X_0$ (where $X_0 \subset X$ has full measure), which is measurable and has the *non-singularity property* that for $A \in \mathcal{B}$, $m(T^{-1}A) = 0$ if and only if $m(A) = 0$. A *measure preserving transformation* of X is a non-singular transformation T with the additional property that $m(T^{-1}A) = m(A) \ \forall \ A \in \mathcal{B}$.

If T is a non-singular transformation of a σ-finite measure space $(X, \mathcal{B}, m)$, and p is another measure on $(X, \mathcal{B})$ *equivalent* to m (denoted $p \sim m$ and meaning that p and m have the same nullsets), then T is a non-singular transformation of $(X, \mathcal{B}, p)$.

Thus, a non-singular transformation of a σ-finite measure space is actually a non-singular transformation of a probability space.

Considering a non-singular transformation $(X, \mathcal{B}, m, T)$ of a probability space as a *dynamical system* (see [23], [29]), the measure space X represents the set of *"configurations"* of the system, and T represents the change under *"passage of time"*. The non-singularity of T reflects the assumed property of the system that configuration sets that are impossible sometimes are always impossible. A probability preserving transformation would describe a system in a "steady state", where configuration sets occur with the same likelihood at all times.

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1 Invariant Measures

Given a particular non-singular transformation, one of the first tasks is to ascertain whether it could have been obtained by starting with a measure preserving transformation, and then "passing" to some equivalent measure.

If $T : X \to X$ is non-singular then $f \to f \circ T$ defines a linear isometry of $L^\infty(m)$. There is a predual called the *Frobenius-Perron* or *transfer* operator, $\widehat{T} : L^1(m) \to L^1(m)$, which is defined by

$$f \mapsto \nu_f(\cdot) = \int_\cdot f dm \mapsto \widehat{T} f = \frac{d\nu_f \circ T^{-1}}{dm}$$

and satisfies

$$\int_X \widehat{T} f.g dm = \int_X f.g \circ T dm \quad f \in L^1(m),\ g \in L^\infty(m).$$

Note that the domain of definition of $\widehat{T}$ can be extended to all non-negative measurable functions. This definition can be made when m is infinite, but σ-finite.

Evidently the density $h \geq 0$ of an absolutely continuous invariant measure μ satisfies $\widehat{T} h = h$, since for any $g \geq 0$ measurable,

$$\int_X \widehat{T} h g dm = \int_X h g \circ T dm = \int_X g \circ T d\mu = \int_X g d\mu = \int_X h g dm.$$

Clearly, if T is invertible, then

$$\widehat{T} f = \frac{dm \circ T^{-1}}{dm} f \circ T^{-1},$$

and if the non-singular transformation T of X is *locally invertible* in the sense that there are disjoint measurable sets $\{A_j : j \in J\}$ (J finite or countable) such that $m(X \setminus \bigcup_{j\in J} A_j) = 0$, and T is invertible on each A_j, then

$$\widehat{T} f = \sum_{j \in J} 1_{TA_j} \frac{dm \circ v_j}{dm} f \circ v_j$$

where $v_j : TA_j \to A_j$ is measurable and satisfies $T \circ v_j \equiv \mathrm{Id}$.

Boole's transformations I

For some locally invertible non-singular transformations T and measurable functions f, $\widehat{T} f$ can be computed explicitly. For example, consider the transformations $T : \mathbb{R} \to \mathbb{R}$ defined by

$$Tx = \alpha x + \beta + \sum_{k=1}^{n} \frac{p_k}{t_k - x} \tag{1.1}$$

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where $n \geq 1$, α, $p_1, \ldots, p_n \geq 0$ (not all zero) and $\beta, t_1, \ldots, t_n \in \mathbb{R}$.

These transformations (called *Boole's transformations*) were considered by G. Boole in [15]. They are non-singular transformations of $\mathbb{R}$ equipped with Lebesgue measure $m_{\mathbb{R}}$, and for $h : \mathbb{R} \to \mathbb{R}$ a non-negative measurable function,

$$\widehat{T}h(x) = \sum_{y \in \mathbb{R},\ Ty=x} \frac{h(y)}{T'(y)}.$$

Note that the $1-1$ Boole's transformations are the real Möbius transformations.

1.1 Boole's Formula [15] *For T as above, $x \in \mathbb{R}$ and $\omega \in \mathbb{C}$, $T\omega \neq x$,*

$$\sum_{y \in \mathbb{R},\ Ty=x} \frac{1}{(y-\omega)T'(y)} = \frac{1}{x - T(\omega)}.$$

If $\omega \in \mathbb{R}^{2+}$, the upper half plane, and $\omega = a + ib$, $a, b \in \mathbb{R}$, $b > 0$ then

$$\operatorname{Im} \frac{1}{x-\omega} = \frac{b}{(x-a)^2 + b^2} = \pi \varphi_\omega(x)$$

where φ_ω is the well known *Cauchy density* and Boole's formula has the immediate corollary:

$$\widehat{T}\varphi_\omega = \varphi_{T(\omega)}, \text{ or } P_\omega \circ T^{-1} = P_{T\omega} \tag{1.2}$$

where $dP_\omega := \varphi_\omega dm_{\mathbb{R}}$.

The original proof of Boole's formula in [15] uses the

1.2 Proposition [15] *Suppose that $F : \mathbb{C} \to \overline{\mathbb{C}}$ is rational, and $E : \mathbb{C} \to \mathbb{C}$ is a polynomial. Then*

$$\sum_{x \in \mathbb{C}:\ E(x)=0} F(x) = - \sum_{a\ a \text{ pole of } F} \operatorname{Res}(F(\log E)'; a) + \operatorname{Res}(F(\log E)'; 0).$$

Modern proofs of (1.2) use the fact that Boole's transformations are $\mathbb{R}$-restrictions of analytic endomorphisms of $\mathbb{R}^{2+}$ (*inner functions*). Many of the results given here for Boole's transformations remain valid for arbitrary inner functions ([4], [38]).

Bounded analytic functions on $\mathbb{R}^{2+}$ are Cauchy integrals of their restrictions to $\mathbb{R}$ and writing $dP_\omega := \varphi_\omega dm_{\mathbb{R}}$, one sees ([36]) that for $t > 0$,

$$\widehat{P_\omega \circ T^{-1}}(t) = \int_{\mathbb{R}} e^{itT(x)} dP_\omega(x) = e^{itT(\omega)} = \widehat{P_{T(\omega)}}(t),$$

whence (1.2).

As a consequence of (1.2), we see that the Boole transformation T has an absolutely continuous invariant probability if $\exists\ \omega \in \mathbb{R}^{2+}$ with $T(\omega) = \omega$ (in which case P_ω is T-invariant). It turns out that this is the only way it can admit an absolutely continuous invariant probability ([38]).

If $\alpha > 0$ in (1.1), then ([36]) using the fact that $\pi b \varphi_{a+ib} \to 1$ as $b \to \infty$, $\frac{a}{b} \to 0$ we see that $m_{\mathbb{R}} \circ T^{-1} = \alpha^{-1} m_{\mathbb{R}}$, whence T preserves $m_{\mathbb{R}}$ if $\alpha = 1$ in (1.1). By considering other analogous limits, one also sees that:

$Tx = \tan x$ preserves the measure $d\mu_0(x) := \frac{dx}{x^2}$ (see [4]);

and

if $f(\omega) = \int_{\mathbb{R}} \frac{d\nu(t)}{t-\omega}$ (the complex Hilbert transform of ν) where $\nu \perp m_{\mathbb{R}}$, and $T(x) := \lim_{y \downarrow 0} f(x+iy)$ a.e. (the Hilbert transform of ν), then $m_{\mathbb{R}} \circ T^{-1} = \nu(\mathbb{R})\mu_0$ (see [22]).

CONDITIONS FOR EXISTENCE OF INVARIANT MEASURES

Unfortunately, one cannot expect always to be able to identify absolutely continuous invariant measures by explicit computation. To help remedy this situation, there are many conditions for existence of such (see [35]). Some conditions for existence of absolutely continuous invariant probabilities depend on the following:

1.3 Proposition *Let T be a non-singular transformation of $(X, \mathcal{B}, m)$. If $\exists\ f \in L^1(m)$, $f \geq 0$, $\int_X f dm > 0$ such that $\{\frac{1}{n}\sum_{k=1}^n \hat{T}^k f : n \geq 1\}$ is a uniformly integrable family, then $\exists$ a T-invariant probability $P << m$.*

The invariant probability's density $h \in L^1(m)$, $h \geq 0$ is found as a weak limit point of $\{\frac{1}{n}\sum_{k=1}^n \hat{T}^k f : n \geq 1\}$ which is weakly sequentially precompact in $L^1(m)$ owing to the assumed uniform integrability.

Expanding interval maps I

Let $I = [0,1]$, m_I be Lebesgue measure on I, and α be a collection of disjoint open subintervals of I such that

$$m(I \setminus U_\alpha) = 0 \text{ where } U_\alpha = \bigcup_{a \in \alpha} a.$$

A piecewise onto, C^2 *interval map* with *basic partition* α is a map $T : I \to I$ is such that

$$\text{For each } a \in \alpha,\ T|_a \text{ extends to a } C^2 \text{ diffeomorphism } T : \overline{a} \to I. \tag{1.3}$$

Note that if $a_1, ..., a_n \in \alpha$ then $a = \bigcap_{k=1}^n T^{-(k-1)} a_k$ is an open interval, and $T^n : \overline{a} \to I$ is a C^2 diffeomorphism. Hence, if T is a piecewise onto, C^2 interval map with basic partition α then, for $n \geq 1$, T^n is an interval map with basic partition

$$\alpha_0^{n-1} := \bigvee_{k=0}^{n-1} T^{-k}\alpha = \{\bigcap_{k=1}^{n} T^{-(k-1)}a_k : \ a_1, ..., a_n \in \alpha\}.$$

The piecewise onto, C^2 interval map T is called *expanding* if

$$\exists \lambda > 1 \ \ni \ |T'x| \geq \lambda \ \forall \ x \in I. \tag{1.4}$$

The following condition limits the multiplicative variation (or *distortion*) of v_a' $(a \in \alpha)$. It is known as *Adler's condition* or *bounded distortion*. It follows from (1.3) and (1.4) in case α is finite.

$$\exists M > 1 \ \ni \ \frac{|T''x|}{|T'x|^2} \leq M \ \forall \ x \in I.$$

Given a piecewise onto, C^2 interval map T with basic partition α, and $a \in \alpha_0^{n-1}$; denote by v_a the C^2 diffeomorphism $v_a : I \to \bar{a}$ satisfying $T^n \circ v_a(x) = x$.

The basic result concerning piecewise onto, C^2 expanding interval maps satisfying Adler's condition is that Adler's condition holds uniformly for all powers of T ([12]):

$$\frac{|T^{n\prime\prime}x|}{|T^{n\prime}x|^2} \leq K = \frac{\lambda M}{\lambda - 1} \ \forall \ x \in I, \ n \geq 1,$$

whence Renyi's distortion property (see [41]):

$$|v_a'(x)| = e^{\pm K} m_I(a) \ \forall \ x \in I, \ \ a \in \bigcup_{n=1}^{\infty} \alpha_0^{n-1}. \tag{1.5}$$

As a consequence of Renyi's distortion property, we have that

$$\widehat{T}^n 1 = \sum_{a \in \alpha_0^{n-1}} |v_a'| = e^{\pm K},$$

and so by proposition 1.3 (a uniformly bounded sequence being uniformly integrable) $\exists$ an absolutely continuous, invariant probability with density h satisfying $h = \widehat{T}h = e^{\pm K}$.

It is shown in [27] that h is Lipschitz continuous. To see this using [21] note that for f Lipschitz continuous (hence differentiable a.e.),

$$(\widehat{T}^n f)' = \sum_{a \in \alpha_0^{n-1}} v_a'' f \circ v_a + \sum_{a \in \alpha_0^{n-1}} v_a'^2 f' \circ v_a$$

whence

$$\|(\widehat{T}^n f)'\|_\infty \le \frac{e^K}{\lambda^n}\|f'\|_\infty + e^{2K}\|f\|_\infty$$

and it follows from [21] that $\exists\ h_0$ Lipschitz continuous and $M > 0,\ r \in (0,1)$ such that $\widehat{T}h_0 = h_0$ and $\|\widehat{T}^n f - h_0 \int_I f dm\|_L \le Mr^n\|f\|_L \quad \forall\ f$ Lipschitz continuous where $\|f\|_L := \|f\|_\infty + \|f'\|_\infty$. In particular, $h = h_0$ mod m.

The assumption that the interval map is expanding is not crucial. Gauss' continued fraction map $Tx = \{\frac{1}{x}\}$ is not expanding, but $(T^2)' \ge 4$ and the above applies.

However the piecewise onto, C^2 interval map $Tx = \{\frac{1}{1-x}\}$ satisfies Adler's condition, and no power is expanding, as $T(0) = 0,\ T'(0) = 1$. In fact T admits no absolutely continuous, invariant probability, the infinite measure $d\nu(x) := \frac{dx}{x}$ being T-invariant.

Conditions for the existence of absolutely continuous, infinite, invariant measures depend on recurrence properties.

2 Recurrence and Conservativity

There are non-singular transformations T of $(X, \mathcal{B}, m)$ which are *recurrent* in the sense that

$$\liminf_{n\to\infty} |h \circ T^n - h| = 0 \text{ a.e. } \forall\ h : X \to \mathbb{R} \text{ measurable.}$$

One extreme form of non-recurrent (or *transient*) behaviour is exhibited by *wandering sets*.

Let T be a non-singular transformation of the standard measure space $(X, \mathcal{B}, m)$.

A set $W \subset X$ is called a *wandering* set (for T) if the sets $\{T^{-n}W\}_{n=0}^\infty$ are disjoint. Let $\mathcal{W} = \mathcal{W}(T)$ denote the collection of measurable wandering sets.

Evidently, the collection of measurable wandering sets is a hereditary collection (any subset of a wandering set is also wandering), and T-invariant (W a wandering set $\implies$ $T^{-1}W$ a wandering set).

Using a standard exhaustion argument it can be shown that $\exists$ a countable union of wandering sets $\mathfrak{D}(T) \in \mathcal{B}$ with the property that any wandering set $W \in \mathcal{B}$ is contained in $\mathfrak{D}(T)$ mod m (i.e. $m(W \setminus \mathfrak{D}(T)) = 0$). Evidently $\mathfrak{D}(T)$ is unique mod m and $T^{-1}\mathfrak{D} \subseteq \mathfrak{D}$ mod m. It is called the *dissipative part* of the non-singular transformation T. In case T is invertible, it can be shown that $\exists$ a wandering set $W \in \mathcal{B}$ such that $\mathfrak{D}(T) = \bigcup_{n\in\mathbb{Z}} T^n W$, whence $T^{-1}\mathfrak{D} = \mathfrak{D}$.

The *conservative part* of T is defined to be $\mathfrak{C}(T) := X \setminus \mathfrak{D}(T)$ and the partition $\{\mathfrak{C}(T),\ \mathfrak{D}(T)\}$ is called the *Hopf decomposition* of T.

The non-singular transformation T is called *conservative* if $\mathfrak{C}(T) = X$ mod m, and (totally) *dissipative* if $\mathfrak{D}(T) = X$ mod m.

Using

Halmos' recurrence theorem [26] *If T is conservative, then*

$$\sum_{n=1}^{\infty} 1_B \circ T^n = \infty \text{ a.e. on } B, \quad \forall B \in \mathcal{B}.$$

one can show that a non-singular transformation is recurrent if and only if it is conservative.

CONDITIONS FOR CONSERVATIVITY

If there exists a finite, T-invariant measure $q << m$, then clearly there can be no wandering sets with positive q-measure, whence $q(\mathfrak{D}) = 0$ and $[\frac{dq}{dm} > 0] \subseteq \mathfrak{C}$ mod m. In particular ([40]) any probability preserving transformation is conservative.

A measure preserving transformation of a σ-finite, infinite measure space is not necessarily conservative. For example $x \mapsto x+1$ is a measure preserving transformation of $\mathbb{R}$ equipped with Borel sets, and Lebesgue measure, which is totally dissipative.

2.1 Proposition

1) If $T : X \to X$ is non-singular, then

$$\mathfrak{C}(T) = [\sum_{n=1}^{\infty} \widehat{T}^k f = \infty] \quad \text{mod } m, \ \forall f \in L^1(m), f > 0.$$

2) If $T : X \to X$ is a measure preserving transformation, then $T^{-1}\mathfrak{C}(T) = \mathfrak{C}(T)$ mod m. Indeed

$$\mathfrak{C}(T) = \left[\sum_{n=1}^{\infty} f \circ T^n = \infty\right] \quad \text{mod } m, \ \ \forall f \in L^1(m), f > 0 \text{ a.e.}.$$

Boole's transformations II

If

$$Tx = \alpha x + \beta + \sum_{k=1}^{n} \frac{p_k}{t_k - x}$$

where $n \geq 1$, $\alpha > 0$, $p_1, \ldots, p_n \geq 0$ and β, $t_1, \ldots, t_n \in \mathbb{R}$ then (as deduced from Boole's formula) $m_{\mathbb{R}} \circ T^{-1} = \alpha^{-1} m_{\mathbb{R}}$.

When $\alpha > 1$,

$$\sum_{n=1}^{\infty} \widehat{T}^n f < \infty \text{ a.e. } \forall \ f \in L^1(m) \cap L^\infty(m), \ f > 0 \text{ a.e.},$$

whence by proposition 2.1(1), T is totally dissipative.

Now let

$$Tx = x + \beta + \sum_{k=1}^{n} \frac{p_k}{t_k - x} \tag{2.1}$$

where $n \geq 1$, $p_1, \ldots, p_n \geq 0$ (not all zero) and $\beta, t_1, \ldots, t_n \in \mathbb{R}$. As above, $m_{\mathbb{R}} \circ T^{-1} = m_{\mathbb{R}}$.

To see when T is conservative, we use proposition 2.1. For T defined by (2.1) and $\omega \in \mathbb{R}^{2+}$, $T^n(\omega) \to \infty$. Write $T^n(\omega) := u_n + iv_n \to \infty$, whence $\pi\widehat{T}^n\varphi_\omega(x) = \text{Im}\,\frac{1}{x - T^n(\omega)} = \frac{v_n}{(x-u_n)^2 + v_n^2} \sim \frac{v_n}{u_n^2 + v_n^2}$ and T is conservative iff

$$\sum_{n=1}^{\infty} \frac{v_n}{u_n^2 + v_n^2} = \infty.$$

We see (as in [4], [5]) that when $\beta \neq 0$,

$$v_n \uparrow v_\infty < \infty, \quad u_n = \beta n - \frac{\nu}{\beta} \log n + O(1) \text{ as } n \to \infty; \tag{2.2}$$

and when $\beta = 0$,

$$\sup_{n \geq 1} |u_n| < \infty, \quad v_n \sim \sqrt{2\nu n} \text{ as } n \to \infty \tag{2.3}$$

where $\nu := \sum_{k=1}^{n} p_k$.

It follows that T is conservative when $\beta = 0$ ($\sum_{n=1}^{\infty} \frac{v_n}{u_n^2+v_n^2} = \infty$); and totally dissipative when $\beta \neq 0$ ($\sum_{n=1}^{\infty} \frac{v_n}{u_n^2+v_n^2} < \infty$).

INDUCED TRANSFORMATIONS, CONSERVATIVITY AND INVARIANT MEASURES

Suppose T is conservative and non–singular, and let $A \in \mathcal{B}_+$, then m–a.e. point of A returns infinitely often to A under iterations of T, and in particular the *return time* function to A, defined for $x \in A$ by $\varphi_A(x) := \min\{n \geq 1 : T^n x \in A\}$ is finite m–a.e. on A.

The *induced transformation* ([33]) on A is defined by $T_A x = T^{\varphi_A(x)} x$, and can be defined whenever the return time function is finite m–a.e. on A (whether T is conservative, or not).

The first key observation is that $m|_A \circ T_A^{-1} << m|_A$. This is because

$$T_A^{-1} B = \bigcup_{n=1}^{\infty} [\varphi = n] \cap T^{-n} B.$$

It follows that $\varphi_A \circ T_A$ is defined a.e. on A and an induction now shows that all powers $\{T_A^k\}_{k \in \mathbb{N}}$ are defined a.e. on A, and satisfy

$$T_A^k x = T^{(\varphi_A)_k(x)} x \text{ where } (\varphi_A)_1 = \varphi_A, \ (\varphi_A)_k = \sum_{j=0}^{k-1} \varphi_A \circ T_A^j.$$

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2.2 Proposition (c.f. [33]) *Let T be a non-singular transformation of $(X, \mathcal{B}, m)$, and suppose that $A \in \mathcal{B}$, $m(A) > 0$ satisfies $\varphi_A < \infty$ a.e. on A.*

1) If T is conservative, then the induced transformation T_A is a conservative, non-singular transformation of $(A, \mathcal{B} \cap A, m|_A)$.

2) If T is a measure preserving transformation, then T_A is a measure preserving transformation of $(A, \mathcal{B} \cap A, m|_A)$, and in case $\bigcup_{n=0}^{\infty} T^{-n}A = X$ mod m, T_A a conservative iff T is conservative.

2.3 Proposition (c.f. [32]) *Let T be a non-singular transformation of $(X, \mathcal{B}, m)$, and suppose that the return time function to $A \in \mathcal{B}$, $m(A) > 0$ is finite m-a.e. on A.*

Let $A \in \mathcal{B}_+$, and suppose that $q << m|_A$ is a T_A-invariant measure. Set, for $B \in \mathcal{B}$,

$$\mu(B) = \sum_{k=0}^{\infty} q(A \cap T^{-k}B \setminus \bigcup_{j=1}^{k} T^{-j}A).$$

Then $\mu << m$ is a T-invariant measure.

Non-expanding interval maps I

Let T be a piecewise onto, C^2 interval map with basic partition $\alpha = \{(0, u)\} \cup \alpha_0$ satisfying Adler's condition. Suppose that

$$Tx = x + cx^{1+p} + o(x^{1+p}) \quad \text{as } x \to 0$$

where $c > 0$, $p \geq 1$, $T(u) = 1$, $T'' \geq 0$ on $[0, u]$ and $\exists\ \kappa > 1$ such that

$$T'x \geq \kappa \ \ \forall\ x \in \ a \in \alpha_0$$

(e.g. $Tx = \{\frac{1}{1-x}\}$ where $p = c = 1$).

Evidently the return time function to $[u, 1]$ is finite on $[u, 1]$, and $T_{[u,1]}$ is an expanding, piecewise onto, C^2 interval map of $[u, 1]$.

It turns out that $T_{[u,1]}$ also satisfies Adler's condition. We sketch a way to see this (the full proof is in [45]). Let $x \in [u, 1]$, then $T_{[u,1]}x = v_0^{-n} \circ (x)$ for some $n \geq 0$, where $v_0 : I \to [0, u]$, $T \circ v_0 = \mathrm{Id}$. It follows that

$$\frac{|T''_{[u,1]}x|}{(T'_{[u,1]}x)^2} \leq \frac{|v_0^{-n\prime\prime}(Tx)|}{v_0^{-n\prime}(Tx)^2} + 1.$$

Adler's condition for $T_{[u,1]}$ will now follow from

$$\sup_{y \in\ (u,1),\ n \geq 1} \frac{|v_0^{n\prime\prime}(y)|}{v_0^{n\prime}(y)} < \infty.$$

To show this, calculate first that

$$v_0^{n\prime\prime}(y) = v_0^{n\prime}(y) \sum_{k=0}^{n-1} \frac{v_0''(v_0^k y)}{v_0'(v_0^k y)} v_0^{k\prime}(y),$$