UNIVERSALITY AND RENORMALISATION IN DYNAMICAL SYSTEMS.

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CONTENTS

INTRODUCTION.

The renormalisation group formalism has lead to a number of fruitful developments in our understanding of the "transition to chaos". The best known examples concern the quantitative universality of period-doubling cascades and the breakdown of invariant circles in dissipative and area-preserving maps. This paper is meant to be an introduction to, and biased review of, these ideas.

On period-doubling, I just give a relatively brief review of the basic ideas for unimodal maps of the interval. I do not touch upon period-doubling in area-preserving maps because the theory is so similar. The interested reader is referred to Bountis (1981), Benettin, Cercignani, Galgani & Giorgilli (1980), Benettin, Galgani & Giorgilli (1980), Collet, Eckmann & Koch (1981a), Eckmann, Koch & Wittwer (1982), and Greene, MacKay, Vivaldi & Feigenbaum (1981).

After dealing with period-doubling, I discuss the theory of critical circle maps, especially those with golden-mean rotation number. This leads in turn to a theory for the breakup of invariant circles of dissipative maps with golden-mean rotation number. A natural extension is then to the case of area-preserving twist maps and in Section 4 the theory in Sections 2 and 3 is applied to study the breakdown of the last homotopically non-trivial invariant circle. There is an interesting related theory due to Manton and Nauenberg (1983) which deals with the universal small-scale structure found in the boundaries of Siegel domains of rational maps of the Riemann sphere. I shall not discuss it here because of lack of space and because the underlying ideas are similar. The interested reader is referred to Manton & Nauenberg (1983) and Widom (1983).

The results of the next section are not so well-known. In it I describe a general formalism of renormalisation strange sets which, for example, handles the case of general irrational rotation numbers. Predictions about the universal fractal structure of fractal bifurcation sets in parameter space come from this theory. I believe it will be of much more general use in physical problems. Some open problems are discussed in Section 6.

1. PERIOD-DOUBLING CASCADES.

This section contains some introductory remarks on maps of the interval, and a discussion of the Feigenbaum conjectures about period-doubling cascades in unimodal maps of the interval and the associated renormalisation group formalism. The explanation of how to use the renormalisation structure constructed to get similar results for diffeomorphisms and flows is similar to that given in Section 3.2 (see Collet & Eckmann (1980)).

1.1 Unimodal maps of the interval.

1.1.1 Unimodal maps of $I = [-1,1]$.

Definition. A map $f : I \to I$ is unimodal if (a) f is C^1, (b) $f(0) = 1$, (c) f is strictly increasing on $[-1,0]$ and strictly decreasing on $[0,1]$, and (d) f is even.

Remark 1.1. Condition (d) is not really necessary for what follows, but its assumption will make a number of computations considerably easier.

Important Example 1.1 Consider the one-parameter family of unimodal maps given by $f_\mu(x) = 1 - \mu x^2$, $0 < \mu \le 2$. Clearly, when $\mu > 0$ is small the fixed point is a global attractor. On the other hand, if $y = \phi(x) = (4/\pi)(\sin^{-1}\sqrt{(x+1)/2}) - 1$ then $F = \phi \circ f_2 \circ \phi^{-1} = 1 - 2|y|$. The Lebesgue measure dx is an invariant ergodic measure for F whence $dx/\pi\sqrt{(1-x^2)}$ is one for f_2.

$\|f\|_1 = \sup_{x \in I}(|f(x)| + |f'(x)|)$. This topology is called the C^1*–topology*.

Definition. (a) f is p-superstable if 0 is in a p–cycle of f. (b) f is p-filling if there exists p disjoint closed sub-intervals $I_1, \ldots, I_p$ such that f is a homeomorphism from I_j to I_{j+1} for $1 \le j < p$, $f(I_p) \subseteq I_1$ and $g = f^p|I_1$ is a unimodal map of I_1 such that $g(I_1) = I_1$ (i.e. with respect to I_1, g is 1-filling).

Remark 1.4. If f is 1-superstable, points near the orbit of 0 converge to it at a quadratic rate, hence the term *superstable*. The map $f_2(x) = 1 - 2x^2$ is 1-filling.

Definition. A C^1 *family* is a 1-parameter family f_μ in $C^1(I, I)$, $\mu \in (\alpha, \beta)$, which is continuous in the C^1-topology. It is *full* on (α, β) if $f_\mu(1) \to 1$ (resp. $\to -1$) as $\mu \to \alpha^+$ (resp. $\to \beta^-$). For example, $1 - \mu x^2$ is full on $(0, 2)$.

Theorem 1.1. If f_μ is a full C^1-family on (α, β) then there exists $\alpha < \alpha_1 < \alpha_2 < \cdots < \beta_2 < \beta_1 < \beta$ such that 1. f_{α_i} is 2^i-superstable, 2. f_{β_i} is 2^i-filling, and 3. $\lim_{i\to\infty} \alpha_i = \lim_{i\to\infty} \beta_i$.

Remark 1.5. Clearly, there must be a bifurcation point between α_i and α_{i+1} Typically, this will be a generic period-doubling bifurcation as described in the next subsection.

Proof of Theorem 1.1. Consider a full C^1 family on (α, β). Let $\alpha_1 = \sup\{\mu : f_\mu(1) = 0\} = \sup\{\mu : a(f_\mu) = 0\}$. Then there exists $\varepsilon > 0$ such that if $\mu \in (\alpha, \alpha + \varepsilon)$ then $f_\mu \in D(T)$. Let $\beta_1 = \inf\{\mu > \alpha : f_\mu \in D(T)\} < \beta$. Then $\alpha(f_{\beta_1}) \ne 0$ so $f^2_{\beta_1}(\alpha_{\beta_1}) = \alpha_{\beta_1}$. It easily follows that $\mu \to Tf_\mu$ is a full family on (α_1, β_1). Repeating this construction one deduces the existence of sequences $\alpha < \alpha_1 < \alpha_2 < \cdots < \beta_2 < \beta_1 < \beta$ such that for $\alpha_i < \mu < \beta_i$, $f_\mu \in D(T^i)$ and $\mu \to T^i f_\mu$ is full on (α_i, β_i). These have been constructed so that $T^{j-1} f_{\alpha_j}(1) = 0$. Therefore $T^{j-1} f_{\alpha_j}$ is 2-superstable, i.e. f_{α_j} is 2^j-superstable. Moreover, $T^j f_{\beta_j}(1) = -1$ so $T^j f_{\beta_j}$ is 1-filling whence f_{β_j} is 2^j-filling. A simple calculation shows that α_i and β_i satisfy 3. ■

Since $T^{j-1} f_{\alpha_j}$ is 2-superstable, $T^{j-1} f_{\alpha_j+\varepsilon}$ has a 2-sink for $|\varepsilon|$ sufficiently small. Let $\mu_j = \sup\{\mu \in (\alpha_{j-1}, \alpha_j) \mid T^{j-1} f_\mu$ has a 2–sink $\}$. Then μ_j is the bifurcation value associated with the bifurcation 2^j-sink $\to 2^{j+1}$-sink, though this is not necessarily a generic bifurcation.

1.2 The period-doubling bifurcation.

Proposition 1.1. If $f : I \times [-1, 1] \to I$ is a family of maps, not necessarily unimodal and $f_\mu(x) = f(x, \mu)$ is such that: $f_0(p) = p$; $f'_0(p) = -1$; $(f_0^2)'''(p) < 0$; $\partial^2 f/\partial\mu\partial x|_{x=p, \mu=0} < 0$; then there exist intervals $(\nu_1, 0)$ and $(0, \nu_2)$ and $\varepsilon > 0$ such that: (a) if $\mu \in (\nu_1, 0)$ then f_μ^2 has exactly one fixed point in $(p-\varepsilon, p+\varepsilon)$, and this is a sink; (b) if $\mu \in (0, \nu_2)$ then f_μ^2 has three fixed points in $(p-\varepsilon, p+\varepsilon)$, and the largest and smallest are sinks, the middle a source.

Remark 1.6. To deal with $2^n \to 2^{n+1}$ bifurcation, replace f by f^{2^n}.

To get a better picture of how f_μ bifurcates to get from the simple picture for small μ to the complex one for $\mu = 2$ let us concentrate on those aspects brought out by the doubling transformation. For other results the reader should consult Guckenheimer (1977) and (1979) and Jonker & Rand (1981).

1.1.2 Doubling transformation.

Throughout this section f denotes a unimodal map. Let $a = a(f) = -f(1), b = b(f) = f(a)$. Let $D(T)$ denote the set of f's such that (i) $a > 0$ (ii) $b > a$, and (iii) $f(b) \le a$. For $f \in D(T)$ define the *doubling transformation* T by

$$Tf(x) = -a.f^2(-ax).$$

Remark 1.2. Figure 1 shows the graphs of f and f^2 when $f = 1 - 1.52x^2 + 0.104x^4$. To get Tf one just scales the coordinate using $x_{new} = -ax_{old}$ to scale the small box to have length one and turn the graph upside down. Note that f^2 will look as shown if $f \in D(T)$ because then

$$[-a, a] \to [b, 1] \to [-a = f(1), f(b)] \subseteq [-a, a].$$

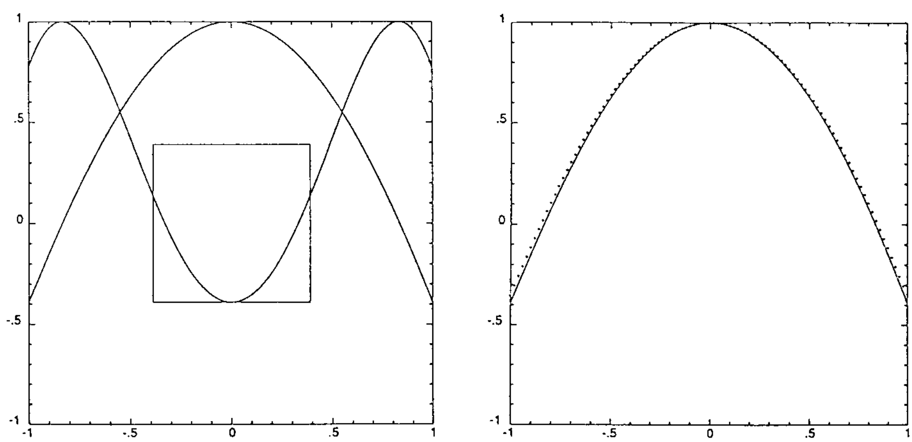

Figure 1. (a) the graphs of f and f^2 where $f = 1 - 1.52x^2 + .104x^4$; the box is $-a \le x,y \le a$; (b) points from the graph of Tf superimposed on the graph of f.

Remark 1.3. The boundary of $D(T)$ consists of the surfaces $a = 0$ and $f^2(a) = a$ because, in moving from $D(T)$ to a point where (ii) fails, $b = a$ is encountered i.e. $f^2(a) = a$. Condition (iii) can fail when (ii) is true.

1.1.3 2^n-superstable and 2^n-filling functions.

To get some feeling for the doubling transformation I discuss here an elementary application which is taken from Collet, Eckmann and Lanford (1980). Let $C^1(I, I)$ denote the space of C^1 maps of I to I endowed with the topology defined by the C^1-norm :

Proof. The proof is a simple calculation using the implicit function theorem. It can be found in Guckenheimer (1977) or Whitley (1983) for example. ■

1.3 Feigenbaum conjectures.

1.3.1 Motivation.

I give a heuristic introduction to the Feigenbaum conjectures. Consider the 1-parameter family $f_\mu = 1 - \mu x^2$, $0 < \mu \le 2$, discussed in the previous section. Recall the meaning of the parameter values α_i, β_i of Theorem 1.1. Using a pocket calculator one finds

$$\lim_{n\to\infty} (\alpha_n - \alpha_{n-1})/(\alpha_{n+1} - \alpha_n) = \lim_{n\to\infty} (\beta_n - \beta_{n-1})/(\beta_{n+1} - \beta_n) = \delta = 4.669...$$

and using something a bit more powerful, it appears that there exists $\lambda(\sim -.3995...)$ such that if $\alpha_\infty = \lim_{n\to\infty}\alpha_n$,

$$\zeta = \lim_{n\to\infty} \lambda^{-n} f^{2^n}_{\alpha_\infty} \lambda^n$$

exists and is an analytic function of x^2. Moreover, if one takes any other 1-parameter family close to this one (and, in practice, many not so close) one gets the same experimental values for δ and λ and, (up to a scale change), the same function ζ. This is an example of *universality*.

1.3.2 Feigenbaum Conjectures.

The explanation (essentially proposed by Feigenbaum (1978, 1979) and independently by Coullet and Tresser (1979)) goes as follows: Consider the doubling transformation defined on some suitable subspace of $D(T)$ consisting of analytic functions. Assume that the following facts are true.

Conjecture 1. T has a fixed point f_* with the property that $f_*'(0) = 0$ and $f_*''(0) \neq 0$.

Conjecture 2. The only element of the spectrum of $dT(f_*)$ outside the disk $|z| < 1$ is a simple eigenvalue $\delta = 4.669...$. The rest of the spectrum is contained inside a disk of radius strictly less than 1.

Conjecture 3. The unstable manifold of f_* intersects and is transverse to the submanifolds Σ_n and Λ_n of bifurcation and superstable maps defined as

$$\Sigma_n = \{ f : \text{for some } p \text{ in a } 2^n\text{–cycle of } f,\ (f^{2^n})'(p) = -1 \text{ and } (f^{2^n})'''(p) \neq 0 \}$$

$$\Lambda_n = \{ f : f^{2^n}(0) = 0 \text{ and } f^m(0) \neq 0 \text{ for } 0 < m < 2^n \}.$$

These imply :

1. Since $Tf_* = f_*$,

$$f_*^2(\lambda x) = \lambda f_*(x) \tag{1.1}$$

where $\lambda = -a(f_*) = f_*(1)$. Equation (1.1) is sometimes known as the Cvitanovic-Feigenbaum functional equation.

2. $T^n f \to f_*$ as $n \to \infty$ implies there exists $\beta > 0$ such that

$$\lambda^{-n} f^{2^n}(\lambda^n x) \to \beta^{-1}(f_*(\beta x))$$

uniformly in x as $n \to \infty$.

3. Conjecture 2 implies that, with respect to T, f_* is a saddle point with a 1-

dimensional unstable manifold W^u and a stable manifold W^s of codimension one (see Figure 2); W^u defines a universal 1-parameter family of maps $f_{*,\mu}$. For a 1-parameter family f_μ near f_* with $f_0 \in W^s$ one has

$$\lambda^{-n} f^{2^n}_{\mu\delta^{-n}} \circ \lambda^n \to \beta^{-1} f_{*,\mu} \circ \beta$$

for some $\beta > 0$.

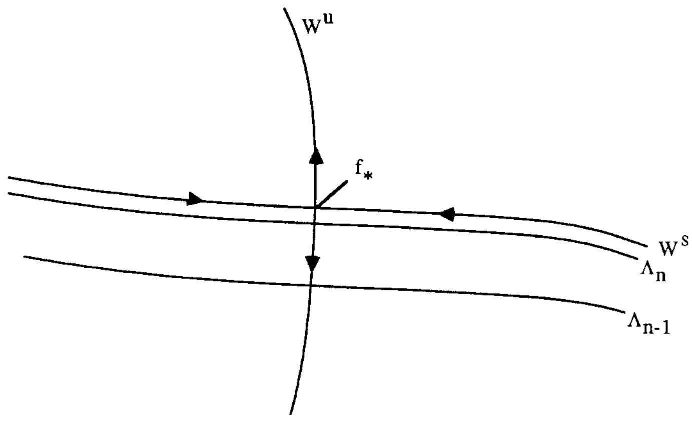

Figure 2. The saddle-point structure of the fixed point and the accumulation of the sets Λ_n.

4. Obviously, $T(\Sigma_n) \subsetneq \Sigma_{n-1}$ and $T(\Lambda_n) \subseteq \Lambda_{n-1}$. Thus, Conjecture 3 implies that the Σ_n and Λ_n accumulate on W^s exponentially fast with the distances from W^s decreasing like δ^{-n} (to prove this one one uses the fact that T can be linearised along the unstable direction). If f_μ is a 1-parameter family near $f_{*,\mu}$ and transverse to W^s with say $f_0 \in W^s$ then $f_{\alpha_i} \in \Lambda_i$ and therefore

$$\lim_{i \to \infty} (\alpha_i - \alpha_\infty)/(\alpha_{i+1} - \alpha_\infty) = \delta,$$

if $\alpha_\infty = \lim_{n \to \infty} \alpha_n$. In the same fashion, if μ_n denotes the parameter value at which a $2^n \to 2^{n+1}$ period-doubling occurs then

$$\lim_{i \to \infty} (\mu_i - \mu_\infty)/(\mu_{i+1} - \mu_\infty) = \delta$$

where $\mu_\infty = \lim_{n \to \infty} \mu_n = \alpha_\infty$.

This explains where the simpler universal quantities δ , λ $(= -a(f_*))$ and ζ $(= f_*$ scaled) come from. This sort of a set up is known to physicists as a renormalisation group: a notion which in its most powerful form was invented by Wilson to study problems such as phase transitions in statistical mechanics and quantum field theory (Wilson (1971, 1975)). For this he received the 1983 Physics Nobel Prize.

Lanford has given a proof of Conjectures 1 and 2. His proof makes essential use of rigorous computer-generated estimates.[1] The idea is to cleverly approximate the doubling operator and some related transformations by their action on certain polynomial functions, using the computer to do the more tedious calculations which because of their length are not really accessible to pen and paper, and to keep a rigorous check on the errors. Campanino, Epstein and Ruelle (1981, 1982) previously gave a proof of Conjecture 1 alone which only used very simple numerical estimates and was basically analytical. Very recently, Epstein and Eckmann have proved a result which is essentially, but not quite, Conjecture 1 using the fact that the inverse of (one half of) the fixed point function is a Herglotz function (Epstein (1986), Epstein & Eckmann (1986, 1987)). An attractive aspect of their approach is that it also applies to the related conjecture about golden circle maps (see Section 3) and brings out the similarities between the two problems. However, I should stress at this point that none of these proofs casts any light on the ubiquity of such renormalisation schemes. Why do such low-codimensional renormalisation structures occur so often and in such widely differing situations as 1-dimensional maps and area-preserving twist maps? There must be some general principle underlying this phenomenon which the right proof would reveal and an approach that would unify the seemingly disparate applications. To find these is the main open problem in this area.

An important recent development is a proof by Sullivan that the stable manifold contains all analytic unimodal maps which are topologically conjugate to the fixed point and whose extension to the complex plane is a *quadratic-like map* in the sense of Douady and Hubbard (1985). In particular, this implies that there is a unique quadratic-like fixed point. The proof which is outlined in Sullivan (1987), depends upon the Douady-Hubbard theory for such maps. This theory also gives an important insight into the universality of δ. According to it every quadratic-like map can be quasi-conformally conjugated to a uniquely determined member of the family $f_\mu(x) = 1 - \mu x^2$. In particular, if f_μ is in $D(T)$ then $T(f_\mu)$ can be quasi-conformally conjugated to $f_{h(\mu)}$ for some unique $h(\mu)$. The function h has derivative δ at μ_∞, the parameter value at which the period doublings accumulate.

1.4 The Feigenbaum attractor, the scaling spectrum and Feigenbaum's scaling function.

Firstly, I discuss Feigenbaum's scaling function σ. The definition of σ is from Feigenbaum (1982). Consider a flow $f^t = f^t_\mu$ depending upon a parameter μ. Assume that as μ is increased the system undergoes a cascade of period-doubling bifurcations of the form discussed above with the nth of these occurring at $\mu = \mu_n$ and the μ_n accumulating on μ_∞. Let $f = f_\mu$ be a local Poincaré map for the system obtained as the return map to some 1-codimensional submanifold which is transverse to all the relevant periodic orbits. Let $x(t) = f^t(x)$, $t \in \mathbf{Z}$, denote a solution starting on the appropriate periodic attractor. Then $T_n = 2^{n-1}$ is the period just before the nth period doubling. Thus if $\mu < \mu_n$ is close to μ_n, $x(t + T_n) - x(t) = 0$. Let

$$\psi^{(n)}(t) = x(t) - x(t+T_{n-1}).$$

Then, at least for large n,

$$\psi^{(n+1)}(t) \sim \sigma(t/T_{n+1})\psi^{(n)}(t)$$

where $\sigma(x+1) = \sigma(x)$ and $\sigma(x+\frac{1}{2}) = -\sigma(x)$. Also, defining $r_n(t) = \psi^{(n)}(t)/\psi^{(n-1)}(t)$,

[1] **Note added in proof.** A survey of the ideas behind such computer-assisted proofs is contained in Lanford (1987).

$$r_{n+1}(2t) = \frac{\psi^{(n+1)}(2t)}{\psi^{(n)}(2t)} \sim \sigma(2t/T_{n+1}) = \sigma(t/T_n) \sim \frac{\psi^{(n)}(t)}{\psi^{(n-1)}(t)} = r_n(t)$$

which looks as in Figure 3 (at least for large n ; the picture is for $n = 4,5$ in the case of Duffing's equation and is taken from Feigenbaum (1982)). The universal structure of the power spectrum associated with $x(t)$ can be deduced from a knowledge of σ but this is really secondary; as Feigenbaum has pointed out, the sensible thing to look at is σ not the spectrum.

To see in what sense σ is universal I firstly consider the nature of the attractor of the quadratic fixed point $g = f_* \sim 1 - 1.527x^2 + 0.1048x^4 + 0.0267x^6 + \cdots$ of the doubling transformation and then relate it to the *cookie-cutter* F given by (1.2) below.

Let x_0 denote the critical point of g and $x_n = g^n(x_0)$. For $n \geq 1$, let $J_{0,n}$ be the closed interval between x_{2^n} and $x_{2^{n+1}}$ and $J_{i,n}$, $1 \leq i < 2^n$, be the closed interval between x_i and x_{i+2^n}. Then elementary computations using the fixed-point equation $g = \lambda^{-1}.g^2_\circ\lambda$ prove:

(a) $g^{2^n}(J_{0,n}) \subseteq J_{0,n}$;
(b) for each $n \geq 1$ the $J_{i,n}$ are pairwise disjoint;
(c) $J_{i,n-1}$ contains $J_{i,n}$ and $J_{i+2^{n-1},n}$ and no other of the $J_{k,n}$;
(d) $J_{2k+1,n} \subseteq J_{1,1}$ and $J_{2k,n} \subseteq J_{0,1}$; and
(e) $|J_{i,n}| \leq (1 + |\lambda|)|\lambda|^n$.

The map g has a unique fixed point which is contained in the gap G between the intervals $J_{0,1}$ and $J_{1,1}$. Since $|g'| > 1$ on $\overline{G}$, if $x \in G$ and $g(x) \neq x$ then $g^m(x) \in J_{0,1}$ for some $m \geq 1$. Using the self-similarity given by $g = \lambda^{-1}.g^2_\circ\lambda$, if $x \in G$ where G is any gap between two adjacent cylinders of the form $J_{k,n}$ then either $g^m(x) = x$ for some $m = 2^k$ with $k \leq n$, x is the image of such a point, or $g^m(x) \in J_{0,n}$ for some $m \geq 1$. Thus each $x \in [-1, 1]$ is either an unstable periodic point of period 2^k for some $k \geq 0$, a preimage of such a point, or $g^n(x)$ converges to

$$\Lambda_\infty(g) = \bigcap_{n\geq 1} \bigcup_{i=0}^{2^n-1} J_{i,n}.$$

The set $\Lambda_\infty(g)$ is clearly invariant. Since $g(J_{i,n}) = J_{i+1,n}$ for $0 \leq i < 2^n$, every orbit in $\Lambda_\infty(g)$ is dense in $\Lambda_\infty(g)$. Thus, $\Lambda_\infty(g)$ is *minimal*.

For each $x \in \Lambda_\infty(g)$ there exists a sequence $k(n)$ such that $x \in J_{k(n),n}$ for all $n \geq 1$. But, by (c), $k(n+1) = k(n) + \varepsilon_n 2^n$ where ε_n equals 0 or 1. Thus there exists $\underline{\varepsilon} = \underline{\varepsilon}(x) = \varepsilon_0\varepsilon_1 \cdots \in \{0,1\}^{N\cup 0}$ such that for all $n \geq 1$, $k(n) = \varepsilon_0 + \cdots + \varepsilon_{n-1}2^{n-1}$. Clearly, x is completely determined by $\underline{\varepsilon}$ and vice versa, and $\underline{\varepsilon}$ gives a homeomorphism from $\Lambda_\infty(g)$ to $\{0,1\}^{N\cup 0}$ endowed with the usual product topology. But if $x \in J_{k,n}$, $g(x) \in J_{k',n}$ where $k' = k + 1 \bmod 2^n$. Thus, $g|\Lambda_\infty(g)$ is conjugated to the *adding transformation* $(\varepsilon_0, \varepsilon_1, \ldots) \to (\varepsilon_0', \varepsilon_1', \ldots)$ of $\{0,1\}^{N\cup 0}$ where the ε_n' are defined by $\varepsilon_0' + \cdots + \varepsilon_{n-1}'2^{n-1} = 1 + \varepsilon_0 + \cdots + \varepsilon_{n-1}2^{n-1} \bmod 2^n$ for all $n \geq 0$.

If f is in the stable manifold of g, then the f-orbit $y_n = f^n(y_0)$ of the critical point y_0 is ordered on the interval exactly as the x_n. Consequently, the intervals $J_{k,n} = [y_k, y_{k+2^n}]$ have properties analogous to (a) to (d) above. Moreover, since f is in the stable manifold, $|J_{k,n}| \leq c.|\lambda|^n$ for some constant $c > 0$. Thus, as above, the attractor for f given by

$$\Lambda_\infty(f) = \bigcap_{n\geq 1} \bigcup_{i=0}^{2^n-1} J_{i,n}.$$

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can be conjugated to the adding transformation on $\{0,1\}^{\mathbb{N}\cup 0}$. Consequently, $f|\Lambda_\infty(f)$ and $g|\Lambda_\infty(g)$ are conjugate.

Before proceeding further it will be useful to introduce a hyperbolic dynamical system (a cookie-cutter) which generates the $J_{i,n}$. Define

$$F(x) = \begin{cases} \lambda^{-1}x \text{ on } I_0 = J_{1,0} = [x_2, x_4] \\ \lambda^{-1}g(x) \text{ on } I_1 = J_{1,1} = [x_3, x_1] \end{cases} \tag{1.2}$$

Then $F(I_0) = F(I_1) = [x_2, x_1]$ and $|F'| > 1$. Moreover, the n-cylinders of the cookie-cutter F which are defined by

$$I_{a_0,\ldots,a_{n-1}} = \{x : F^j(x) \in I_{a_j}\}\ ,\ a_0,\ldots,a_{n-1} \in \{0,1\}$$

are the 2^n intervals $J_{i,n}$ whose end-points are x_i and x_{i+2^n}, $i = 1,\ldots,2^n$. Thus if

$$\Lambda = \bigcap_{n\geq 0} \bigcup_{a_0,\ldots,a_{n-1}} I_{a_0,\ldots,a_{n-1}}$$

then $\Lambda = \Lambda_\infty(g)$ the attractor for g. Of course, the dynamics of $F|\Lambda$ and $g|\Lambda_\infty(g)$ are completely different.

Now suppose that f is in the stable manifold of g and $\|T^n(f) - g\| < c.\tau^n$ for some constant $c > 0$ and some $0 < \tau < 1$. Again let y_n denote the f-orbit of the critical point of f. As was already noted the intervals whose end-points are y_i and y_{i+2^n} have the same ordering as and are in one-to-one correspondence with those given by the x_i. Denote the interval corresponding to $I_{a_0,\ldots,a_{n-1}}$ by $I^f_{a_0,\ldots,a_{n-1}}$. Let $f_i = T^i(f)$ and $\alpha_i = a(f_i)^{-1}$. Then, since the map $\alpha_{n-1}f_{n-1}^{a_{n-1}}\circ\cdots\circ\alpha_0 f_0^{a_0}$ sends $I^f_{a_0,\ldots,a_{n-1}}$ injectively onto $I^{f_n}_{a_0,\ldots,a_{n-1}}$ it is not too difficult to deduce that there exists a $\kappa > 0$ depending only upon f such that

$$\kappa^{-1} < |I^f_{a_0,\ldots,a_{n-1}}|/|I_{a_0,\ldots,a_{n-1}}| < \kappa \tag{1.3}$$

independently of n and $a_0,\ldots,a_{n-1}$. This implies that the conjugacy from $\Lambda_\infty(f)$ to the universal attractor $\Lambda_\infty(g)$ is Lipschitz and has a Lipschitz inverse. Geometric universality follows from this and we see in what sense σ is universal.

In particular, this result implies that the Hausdorff dimension d of $\Lambda_\infty(f)$ and $\Lambda_\infty(g)$ are the same. In fact, if $P(\beta)$ denotes the growth rate

$$\lim_{n\to\infty} \log \sum_{a_0,\ldots,a_{n-1}} |I_{a_0,\ldots,a_{n-1}}|^\beta \tag{1.4}$$

then the Hausdorff dimension d is determined by the equation $P(d) = 0$.

Now, $|I_{a_0,\ldots,a_{n-1}}| = (x_1 - x_2)|(F^n)'(x)|^{-1}$ for some $x \in I_{a_0,\ldots,a_{n-1}}$ because $F^n(I_{a_0,\ldots,a_{n-1}}) = [x_2, x_1]$ and $F^n|I_{a_0,\ldots,a_{n-1}}$ is a homeomorphism. Therefore if $r = \lambda^{-1}$ and s (resp. t) denotes the infimum (resp. supremum) of $|a^{-1}g'(x)|$ on $I_1 = [x_2, x_1]$, then $r^{-(n-k)}t^{-k} < |I_{a_0,\ldots,a_{n-1}}| < r^{-(n-k)}s^{-k}$ where $k = a_0 + \cdots + a_{n-1}$. Thus the sum in (1.4) is bounded below and above by $(r^{-\beta} + t^{-\beta})^n$ and $(r^{-\beta} + s^{-\beta})^n$ respectively. Consequently, $(r^{-\beta} + t^{-\beta}) < P(\beta) < (r^{-\beta} + s^{-\beta})$. Inserting estimates for r, s and t one finds

$$0.5345 < d < 0.5544,$$

the estimate given in Falconer (1985). This estimate can be successively improved by estimating the derivatives of F^n on the intervals $I_{a_0, \ldots, a_{n-1}}$.

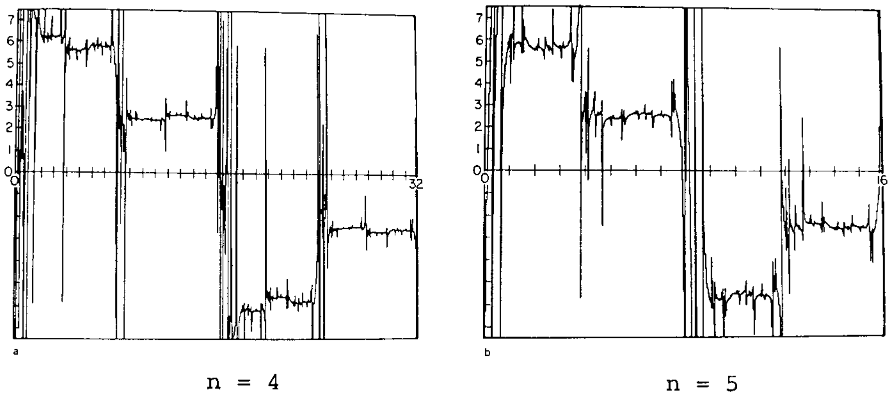

Figure 3. A numerical approximation of r_n obtained from Duffing's equation for $n = 4,5$. (From Feigenbaum (1982).)

The problem with the scaling function σ is that it is not smooth and therefore difficult to specify. Now I introduce a related invariant which is analytic and therefore easier to specify. In this context it was introduced by Halsey, Jensen, Kadanoff, Procaccia & Shraiman (1985), but the form presented here is closer to Bohr & Rand (1987). For $i = 0, \ldots, 2^{n-1} - 1$ consider the lengths

$$l_{i,n} = dist\ (x(T_{n-1} + i), x(i))$$

and their rates of decrease

$$\zeta_i = \lim_{n \to \infty} \log l_{i,n.}$$

There are functions s and p such that if the cascade of bifurcations is in the Feigenbaum universality class then the following are true.

(i) Let $N_n(\zeta)$ be the number of $l_{i,n}$ in $[\zeta, \zeta + d\zeta]$. Then $N_n(\zeta) \sim e^{ns(\zeta)}$.

(ii) Let $p(\beta)$ denote the growth rate of the sums $\sum_{i=1}^{2^{n-1}} l_{i,n}^{\beta}$. Then $p(\beta)$ is real-analytic and strictly convex and s and p are the Legendre transforms of each other.

(iii) If $d(\zeta) = -s(\zeta)/\zeta$, then $d(\zeta)$ is the Hausdorff dimension of the set of points x in the attractor Λ_∞ at $\mu = \mu_\infty$ such that $\lim_{n \to \infty} l_n(x) = \zeta$ where $l_n(x)$ is defined as follows. It follows from an argument similar to that of Proposition 3.2 that the set $\Lambda_\infty' = \overline{\{x(n) : n \in \mathbf{Z}\}}$ is the graph of a Lipschitz function over some coordinate axis. Let π be the projection onto this coordinate axis. Then $l_n(x) = dist\ (x(T_{n-1} + i), x(i))$ where i is such that $\pi(x)$ lies between $\pi(x(T_{n-1} + i))$ and $\pi(x(i))$.

Of course, the definition of the number $N_n(\zeta)$ is somewhat heuristic. This is put on a sounder basis as follows. If J is an interval define $N_n(J)$ to be the number of $l_{i,n}$ in J. Let $s(J)$ be the growth rate of these numbers and let $s(\zeta) = \inf\{s(J) : \zeta \in J\}$.