Chapter 2

Eigenvalue and Singular Value Inequalities of Schur Complements

2.0 Introduction

The purpose of this chapter is to study inequalities involving eigenvalues and singular values of products and sums of matrices.

In addition to denoting the $m \times n$ matrices with complex (real) entries by $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$), we denote by $\mathbb{H}_n$ the set of $n \times n$ Hermitian matrices, and for an $A \in \mathbb{H}_n$, we arrange the eigenvalues of A in a decreasing order:

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

The singular values of a matrix $A \in \mathbb{C}^{m\times n}$ are defined to be the square roots of the eigenvalues of the matrix A^*A, denoted and arranged as

$$\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A).$$

For a set of subscript indices $i_1, i_2, \ldots, i_k$, we always assume that $i_1 \leq i_2 \leq \cdots \leq i_k$. Furthermore, if $A \in \mathbb{H}_n$, then $\lambda_{i_t}(A)$ indicates $1 \leq i_t \leq n$.

One of the most important results in matrix analysis is the *Cauchy (eigenvalue) interlacing theorem* (see, e.g., [272, p. 294]). It asserts that the eigenvalues of any principal submatrix of a Hermitian matrix interlace those of the Hermitian matrix. To be precise, if $H \in \mathbb{H}_n$ is partitioned as

$$H = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

in which A is an $r \times r$ principal submatrix, then for each $i = 1, 2, \ldots, r$,

$$\lambda_i(H) \geq \lambda_i(A) \geq \lambda_{i+n-r}(H).$$

Eigenvalue and singular value problems are a central topic of matrix analysis and have reached out to many other fields. A great number of inequalities on eigenvalues and singular values of matrices are seen in the literature (see, e.g., [228, 230, 272, 301, 438, 452]). Here, we single some of these out for later use.

Let A and B be $n \times n$ complex matrices. Let l be an integer such that $1 \leq l \leq n$. Then for any index sequence $1 \leq i_1 \leq \cdots \leq i_l \leq n$,

$$\prod_{t=1}^{l} \sigma_t(AB) \geq \prod_{t=1}^{l} \sigma_{i_t}(A)\sigma_{n-i_t+1}(B), \tag{2.0.1}$$

$$\prod_{t=1}^{l} \sigma_{i_t}(A)\sigma_t(B) \geq \prod_{t=1}^{l} \sigma_{i_t}(AB) \geq \prod_{t=1}^{l} \sigma_{i_t}(A)\sigma_{n-t+1}(B), \tag{2.0.2}$$

and

$$\min_{i+j=t+1}\{\sigma_i(A)\sigma_j(B)\} \geq \sigma_t(AB) \geq \max_{i+j=t+n}\{\sigma_i(A)\sigma_j(B)\}. \tag{2.0.3}$$

The inequalities on the product ($\prod$) yield the corresponding inequalities on the sum ($\sum$). This is done by majorization in the following sense.

Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two sequences of nonnegative numbers in the order $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. Then

$$\prod_{i=1}^{k} x_i \leq \prod_{t=1}^{k} y_i, \ \ k \leq n \ \ \Rightarrow \ \ \sum_{i=1}^{k} x_i \leq \sum_{t=1}^{k} y_i, \ \ k \leq n \tag{2.0.4}$$

and

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{t=1}^{k} y_{(i)}, \ \ k \leq n \Rightarrow \prod_{i=1}^{k} x_{(i)} \leq \prod_{t=1}^{k} y_{(i)}, \ \ k \leq n, \tag{2.0.5}$$

where $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)}$ are rearrangements of $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, respectively.

Translations from product to sum or vice versa are often done through (2.0.4) and (2.0.5). For example, by (2.0.2) and (2.0.4), we can get

$$\sum_{t=1}^{l} \sigma_{i_t}(A)\sigma_t(B) \geq \sum_{t=1}^{l} \sigma_{i_t}(AB) \geq \sum_{t=1}^{l} \sigma_{i_t}(A)\sigma_{n-t+1}(B). \tag{2.0.6}$$

We point out that all the above singular value inequalities remain valid when AB is changed to BA; even though $\sigma_i(AB) \neq \sigma_i(BA)$ in general. Moreover they all hold with the replacement of the eigenvalues (λ) by the singular values (σ) when A and B are positive semidefinite. For instance,

$$\sum_{t=1}^{l} \lambda_{i_t}(AB) \geq \sum_{t=1}^{l} \lambda_{i_t}(A)\lambda_{n-t+1}(B). \tag{2.0.7}$$

For the sum of Hermitian matrices, two existing parallel results are

$$\sum_{t=1}^{l} \lambda_{i_t}(A) + \sum_{t=1}^{l} \lambda_t(B) \geq \sum_{t=1}^{l} \lambda_{i_t}(A+B) \geq \sum_{t=1}^{l} \lambda_{i_t}(A) + \sum_{t=1}^{l} \lambda_{n-t+1}(B) \quad (2.0.8)$$

and

$$\min_{i+j=t+1} \{\lambda_i(A) + \lambda_j(B)\} \geq \lambda_t(A+B) \geq \max_{i+j=t+n} \{\lambda_i(A) + \lambda_j(B)\}. \quad (2.0.9)$$

All the above inequalities appear explicitly in Chapter 2 of [451]. We note that the second inequality in (2.0.8) does not hold in general for singular values (σ) [451, p. 113].

2.1 The interlacing property

The Cauchy interlacing theorem states that the eigenvalues of any principal submatrix of a Hermitian matrix interlace those of the grand matrix. Does a Schur complement possess a similar property? That is, do the eigenvalues of a Schur complement in a Hermitian matrix interlace the eigenvalues of the original Hermitian matrix? The answer is negative in general: Take

$$H = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \alpha = \{1\}.$$

Then $H/\alpha = (-3)$, while the eigenvalues of H are -1 and 3.

In what follows, we show that with a slight modification of the Schur complement (augmented by 0s) the analogous interlacing property holds.

Theorem 2.1 *Let $H \in \mathbb{H}_n$ and let α be an index set with k elements, $1 \leq k < n$. If the principal submatrix $H[\alpha]$ is positive definite, then*

$$\lambda_i(H) \geq \lambda_i(H/\alpha \oplus 0) \geq \lambda_{i+k}(H), \quad i = 1, 2, \ldots, n-k, \quad (2.1.10)$$

and if $H[\alpha]$ is negative definite, i.e., $-H[\alpha]$ is positive definite, then

$$\lambda_i(H) \geq \lambda_{i+k}(H/\alpha \oplus 0) \geq \lambda_{i+k}(H), \quad i = 1, 2, \ldots, n-k. \quad (2.1.11)$$

Proof. Since permutation similarity preserves the eigenvalues, we may assume that $\alpha = \{n-k+1, \ldots, n\}$. With $\alpha^c = \{1, 2, \ldots, n-k\}$, we have

$$H = \begin{pmatrix} H/\alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} H[\alpha^c, \alpha](H[\alpha])^{-1} H[\alpha, \alpha^c] & H[\alpha^c, \alpha] \\ H[\alpha, \alpha^c] & H[\alpha] \end{pmatrix} \equiv E + F.$$

Let

$$P = \begin{pmatrix} I_{n-k} & -H[\alpha^c, \alpha](H[\alpha])^{-1} \\ 0 & I_k \end{pmatrix}.$$

Then $PFP^* = 0 \oplus H[\alpha]$, so F is positive semidefinite if $H[\alpha] \geq 0$. Moreover

$$\operatorname{rank}(F) = \operatorname{rank}(H[\alpha]) = k < n.$$

Now using (2.1.10) and by (2.0.9), we have

$$\lambda_{i+k}(H) = \lambda_{i+k}(E+F) \leq \lambda_i(E) + \lambda_{k+1}(F) = \lambda_i \left[\begin{pmatrix} H/\alpha & 0 \\ 0 & 0 \end{pmatrix} \right]$$

and

$$\lambda_i(H) = \lambda_i(E+F) \geq \lambda_i(E) + \lambda_n(F) = \lambda_i \left[\begin{pmatrix} H/\alpha & 0 \\ 0 & 0 \end{pmatrix} \right].$$

The inequalities (2.1.11) are proven in a similar manner. Note that if A is a Hermitian matrix, then $\lambda_i(-A) = -\lambda_{n-i+1}(A)$, $i = 1, 2, \ldots, n$. ∎

The theorem immediately yields the following results for positive semidefinite matrices; see [160, 288, 421].

Corollary 2.3 *Let H (or $-H$) be an $n \times n$ positive semidefinite matrix and let $H[\alpha]$ be a $k \times k$ nonsingular principal submatrix, $1 \leq k < n$. Then*

$$\lambda_i(H) \geq \lambda_i(H/\alpha) \geq \lambda_{i+k}(H), \quad i = 1, 2, \ldots, n-k. \tag{2.1.12}$$

Proof. When H is positive semidefinite, H/α is positive semidefinite. It is sufficient to notice that $\lambda_i(H/\alpha \oplus 0) = \lambda_i(H/\alpha)$ for $i = 1, 2, \ldots, n-k$. ∎

Corollary 2.4 *Let H be an $n \times n$ positive semidefinite matrix and let $H[\alpha]$ be a $k \times k$ nonsingular principal submatrix of H, $1 \leq k < n$. Then*

$$\lambda_i(H) \geq \lambda_i(H[\alpha^c]) \geq \lambda_i(H/\alpha) \geq \lambda_{i+k}(H), \quad i = 1, 2, \ldots, n-k. \tag{2.1.13}$$

Proof. Since H, $H[\alpha]$, and $H[\alpha^c]$ are all positive semidefinite, we obtain

$$H[\alpha^c] \geq H[\alpha^c] - H[\alpha^c, \alpha](H[\alpha])^{-1} H[\alpha, \alpha^c] = H/\alpha.$$

The second inequality in (2.1.13) follows at once, while the first inequality is the Cauchy interlacing theorem and the last one is (2.1.12). ∎

Corollary 2.5 *Let H be an $n \times n$ positive semidefinite matrix and let α and α' be nonempty index sets such that $\alpha' \subset \alpha \subset \{1, 2, \ldots, n\}$. If $H[\alpha]$ is nonsingular, then for every $i = 1, 2, \ldots, n - |\alpha|$,*

$$\lambda_i(H/\alpha') \geq \lambda_i(H[\alpha' \cup \alpha^c]/\alpha') \geq \lambda_i(H/\alpha) \geq \lambda_{i+|\alpha|-|\alpha'|}(H/\alpha'). \tag{2.1.14}$$

Proof. Note that $H[\alpha'] > 0$ since it is a principal submatrix of $H[\alpha] > 0$. By the quotient formula on the Schur complement (see Theorem 1.4),

$$H/\alpha = (H/\alpha')/(H[\alpha]/\alpha').$$

With H/α' and $H[\alpha]/\alpha'$ in place of H and $H[\alpha]$, respectively, in Corollary 2.4 and since $(H/\alpha')[\alpha^c] = H[\alpha' \cup \alpha^c]/\alpha'$, (2.1.14) follows. ∎

For the case where H is negative definite, we have the analogs:

$$\lambda_i(H) \geq \lambda_i(H/\alpha) \geq \lambda_i\left(H[\alpha^c]\right) \geq \lambda_{i+k}(H)$$

and

$$\lambda_i(H/\alpha') \geq \lambda_i(H/\alpha) \geq \lambda_i(H[\alpha' \cup \alpha^c]/\alpha') \geq \lambda_{i+|\alpha|-|\alpha'|}(H/\alpha').$$

As we saw, the Cauchy eigenvalue interlacing theorem does not hold for the Schur complement of a Hermitian matrix. We show, however, and interestingly, that it holds for the reciprocals of nonsingular Hermitian matrices. This is not surprising in view of the representation of a Schur complement in terms of a principal submatrix (see Theorem 1.2).

Lemma 2.3 *Let H be an $n \times n$ nonsingular Hermitian matrix and let A be a $k \times k$ nonsingular principal submatrix of H, where $1 \leq k < n$. Then*

$$\lambda_i(H^{-1}) \geq \lambda_i[(H/A)^{-1}] \geq \lambda_{i+k}(H^{-1}), \quad i = 1, 2, \ldots, n-k.$$

Proof. It is sufficient to notice, by Theorem 1.2, that $(H/A)^{-1}$ is a principal submatrix of the Hermitian matrix H^{-1}. ∎

We now extend this to a singular H. That is, we show that if H is any Hermitian matrix and A is a nonsingular principal submatrix of H, then the eigenvalues of $(H/A)^\dagger$ interlace the eigenvalues of $H^\dagger$.

Let $\mathrm{In}(H) = (p,\ q,\ z)$. The eigenvalues $H^\dagger$ are, in decreasing order,

$$\lambda_i(H^\dagger) = \begin{cases} \lambda_{p+1-i}^{-1}(H), & i = 1, \ldots, p, \\ 0, & i = p+1, \ldots, p+z, \\ \lambda_{n+p+z+1-i}^{-1}(H), & i = p+z+1, \ldots, n. \end{cases}$$

Since the eigenvalues of a matrix are continuous functions of the entries of the matrix, the eigenvalues of the Moore–Penrose inverse of a matrix are also continuous functions of the entries of the original matrix.

To establish the interlacing property for any Hermitian H, we need to use the usual trick – continuity argument. Let $H \in \mathbb{H}_n$ and $H_\varepsilon = H + \varepsilon I_n$, where ε is a positive number. Let A be a $k \times k$ nonzero principal submatrix of H and denote $A_\varepsilon = A + \varepsilon I_k$. Choose ε such that it is less than the absolute value of any nonzero eigenvalue of H and A. Thus H_ε, A_ε, and $H_\varepsilon/A_\varepsilon$ are all invertible. It follows that if $\lambda_s(H^\dagger) \neq 0$ and $\lambda_t\left[(H/A)^\dagger\right] \neq 0$,

$$\lim_{\varepsilon\to 0} \lambda_s(H_\varepsilon^{-1}) = \lambda_s(H^\dagger)$$

and

$$\lim_{\varepsilon\to 0} \lambda_t\left[(H_\varepsilon/A_\varepsilon)^{-1}\right] = \lambda_t\left[(H/A)^\dagger\right].$$

Now we are ready to present the following interlacing theorem [421].

Theorem 2.2 *Let H be an $n\times n$ Hermitian matrix and let A be a $k\times k$ nonsingular principal submatrix of H. Then for $i = 1, 2, \ldots, n-k$,*

$$\lambda_i(H^\dagger) \geq \lambda_i[(H/A)^\dagger] \geq \lambda_{i+k}(H^\dagger). \tag{2.1.15}$$

Proof. Let $\mathrm{In}(H) = (p,\ q,\ z)$ and $\mathrm{In}(A) = (p_1,\ q_1,\ 0)$. Consequently, $\mathrm{In}(H/A) = (p - p_1,\ q - q_1,\ z)$ by Theorem 1.6. Without loss of generality, we write $H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. Let

$$H_\varepsilon = H + \varepsilon I_n = \begin{pmatrix} A + \varepsilon I_k & B \\ B^* & C + \varepsilon I_{n-k} \end{pmatrix} \equiv \begin{pmatrix} A_\varepsilon & B \\ B^* & C_\varepsilon \end{pmatrix},$$

in which ε is such a small positive number that both H_ε and A_ε are nonsingular. Note that $\mathrm{In}(H_\varepsilon) = (p + z,\ q,\ 0)$, $\mathrm{In}(A_\varepsilon) = \mathrm{In}(A)$, and also $\mathrm{In}(K) = \mathrm{In}(K^\dagger)$ for any Hermitian matrix K. Moreover, upon computation, we have $H_\varepsilon/A_\varepsilon = C_\varepsilon - B^* A_\varepsilon^{-1} B$, and thus $\lim_{\varepsilon\to 0} H_\varepsilon/A_\varepsilon = H/A$.

To show that $\lambda_i(H^\dagger) \geq \lambda_i[(H/A)^\dagger]$ for $i = 1, 2, \ldots, n-k$, we consider a set of exhaustive cases on the index i:

Case (1) If $i \leq p - p_1$, then $\lambda_i[(H/A]^\dagger] > 0$. By Lemma 2.3,

$$\lambda_i(H_\varepsilon^{-1}) \geq \lambda_i[(H_\varepsilon/A_\varepsilon)^{-1}] > 0.$$

The desired inequalities follow by taking the limits as $\varepsilon \to 0$.

Case (2) If $p - p_1 < i \leq p + z$, then $\lambda_i(H^\dagger) \geq 0 \geq \lambda_i\left[(H/A)^\dagger\right]$.

Case (3) If $p + z < i \leq n - k$, then, by Lemma 2.3,

$$0 > \lambda_i(H_\varepsilon^{-1}) \geq \lambda_i[(H_\varepsilon/A_\varepsilon)^{-1}].$$

By continuity, we arrive at $0 > \lambda_i(H^\dagger) \geq \lambda_i\left[(H/A)^\dagger\right]$.

To establish the second inequality in (2.1.15), we proceed by exhausting the cases of the index $i + k$:

Case (i) If $i + k \leq p$, i.e., $i \leq p - k \leq n - k$, then, by Lemma 2.3, $\lambda_i[(H_\varepsilon/A_\varepsilon)^{-1}] \geq \lambda_{i+k}(H_\varepsilon^{-1}) > 0$. Letting $\varepsilon \to 0$ yields the inequalities.

Case (ii) If $p + 1 < i + k \leq p + z$, then $i \leq p - k + z \leq p - p_1 + z$, so $\lambda_i[(H/A)^\dagger] \geq 0$ and $\lambda_{i+k}(H^\dagger) = 0$. The inequality then follow.

Case (iii) If $p + z < i + k \leq p + k - p_1 + z = p + q_1 + z \leq n$, then $i \leq p - p_1 + z$, so $\lambda_i[(H/A)^\dagger] \geq 0$ and $\lambda_{i+k}(H^\dagger) < 0$ since $i + k > p + z$.

Case (iv) If $p+z \leq p+k-p_1+z < i+k \leq n$, then $p-p_1+z < i \leq n-k$. By Lemma 2.3, $0 > \lambda_i[(H_\varepsilon/A_\varepsilon)^{-1}] \geq \lambda_{i+k}(H_\varepsilon^{-1})$. Letting $\varepsilon \to 0$ shows that $0 > \lambda_i[(H/A)^\dagger] \geq \lambda_{i+k}(H^\dagger)$. ∎

At the end of this section we note that the converse of the previous theorem is discussed by Hu and Smith in [235].

2.2 Extremal characterizations

The Courant–Fischer min-max principles, or the extremal characterizations, of eigenvalues for Hermitian matrices play an important role in deducing eigenvalue inequalities. For instance, the representation of the minimum eigenvalue $\lambda_{\min}(H)$ of a Hermitian matrix $H \in \mathbb{H}_n$

$$\lambda_{\min}(H) = \min_{x \in \mathbb{C}^n} \{x^* H x : \ x^* x = 1\}$$

leads immediately to the eigenvalue inequalities: For $A,\ B \in \mathbb{H}_n$

$$\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B).$$

We now show extremal characterizations [280] for Schur complements.

Theorem 2.3 *Let H be an $n \times n$ positive semidefinite matrix partitioned as*

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where H_{11} is a $k \times k$ leading principal submatrix of H, $1 \leq k < n$. Then

$$H/H_{11} = \max_{X \in \mathbb{C}^{(n-k)\times(n-k)}} \{X : \ H - (0_k \oplus X) \geq 0, \ X = X^*\}$$

and

$$H/H_{11} = \min_{Y \in \mathbb{C}^{(n-k)\times k}} \{Y : \ (Y, I_{n-k}) H (Y, I_{n-k})^*\}. \tag{2.2.16}$$

Proof. Let X be an $(n-k) \times (n-k)$ Hermitian matrix and set

$$\hat{X} = \begin{pmatrix} 0_k & 0 \\ 0 & X \end{pmatrix}, \qquad T = \begin{pmatrix} I_k & 0 \\ -H_{21} H_{11}^\dagger & I_{n-k} \end{pmatrix}.$$

Since H_{11} is positive semidefinite, we have $(H_{11}^{1/2})^\dagger = (H_{11}^\dagger)^{1/2}$. Since H is positive semidefinite, by Theorem 1.19, we have $H_{11}H_{11}^\dagger H_{12} = H_{12}$. Thus,

$$T(H - \hat{X})T^* = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} - X - H_{21}H_{11}^\dagger H_{12} \end{pmatrix}.$$

So $H \geq \hat{X}$ if and only if the matrix on the right-hand side is positive semidefinite, and this occurs if and only if $H/H_{11} - X \geq 0$.

The maximum is attained when $X = H/H_{11}$ due to the fact that

$$H - \begin{pmatrix} 0 & 0 \\ 0 & H/H_{11} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{21}H_{11}^\dagger H_{12} \end{pmatrix} \geq 0.$$

To show the minimum representation (2.2.16), observe that

$$(Y, I_{n-k})H(Y, I_{n-k})^* = H/H_{11} + (Y + H_{21}H_{11}^\dagger)H_{11}(H_{11}^\dagger H_{12} + Y^*).$$

It follows that

$$(Y, I_{n-k})H(Y, I_{n-k})^* \geq H/H_{11},$$

and equality holds if and only if

$$(Y + H_{21}H_{11}^\dagger)H_{11}(H_{11}^\dagger H_{12} + Y^*) = 0,$$

equivalently, $(Y + H_{21}H_{11}^\dagger)H_{11} = 0$. One may take $Y = -H_{21}H_{11}^\dagger$. ∎

The following corollary will be used repeatedly in later sections.

Corollary 2.6 *Let H be $n \times n$ Hermitian. If $\alpha = \{1, 2, \ldots, k\}$, then*

$$H/\alpha = (Z, I)H(Z, I)^*$$

and if $\alpha = \{k+1, k+2, \ldots, n\}$, then

$$H/\alpha = (I, Z)H(I, Z)^*,$$

where, for both cases,

$$Z = -H[\alpha^c, \alpha]H[\alpha]^\dagger.$$

As consequences of the theorem, we have, for positive semidefinite A, B,

$$(A \star B)/\alpha \geq A/\alpha \star B/\alpha,$$

where α is an index set and $\star$ denotes sum $+$ or the Hadamard product $\circ$.

We now show a minimum representation for the product of the eigenvalues of a Schur complement [289]. Let integers l and k be such that $1 \leq l \leq k \leq n$. We consider the product of the eigenvalues of the Schur complement indexed by an increasing sequence $1 \leq i_1 \leq i_2 \leq \cdots \leq i_l \leq k$.

Theorem 2.4 *Let A be an $n \times n$ positive semidefinite matrix partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

in which A_{22} is an $(n-k) \times (n-k)$ principal submatrix. Then

$$\prod_{t=1}^{l} \lambda_{i_t}(A/A_{22}) = \min_{Z \in \mathbb{C}^{k \times (n-k)}} \prod_{t=1}^{l} \lambda_{i_t}[(I_k, Z)A(I_k, Z)^*]. \tag{2.2.17}$$

Proof. For any $Z \in \mathbb{C}^{k \times (n-k)}$, by (2.2.16), we have

$$(I_k, Z)A(I_k, Z)^* \geq A/A_{22}$$

which yields

$$\lambda_{i_t}[(I_k, Z)A(I_k, Z)^*] \geq \lambda_{i_t}(A/A_{22})$$

for each i_t, $t = 1, 2, \ldots, l$, and equality holds by setting $Z = -A_{12}A_{22}^{\dagger}$. ∎

Putting $l = 1$ results in, for any $t = 1, 2, \ldots, k$,

$$\lambda_t(A/A_{22}) = \min_{Z \in \mathbb{C}^{k \times (n-k)}} \lambda_t[(I_k, Z)A(I_k, Z)^*]. \tag{2.2.18}$$

In a similar fashion, one proves that for positive $\theta_1, \theta_2, \ldots, \theta_l \in \mathbb{R}$

$$\begin{aligned} \sum_{t=1}^{l} \lambda_{i_t}(A/A_{22})\theta_t &= \min_{Z \in \mathbb{C}^{k \times (n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(I_k, Z)A(I_k, Z)^*]\theta_t \\ &= \sum_{t=1}^{l} \min_{Z \in \mathbb{C}^{k \times (n-k)}} \lambda_{i_t}[(I_k, Z)A(I_k, Z)^*]\theta_t. \end{aligned} \tag{2.2.19}$$

2.3 Eigenvalues of the Schur complement of a product

This section, based on [289], is focused on the eigenvalue inequalities of Schur complements concerning the product of positive semidefinite matrices that resemble those of Section 2.0.

Theorem 2.5 *Let A be $n \times n$ positive semidefinite. Let $\alpha \subset \{1, 2, \ldots, n\}$ denote an index set and $1 \leq i_1 < \cdots < i_l \leq k \equiv n - |\alpha|$, where l and k are positive integers such that $1 \leq l \leq k < n$. Then for any $B \in \mathbb{C}^{n \times n}$,*

$$\prod_{t=1}^{l} \lambda_{i_t}[(BAB^*)/\alpha] \geq \prod_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]\lambda_{n-t+1}(A), \tag{2.3.20}$$

$$\prod_{t=1}^{l} \lambda_t[(BAB^*)/\alpha] \geq \prod_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]\lambda_{n-i_t+1}(A), \tag{2.3.21}$$

and

$$\prod_{t=1}^{l} \lambda_{i_t}[(BAB^*)/\alpha] \le \prod_{t=1}^{l} \lambda_{i_t}(A)\lambda_t[(BB^*)/\alpha]. \tag{2.3.22}$$

Proof. There exists an $n \times n$ permutation matrix U such that

$$UAU^* = \begin{pmatrix} A[\alpha^c] & A[\alpha^c, \alpha] \\ A[\alpha, \alpha^c] & A[\alpha] \end{pmatrix}, \quad UBU^* = \begin{pmatrix} B[\alpha^c] & B[\alpha^c, \alpha] \\ B[\alpha, \alpha^c] & B[\alpha] \end{pmatrix}.$$

Let $\beta = \{k+1, \ldots, n\}$. Notice that for any $P \in \mathbb{C}^{k\times n}$, $Q \in \mathbb{C}^{n\times k}$, PQ and QP have the same nonzero eigenvalues. Using (2.2.17) and (2.0.2), we have

$$\begin{aligned}
&\prod_{t=1}^{l} \lambda_{i_t}[(BAB^*)/\alpha] \\
&\quad = \prod_{t=1}^{l} \lambda_{i_t}[(UBAB^*U^*)/\beta] \\
&\quad = \prod_{t=1}^{l} \lambda_{i_t}[(UBU^*UAU^*UB^*U^*)/\beta] \\
&\quad = \min_{Z\in\mathbb{C}^{k\times(n-k)}} \prod_{t=1}^{l} \lambda_{i_t}[(I_k, Z)UBU^*UAU^*UB^*U^*(I_k, Z)^*] \\
&\quad = \min_{Z\in\mathbb{C}^{k\times(n-k)}} \prod_{t=1}^{l} \lambda_{i_t}[(UAU^*)UB^*U^*(I_k, Z)^*(I_k, Z)UBU^*] \\
&\quad \ge \min_{Z\in\mathbb{C}^{k\times(n-k)}} \prod_{t=1}^{l} \lambda_{n-t+1}(UAU^*)\lambda_{i_t}[UB^*U^*(I_k, Z)^*(I_k, Z)UBU^*] \\
&\quad = \min_{Z\in\mathbb{C}^{k\times(n-k)}} \prod_{t=1}^{l} \lambda_{n-t+1}(A)\lambda_{i_t}[(I_k, Z)UBU^*UB^*U^*(I_k, Z)^*] \\
&\quad = \prod_{t=1}^{l} \lambda_{n-t+1}(A) \min_{Z\in\mathbb{C}^{k\times(n-k)}} \lambda_{i_t}[(I_k, Z)UBB^*U^*(I_k, Z)^*] \\
&\quad = \prod_{t=1}^{l} \lambda_{n-t+1}(A)\lambda_{i_t}[(UBB^*U^*)/\beta] \\
&\quad = \prod_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]\lambda_{n-t+1}(A).
\end{aligned}$$

This proves (2.3.20). (2.3.21) and (2.3.22) can be proved similarly. ∎

An analogous result for (2.3.22) is

$$\prod_{t=1}^{l} \lambda_{i_t}[(BAB^*)/\alpha] \le \prod_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]\lambda_t(A).$$

Setting $B = I$ in (2.3.20), (2.3.22), and (2.3.21), respectively, we obtain

$$\prod_{t=1}^{l} \lambda_{n-t+1}(A) \le \prod_{t=1}^{l} \lambda_{i_t}(A/\alpha) \le \prod_{t=1}^{l} \lambda_{i_t}(A)$$

and

$$\prod_{t=1}^{l} \lambda_t(A/\alpha) \ge \prod_{t=1}^{l} \lambda_{n-i_t+1}(A).$$

Putting $l = k$ in Theorem 2.5 reveals the inequalities

$$\prod_{t=1}^{k} \lambda_{n-t+1}(A) \det((BB^*)/\alpha) \le \det((BAB^*)/\alpha) \le \prod_{t=1}^{k} \lambda_t(A) \det((BB^*)/\alpha).$$

We point out that every matrix can be regarded as a Schur complement of some matrix. For instance, we may embed an $n \times n$ matrix A in

$$\tilde{A} \equiv \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

If we take $\alpha = \{1\}$, then $\tilde{A}/\alpha = A$. With this observation, many of our inequalities on the Schur complements reduce to certain existing results on regular matrices (without involving the Schur complements).

Theorem 2.6 *Let A be $n \times n$ positive semidefinite. Let $\alpha \subset \{1, 2, \dots, n\}$ denote an index set and $1 \le i_1 < \cdots < i_l \le k \equiv n - |\alpha|$, where l and k are positive integers such that $1 \le l \le k < n$. Then for any $B \in \mathbb{C}^{n\times n}$*

$$\sum_{t=1}^{l} \lambda_{i_t}[(BAB^*)/\alpha] \ge \sum_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]\lambda_{n-t+1}(A), \tag{2.3.23}$$

$$\sum_{t=1}^{l} \lambda_t[(BAB^*)/\alpha] \ge \sum_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]\lambda_{n-i_t+1}(A), \tag{2.3.24}$$

and

$$\sum_{t=1}^{l} \lambda_{i_t}[(BAB^*)/\alpha] \le \sum_{t=1}^{l} \lambda_{i_t}(A)\lambda_t[(BB^*)/\alpha]. \tag{2.3.25}$$

Proof. This follows from (2.0.4) immediately. Following is a proof based upon (2.2.19) and (2.0.7). We may take $\alpha = \{k+1, \dots, n\}$. Then

$$\begin{aligned}
&\sum_{t=1}^{l} \lambda_{i_t}[(BAB^*)/\alpha] \\
&\quad = \min_{Z \in \mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(I_k, Z)BAB^*(I_k, Z)^*] \\
&\quad = \min_{Z \in \mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[AB^*(I_k, Z)^*(I_k, Z)B] \\
&\quad \geq \min_{Z \in \mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{n-t+1}(A)\lambda_{i_t}[B^*(I_k, Z)^*(I_k, Z)B] \\
&\quad = \min_{Z \in \mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{n-t+1}(A)\lambda_{i_t}[(I_k, Z)BB^*(I_k, Z)^*] \\
&\quad = \sum_{t=1}^{l} \lambda_{n-t+1}(A)\lambda_{i_t}[(BB^*)/\alpha]. \blacksquare
\end{aligned}$$

The following is a parallel result to the inequality (2.3.25):

$$\sum_{t=1}^{l} \lambda_{i_t}[(BAB^*)/\alpha] \leq \sum_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]\lambda_t(A).$$

Setting $B = I$ in (2.3.23), (2.3.25), and (2.3.24), respectively, we obtain

$$\sum_{t=1}^{l} \lambda_{n-t+1}(A) \leq \sum_{t=1}^{l} \lambda_{i_t}(A/\alpha) \leq \sum_{t=1}^{l} \lambda_{i_t}(A)$$

and

$$\sum_{t=1}^{l} \lambda_t(A/\alpha) \geq \sum_{t=1}^{l} \lambda_{n-i_t+1}(A).$$

Putting $l = k$ in Theorem 2.6, since $(BAB^*)/\alpha$ is $k \times k$, we have

$$\sum_{t=1}^{k} \lambda_t[(BB^*)/\alpha]\lambda_{n-t+1}(A) \leq \operatorname{tr}[(BAB^*)/\alpha)] \leq \sum_{t=1}^{k} \lambda_t[(BB^*)/\alpha]\lambda_t(A).$$

Theorem 2.7 *Let A be an $n \times n$ positive semidefinite matrix and let α be an index set of k elements. Then for any $B \in \mathbb{C}^{n\times n}$ and $t = 1, 2, \dots, n-k$,*

$$\min_{i+j=t+1} \lambda_i(A)\lambda_j[(BB^*)/\alpha] \geq \lambda_t[(BAB^*)/\alpha] \geq \max_{i+j=t+n} \lambda_i(A)\lambda_j[(BB^*)/\alpha].$$

Proof. Taking $\alpha = \{n-k+1, \ldots, n\}$, by (2.2.18) and (2.0.3), we have

$$\begin{aligned}
&\lambda_t[(BAB^*)/\alpha] \\
&= \min_{Z \in \mathbb{C}^{(n-k)\times k}} \lambda_t[(I_{n-k}, Z)BAB^*(I_{n-k}, Z)^*] \\
&\geq \min_{Z \in \mathbb{C}^{(n-k)\times k}} \max_{i+j=t+n} \lambda_i(A)\lambda_j[(I_{n-k}, Z)BB^*(I_{n-k}, Z)^*] \\
&= \max_{i+j=t+n} \lambda_i(A) \min_{Z \in \mathbb{C}^{(n-k)\times k}} \lambda_j[(I_{n-k}, Z)BB^*(I_{n-k}, Z)^*] \\
&= \max_{i+j=t+n} \lambda_i(A)\lambda_j[(BB^*)/\alpha].
\end{aligned}$$

By (2.2.19), along with the first inequality in (2.0.3),

$$\begin{aligned}
&\lambda_t[(BAB^*)/\alpha] \\
&\leq \min_{Z \in \mathbb{C}^{(n-k)\times k}} \min_{i+j=t+1} \lambda_i(A)\lambda_j[(I_{n-k}, Z)BB^*(I_{n-k}, Z)^*] \\
&= \min_{i+j=t+1} \lambda_i(A) \min_{Z \in \mathbb{C}^{(n-k)\times k}} \lambda_j[(I_{n-k}, Z)BB^*(I_{n-k}, Z)^*] \\
&= \min_{i+j=t+1} \lambda_i(A)\lambda_j[(BB^*)/\alpha].
\end{aligned}$$

As we are interested in relating the eigenvalues of the matrix product AB to those of individual matrices A and B, our next result shows lower bounds for the eigenvalues of the Schur complement of the matrix product BAB^* in terms of the eigenvalues of the Schur complements of BB^* and A. The proof of the theorem is quite technical.

Theorem 2.8 *Let A be $n \times n$ positive semidefinite of rank r, $B \in \mathbb{C}^{m\times n}$, and $\alpha \subset \{1, 2, \ldots, m\}$. If* $\operatorname{rank}[(BAB^*)/\alpha] = s$, *then for each $l = 1, 2, \ldots, s$,*

$$\lambda_l[(BAB^*)/\alpha] \geq \max_{\substack{1\leq t\leq s-l+1 \\ 1\leq u\leq t}} [\lambda_{l+t+r-s-1}(A)\lambda_{r-u+1}(A)]^{\frac{1}{2}} \lambda_{s-t+u+n-r}[(BB^*)/\alpha].$$

Proof. Let $k = m - |\alpha|$. We may assume $\alpha = \{k+1, \ldots, m\}$. Then $\alpha^c = \{1, 2, \ldots, k\}$. Since $\operatorname{rank}(A) = r$, there exists unitary $U \in \mathbb{C}^{n\times n}$ such that

$$UAU^* = D \oplus 0 \equiv \operatorname{diag}(D, 0), \quad \text{where } D = \operatorname{diag}(\lambda_1(A), \ldots, \lambda_r(A)) \geq 0.$$

Let

$$X = -[(BAB^*)[\alpha^c, \alpha]]\,[(BAB^*)[\alpha]]^\dagger.$$

Then

$$\lambda_l[(BAB^*)/\alpha] = \lambda_l[(I_k, X)BAB^*(I_k, X)^*] = \lambda_l[AB^*(I_k, X)^*(I_k, X)B].$$

Thus

$$\operatorname{rank}[AB^*(I_k, X)^*(I_k, X)B] = \operatorname{rank}[(BAB^*)/\alpha] = s.$$

Let

$$\tilde{B} = B^*(I_k, X)^*(I_k, X)B, \quad U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \; U_1 \in \mathbb{C}^{r\times m}.$$

Then

$$\begin{aligned}
\operatorname{rank}(A\tilde{B}) &= \operatorname{rank}(UAU^*U\tilde{B}U^*) \\
&= \operatorname{rank}[\operatorname{diag}(D, 0)(U\tilde{B}U^*)] \\
&= \operatorname{rank}[\operatorname{diag}(D^{\frac{1}{2}}, 0)(U\tilde{B}U^*) \operatorname{diag}(D^{\frac{1}{2}}, 0)] \\
&= \operatorname{rank}(D^{\frac{1}{2}}U_1\tilde{B}U_1^*D^{\frac{1}{2}}) \\
&= \operatorname{rank}(U_1\tilde{B}U_1^*).
\end{aligned}$$

Since $U_1\tilde{B}U_1^*$ is $r \times r$ positive semidefinite and $\operatorname{rank}[(BAB^*)/\alpha] = s$, there exists an $r \times r$ unitary matrix V_1 such that

$$V_1U_1\tilde{B}U_1^*V_1^* = \operatorname{diag}(G, 0),$$

where

$$G = \operatorname{diag}(\lambda_1(U_1\tilde{B}U_1^*), \ldots, \lambda_s(U_1\tilde{B}U_1^*)).$$

Set $\tilde{D} = V_1DV_1^*$ and partition it as $\begin{pmatrix} D_1 & D_2 \\ D_2^* & D_3 \end{pmatrix}$ with D_1 of order $s \times s$. Let

$$L = \begin{pmatrix} I & 0 \\ -D_2^*D_1^\dagger & I \end{pmatrix}.$$

Then

$$LV_1DV_1^*L^* = \operatorname{diag}(D_1, D_3 - D_2^*D_1^\dagger D_2).$$

Let $\tilde{B}_1 = D^{\frac{1}{2}}U_1\tilde{B}U_1^*D^{\frac{1}{2}}$. Then

$$\begin{aligned}
&(L^{*-1}V_1D^{-\frac{1}{2}})\tilde{B}_1(L^{*-1}V_1D^{-\frac{1}{2}})^{-1} \\
&\quad = L^{*-1}V_1D^{-\frac{1}{2}}D^{\frac{1}{2}}U_1\tilde{B}U_1^*D^{\frac{1}{2}}D^{\frac{1}{2}}V_1^*L^* \\
&\quad = L^{*-1}V_1U_1\tilde{B}U_1^*DV_1^*L^* \\
&\quad = L^{*-1}V_1U_1\tilde{B}U_1^*V_1^*L^{-1}(LV_1DV_1^*L^*) \\
&\quad = (L^{-1})^* \operatorname{diag}(G, 0)L^{-1} \operatorname{diag}(D_1, D_3 - D_2^*D_1^\dagger D_2) \\
&\quad = \operatorname{diag}(GD_1, 0).
\end{aligned}$$

So $\tilde{B}_1$ and GD_1 have the same nonzero eigenvalues. On the other hand,

$$\begin{aligned}
\lambda_l[(BAB^*)/\alpha] &= \lambda_l[AB^*(I_k, X)^*(I_k, X)B] \\
&= \lambda_l(A\tilde{B}) \\
&= \lambda_l[(UAU^*)(U\tilde{B}U^*)] \\
&= \lambda_l[\mathrm{diag}(D, 0)(U\tilde{B}U^*)] \\
&= \lambda_l[\mathrm{diag}(D^{\frac{1}{2}}, 0)\,\mathrm{diag}(D^{\frac{1}{2}}, 0)(U\tilde{B}U^*)] \\
&= \lambda_l[\mathrm{diag}(D^{\frac{1}{2}}, 0)(U\tilde{B}U^*)\,\mathrm{diag}(D^{\frac{1}{2}}, 0)] \\
&= \lambda_l(D^{\frac{1}{2}}U_1\tilde{B}U_1^*D^{\frac{1}{2}}) \\
&= \lambda_l(\tilde{B}_1).
\end{aligned}$$

Noticing that

$$D_1^{\frac{1}{2}}(GD_1)D_1^{-\frac{1}{2}} = D_1^{\frac{1}{2}}GD_1^{\frac{1}{2}}$$

and

$$G^{-\frac{1}{2}}(GD_1)G^{\frac{1}{2}} = G^{\frac{1}{2}}D_1G^{\frac{1}{2}},$$

we see that $\tilde{B}_1$, $D_1^{\frac{1}{2}}GD_1^{\frac{1}{2}}$, and $G^{\frac{1}{2}}D_1G^{\frac{1}{2}}$ have the same nonzero eigenvalues, including multiplicities. It follows that, for $l = 1, 2, \ldots, s$,

$$\lambda_l[(BAB^*)/\alpha] = \lambda_l(\tilde{B}_1) = \lambda_l(D_1^{\frac{1}{2}}GD_1^{\frac{1}{2}}) = \lambda_l(G^{\frac{1}{2}}D_1G^{\frac{1}{2}}).$$

For $l = s+1, \ldots, k$, since $\mathrm{rank}[((BAB^*)/\alpha] = s$, we have

$$\lambda_l[(BAB^*)/\alpha] = 0.$$

By the Cauchy interlacing theorem, we have, for $i = 1, 2, \ldots, s$,

$$\lambda_i(D_1) \geq \lambda_{i+r-s}(V_1DV_1^*) = \lambda_{i+r-s}(A) \tag{2.3.26}$$

and for $i = 1, 2, \ldots, r$,

$$\lambda_i(U_1\tilde{B}U_1^*) = \lambda_i(U_1\tilde{B}U_1^*) \geq \lambda_{i+n-r}(\tilde{B}). \tag{2.3.27}$$

By (2.0.3) and (2.2.17), we have, for $t = 1, \ldots, s-l+1$, $u = 1, \ldots, t$,

$$\begin{aligned}
&\lambda_l[(BAB^*)/\alpha] \\
&= \lambda_l(D_1^{\frac{1}{2}} G D_1^{\frac{1}{2}}) \\
&\geq \lambda_{l+t-1}(D_1^{\frac{1}{2}})\sigma_{s-t+1}(GD_1^{\frac{1}{2}}) \quad \text{[by (2.0.3)]} \\
&\geq \lambda_{l+t-1}(D_1^{\frac{1}{2}})\lambda_{s-t+1+u-1}(G)\lambda_{s-u+1}(D_1^{\frac{1}{2}}) \quad \text{[by (2.0.3)]} \\
&\geq [\lambda_{l+t-1+r-s}(A)\lambda_{r-u+1}(A)]^{\frac{1}{2}} \\
&\quad \cdot\lambda_{s-t+u+n-r}(\tilde{B}) \quad \text{[by (2.3.26) and (2.3.27)]} \\
&= [\lambda_{l+t+r-s-1}(A)\lambda_{r-u+1}(A)]^{\frac{1}{2}} \\
&\quad \cdot\lambda_{s-t+u+n-r}[B^*(I_k\ X)^*(I_k\ X)B] \\
&= [\lambda_{l+t+r-s-1}(A)\lambda_{r-u+1}(A)]^{\frac{1}{2}} \\
&\quad \cdot\lambda_{s-t+u+n-r}[(I_k\ X)BB^*(I_k\ X)^*] \\
&\geq [\lambda_{l+t+r-s-1}(A)\lambda_{r-u+1}(A)]^{\frac{1}{2}} \\
&\quad \cdot \min_{Z\in C^{k\times(m-k)}} \lambda_{s-t+u+n-r}[(I_k\ Z)BB^*(I_k\ Z)^*] \\
&= [\lambda_{l+t+r-s-1}(A)\lambda_{r-u+1}(A)]^{\frac{1}{2}} \\
&\quad \cdot\lambda_{s-t+u+n-r}[(BB^*)/\alpha] \quad \text{[by (2.2.17)].} \ \blacksquare
\end{aligned}$$

In a similar manner, one can obtain the following additional inequalities

$$\lambda_l[(BAB^*)/\alpha] \geq$$

$$\max_{\substack{t=1,\ldots,s-l+1\\ u=1,\ldots,t}} \left\{ \begin{array}{l}
\{\lambda_{l+t-1+n-r}[(BB^*)/\alpha]\lambda_{s-u+1+n-r}[(BB^*)/\alpha]\}^{\frac{1}{2}}\lambda_{r-t+u}(A), \\
[\lambda_{r-t+u}(A)\lambda_{l+t+r-s-1}(A)]^{\frac{1}{2}}\lambda_{s-u+1+n-r}[(BB^*)/\alpha], \\
[\lambda_{r-u+1}(A)\lambda_{r-t+1}(A)]^{\frac{1}{2}}\lambda_{l+t+u-2+n-r}[(BB^*)/\alpha], \\
[\lambda_{l+t+u-2+r-s}(A)\lambda_{r-t+1}(A)]^{\frac{1}{2}}\lambda_{s-u+1+n-r}[(BB^*)/\alpha], \\
\{\lambda_{l+t-1+n-r}[(BB^*)/\alpha]\lambda_{s-t+u+n-r}[(BB^*)/\alpha]\}^{\frac{1}{2}}\lambda_{r-u+1}(A), \\
\{\lambda_{s-t+1+n-r}[(BB^*)/\alpha]\lambda_{s-u+1+n-r}[(BB^*)/\alpha]\}^{\frac{1}{2}}\lambda_{r+t+u-2}(A), \\
\{\lambda_{s-t+1+n-r}[(BB^*)/\alpha]\lambda_{s+t+u-2+n-r}[(BB^*)/\alpha]\}^{\frac{1}{2}}\lambda_{r-u+1}(A).
\end{array} \right.$$

Setting $r = n$ in the first inequality above, we arrive at

$$\lambda_l[(BAB^*)/\alpha] \geq \max_{\substack{t=1,\ldots,s-l+1\\ u=1,\ldots,t}} \{\lambda_{l+t-1}[(BB^*)/\alpha]\lambda_{s-u+1}[(BB^*)/\alpha]\}^{\frac{1}{2}}\lambda_{n-t+u}(A).$$

In particular, letting $t = u = 1$ reveals that

$$\lambda_l[(BAB^*)/\alpha] \geq [\lambda_l[(BB^*)/\alpha]\lambda_s[(BB^*)/\alpha]]^{\frac{1}{2}}\,\lambda_n(A).$$

If we take $\alpha = \{1\}$ and set $\tilde{X} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & X \end{smallmatrix}\right)$ for the matrix X, then for any $n \times n$ matrices A and B, we obtain

$$\lambda_l(BAB^*) \geq [\lambda_l(BB^*)\lambda_s(BB^*)]^{\frac{1}{2}} \lambda_{n+1}(\tilde{A}).$$

The result below presents a lower bound for the product of eigenvalues.

Theorem 2.9 *Let all assumptions of Theorem 2.8 be satisfied, let u be a positive integer with $1 \leq u \leq k$, and let $1 \leq i_1 < \cdots < i_u \leq k$. Then*

$$\prod_{t=1}^{u} \lambda_t[(BAB^*)/\alpha] \geq \prod_{t=1}^{u} [\lambda_{r-i_t+1}(A)\lambda_{r-t+1}(A)]^{\frac{1}{2}} \lambda_{n-r+i_t}[(BB^*)/\alpha].$$

Proof. Following the line of the proof of the previous theorem, we have

$$\begin{aligned}
&\prod_{t=1}^{u} \lambda_t[(BAB^*)/\alpha] \\
&\quad = \prod_{t=1}^{u} \lambda_t(D_1^{\frac{1}{2}} G D_1^{\frac{1}{2}}) \\
&\quad \geq \prod_{t=1}^{u} \lambda_{s-i_t+1}(D_1^{\frac{1}{2}})\lambda_{i_t}(G D_1^{\frac{1}{2}}) \quad \text{[by (2.0.1)]} \\
&\quad \geq \prod_{t=1}^{u} \lambda_{s-i_t+1}(D_1^{\frac{1}{2}})\lambda_{s-t+1}(D_1^{\frac{1}{2}})\lambda_{i_t}(G) \quad \text{[by (2.0.2)]} \\
&\quad \geq \prod_{t=1}^{u} [\lambda_{r-i_t+1}(A)\lambda_{r-t+1}(A)]^{\frac{1}{2}} \\
&\qquad \cdot \lambda_{n-r+i_t}(\tilde{B}) \quad \text{[by (2.3.26) and (2.3.27)]} \\
&\quad = \prod_{t=1}^{u} [\lambda_{r-i_t+1}(A)\lambda_{r-t+1}(A)]^{\frac{1}{2}} \\
&\qquad \cdot \lambda_{n-r+i_t}[(I_k \ X)BB^*(I_k \ X)^*] \\
&\quad \geq \prod_{t=1}^{u} [\lambda_{r-i_t+1}(A)\lambda_{r-t+1}(A)]^{\frac{1}{2}} \\
&\qquad \cdot \min_{Z \in C^{k \times (m-k)}} \lambda_{n-r+i_t}[(I_k \ Z)BB^*(I_k \ Z)^*] \\
&\quad = \prod_{t=1}^{u} [\lambda_{r-i_t+1}(A)\lambda_{r-t+1}(A)]^{\frac{1}{2}} \\
&\qquad \cdot \lambda_{n-r+i_t}[(BB^*)/\alpha] \quad \text{[by (2.2.17)]}. \ \blacksquare
\end{aligned}$$

Similar results are

$$\prod_{t=1}^{u} \lambda_t[(BAB^*)/\alpha] \geq$$

$$\begin{cases} \prod_{t=1}^{u} [\lambda_{r-i_t+1}(A)\lambda_{r-s+i_t}(A)]^{\frac{1}{2}} \lambda_{s-t+1+n-r}[(BB^*)/\alpha], \\ \prod_{t=1}^{u} \{\lambda_{s-i_t+1+n-r}[(BB^*)/\alpha]\lambda_{n-r+i_t}[(BB^*)/\alpha]\}^{\frac{1}{2}} \lambda_{r-t+1}(A), \\ \prod_{t=1}^{u} \{\lambda_{s-i_t+1+n-r}[(BB^*)/\alpha]\lambda_{s-t+1+n-r}[(BB^*)/\alpha]\}^{\frac{1}{2}} \lambda_{r-s+i_t}(A). \end{cases}$$

2.4 Eigenvalues of the Schur complement of a sum

This section is concerned with inequalities involving the eigenvalues of Schur complements of sums of positive semidefinite matrices [289].

Theorem 2.10 *Let A, B be $n\times n$ positive semidefinite. Let $\alpha \subset \{1, 2, \ldots, n\}$ and $k = n - |\alpha|$. If $1 \leq i_1 < \cdots < i_l \leq n$, where $1 \leq l \leq k$, then*

$$\sum_{t=1}^{l} \lambda_{i_t}[(A+B)/\alpha] \geq \sum_{t=1}^{l} \lambda_{i_t}(A/\alpha) + \sum_{t=1}^{l} \lambda_{k-t+1}(B/\alpha).$$

Proof. This actually follows immediately from (2.0.8) and the fact that $(A+B)/\alpha \geq A/\alpha + B/\alpha$. It can also be proven by (2.2.19) as follows. As in the proof of Theorem 2.5, we may take $\alpha = \{k+1, \ldots, n\}$ and have

$$\begin{aligned} &\sum_{t=1}^{l} \lambda_{i_t}[(A+B)/\alpha] \\ &\quad = \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(I_k, Z)(A+B)(I_k, Z)^*] \\ &\quad = \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(I_k, Z)A(I_k, Z)^* + (I_k, Z)B(I_k, Z)^*] \\ &\quad \geq \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(I_k, Z)A(I_k, Z)^*] \\ &\qquad + \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{k-t+1}[(I_k, Z)B(I_k, Z)^*] \\ &\quad = \sum_{t=1}^{l} \lambda_{i_t}(A/\alpha) + \sum_{t=1}^{l} \lambda_{k-t+1}(B/\alpha). \blacksquare \end{aligned}$$

Note that $\prod_{i=1}^{l}(x_i+y_i)^{1/l} \geq (\prod_{i=1}^{l} x_i)^{1/l}+(\prod_{i=1}^{l} y_i)^{1/l}$ for nonnegative x, y's and $(a+b)^p \geq a^p + b^p$ for a, $b \geq 0$, $p \geq 1$. Setting $i_t = k-t+1$, $x_t = \lambda_t[(A+B)/\alpha]$ and $y_t = \lambda_t(A/\alpha)+\lambda_t(B/\alpha)$ and by (2.0.5), we have

Corollary 2.7 *Let A, B be $n\times n$ positive semidefinite. Let $\alpha \subset \{1,2,\ldots,n\}$ and $k = n-|\alpha|$. Then for any integer l, $1 \leq l \leq k$, and real number $p > 1$,*

$$\prod_{t=1}^{l} \lambda_{k-t+1}^{p/l}[(A+B)/\alpha] \geq \prod_{t=1}^{l} \lambda_{k-t+1}^{p/l}(A/\alpha) + \prod_{t=1}^{l} \lambda_{k-t+1}^{p/l}(B/\alpha).$$

Putting $l = k$ and $p = 1$ in the corollary reveals the known result:

$$\left(\frac{\det(A+B)}{\det(A+B)[\alpha]}\right)^{1/k} \geq \left(\frac{\det A}{\det A[\alpha]}\right)^{1/k} + \left(\frac{\det B}{\det B[\alpha]}\right)^{1/k}.$$

By mathematical induction, we may extend our results to multiple copies of positive semidefinite matrices.

Corollary 2.8 *Let $A_1,\ldots,A_m$ be $n\times n$ positive semidefinite matrices. Let $\alpha \subset \{1,2,\ldots,n\}$ and $k = n-|\alpha|$. Then for any integer l, $1 \leq l \leq k$, and real number $p > 1$,*

$$\sum_{t=1}^{l} \lambda_{k-t+1}\left[\left(\sum_{j=1}^{m} A_j\right)\Big/\alpha\right] \geq \sum_{t=1}^{l}\sum_{j=1}^{m} \lambda_{k-t+1}(A_j/\alpha)$$

and

$$\prod_{t=1}^{l} \lambda_{k-t+1}^{p/l}\left[\left(\sum_{j=1}^{m} A_j\right)\Big/\alpha\right] \geq \sum_{j=1}^{m}\prod_{t=1}^{l} \lambda_{k-t+1}^{p/l}(A_j/\alpha).$$

The next theorem presents a sum-product to product-sum inequality on Schur complements. For this purpose, we recall the Hölder inequality [22]: Let $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ be nonnegative numbers, let p be a nonzero number, $p < 1$, and let $\frac{1}{p}+\frac{1}{p'} = 1$. Then, assuming x, $y > 0$ if $p < 0$,

$$\sum_{t=1}^{n} x_i y_i \geq \left(\sum_{t=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{t=1}^{n} y_i^{p'}\right)^{\frac{1}{p'}}.$$

We note here that if we take in the following theorem $\alpha = \{1\}$ or $\{1, \ldots, n\}$ and embed matrices A in

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \quad \text{or} \quad \tilde{A} = \begin{pmatrix} A & 0 \\ 0 & \operatorname{tr} A \end{pmatrix}$$

respectively, we may arrive at many matrix (trace) and scalar inequalities.

Theorem 2.11 *Let A_{pq}, $p = 1, 2, \ldots, \mu$, $q = 1, 2, \ldots, \nu$, be $n \times n$ positive semidefinite matrices. Let $\alpha \subset \{1, 2, \ldots, n\}$, $k = n - |\alpha|$, and l be an integer, $1 \le l \le k$. Then for any nonzero real $r < 1$ and ω, $0 < \omega \le l$, conventionally assuming that all A_{pq} are positive definite if $r < 0$, we have*

$$\left(\sum_{p=1}^{\mu} \left\{ \prod_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum_{q=1}^{\nu} A_{pq} \right) / \alpha \right] \right\}^{r/\omega} \right)^{1/r}$$

$$\geq \sum_{q=1}^{\nu} \left\{ \sum_{p=1}^{\mu} \left[\prod_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{r/\omega} \right\}^{1/r}.$$

Proof. Let s be the number so that $1/r + 1/s = 1$. Then $(r-1)s = r$. Set

$$C_{pq} \equiv \left[\prod_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{1/\omega}$$

and

$$B_p \equiv \left\{ \prod_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum_{q=1}^{\nu} A_{pq} \right) / \alpha \right] \right\}^{(r-1)/\omega}.$$

Then we need to show

$$\left(\sum_{p=1}^{\mu} B_p^s \right)^{1/r} \geq \sum_{q=1}^{\nu} \left(\sum_{p=1}^{\mu} C_{pq}^r \right)^{1/r}.$$

Note that $1/\omega \geq 1/l$. By Corollary 2.8, we have

$$\left\{ \prod_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum_{q=1}^{\nu} A_{pq} \right) / \alpha \right] \right\}^{1/\omega} \geq \sum_{q=1}^{\nu} \left[\prod_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{1/\omega} = \sum_{q=1}^{\nu} C_{pq},$$

from which, and by the Hölder inequality, we have

$$\begin{aligned}
\sum_{p=1}^{\mu} B_p^s &= \sum_{p=1}^{\mu} \left\{ \prod_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum_{q=1}^{\nu} A_{pq} \right) / \alpha \right] \right\}^{r/\omega} \\
&= \sum_{p=1}^{\mu} \left\{ \prod_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum_{q=1}^{\nu} A_{pq} \right) / \alpha \right] \right\}^{1/\omega} B_p \\
&\geq \sum_{p=1}^{\mu} \left(\sum_{q=1}^{\nu} C_{pq} \right) B_p \\
&= \sum_{q=1}^{\nu} \sum_{p=1}^{\mu} C_{pq} B_p \\
&\geq \sum_{q=1}^{\nu} \left[\left(\sum_{p=1}^{\mu} C_{pq}^r \right)^{1/r} \left(\sum_{p=1}^{\mu} B_p^s \right)^{1/s} \right] \\
&= \left[\sum_{q=1}^{\nu} \left(\sum_{p=1}^{\mu} C_{pq}^r \right)^{1/r} \right] \left(\sum_{p=1}^{\mu} B_p^s \right)^{1/s}.
\end{aligned}$$

Since $1 - \frac{1}{s} = \frac{1}{r}$, dividing by $\left(\sum_{p=1}^{\mu} B_p^s \right)^{1/s}$ yields the desired result. ∎

If we set $\omega = 1$ in the theorem, we then obtain

$$\left(\sum_{p=1}^{\mu} \left\{ \prod_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum_{q=1}^{\nu} A_{pq} \right) / \alpha \right] \right\}^{r} \right)^{1/r}$$

$$\geq \quad \sum_{q=1}^{\nu} \left\{ \sum_{p=1}^{\mu} \left[\prod_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{r} \right\}^{1/r}.$$

Theorem 2.12 *Let A_{pq}, $p = 1, 2, \ldots, \mu$, $q = 1, 2, \ldots, \nu$, be $n \times n$ positive semidefinite matrices. Let $\alpha \subset \{1, 2, \ldots, n\}$ and denote $k = n - |\alpha|$. Let l be an integer such that $1 \leq l \leq k$ and $c_1, c_2, \ldots, c_\mu$ be positive numbers such that $c_1 + c_2 + \cdots + c_\mu \geq 1/l$. Then*

$$\sum_{q=1}^{\nu} \sum_{p=1}^{\mu} \left[\prod_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{c_p} \leq \prod_{p=1}^{\mu} \left\{ \prod_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum_{q=1}^{\nu} A_{pq} \right) / \alpha \right] \right\}^{c_p}.$$

Proof. All we need to show is that

$$L \equiv \frac{\sum\limits_{q=1}^{\nu} \prod\limits_{p=1}^{\mu} \left[\prod\limits_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{c_p}}{\prod\limits_{p=1}^{\mu} \left\{ \prod\limits_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum\limits_{q=1}^{\nu} A_{pq} \right) /\alpha \right] \right\}^{c_p}} \leq 1.$$

Let $c = \sum_{p=1}^{\mu} c_p$, $c'_p = c_p/c$, $p = 1, 2, \ldots, \mu$. Then $c \geq 1/l$ and

$$\sum_{p=1}^{\mu} c'_p = \sum_{p=1}^{\mu} \frac{c_p}{c} = \frac{1}{c} \sum_{p=1}^{\mu} c_p = 1.$$

By the weighted arithmetic-geometric mean inequality and Corollary 2.8,

$$\begin{aligned}
L &= \sum_{q=1}^{\nu} \prod_{p=1}^{\mu} \frac{\left[\prod\limits_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{c_p}}{\left\{ \prod\limits_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum\limits_{q=1}^{\nu} A_{pq} \right) /\alpha \right] \right\}^{c_p}} \\
&= \sum_{q=1}^{\nu} \prod_{p=1}^{\mu} \left(\frac{\left[\prod\limits_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{c}}{\left\{ \prod\limits_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum\limits_{q=1}^{\nu} A_{pq} \right) /\alpha \right] \right\}^{c}} \right)^{c'_p} \\
&\leq \sum_{q=1}^{\nu} \sum_{p=1}^{\mu} c'_p \frac{\left[\prod\limits_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{c}}{\left\{ \prod\limits_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum\limits_{q=1}^{\nu} A_{pq} \right) /\alpha \right] \right\}^{c}} \\
&= \sum_{p=1}^{\mu} c'_p \frac{\sum\limits_{q=1}^{\nu} \left[\prod\limits_{t=1}^{l} \lambda_{k-t+1}(A_{pq}/\alpha) \right]^{c}}{\left\{ \prod\limits_{t=1}^{l} \lambda_{k-t+1} \left[\left(\sum\limits_{q=1}^{\nu} A_{pq} \right) /\alpha \right] \right\}^{c}} \\
&\leq \sum_{p=1}^{\mu} c'_p \quad \text{[by Corollary 2.8]} \\
&= 1. \blacksquare
\end{aligned}$$

2.5 The Hermitian case

In the previous sections, we presented some eigenvalue inequalities for the Schur complements of positive semidefinite matrices. In particular, we paid attention to the matrices in the form BAB^*, where A is positive semidefinite. We now study the inequalities for the Hermitian case of matrix A. Unless otherwise stated, we arrange the eigenvalues of $A \in \mathbb{H}_n$ in the order

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

Theorem 2.13 *Let $A \in \mathbb{H}_n$, $B \in \mathbb{C}^{m\times n}$, and $\alpha \subset \{1, 2, \ldots, m\}$. Denote $k = m - |\alpha|$. Then for every $t = 1, 2, \ldots, k$,*

$$\lambda_t[(BAB^*)/\alpha] \geq \max_{t\leq r\leq k} \{\lambda_{n-r+t}(A)\lambda_r[(BB^*)/\alpha] : \lambda_{n-r+t}(A) \geq 0\}$$

and

$$\lambda_t[(BAB^*)/\alpha] \leq \min_{1\leq r\leq t} \{\lambda_r(A)\lambda_{k+r-t}[(BB^*)/\alpha] : \lambda_r(A) \leq 0\}.$$

Proof. Without loss of generality, we assume that $\alpha = \{k+1, \ldots, m\}$. Let

$$X = -[(BAB^*)[\alpha^c, \alpha]][(BAB^*)[\alpha]]^\dagger, \quad C = (I_k, X)B.$$

On one hand, for any integer r, $1 \leq r \leq k$, we have

$$CAC^* = C[A - \lambda_{n-r+t}(A)I_n]C^* + \lambda_{n-r+t}(A)CC^*,$$

where $A - \lambda_{n-r+t}(A)I_n$ is $n \times n$ Hermitian and $\lambda_{n-r+t}(A)CC^*$ is $k \times k$ Hermitian. Thus, there exists an $n \times n$ unitary matrix U such that

$$A - \lambda_{n-r+t}(A)I_n = U \operatorname{diag}(\lambda_1(A) - \lambda_{n-r+t}(A), \ldots, \lambda_n(A) - \lambda_{n-r+t}(A))U^*.$$

On the other hand, putting $P = CU$, we have

$$\begin{aligned} &C[A - \lambda_{n-r+t}(A)I_n]C^* \\ &\quad = P \operatorname{diag}(\lambda_1(A) - \lambda_{n-r+t}(A), \ldots, \lambda_n(A) - \lambda_{n-r+t}(A))P^* \\ &\quad \geq P \begin{pmatrix} 0 & 0 \\ 0 & [\lambda_n(A) - \lambda_{n-r+t}(A)]I_{r-t} \end{pmatrix} P^* \equiv D. \end{aligned}$$

Since $-D$ is $k \times k$ positive semidefinite and $\operatorname{rank}(-D) \leq r - t$, we see that

$$-\lambda_{k-r+t}(D) = \lambda_{r-t+1}(-D) = 0$$

and

$$\lambda_{k-r+t}[C(A - \lambda_{n-r+t}(A)I_n)C^*] \geq \lambda_{k-r+t}(D) = 0.$$

Thus

$$\begin{aligned}
\lambda_t[(BAB^*)/\alpha] &= \lambda_t[(I_k, X)BAB^*(I_k, X)^*] \\
&= \lambda_t(CAC^*) \\
&= \lambda_t[C(A - \lambda_{n-r+t}(A)I_n)C^* + \lambda_{n-r+t}(A)CC^*] \\
&\geq \max_{r+s=k+t} \{\lambda_s[C(A - \lambda_{n-r+t}(A)I_n)C^*] \\
&\quad +\lambda_r[\lambda_{n-r+t}(A)CC^*]\} \quad \text{[by (2.0.9)]} \\
&= \max_{t \leq r \leq k} \{\lambda_{k-r+t}[C(A - \lambda_{n-r+t}(A)I_n)C^*] \\
&\quad +\lambda_r[\lambda_{n-r+t}(A)CC^*]\} \\
&\geq \max_{t \leq r \leq k} \{\lambda_{n-r+t}(A)\lambda_r(CC^*)\}.
\end{aligned}$$

It follows that, if $\lambda_{n-r+t}(A) \geq 0$, by (2.2.17), we have

$$\begin{aligned}
\lambda_t[(BAB^*)/\alpha] &\geq \max_{t \leq r \leq k} \{\lambda_{n-r+t}(A)\lambda_r(CC^*)\} \\
&= \max_{t \leq r \leq k} \{\lambda_{n-r+t}(A)\lambda_r[(I_k, X)BB^*(I_k, X)^*]\} \\
&\geq \max_{t \leq r \leq k} \{\lambda_{n-r+t}(A) \min_{Z \in \mathbb{C}^{k \times (n-k)}} \lambda_r[(I_k, Z)BB^*(I_k, Z)^*]\} \\
&= \max_{t \leq r \leq k} \{\lambda_{n-r+t}(A)\lambda_r[(BB^*)/\alpha]\}.
\end{aligned}$$

This completes the proof of the first inequality. The second inequality on the minimum can be similarly dealt with by substituting $-A$ for A. ∎

As an application of the theorem, setting $B = I_n, r = k$ and $B = I_n, r = t$, respectively, we see an interlacing-like result for the Hermitian case:

$$\lambda_t(A/\alpha) \geq \lambda_{n-k+t}(A), \quad \text{if } \lambda_{n-k+t}(A) \geq 0$$

and

$$\lambda_t(A/\alpha) \leq \lambda_t(A), \quad \text{if } \lambda_t(A) \leq 0.$$

In the following two theorems, Theorem 2.14 and Theorem 2.15, for a Hermitian $A \in \mathbb{H}_n$, we arrange and label the eigenvalues of A in the order so that $|\lambda_1(A)| \geq |\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$. Our next theorem, like Theorem 2.8, gives lower bounds for the eigenvalues of the Schur complement of matrix product in terms of those of the individual matrices.

Theorem 2.14 *Let $A \in \mathbb{H}_n, B \in \mathbb{C}^{m\times n}$, and $\alpha \subset \{1, 2, \ldots, m\}$. Denote $k = m - |\alpha|$. Let the rank of A be r. Then for each $l = 1, 2, \ldots, k$,*

$$|\lambda_l[(BAB^*)/\alpha]| \geq \max_{\substack{t=1,\ldots,r-l+1\\ u=1,\ldots,t}} \left\{ \begin{array}{l} \{\lambda_{l+t-1+n-r}[(BB^*)/\alpha]\lambda_{n-u+1}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{r+u-t}(A)| \\ \{\lambda_{l+t-1+n-r}[(BB^*)/\alpha]\lambda_{n-t+u}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{r-u+1}(A)| \\ \{\lambda_{n-t+1}[(BB^*)/\alpha]\lambda_{n-u+1}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{l+t+u-2}(A)| \\ \{\lambda_{n-t+1}[(BB^*)/\alpha]\lambda_{l+t+u-2+n-r}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{r-u+1}(A)|. \end{array} \right.$$

Proof. We may assume that $\alpha = \{k + 1, \ldots, m\}$. Since $A \in \mathbb{H}_n$ and $\operatorname{rank}(A) = r$, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $UAU^* = \operatorname{diag}(D, 0)$, where $D = \operatorname{diag}(\lambda_1(A), \ldots, \lambda_r(A))$, and D is nonsingular. Let

$$X = -[(BAB^*)[\alpha^c, \alpha]][(BAB^*)[\alpha]]^\dagger.$$

Then

$$\begin{aligned} |\lambda_l[(BAB^*)/\alpha]| &= |\lambda_l[(I_k, X)BAB^*(I_k, X)^*]| \\ &= |\lambda_l[AB^*(I_k, X)^*(I_k, X)B]| \\ &= |\lambda_l(A\tilde{B})|, \end{aligned}$$

where $\tilde{B} = B^*(I_k, X)^*(I_k, X)B$. Partition

$$U\tilde{B}U^* = \begin{pmatrix} B_1 & B_2 \\ B_2^* & B_3 \end{pmatrix},$$

where B_1 is $r \times r$ positive semidefinite. Take

$$L = \begin{pmatrix} I_r & 0 \\ -B_2^*B_1^\dagger & I_{n-r} \end{pmatrix}.$$

Then

$$LU\tilde{B}U^*L^* = \operatorname{diag}(B_1, B_3 - B_2^*B_1^\dagger B_2)$$

and

$$L^{*-1}UAU^*L^{-1} = \operatorname{diag}(D, 0).$$

Thus

$$\begin{aligned} (L^{*-1}U)(A\tilde{B})(L^{*-1}U)^{-1} &= L^{*-1}UAU^*L^{-1}LU\tilde{B}U^*L^* \\ &= \operatorname{diag}(D, 0)\operatorname{diag}(B_1, B_3 - B_2^*B_1^\dagger B_2) \\ &= \operatorname{diag}(DB_1, 0). \end{aligned}$$

That is, $A\tilde{B}$ and $\operatorname{diag}(DB_1, 0)$ have the same set of eigenvalues. Thus

$$|\lambda_l[(BAB^*)/\alpha]| = |\lambda_l(A\tilde{B})| = |\lambda_l(\operatorname{diag}(DB_1, 0))|.$$

It follows that, for $l > r$,

$$|\lambda_l[(BAB^*)/\alpha]| = 0$$

and that, for $l = 1, 2, \ldots, r$,

$$|\lambda_l[(BAB^*)/\alpha]| = |\lambda_l(DB_1)|.$$

Notice that the eigenvalue interlacing theorem shows that, for $i = 1, 2, \ldots, r$,

$$\lambda_i(B_1) \geq \lambda_{i+n-r}(U\tilde{B}U^*) = \lambda_{i+n-r}(\tilde{B}).$$

We have, for $t = 1, \ldots, r - l + 1$ and $u = 1, \ldots, t$,

$$\begin{aligned}
&|\lambda_l[(BAB^*)/\alpha]| \\
&\quad = \lambda_l(B_1^{\frac{1}{2}} D B_1^{\frac{1}{2}}) \\
&\quad \geq \lambda_{l+t-1}(B_1^{\frac{1}{2}})\lambda_{r-t+1}(DB_1^{\frac{1}{2}}) \ \ \text{[by (2.0.3)]} \\
&\quad \geq \lambda_{l+t-1}(B_1^{\frac{1}{2}})\lambda_{r-t+u}(D)\lambda_{r-u+1}(B_1^{\frac{1}{2}}) \ \ \text{[by (2.0.3)]} \\
&\quad \geq [\lambda_{l+t-1+n-r}(\tilde{B})\lambda_{n-u+1}(\tilde{B})]^{\frac{1}{2}}|\lambda_{r+u-t}(A)| \\
&\quad = \Big\{\lambda_{l+t-1+n-r}[(I_k \ X)BB^*(I_k \ X)^*] \\
&\qquad \cdot\lambda_{n-u+1}[(I_k \ X)BB^*(I_k \ X)^*]\Big\}^{\frac{1}{2}}|\lambda_{r+u-t}(A)| \\
&\quad \geq \{\min_{Z\in\mathbb{C}^{k\times(m-k)}} \lambda_{l+t-1+n-r}[(I_k \ Z)BB^*(I_k \ Z)^*] \\
&\qquad \cdot \min_{Z\in\mathbb{C}^{k\times(m-k)}} \lambda_{n-u+1}[(I_k \ Z)BB^*(I_k \ Z)^*]\}^{\frac{1}{2}}|\lambda_{r+u-t}(A)| \\
&\quad = \{\lambda_{l+t-1+n-r}[(BB^*)/\alpha]\lambda_{n-u+1}[(BB^*)/\alpha]\}^{\frac{1}{2}} \\
&\qquad \cdot|\lambda_{r+u-t}(A)| \ \ \text{[by (2.2.17)]}.
\end{aligned}$$

The other remaining inequalities can be proved similarly. ∎

Setting $r = n$ in Theorem 2.14 yields the following

$$|\lambda_l[(BAB^*)/\alpha]| \geq$$

$$\max_{\substack{t=1,\ldots,n-l+1 \\ u=1,\ldots,t}} \left\{ \begin{array}{l}
\{\lambda_{l+t-1}[(BB^*)/\alpha]\lambda_{n-u+1}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{n-t+u}(A)|, \\
\{\lambda_{l+t-1}[(BB^*)/\alpha]\lambda_{n-t+u}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{n-u+1}(A)|, \\
\{\lambda_{n-t+1}[(BB^*)/\alpha]\lambda_{n-u+1}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{l+t+u-2}(A)|, \\
\{\lambda_{n-t+1}[(BB^*)/\alpha]\lambda_{l+t+u-2}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{n-u+1}(A)|.
\end{array} \right.$$

Our next theorem is a version of Theorem 2.9 for Hermitian matrices.

Theorem 2.15 *Let $A \in \mathbb{H}_n$ with* $\operatorname{rank} A = r$ *and* $B \in \mathbb{C}^{m\times n}$. *Let* $\alpha \subset \{1,2,\ldots,m\}$ *and denote* $k = m - |\alpha|$. *Then for any* $1 \le i_1 < \cdots < i_u \le k$,

$$\prod_{t=1}^{u} |\lambda_t[(BAB^*)/\alpha]| \ge \prod_{t=1}^{u}\{\lambda_{n-i_t+1}[(BB^*)/\alpha]\lambda_{n-r+i_t}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{r-t+1}(A)|.$$

Proof. Following the line of the proof of Theorem 2.14, we have

$$\begin{aligned}
&\prod_{t=1}^{u}|\lambda_t[(BAB^*)/\alpha]|\\
&= \prod_{t=1}^{u}\lambda_t(B_1^{\frac{1}{2}}DB_1^{\frac{1}{2}})\\
&\ge \prod_{t=1}^{u}\lambda_{r-i_t+1}(B_1^{\frac{1}{2}})\sigma_{i_t}(DB_1^{\frac{1}{2}}) \quad \text{[by (2.0.1)]}\\
&\ge \prod_{t=1}^{u}\lambda_{r-i_t+1}(B_1^{\frac{1}{2}})\lambda_{i_t}(B_1^{\frac{1}{2}})\lambda_{r-t+1}(D) \quad \text{[by (2.0.2)]}\\
&\ge \prod_{t=1}^{u}\lambda_{r-i_t+1}(\tilde{B}^{\frac{1}{2}})\lambda_{n-r+i_t}(\tilde{B}^{\frac{1}{2}})\lambda_{r-t+1}(A)\\
&= \prod_{t=1}^{u}[\lambda_{n-i_t+1}(\tilde{B})\lambda_{n-r+i_t}(\tilde{B})]^{\frac{1}{2}}|\lambda_{r-t+1}(A)|\\
&= \prod_{t=1}^{u}\{\lambda_{n-i_t+1}[(I_k\ X)BB^*(I_k\ X)^*]\\
&\quad \cdot\lambda_{n-r+i_t}[(I_k\ X)BB^*(I_k\ X)^*]\}^{\frac{1}{2}}|\lambda_{r-t+1}(A)|\\
&\ge \prod_{t=1}^{u}\{\min_{Z\in\mathbb{C}^{k\times(m-k)}}\lambda_{n-i_t+1}[(I_k\ Z)BB^*(I_k\ Z)^*]\\
&\quad \cdot\min_{Z\in\mathbb{C}^{k\times(m-k)}}\lambda_{n-r+i_t}[(I_k\ Z)BB^*(I_k\ Z)^*]\}^{\frac{1}{2}}|\lambda_{r-t+1}(A)|\\
&= \prod_{t=1}^{u}\{\lambda_{n-i_t+1}[(BB^*)/\alpha]\lambda_{n-r+i_t}[(BB^*)/\alpha]\}^{\frac{1}{2}}\\
&\quad \cdot|\lambda_{r-t+1}(A)| \quad \text{[by (2.2.17)]}. \blacksquare
\end{aligned}$$

Setting $r = n$ in Theorem 2.15, we arrive at

$$\prod_{t=1}^{u}|\lambda_t[(BAB^*)/\alpha]| \ge \prod_{t=1}^{u}\{\lambda_{n-i_t+1}[(BB^*)/\alpha]\lambda_{i_t}[(BB^*)/\alpha]\}^{\frac{1}{2}}|\lambda_{n-t+1}(A)|.$$

Theorem 2.16 *Let $A \in \mathbb{H}_n$, $B \in \mathbb{C}^{n\times n}$, and $\alpha \subset \{1,2,\dots,n\}$. Denote $k = n - |\alpha|$. Then for any integer l with $1 < l < k$,*

$$\sum_{t=1}^{l} \lambda_{k-t+1}[(BAB^*)/\alpha] \le \sum_{t=1}^{l} \lambda_t(A)\lambda_{i_t}[(BB^*)/\alpha] \tag{2.5.28}$$

and

$$\sum_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]\lambda_{n-t+1}(A) \le \sum_{t=1}^{l} \lambda_t[(BAB^*)/\alpha]. \tag{2.5.29}$$

Proof. If $A \geq 0$, the inequalities follow immediately from Theorem 2.6. So we consider the case where A has negative eigenvalues. Let $\lambda_n(A) < 0$. Without loss of generality, we take $\alpha = \{k+1,\dots,m\}$. Let

$$X = -[(BB^*)[\alpha^c,\alpha]][(BB^*)[\alpha]]^\dagger.$$

By (2.2.19), (2.0.6), we have

$$\begin{aligned}
&\sum_{t=1}^{l} \lambda_{i_t}\{[B(A-\lambda_n(A)I_n)B^*]/\alpha\} \\
&= \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(I_k,Z)B(A-\lambda_n(A)I_n)B^*(I_k,Z)^*] \\
&= \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(A-\lambda_n(A)I_n)B^*(I_k,Z)^*(I_k,Z)B] \\
&\le \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_t[A-\lambda_n(A)]\lambda_{i_t}[B^*(I_k,Z)^*(I_k,Z)B] \\
&= \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} [\lambda_t(A)-\lambda_n(A)]\lambda_{i_t}[(I_k,Z)BB^*(I_k,Z)^*] \\
&\le \sum_{t=1}^{l} [\lambda_t(A)-\lambda_n(A)]\lambda_{i_t}[(I_k,X)BB^*(I_k,X)^*] \\
&= \sum_{t=1}^{l} [\lambda_t(A)-\lambda_n(A)]\lambda_{i_t}[(BB^*)/\alpha] \\
&= \sum_{t=1}^{l} \lambda_t(A)\lambda_{i_t}[(BB^*)/\alpha] - \lambda_n(A)\sum_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]. \qquad (2.5.30)
\end{aligned}$$

By (2.2.19) and (2.0.8), noticing that $-\lambda_n(A) \geq 0$, we have

$$\begin{aligned}
&\sum_{t=1}^{l} \lambda_{i_t}\{[B(A-\lambda_n(A)I_n)B^*]/\alpha\} \\
&= \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(I_k,Z)B(A-\lambda_n(A)I_n)B^*(I_k,Z)^*] \\
&= \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(I_k,Z)BAB^*(I_k,Z)^* - \lambda_n(A)(I_k,Z)BB^*(I_k,Z)^*] \\
&\geq \min_{Z\in\mathbb{C}^{k\times(n-k)}} \Big\{\sum_{t=1}^{l} \lambda_{k-t+1}[(I_k,Z)BAB^*(I_k,Z)^*] \\
&\quad + \sum_{t=1}^{l} \lambda_{i_t}[-\lambda_n(A)(I_k,Z)BB^*(I_k,Z)^*]\Big\} \text{ [by (2.0.8)]} \\
&= \min_{Z\in\mathbb{C}^{k\times(n-k)}} \Big\{\sum_{t=1}^{l} \lambda_{k-t+1}[(I_k,Z)BAB^*(I_k,Z)^*] \\
&\quad -\lambda_n(A)\sum_{t=1}^{l} \lambda_{i_t}[(I_k,Z)BB^*(I_k,Z)^*]\Big\} \\
&\geq \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{k-t+1}[(I_k,Z)BAB^*(I_k,Z)^*] \\
&\quad -\lambda_n(A) \min_{Z\in\mathbb{C}^{k\times(n-k)}} \sum_{t=1}^{l} \lambda_{i_t}[(I_k,Z)BB^*(I_k,Z)^*] \\
&= \sum_{t=1}^{l} \lambda_{k-t+1}[(BAB^*)/\alpha] - \lambda_n(A)\sum_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]. \qquad (2.5.31)
\end{aligned}$$

Combining (2.5.30) and (2.5.31) reveals (2.5.28). Likewise, by making use of (2.0.8) and (2.2.19) in the proof of (2.5.31), we have

$$\begin{aligned}
&\sum_{t=1}^{l} \lambda_{i_t}\{[B(A-\lambda_n(A)I_n)B^*]/\alpha\} \leq \\
&\qquad \sum_{t=1}^{l} \lambda_t[(BAB^*)/\alpha] - \lambda_n(A)\sum_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]. \qquad (2.5.32)
\end{aligned}$$

Using (2.2.19), (2.0.7) and as in the proof of (2.5.30), we have

$$\sum_{t=1}^{l} \lambda_{i_t}\{[B(A-\lambda_n(A)I_n)B^*]/\alpha\} \geq$$
$$\sum_{t=1}^{l} \lambda_{n-t+1}(A)\lambda_{i_t}[(BB^*)/\alpha] - \lambda_n(A)\sum_{t=1}^{l} \lambda_{i_t}[(BB^*)/\alpha]. \quad (2.5.33)$$

Combining (2.5.32) and (2.5.33), we obtain the inequality (2.5.29). ∎

2.6 Singular values of the Schur complement of product

Singular values are, in many aspects, as important as the eigenvalues for matrices. This section, based upon [285], is devoted to the inequalities on singular values of the Schur complements of products of general matrices.

Theorem 2.17 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Let $\alpha \subset \{1,2,\ldots,l\}$, where $l = \min\{m,n,p\}$. If B^*B is nonsingular, then for $s = 1,2,\ldots,l-|\alpha|$,*

$$\sigma_s^2[(AB)/\alpha] \geq \max_{1\leq i\leq m-|\alpha|+p-n} \lambda_{p-|\alpha|-i+s}\left[(B^*B)/\alpha\right]\lambda_{n-p+i}\left[(AA^*)/\alpha\right].$$

Proof. We first claim that we may take $\alpha = \{1,2,\ldots,|\alpha|\}$. To see this, let $\alpha^c = \{1,\ldots,m\}-\alpha$, $\beta^c = \{1,\ldots,n\}-\alpha$, and $\gamma^c = \{1,\ldots,p\}-\alpha$. There exist permutation matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$, $W \in \mathbb{C}^{p\times p}$ such that

$$UAV = \begin{pmatrix} A[\alpha] & A[\alpha,\beta^c] \\ A[\alpha^c,\alpha] & A[\alpha^c,\beta^c] \end{pmatrix},$$

$$V^TBW = \begin{pmatrix} B[\alpha] & B[\alpha,\gamma^c] \\ B[\beta^c,\alpha] & B[\beta^c,\gamma^c] \end{pmatrix},$$

and

$$(UAV)(V^TBW) = UABW = \begin{pmatrix} (AB)[\alpha] & (AB)[\alpha,\gamma^c] \\ (AB)[\alpha^c,\alpha] & (AB)[\alpha^c,\gamma^c] \end{pmatrix}.$$

Let $\bar{\alpha} = \{1,2,\ldots,|\alpha|\}$. Then

$$(AB)/\alpha = (UABW)/\bar{\alpha},$$

$$(B^*B)/\alpha = (W^*B^*BW)/\bar{\alpha},$$

$$(AA^*)/\alpha = (UAA^*U^*)/\bar{\alpha}.$$

So we may replace A with UAV and B with V^TBW in the theorem so that the submatrices indexed by α are now located in the upper left corners. Thus, without loss of generality, we may assume that $\alpha = \{1,2,\ldots,|\alpha|\}$.

The idea of the proof of the inequality is to obtain two quantities, one of which bounds $\lambda_i\{[AB(B^*B)^{-1}B^*A^*]/\alpha\}$ from above, and the other from below; combining the two inequalities will yield the desired inequality.

We shall make heavy use of Corollary 2.6. Let

$$C = AB(B^*B)^{-1}B^*A^*,$$

$$X = -C[\alpha^c,\ \alpha][C[\alpha]]^\dagger,$$

$$Y = -[(AB)[\alpha^c, \alpha]][(AB)[\alpha]]^\dagger,$$

and

$$Z = -[(AA^*)[\alpha^c, \alpha]][(AA^*)[\alpha]]^\dagger.$$

Using (2.0.3) and upon computation, we have

$$\begin{aligned}
&\lambda_i\{[AB(B^*B)^{-1}B^*A^*]/\alpha\}\\
&= \lambda_i[C[\alpha^c] + XC[\alpha,\ \alpha^c]]\\
&= \lambda_i\{(Y,\ I_{m-|\alpha|})C(Y,\ I_{m-|\alpha|})^* - Y(C[\alpha,\ \alpha^c])\\
&\quad -(C[\alpha^c,\ \alpha])Y^* - Y(C[\alpha])Y^* + XC[\alpha,\ \alpha^c]\}\\
&= \lambda_i\{(Y,\ I_{m-|\alpha|})C(Y,\ I_{m-|\alpha|})^*\\
&\quad +(X - Y)(C[\alpha])(C[\alpha])^\dagger(C[\alpha,\ \alpha^c])\\
&\quad -(C[\alpha^c,\ \alpha])(C[\alpha])^\dagger(C[\alpha])Y^* - Y(C[\alpha])Y^*\}\\
&= \lambda_i\{(Y,\ I_{m-|\alpha|})C(Y,\ I_{m-|\alpha|})^* - (X - Y)(C[\alpha])X^*\\
&\quad +X(C[\alpha])Y^* - Y(C[\alpha])Y^*\}\\
&= \lambda_i\{(Y,\ I_{m-|\alpha|})C(Y,\ I_{m-|\alpha|})^*\\
&\quad -(X - Y)(C[\alpha])(X - Y)^*\} &(2.6.34)\\
&\le \lambda_i\left[(Y,\ I_{m-|\alpha|})C(Y,\ I_{m-|\alpha|})^*\right]\\
&= \lambda_i\{[(Y,\ I_{m-|\alpha|})AB](B^*B)^{-1}[(Y,\ I_{m-|\alpha|})AB]^*\}\\
&= \lambda_i\left[(0,\ (AB)/\alpha)(B^*B)^{-1}(0,\ (AB)/\alpha)^*\right]\\
&= \lambda_i\{[(AB)/\alpha][(B^*B)^{-1}[\gamma^c]][(AB)^*/\alpha]\}\\
&= \lambda_i\{[(B^*B)^{-1}[\gamma^c]][(AB)/\alpha]^*[(AB)/\alpha]\} &(2.6.35)\\
&\le \min_{\substack{t+s=i+1\\ t=1,\ldots,p-|\alpha|\\ s=1,\ldots,p-|\alpha|}} \lambda_t[(B^*B)^{-1}[\gamma^c]]\lambda_s\{[(AB)/\alpha]^*[(AB)/\alpha]\}\\
&\le \min_{\substack{t+s=i+1\\ t=1,\ldots,l-|\alpha|\\ s=1,\ldots,l-|\alpha|}} \lambda_t[(B^*B)^{-1}[\gamma^c]]\sigma_s^2[(AB)/\alpha]. &(2.6.36)
\end{aligned}$$

On the other hand, by (2.0.6), for every $i = 1, 2, \ldots, m - |\alpha| + p - n$,

$$\begin{aligned}
&\lambda_i\{[AB(B^*B)^{-1}B^*A^*]/\alpha\} \\
&= \lambda_i\left[(X,\ I_{m-|\alpha|})AB(B^*B)^{-1}B^*A^*(X,\ I_{m-|\alpha|})^*\right] \\
&= \lambda_i\{[B(B^*B)^{-1}B^*][A^*(X,\ I_{m-|\alpha|})^*(X,\ I_{m-|\alpha|})A]\} \\
&\geq \max_{t+s=n+i} \lambda_t[B(B^*B)^{-1}B^*]\lambda_s[A^*(X,\ I_{m-|\alpha|})^*(X,\ I_{m-|\alpha|})A] \\
&\geq \lambda_p[B(B^*B)^{-1}B^*]\lambda_{n-p+i}[(X,\ I_{m-|\alpha|})AA^*(X,\ I_{m-|\alpha|})^*] \\
&= \lambda_p[(B^*B)^{-1}B^*B]\lambda_{n-p+i}[(X,\ I_{m-|\alpha|})AA^*(X,\ I_{m-|\alpha|})^*] \\
&= \lambda_{n-p+i}[(X,\ I_{m-|\alpha|})AA^*(X,\ I_{m-|\alpha|})^*] \\
&= \lambda_{n-p+i}\{(Z,\ I_{m-|\alpha|})AA^*(Z,\ I_{m-|\alpha|})^* \\
&\quad +(X - Z)[(AA^*)[\alpha]](X - Z)^*\} \quad \text{[by (2.6.35)]} \qquad (2.6.37) \\
&\geq \lambda_{n-p+i}\left[(Z,\ I_{m-|\alpha|})AA^*(Z,\ I_{m-|\alpha|})^*\right] \\
&= \lambda_{n-p+i}[(AA^*)/\alpha]. \qquad (2.6.38)
\end{aligned}$$

By Theorem 1.2, $(B^*B)^{-1}[\gamma^c] = [(B^*B)/\alpha]^{-1}$, so for $t = 1, 2, \ldots, p - |\alpha|$,

$$\lambda_t^{-1}\{[(B^*B)/\alpha]^{-1}\} = \lambda_{p-|\alpha|-t+1}[(B^*B)/\alpha],$$

it follows that, by using (2.6.36) and (2.6.38), for $s = 1, 2, \ldots, l - |\alpha|$,

$$\begin{aligned}
\sigma_s^2[(AB)/\alpha] &\geq \max_{\substack{1\leq i\leq m-|\alpha|+p-n \\ t=i-s+1}} \lambda_{p-|\alpha|-t+1}[(B^*B)/\alpha]\lambda_{n-p+i}[(AA^*)/\alpha] \\
&= \max_{1\leq i\leq m-|\alpha|+p-n} \lambda_{p-|\alpha|-i+s}[(B^*B)/\alpha]\lambda_{n-p+i}[(AA^*)/\alpha]. \blacksquare
\end{aligned}$$

Corollary 2.9 *Let* $A \in \mathbb{C}^{m\times n}$, $l = \min\{m,\ n\}$, $\alpha \subset \{1, 2, \ldots, l\}$. *Then*

$$\sigma_i(A/\alpha) \geq \lambda_i^{\frac{1}{2}}[(AA^*)/\alpha] \geq \sigma_{i+|\alpha|}(A), \quad i = 1, 2, \ldots, l - |\alpha|.$$

Proof. Set $B = I$, $i = s$ in Theorem 2.17 and use Corollary 2.4. ∎

Corollary 2.10 *If* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $l = \min\{m, n, p\}$, *and let* $\alpha \subset \{1, 2, \ldots, l\}$. *If* B^*B *is nonsingular, then for* $s = 1, 2, \ldots, l - |\alpha|$,

$$\sigma_s^2[(AB)/\alpha] \geq \max_{i=1,2,\ldots,m-|\alpha|+p-n} \lambda_{n-p+i}[(AA^*)/\alpha]\sigma_{p-i+s}^2(B).$$

Proof. This follows from (2.1.12) and Theorem 2.17. ∎

Setting $m = n = p$ in Corollary 2.10 shows, for each $s = 1, 2, \ldots, n-|\alpha|$,

$$\sigma_s[(AB)/\alpha] \geq \max_{i+j=n+s} \lambda_i^{\frac{1}{2}}[(AA^*)/\alpha]\sigma_j(B).$$

In a similar manner, we obtain lower bounds for products of singular values of Schur complements of matrix products.

Theorem 2.18 *Let the assumptions of Theorem 2.17 be satisfied. Let u be an integer, $1 \le u \le l - |\alpha|$, and $n - p + 1 \le i_1 < \cdots < i_u \le l - |\alpha|$. Then*

$$\prod_{t=1}^{u} \sigma_t[(AB)/\alpha] \ge \prod_{t=1}^{u} \lambda_{i_t}^{\frac{1}{2}}[(AA^*)/\alpha]\lambda_{p-|\alpha|-t+1}^{\frac{1}{2}}[(B^*B)/\alpha].$$

Proof. Following the proof of Theorem 2.17, by (2.6.35) and using (2.0.1), we have, for every $n \ge i_t \ge n - p + 1$, $t = 1, 2, \ldots, u$,

$$\begin{aligned}
&\prod_{t=1}^{u} \lambda_t\{[AB(B^*B)^{-1}B^*A^*]/\alpha\} \\
&\quad = \prod_{t=1}^{u} \lambda_t \left[(X,\ I_{m-|\alpha|})AB(B^*B)^{-1}B^*A^*(X,\ I_{m-|\alpha|})^*\right] \\
&\quad \ge \prod_{t=1}^{u} \lambda_{i_t}[A^*(X,\ I_{m-|\alpha|})^*(X,\ I_{m-|\alpha|})A] \\
&\qquad \cdot\lambda_{n-i_t+1}[B(B^*B)^{-1}B^*] \ \ [\text{by } 2.0.1)] \\
&\quad = \prod_{t=1}^{u} \lambda_{i_t}\{(Z,\ I_{m-|\alpha|})AA^*(Z,\ I_{m-|\alpha|})^* \\
&\qquad +(X-Z)[(AA^*)[\alpha]](X-Z)^*\} \ \ [\text{see } (2.6.35)] \\
&\quad \ge \prod_{t=1}^{u} \lambda_{i_t} \left[(Z,\ I_{m-|\alpha|})AA^*(Z,\ I_{m-|\alpha|})^*\right] \\
&\quad = \prod_{t=1}^{u} \lambda_{i_t}[(AA^*)/\alpha].
\end{aligned} \tag{2.6.39}$$

On the other hand, by (2.6.35) and (2.0.2), we have

$$\begin{aligned}
&\prod_{t=1}^{u} \lambda_t \left\{[AB(B^*B)^{-1}B^*A^*]/\alpha\right\} \\
&\quad = \prod_{t=1}^{u} \lambda_t\{(Y,\ I_{m-|\alpha|})C(Y,\ I_{m-|\alpha|})^* \\
&\qquad -(X-Y)[C[\alpha]](X-Y)^*\} \ \ [\text{by } (2.6.35)] \\
&\quad \le \prod_{t=1}^{u} \lambda_t \left[(Y,\ I_{m-|\alpha|})C(Y,\ I_{m-|\alpha|})^*\right] \\
&\quad \le \prod_{t=1}^{u} \lambda_t[(B^*B)^{-1}(\gamma^c)]\sigma_t^2[(AB)/\alpha] \ \ [\text{by } (2.6.35)].
\end{aligned} \tag{2.6.40}$$

Combining (2.6.39) and (2.6.40) we obtain the desired inequality. ∎

The proof of the next theorem is similar to the above, thus omitted.

Theorem 2.19 *Let $A \in \mathbb{C}^{m\times n}$ and B be $n \times n$ nonsingular. Let $l = \min\{m,\ n\}$, $\alpha \subset \{1, 2, \ldots, l\}$, and u be an integer with $1 \le u \le l - |\alpha|$. Then for $1 \le i_1 < \cdots < i_u \le l - |\alpha|$,*

$$\prod_{t=1}^{u} \sigma_{i_t}[(AB)/\alpha] \ge \prod_{t=1}^{u} \lambda_{i_t}^{\frac{1}{2}}[(AA^*)/\alpha]\lambda_{n-|\alpha|-t+1}^{\frac{1}{2}}[(B^*B)/\alpha]$$

and

$$\prod_{t=1}^{u} \sigma_{t}[(AB)/\alpha] \ge \prod_{t=1}^{u} \lambda_{i_t}^{\frac{1}{2}}[(AA^*)/\alpha]\lambda_{n-|\alpha|-i_t+1}^{\frac{1}{2}}[(B^*B)/\alpha].$$

Corollary 2.11 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Let $l = \min\{m, n, p\}$, $\alpha \subset \{1, 2, \ldots, l\}$, and u be an integer such that $1 \le u \le l - |\alpha|$. Then for $n - p + 1 \le i_1 < \cdots < i_u \le l - |\alpha|$,*

$$\prod_{t=1}^{u} \sigma_{t}[(AB)/\alpha] \ge \prod_{t=1}^{u} \lambda_{i_t}^{\frac{1}{2}}[(AA^*)/\alpha]\sigma_{p-t+1}(B) \tag{2.6.41}$$

and if $n = p$, then

$$\prod_{t=1}^{u} \sigma_{i_t}[(AB)/\alpha] \ge \prod_{t=1}^{u} \lambda_{i_t}^{\frac{1}{2}}[(AA^*)/\alpha]\sigma_{n-t+1}(B). \tag{2.6.42}$$

Proof. If B^*B is singular, for $t = 1$ we have $\sigma_{p-t+1}(B) = \sigma_p(B) = 0$. The first inequality holds. If B^*B is nonsingular, then Theorem 2.18, together with Theorem 2.2, yields the first inequality again. The second inequality can be obtained in a similar manner. ∎

Corollary 2.12 *Let all the assumptions of Theorem 2.19 be satisfied. Then*

$$\prod_{t=1}^{u} \sigma_{t}[(AB)/\alpha] \ge \prod_{t=1}^{u} \lambda_{i_t}^{\frac{1}{2}}[(AA^*)/\alpha]\sigma_{n-i_t+1}(B)$$

and

$$\prod_{t=1}^{u} \sigma_{t}[(AB)/\alpha] \ge \prod_{t=1}^{\mu} \sigma_{i_t+|\alpha|}(A)\lambda_{n-|\alpha|-i_t+1}^{\frac{1}{2}}[(B^*B)/\alpha].$$

All the inequalities we obtained so far in this section present lower bounds for the singular values. It is tempting to obtain analogous results on upper bounds. For instance, we may ask if an analog of (2.0.2)

$$\prod_{t=1}^{l} \sigma_{i_t}[(AB)/\alpha] \leq \min\left\{\prod_{t=1}^{l} \sigma_{i_t}(A/\alpha)\sigma_t(B), \ \prod_{t=1}^{l} \sigma_t(A/\alpha)\sigma_{i_t}(B)\right\}$$

or an analog of (2.0.3)

$$\sigma_t[(AB)/\alpha] \leq \min_{i+j=t+1} \{\sigma_i(A/\alpha)\sigma_j(B), \ \sigma_i(B/\alpha)\sigma_j(A)\}$$

holds. The answer is negative as the following example shows: Take

$$A = I_2, \quad B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad \alpha = \{2\}.$$

Then $\sigma_1[(AB)/\alpha] = \sigma_1(5) = 5, \ \sigma_1(A/\alpha)\sigma_1(B) = \sigma_1(B) = \sqrt{5}$. Thus

$$\sigma_1[(AB)/\alpha] \geq \sigma_1(A/\alpha)\sigma_1(B).$$

This says neither of the above two inequalities holds. This comes as no surprise if one reinspects the signs of the second summands in (2.6.35) and (2.6.38). Furthermore, invalidity remains true even if one replaces the pair $\sigma(A/\alpha)$ and $\sigma(B/\alpha)$ or $\sigma(A)$ and $\sigma(B)$ by the pair $\lambda^{\frac{1}{2}}[(AA^*)/\alpha]$ and $\lambda^{\frac{1}{2}}[(BB^*)/\alpha]$.

Finally, we apply the theorems of this section to obtain some new upper bounds for eigenvalues of the Schur complements of BAB^*, where A is $n \times n$ positive semidefinite and B is any $m \times n$ matrix.

Theorem 2.20 *Let A be $n \times n$ positive semidefinite and B be $m \times n$. Let $l = \min\{m, n\}$, $\alpha \subset \{1, 2, \ldots, l\}$, and $\alpha^c = \{1, 2, \ldots, n\} - \alpha$. Then for every $i = 1, 2, \ldots, m - |\alpha|$ and every $t = 1, 2, \ldots, l - |\alpha|$,*

$$\lambda_i[(BAB^*)/\alpha] \leq \min_{s+t=i+1} \lambda_s(A[\alpha^c]){\sigma_t}^2(B/\alpha).$$

Proof. Without loss of generality, assume $\alpha = \{1, 2, \ldots, |\alpha|\}$. By Theorem 2.17, since $(A^{\frac{1}{2}})^* = A^{\frac{1}{2}}$, $(A^{-\frac{1}{2}})^* = A^{-\frac{1}{2}}$, for $t = 1, 2, \ldots, l - |\alpha|$, we have

$$\begin{aligned} {\sigma_t}^2(B/\alpha) &= {\sigma_t}^2\left[(BA^{\frac{1}{2}}A^{-\frac{1}{2}})/\alpha\right] \\ &\geq \max_{i=1,2,\cdots,m-|\alpha|} \lambda_{n-|\alpha|-i+t}(A^{-1}/\alpha)\lambda_i[(BAB^*)/\alpha]. \end{aligned}$$

By Theorem 1.2, we have

$$A^{-1}/\alpha = [(A^{-1}/\alpha)^{-1}]^{-1} = (A[\alpha^c])^{-1}.$$

So, for any $1 \leq j \leq n$, we have

$$\lambda_j^{-1}(A^{-1}/\alpha) = \lambda_{n-|\alpha|-j+1}(A[\alpha^c]).$$

It follows that, for every $i = 1, 2, \ldots, m - |\alpha|$ and $t = 1, 2, \ldots, l - |\alpha|$,

$$\begin{aligned} \lambda_i[(BAB^*)/\alpha] &\leq \min_{t=1,2,\ldots,l-|\alpha|} \sigma_t^2(B/\alpha)\lambda_{i-t+1}(A[\alpha^c]) \\ &= \min_{t+s=i+1} \lambda_s(A[\alpha^c])\sigma_t^2(B/\alpha). \blacksquare \end{aligned}$$

Setting $B = I$ in Theorem 2.20 results in eigenvalue inequalities that may be compared with the ones in Section 2.1: For $i = 1, 2, \ldots, n - |\alpha|$,

$$\lambda_i(A/\alpha) \leq \lambda_i(A[\alpha^c]).$$

By Theorems 2.18 and 2.19, one gets the following result which is proven in a manner similar to that of Theorem 2.20.

Theorem 2.21 *Let all the assumptions of Theorem 2.20 be satisfied. Let u be an integer such that $1 \leq u \leq l - |\alpha|$. Then for $1 \leq i_1 < \cdots < i_t \leq l - |\alpha|$,*

$$\prod_{t=1}^{u} \lambda_{i_t}[(BAB^*)/\alpha] \leq \min \left\{ \prod_{t=1}^{u} \lambda_{i_t}(A[\alpha^c]){\sigma_t}^2(B/\alpha), \prod_{t=1}^{u} \lambda_t(A[\alpha^c]){\sigma_{i_t}}^2(B/\alpha) \right\}.$$