Foundations of Engineering Mechanics

Spatial Contact Problems in Geotechnics

Boundary-Element Method

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Chapter 2
Static Analysis of Contact Problems for an Elastic Half-Space

Abstract The second chapter is devoted to the mathematical formulation of mixed problems of the elasticity theory for a half-space and to the numerical-and-analytical methods of their solution. The results obtained in this chapter on developing the mathematical means are the reference data for BEM-based numerical modeling of the spatial contact interaction. The integrated boundary equations of the spatial contact problem are written for the case when the calculation scheme is accepted in the form of variously deepened punches undergoing the action of the spatial system of forces. It is shown how to reduce the initial integral equation system of the contact problem with respect to the contact stress function and the punch displacement parameters to the appropriate finite-dimensional algebraic analogue. Much attention is paid to calculating the matrix coefficients of the resolving system of algebraic equations. A numerical-and-analytical procedure is given for integrating Mindlin's fundamental solutions over flat triangular and quadrangular boundary elements, arbitrary oriented in the half-space. For convenience, to apply the developed approach in practical calculations, the boundary integral equations of the spatial contact problems for a number of essential special cases are presented. The contact problems at axial loading and torsion of absolutely rigid rotation bodies deepened into the half-space, are considered. Boundary-element formulations of the contact problems for complex-shaped punches with flat and smooth bases (shallow foundations), situated on spatially nonhomogeneous bases of the semi-infinite elastic massif type are presented.

2.1 Boundary Integral Equations of the Contact Problem for an Absolutely Rigid Punch, Deepened into an Elastic Half-Space, Under a Spatial Load System

Consider an elastic homogeneous half-space $z \geq 0$, containing a cavity $S$with a boundary Γ from the side of the surface $z = 0$. The mechanical properties of the half-space are determined by the elastic modulus E and Poisson's ratio ν. We assume the surface $z = 0$, being the boundary of the half-space, to be free from any

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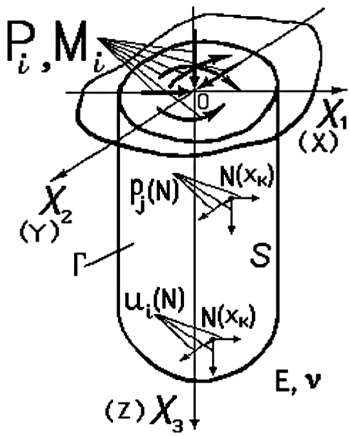

Fig. 2.1 Calculation scheme for the contact problem of a volumetric punch, deepened into an elastic half-space

load. In the cavity *S* an absolutely rigid volumetric punch is deepened, subject to a static load, reduced to a resultant force $P = \{P_1, P_2, P_3\}$ and a resultant moment $M = \{M_1, M_2, M_3\}$ where P_i, M_i $(i = 1, 2, 3)$ are the projections of the corresponding vectors onto the axes of the Cartesian coordinate system $OX_1X_2X_3$ *(OXYZ)* (Fig. 2.1). The contact problem of spatial theory of elasticity for the deepened punch consists in the determination of contact stress on the surface of interaction of the elastic medium with the punch as well as the determination of the parameters of its displacement as a rigid solid. We assume the punch to be welded with the elastic half-space, i.e. on the contact surface of the punch and the base the displacements coincide (the second-order boundary conditions according to Galin [14] are fulfilled). In order to derive the main equations of the contact problem we follow a rather demonstrative method, first considered by Kovneristov [20] and later applied by Shishov [29] while solving the problems in an axisymmetric arrangement. The method suggests the involvement of Betti's theorem of reciprocity [24] that requires the concept of basic and auxiliary states of an elastic body to be introduced.

As a basic state we consider an elastic half-space with a cavity *S*, in each point of whose surface the displacements $u_i\,(N)$ are given and the stresses $P_j\,(N)$ are acting, being a distributed reaction from the side of the punch (Fig. 2.1). The stress-strained state of the base in the initial contact problem will be equivalent to the basic state introduced into the consideration.

In order to build auxiliary states consider a solid elastic half-space, loaded in a point K (ξ,η,ζ) by unit concentrated forces

$$A_j = \delta(x - \xi, y - \eta, z - \zeta), \quad i = 1,2,3 \tag{2.1}$$

directed along the coordinate axes, respectively. The load point K (ξ,η,ζ) is chosen outside the domain bounded by the surface Γ. Virtually remove the elastic body S' from the half-space, the surface Γ of the body S' being identical to that of the rigid punch. In order to keep the half-space, weakened by the cavity, in equilibrium,

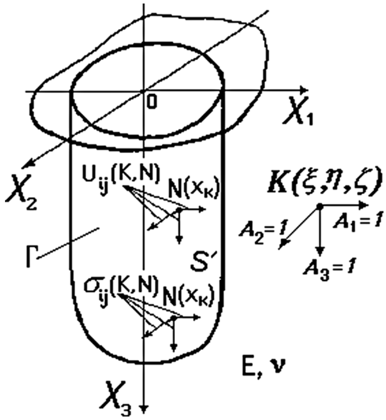

Fig. 2.2 Elastic half-space, weakened by a cavity, corresponding to the deepened punch shape

forces σ_{ij} (N, K) and displacements U_{ij} (N, K), being the fundamental Mindlin's solutions, should be distributed over the surface Γ (Fig. 2.2).

We take the advantage of the Betti's theorem of reciprocity [24], linking the solution of two different problems for the same domain of an elastic body: the work of the system of forces of the basic state on the displacements of the auxiliary state is equal to the work, performed by the system of forces of the auxiliary state on the displacements of the basic state. The equations of reciprocity of the works for the basic and the auxiliary states considered in this contact problem, are given by

$$\iint\limits_{\Gamma} \left[p_1(N)U_{1i}(K,N) + p_2(N)U_{2i}(K,N) + p_3(N)U_{3i}(K,N)\right] d\Gamma =$$
$$= \iint\limits_{\Gamma} [\sigma_{1i}(K,N)u_1(N) + \sigma_{2i}(K,N)u_2(N) + \sigma_{3i}(K,N)u_3(N)]\, d\Gamma + u_i(K), \tag{2.2}$$
$$i = 1,2,3.$$

Equation (2.2) gives the integral representation of displacements at any point (outside the punch) of the elastic half-space and is known as Somigliana identity for the displacements [10, 24]. This equation could be immediately used formally as an initial one. Note that Eq. (2.2) lacks the integrity over the half-space surface, since the absence of stress at the free surface of the elastic half-space in the basic state had been initially assumed, and the fundamental Mindlin equation was obtained under the same condition. The Somigliana equation explains the main advantage of the boundary integral equation method (and the boundary-element method as a method of its numerical implementation), consisting in the fact the displacement vector components (and, consequently, the stresses) are determined solely by the boundary data at the punch surface. In other words, if the displacement values u_i and forces p_j at the Γ boundary are known, then using the Somigliana identity (2.2) one can always find the displacements and, consequently, deformations and stresses at any internal point $K(\xi, \eta, \zeta)$ of the elastic half-space.

The deformations of the half-space can be determined using Eq. (2.2) in a conventional way after differentiating and using the Cauchy relations

$$\varepsilon_{11} = \frac{\partial u_1}{\partial \xi}(K), \quad \varepsilon_{22} = \frac{\partial u_2}{\partial \eta}(K), \quad \varepsilon_{33} = \frac{\partial u_3}{\partial \zeta}(K),$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial \eta}(K) + \frac{\partial u_2}{\partial \xi}(K)\right), \quad \varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial \zeta}(K) + \frac{\partial u_3}{\partial \eta}(K)\right),$$

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial \zeta}(K) + \frac{\partial u_3}{\partial \xi}(K)\right),$$

which afterwards enables the stress tensor components in the elastic half-space for the basic state to be determined using Hooke equations

$$\sigma_k(K) = 2G\left[\frac{\nu}{1-2\nu}\theta(K) + \varepsilon_k(K)\right], \quad k = 1,2,3,$$

$$\tau_{12} = 2G\varepsilon_{12}(K), \quad \tau_{23} = 2G\varepsilon_{23}(K), \quad \tau_{13} = 2G\varepsilon_{13}(K)$$

where $G = E/2(1+\nu)$ is the shear modulus, $\theta(K) = \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$ is dilatation. The obtained equations are cumbersome and, therefore, not given here in the extended form.

In order to obtain the equations of the contact problem, we direct the point $K(\xi,\eta,\zeta)$ of application of the unit concentrated forces toward the deepened punch surface, i.e. perform a limiting transition from the internal point to the boundary one. The limiting transition results in a system of three boundary integral equations

$$\frac{1}{2}u_i(K) = \iint\limits_{\Gamma}\left[\sum_{j=1}^{3}(p_j(N)U_{ji}(K,N) - u_j(N)\sigma_{ji}(K,N))\right]d\Gamma, \tag{2.3}$$

$$i,j = 1,2,3; \quad K(\xi,\eta,\zeta) \in \Gamma, \quad N(x_1,x_2,x_3) \in \Gamma.$$

The factor 1/2 on the left-hand side of Eq. (2.3) arises due to the fact the unit forces in the point $K(\xi,\eta,\zeta)$ in the auxiliary state are divided by the surface Γ in two parts: one acts at the half-space with the cavity, the other acts at the punch-shaped elastic body being removed. The singularity of the equations consists in an unlimited increase of the integrands at $N \to K$. It will be shown below (Sect. 2.3) that all the integrals, containing functions $U_{ij}(N, K)$ with a weak singularity (of the $1/R$ type), can be calculated for flat integration surfaces numerically-and-analytically with any degree of accuracy. Integrals, containing functions $\sigma_{ij}(N, K)$ with a strong singularity (of the $1/R^2$ type), require special calculation in the sense of the Cauchy principal value. Below it will be shown that for the contact problems considered here the integrals, containing cores with strong singularities, can be excluded out of consideration after the account of displacements of the punch as a rigid solid as well as the application of equilibrium equations for the auxiliary state.

The displacement of the punch as a rigid solid enables the following relation to be written for the points on the contact surface [21]:

$$u_i(K) = \Delta_i - \varepsilon_{ijk}\zeta_j\psi_k \tag{2.4}$$

where Δ_i are translational displacements of the punch, ψ_k are small rotations of the punch with respect to the coordinate axes, $i,j,k = 1,2,3$, $K(\xi,\eta,\zeta) \in \Gamma$, $\zeta_1 = \xi$, $\zeta_2 = \eta$, $\zeta_3 = \zeta$.

The boundary integral equations (2.3) of the contact problem for the deepened punch with the account of Eq. (2.4) take the following form (summation over the repeated indices is assumed):

$$\frac{1}{2}\left(\Delta_i - \varepsilon_{ijk}\zeta_j\psi_k\right) = \iint\limits_{\Gamma} p_j U_{ji} d\Gamma - \Delta_j \iint\limits_{\Gamma} \sigma_{ji} d\Gamma + \psi_k \iint\limits_{\Gamma} \varepsilon_{jlk}\zeta_l\sigma_{ji} d\Gamma, \tag{2.5}$$

$i, j,k,l =1, 2, 3.$

The three obtained equations (2.5) can be essentially simplified by using the equilibrium equations for the elastic body S' in the shape of the deepened punch for the auxiliary states from the action of the unit concentrated forces on the surface Γ:

$$\iint\limits_{\Gamma} \sigma_{ji} d\Gamma = \frac{1}{2}\delta_{ji}, \iint\limits_{\Gamma} \varepsilon_{jlk}\zeta_l\sigma_{ji} d\Gamma = \frac{1}{2}\varepsilon_{ijk}\zeta_j, \quad i,j,k,l = 1,2,3. \tag{2.6}$$

With the account of Eq. (2.6) the boundary integral equations of the spatial contact problem for the rigid punch deepened into an elastic half-space, are given by

$$\iint\limits_{\Gamma} \left[\sum_{j=1}^{3} \left(p_j(N)U_{ji}(K,N)\right)\right] d\Gamma = \Delta_i - \varepsilon_{ijk}\zeta_j\psi_k, \quad i,j,k,l = 1,2,3. \tag{2.7}$$

Equation (2.7) assert that the displacement of any point on the punch contact surface is numerically equal to the work of contact forces $p_j(N)$ in the basic state on the displacements $U_{ji}(K, N)$ of the auxiliary state.

Rigid displacements Δ_i and rotations ψ_k of the punch (i, $k = 1, 2, 3$) are also unknown, and to determine them one should invoke six equations of the punch equilibrium (in the basic state):

$$P_i = \iint\limits_{\Gamma} p_i(N) d\Gamma, \quad M_i = \iint\limits_{\Gamma} \varepsilon_{ijk} x_j p_k(N) d\Gamma, \quad (i,j,k = \overline{1,3}). \tag{2.8}$$

Thus the solution of the spatial contact problem for an absolutely rigid punch of arbitrary shape, deepened into an elastic half-space, under an external static load of a general type is determined by a system of nine integral equations (2.7), (2.8) which for convenience hereinafter can be presented in the following extended form:

$$\begin{cases} \iint\limits_{\Gamma} \left[p_1(N)U_{11}(K,N) + p_2(N)U_{12}(K,N) + p_3(N)U_{13}(K,N)\right] d\Gamma = \Delta_1 + \eta\psi_3 - \zeta\psi_2, \\ \iint\limits_{\Gamma} \left[p_1(N)U_{21}(K,N) + p_2(N)U_{22}(K,N) + p_3(N)U_{23}(K,N)\right] d\Gamma = \Delta_2 + \zeta\psi_1 - \xi\psi_3, \\ \iint\limits_{\Gamma} \left[p_1(N)U_{31}(K,N) + p_2(N)U_{32}(K,N) + p_3(N)U_{33}(K,N)\right] d\Gamma = \Delta_3 + \xi\psi_2 - \eta\psi_1, \end{cases} \tag{2.9}$$

$$\begin{cases} \iint\limits_{\Gamma} p_1(N)d\Gamma = P_1, \quad \iint\limits_{\Gamma} p_2(N)d\Gamma = P_2, \quad \iint\limits_{\Gamma} p_3(N)d\Gamma = P_3 \\ \iint\limits_{\Gamma} \left[p_3(N)x_2 - p_2(N)x_3\right] d\Gamma = M_1, \\ \iint\limits_{\Gamma} \left[p_1(N)x_3 - p_3(N)x_1\right] d\Gamma = M_2, \\ \iint\limits_{\Gamma} \left[p_2(N)x_1 - p_1(N)x_2\right] d\Gamma = M_3. \end{cases} \tag{2.10}$$

Having solved the system of Eqs. (2.9), (2.10), one can determine three functions of contact stresses p_i and six parameters Δ_i, $\psi_{\mathrm{i}}\psi_i$ $(i = 1, 2, 3)$ of the punch displacement as a rigid solid, i.e. the stress-strained state at the contact surface Γ is determined.

2.2 Finite-Measure Analogue of the Contact Problem Using Direct Boundary-Element Method

Analytical solutions of the system of integral equations (2.7), (2.8) formulated in Sect. 2.1 for deepened punches of any particular geometrical shape have not been obtained yet even for the simplest loading schemes. The main difficulty here, as has been noted by many authors, concerns the integration of the fundamental Mindlin's solution.

For the numerical solution of the spatial contact problem, formulated in the most general way, we use the boundary-element method in its direct formulation [7, 10, 34] when the unknown function values on the boundary have the physical sense of contact pressures and play the role of source densities determining the stress-strained state inside the stressed domain. The method application to an essentially spatial contact problem under consideration is reduced to the following stages:

(1) discretization of the boundary surface Γ by means of a finite element ensemble;
(2) determination for the boundary elements of a finite set of nodes with respect to which the collocation method is applied, enabling the node values of the unknowns to be related based on a finite-measure analogue of the initial integral equations;
(3) formation of the resolving system of algebraic equations whose coefficients are calculated by analytical and/or numerical integration over each boundary element;
(4) direct or iterative solution of the resolving system of algebraic equations;
(5) finding of the stress-strained state in the given internal points of the stressed medium with invoking the schemes of numerical integration of various orders.

Hereinafter we restrict ourselves to the discretization of the contact surface Γ with a set of boundary elements of polygonal (as a rule, triangular, and/or quadrangular) shape (Fig. 2.3). The overwhelming majority of volumetric deepened punches in the problems of civil engineering (first of all, for geotechnical purposes, see Sects. 3.3 and 3.4) are restricted by fragments of planes or second-order (conical, cylindrical, or spherical) surfaces. A moderate number of flat boundary elements enables the punch boundary of practically any geometrical shape to be approximated with a required accuracy. Therefore, application of non-flat boundary elements in the contact problems of geotechnics is hardly appropriate. Here we note once again that, since in this approach one uses the fundamental Mindlin's solution for the problem of the concentrated force inside the elastic half-space (automatically satisfying the boundary conditions on the stress-free base surface), only the contact surface of the punch and the base can be discretized.

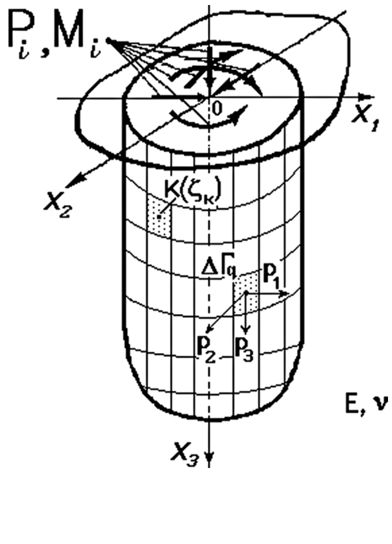

Fig. 2.3 Discretization of the contact surface of the punch and the elastic base using the boundary elements

After the boundary-element discretization of the boundary Γ the integral equation system of the contact problem is given by

$$\begin{cases} \Delta_i = \sum_{q=1}^{m} \iint_{\Delta\Gamma_q} p_j(N)U_{ji}(K_f, N)d\Gamma + \zeta_k\, \psi_j\, \varepsilon_{ijk}; \\ P_i = \sum_{q=1}^{m} \iint_{\Delta\Gamma_q} p_i(N)d\Gamma; \quad M_i = \sum_{q=1}^{m} \iint_{\Delta\Gamma_q} \varepsilon_{ijk} p_j(N) x_k d\Gamma; \\ i,j,k = \overline{1,3},\, f,q = 1,2,...m \end{cases} \tag{2.11}$$

where m is the number of the boundary elements on the punch contact surface, $\Delta\Gamma_q$ is the surface of the q-th boundary element, K_f are the collocation points (the finite-element gravity centres), ζ_k are the coordinates of the point K_f $(\zeta_1 = \xi,\ \zeta_2 = \eta,\ \zeta_3 = \zeta)$.

The system of Eqs. (2.11) is the consequence of the system of Eqs. (2.7), (2.8) where the calculation of 2-D integrals over the surface Γ is substituted by the sum of integrals over the flat surfaces of the introduced boundary elements $\Delta\Gamma_q$, $q = 1, 2 \ldots m$.

Within each boundary element one should assume that the contact forces p_i vary according to a pre-given law. As a rule, polynomial (constant, linear, quadratic, or higher-order) approximation is applied [7, 10, 17, 34, 36]. Application of the boundary-element method to the solution of spatial static problems for finite-size bodies (local strength problems) shows that quite satisfactory results are achieved already at application of piecewise constant or piecewise linear approximation of the unknown densities, in particular, of the stress function. Note that in [16], based on the analysis of the literature, a hypothetic idea is suggested to choose the order of approximation for each boundary element by a unit higher than the order of approximation of the functions to be found. Though this statement has not been proved strictly (it is only confirmed by calculations for the flat and the axisymmetric cases), a conclusion is made that it is appropriate to combine flat boundary elements and constancy of the sought function, second-order elements and linear variation of the sought functions etc. Violation of this correspondence is not justified since it does not result in a guaranteed increase of the accuracy of the approximate solution. Thus, at further application of flat boundary elements the approximation of piecewise constant variation of the sought function of contact stress will be to a certain extent justified. Then, taking into account that stress p_i in theory of elasticity is presented by a derivative of displacement u_i, the application of constant stress on a boundary element corresponds to a linear variation of displacement in the plane of each finite element. This is in agreement with the linear distribution of displacements of the boundary surface of an absolutely rigid punch and is an additional argument for the piecewise constant approximation of the contact stress function in the proposed version of the numerical boundary-element method with application of flat boundary elements.

For m collocation nodes, in which the condition of fulfillment of boundary integral equations is set, we choose points, uniformly distributed over the discretized punch surface. It is quite natural to obtain the first $3m$ equations of the algebraic system of the boundary-element method by locating the unit forces of the auxiliary state in the gravity centres of the boundary elements. Then, in accordance with the approximation applied, the system of integral boundary equations of the spatial contact problem for the absolutely rigid punch deepened into a half-space together with the integral equilibrium equations can be given in the following discrete form:

$$\begin{cases} \Delta_i = \sum\limits_{q=1}^{m} p_j(N_q) \iint\limits_{\Delta\Gamma_q} U_{ji}(K_f, N_q)d\Gamma + \zeta_k\, \psi_j\, \varepsilon_{ijk}; \\ P_i = \sum\limits_{q=1}^{m} p_j(N_q)\Delta s_q; \quad M_i = \sum\limits_{q=1}^{m} p_j(N_q)\varepsilon_{ijk}x_k\Delta s_q; \\ i,j,k = \overline{1,3},\ q,f = 1,2,...m \end{cases} \tag{2.12}$$

where $p_j(N_q) = p_j^{(q)}$ are the averaged values of contact stresses in the j-th direction within the q-th boundary element, $N_q \in \Delta\Gamma_q$, $\Delta\Gamma_q$ is the surface of the q-th boundary element, $\Delta S_q = mes(\Delta\Gamma_q)$ is the surface of the q-th boundary element.

In an extended notation the equation system (2.12) is given by

$$\sum_{q=1}^{m} \left[p_1(N_q) \iint\limits_{\Delta\Gamma_q} U_{1j}(K_f, N_q)d\Gamma + p_2(N_q) \iint\limits_{\Delta\Gamma_q} U_{2j}(K_f, N_q)d\Gamma + \right.$$

$$\left. + p_3(N_q) \iint\limits_{\Delta\Gamma_q} U_{3j}(K_f, N_q)d\Gamma \right] = \begin{cases} \Delta_1 + \zeta_f\psi_2 - \eta_f\psi_3, & j = 1, \\ \Delta_2 + \xi_f\psi_3 - \zeta_f\psi_1, & j = 2, \\ \Delta_3 + \eta_f\psi_1 - \xi_f\psi_2, & j = 3, f = 1,...,m, \end{cases} \tag{2.13}$$

$$\sum_{q=1}^{m} p_1(N_q)\Delta s_q = P_1, \sum_{q=1}^{m} p_2(N_q)\Delta s_q = P_2, \sum_{q=1}^{m} p_3(N_q)\Delta s_q = P_3, \tag{2.14a, b, c}$$

$$\sum_{q=1}^{m} \left[p_3(N_q)y_q - p_2(N_q)z_q\right] \Delta s_q = M_1, \tag{2.14d}$$

$$\sum_{q=1}^{m} \left[p_1(N_q)z_q - p_3(N_q)x_q\right] \Delta s_q = M_2, \tag{2.14e}$$

$$\sum_{q=1}^{m} \left[p_2(N_q)x_q - p_1(N_q)y_q\right] \Delta s_q = M_3 \tag{2.14f}$$

and enables the resolving system of linear algebraic equations of the boundary-element method to be set in a matrix form

$$\mathbf{A}\cdot Z = B \tag{2.15}$$

where $\mathrm{A} = \begin{pmatrix} \mathrm{D}_{3m\times 3m} & \mathrm{C}_{3m\times 6} \\ \mathrm{T}_{6\times 3m} & 0 \end{pmatrix}$ is a square block matrix of the order of $(3m+6)$, $\mathrm{D}_{3m\times 3m} = \left(\delta_{ij}^{(fq)}\right)$ is the influence matrix, $\delta_{ij}^{(fq)} = \begin{pmatrix} \delta_{11}^{(fq)} & \delta_{12}^{(fq)} & \delta_{13}^{(fq)} \\ \delta_{21}^{(fq)} & \delta_{22}^{(fq)} & \delta_{23}^{(fq)} \\ \delta_{31}^{(fq)} & \delta_{32}^{(fq)} & \delta_{33}^{(fq)} \end{pmatrix}$, $f, q = \overline{1,m}$;

$$C_{3m\times 6} = -\begin{pmatrix} 1 & 0 & 0 & 0 & z_1 & -y_1 \\ 0 & 1 & 0 & -z_1 & 0 & x_1 \\ 0 & 0 & 1 & y_1 & -x_1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & z_m & -y_m \\ 0 & 1 & 0 & -z_m & 0 & x_m \\ 0 & 0 & 1 & y_m & -x_m & 0 \end{pmatrix},$$

$$T_{6\times 3m} = \left(\begin{array}{ccc|ccc|c|ccc} \Delta s_1 & 0 & 0 & \Delta s_2 & 0 & 0 & \dots & \Delta s_m & 0 & 0 \\ 0 & \Delta s_1 & 0 & 0 & \Delta s_2 & 0 & \dots & 0 & \Delta s_m & 0 \\ 0 & 0 & \Delta s_1 & 0 & 0 & \Delta s_2 & \dots & 0 & 0 & \Delta s_m \\ 0 & -z_1\Delta s_1 & y_1\Delta s_1 & 0 & -z_2\Delta s_2 & y_2\Delta s_2 & \dots & 0 & -z_m\Delta s_m & y_m\Delta s_m \\ z_1\Delta s_1 & 0 & -x_1\Delta s_1 & z_2\Delta s_2 & 0 & -x_2\Delta s_2 & \dots & z_m\Delta s_m & 0 & -x_m\Delta s_m \\ -y_1\Delta s_1 & x_1\Delta s_1 & 0 & -y_2\Delta s_2 & x_2\Delta s_2 & 0 & \dots & -y_m\Delta s_m & x_m\Delta s_m & 0 \end{array}\right),$$

Z and **B** are column vectors of the size $(3m + 6)$:

$$\mathrm{Z} = (p_1(N_1), p_2(N_1), p_3(N_1)..p_1(N_m), p_2(N_m), p_3(N_m); \Delta_1, \Delta_2, \Delta_3, \psi_1, \psi_2, \psi_3)^{\mathrm{T}},$$

$$\mathrm{B} = (0,0,0,...; P_1, P_2, P_3, M_1, M_2, M_3)^{\mathrm{T}}.$$

The dimensionality of the system of Eq. (2.15) equals $(3m+6)_\times(3m+6)$ where m is the total number of the boundary elements used for the approximation of the contact surface of the punch and the elastic base. The vector of the unknowns Z includes $3m$ components of contact stresses $p_i(N_k) = p_i^{(k)}$ as well as six parameters Δ_i, ψ_i of the punch displacement as a rigid solid (i = 1, 2, 3; k= 1, 2,...m). In a general case, the block $\mathbf{D}_{3m\times 3m}$ of the matrix **A** is non-symmetrical and completely filled. This block is characterized by the diagonal predominance of coefficients.

Application of a conventional Gauss elimination method to solve the linear algebraic equation system (2.15), as shown by a vast experience of calculations performed, results in the numerical solution accuracy and stability, sufficient for the practical purposes. The details of efficient implementation of algorithms of solutions of linear algebraic equation systems of the boundary-element method, appropriate for the specific features of Eq. (2.15), are given below in Sect. 3 (Sect. 3.4).

The coefficients of the main block $\mathbf{D}_{3m\times 3m}$ of the matrix $\mathbf{A}$ are surface integrals of the fundamental Mindlin's solution

$$\delta_{ij}^{(fq)} = \iint\limits_{\Delta\Gamma_q} U_{ij}(K_f, N)d\Gamma(N) \tag{2.16}$$

The analytical calculation of these integrals over flat triangular or quadrangular domains, arbitrarily oriented in an elastic half-space, seems impossible. In practice we have carried out the efficient integration by means of an original numerical-and-analytical approach. It can be assumed that formation of the matrix coefficients of Eq. (2.15) is the key point of the whole boundary-element method since it requires both regular and improper integrals to be calculated with high precision and simultaneously in an optimal way from the point of view of the computation time. These issues need to be considered in more detail what is performed in the following subsection.

2.3 Numerical-and-Analytical Method of Integration of Fundamental Mindlin's Solutions

Formation of the matrix of coefficients of the resolving system (2.15) of linear algebraic equations of the boundary-element method is reduced to the calculations of surface integrals of the fundamental Mindlin's solution for the displacements

$$\delta_{ij} = \iint\limits_{\Delta\Gamma_q} U_{ij}(K_f, N)d\Gamma(N) \tag{2.17}$$

The domains of integration of Eq. (2.17) are the simplest flat polygons (triangles and quadrangles), arbitrarily oriented in the half-space.

The difficulties in the calculation of the integrals of Eq. (2.17) consist in the fact that when the double integrals are reduced to iterated integrals, the primitives cannot be found; besides, near the collocation points (when they belong to the integration domain $\Delta\Gamma$) the integrands become unlimited. In the last case direct application of standard procedures of numerical integration does not lead to the desired results since for 2-D improper integrals it is very difficult to reveal the specific features in the vicinity of the point $K(\xi,\eta,\zeta)$ of application of unit concentrated force by a finite number of summands of the cubature formulae. The experience of numerical calculations has shown that this requires a quite considerable increase of the number of integration points and, simultaneously, their concentration near the integrand singularities (adaptive numerical integration [34]) in order to obtain the result of the desired accuracy. As a result, sufficient accuracy of the approximate values of improper surface integrals requires too much computer time.

Consider the calculation of surface integrals of $U(K,N)$ Mindlin displacements based on the complementary possibilities provided by analytical and numerical integration methods with direct account of the integrand structure.

In Mindlin equations (1.7) for the displacements the singular terms are fundamental Kelvin's solutions for the whole space, other terms have no singularities (since they correspond to an imaginary point $\tilde{K}(\xi,\eta,-\zeta)$ of unit concentrated force application). Hence, similarly to [9, 10], it is natural to present Mindlin formulae in the form

$$U_{ij} = (U_{ij})^K + (U_{ij})^C \tag{2.18}$$

where superscripts K and C correspond to the singular Kelvin's solution and the auxiliary (regular) solution, respectively. As shown by the experience of numerous calculations, analytical determination of improper integrals of the Kelvin functions (containing only R_1 powers, at $K \in \Delta\Gamma$) and numerical integration of complete Mindlin's solutions at $K \notin \Delta\Gamma$ has appeared an efficient (both in accuracy and in speed) combined method of calculation of improper surface integrals in the spatial problems of elasticity theory for a half-space.

Numerical integration was performed using the cubature formulae of various order with the highest accuracy degree. In each separate case the choice of the number of nodes of the cubature formulae was performed on the base of empiric criteria obtained from an extended series of numerical experiments, including the dependences on the discretization degree and the contact surface shape. A common feature of the obtained regularities was an increase of the order of quadratures with the decrease of the distance from the point K to the integration domain. The detailed data on the numerical integration procedures are presented in Appendix B.

In order to determine the improper surface integrals with a weak (integrable) singularity in the centre of gravity of the boundary element one can apply analytical transformations. As mentioned above, the singularities in the integrand expressions are determined by the summands of the fundamental Kelvin's solutions for an unbounded elastic space [7]:

$$(U_{ij})^K(K,N) = \frac{1}{16\pi G(1-\nu)} \cdot \left[\frac{3-4\nu}{R}\delta_{ij} + \frac{z_i z_j}{R^3}\right] \tag{2.19}$$

where $z_i = \zeta_i - \xi_i$, $R = \sqrt{z_1^2 + z_2^2 + z_3^2}$, $N(\zeta_1,\zeta_2,\zeta_3)$ is a point in the integration domain, $K(\xi_1, \xi_2, \xi_3)$ is the point of application of a unit force (source), ξ_i are global Cartesian coordinates. Note that the Kelvin's solution is a special case of the Mindlin's solution and can be obtained from it at $R_1 = R$, $R_2 \rightarrow R$.

From the tensor notation of Eq. (2.19) it follows that the problem is reduced to the exact calculation of the following integrals with a singularity in the centre of gravity of a flat boundary element

$$I_1 = \iint_{\Delta\Gamma} \frac{d\Gamma}{R_1}, I_2 = \iint_{\Delta\Gamma} \frac{z_1 z_3}{R_1^3} d\Gamma, I_3 = \iint_{\Delta\Gamma} \frac{z_2 z_3}{R_1^3} d\Gamma, I_4 = \iint_{\Delta\Gamma} \frac{z_1^2}{R_1^3} d\Gamma, \quad (2.20)$$

$$I_5 = \iint_{\Delta\Gamma} \frac{z_2^2}{R_1^3} d\Gamma, I_6 = \iint_{\Delta\Gamma} \frac{z_3^2}{R_1^3} d\Gamma, I_7 = \iint_{\Delta\Gamma} \frac{z_1 z_2}{R_1^3} d\Gamma.$$

If boundary elements on the surface of the half-space $x_3 = 0$ are used, an additional pair of surface integrals should be included into consideration:

$$I_8 = \iint_{\Delta\Gamma} \frac{z_1}{R_1^2} d\Gamma, I_9 = \iint_{\Delta\Gamma} \frac{z_2}{R_1^2} d\Gamma. \quad (2.21)$$

We connect the point of application of the unit concentrated force $K(\xi,\eta,\zeta) \in \Delta\Gamma$ with the vertices of the boundary element $\Delta\Gamma_j$ (Fig. 2.4) within which this point is located. As a result, the flat integration domain will be divided into $\bar{m}$ additional triangular subelements $\Delta\Gamma_{jk}$ where $k = 1, 2, .., \bar{m}$. Here $\bar{m} = 3$ for a triangular boundary element, $\bar{m} = 4$ for a quadrangular one. It is clear that such an additional mesh, being internal for each boundary element, does not lead to any changes in the general approximating grid on the contact surface. Then, each of the improper integrals considered in Eqs. (2.20) and (2.21) should be substituted by a sum

$$\iint_{\Delta\Gamma_j} \frac{z_1^\alpha z_2^\beta z_3^\gamma}{R_1^\delta} d\Gamma = \sum_{k=1}^{\bar{m}} \iint_{\Delta\Gamma_{jk}} \frac{z_1^\alpha z_2^\beta z_3^\gamma}{R_1^\delta} d\Gamma. \quad (2.22)$$

Here $\alpha, \beta, \gamma, \delta$ are integer powers, determined in accordance with Eqs. (2.18) and (2.19).

Separate terms in the sum of Eq. (2.22) will be reduced to a set of elementary functions by performing calculations according to a uniform procedure. The apices

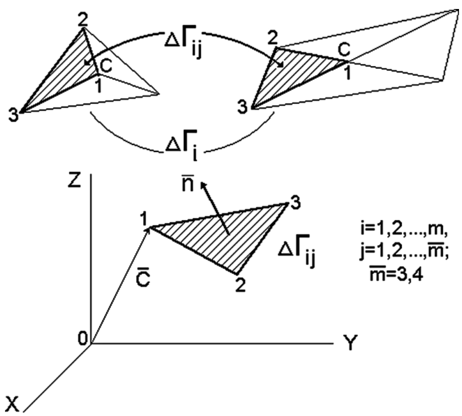

Fig. 2.4 Representation of triangular and quadrangular elements using triangular subelements with a common vertex

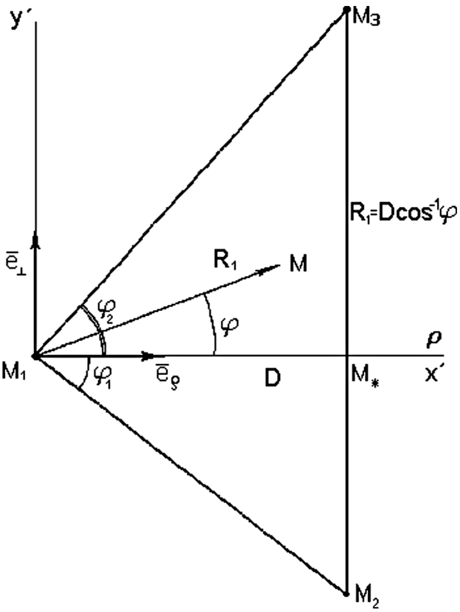

Fig. 2.5 Geometrical representations for integration over a flat triangular domain with a singularity of the integrand in one of the vertices

of a currently considered fragment $\Delta\Gamma_{jk}$ anti-clockwise are put into correspondence with the points $M_1(X_1,\, Y_1,\, Z_1)$, $M_2(X_2,\, Y_2,\, Z_2)$, $M_3(X_3,\, Y_3,\, Z_3)$ in such a way that the first of them be simultaneously the point of application of the unit concentrated force (the boundary-element centre of gravity). In the plane of the triangle $M_1M_2M_3$ we introduce a polar coordinate system with a pole in the point M_1, the polar axis being normal to the M_2M_3 side (Fig. 2.5). We will show that the singularities of the integrands for the integrals under consideration in the point M_1, due to the introduction of the above polar coordinate systems, annihilate.

Denote $\left|\overline{M_1M_*}\right| = |\bar{r}_*| = D,\quad \overline{M_1M_3} = \bar{r}_{13},\quad \overline{M_1M_2} = \bar{r}_{12}$. Then

$$\bar{R}_1 = R_1\left(\cos\varphi\cdot\bar{e}_\rho + \sin\varphi\cdot\bar{e}_\perp\right)$$

where $\bar{e}_\rho = \frac{\bar{r}_*}{|\bar{r}_*|} = \{A_1,A_2,A_3\}\,;\quad \bar{e}_\perp = \frac{\bar{r}_{23}}{|\bar{r}_{23}|} = \{B_1,B_2,B_3\}\,.$

After the transition to the polar coordinates $(R_1,\, \varphi)$ I_1 can be readily calculated:

$$I_1 = \iint\limits_{\Delta\Gamma_{jk}} \frac{d\Gamma}{R_1} = \int\limits_{\varphi_1}^{\varphi_2} d\varphi \int\limits_{0}^{D/\cos\varphi} dR_1 = D\int\limits_{\varphi_1}^{\varphi_2} \frac{d\varphi}{\cos\varphi} =$$
$$= \frac{D}{2}\cdot\ln\left(\frac{1+\sin\varphi_2}{1-\sin\varphi_2}\cdot\frac{1-\sin\varphi_1}{1+\sin\varphi_1}\right) = D\ln\left(\frac{1+\tan\dfrac{\varphi_2}{2}}{1-\tan\dfrac{\varphi_2}{2}}\cdot\frac{1-\tan\dfrac{\varphi_1}{2}}{1+\tan\dfrac{\varphi_1}{2}}\right). \tag{2.23}$$

The components of the vector $\bar{R}_1$ are written as

$$\begin{aligned} z_1 &= R_1\left(A_1 \cdot \cos\varphi + B_1 \cdot \sin\varphi\right), \\ z_2 &= R_1\left(A_2 \cdot \cos\varphi + B_2 \cdot \sin\varphi\right), \\ z_3 &= R_1\left(A_3 \cdot \cos\varphi + B_3 \cdot \sin\varphi\right) \end{aligned} \tag{2.24}$$

or, since $x' = R_1 \cos\varphi$, $y' = R_1 \sin\varphi$ are the components of the $\bar{R}_1$ vector in the plane of the triangle $M_1M_2M_3$ (in the Cartesian coordinate system $OX'Y'$, formed by e_ρ,$e_\perp$ vectors), then

$$\begin{aligned} z_1 &= A_1 \cdot x' + B_1 \cdot y', \\ z_2 &= A_2 \cdot x' + B_2 \cdot y', \\ z_3 &= A_3 \cdot x' + B_3 \cdot y'. \end{aligned} \tag{2.25}$$

Using Eqs. (2.24), (2.25), the sought integrals can be given by

$$I_q = Q_q \cdot J_1 + S_q \cdot J_2 + T_q \cdot J_3, \quad q = 2{,}3{,}...{,}7;$$

$$I_8 = A_1 \cdot J_4 + B_1 \cdot J_5, \quad I_9 = A_2 \cdot J_4 + B_2 \cdot J_5$$

where

$$Q_2 = A_1 \cdot A_3, \quad S_2 = B_1 \cdot B_3, \quad T_2 = A_3 \cdot B_1 + A_1 \cdot B_3;$$

$$Q_3 = A_2 \cdot A_3, \quad S_3 = B_2 \cdot B_3, \quad T_3 = A_3 \cdot B_2 + A_2 \cdot B_3;$$

$$Q_4 = A_1^2, \quad S_4 = B_1^2, \quad T_4 = 2A_1 \cdot B_1;$$

$$Q_5 = A_2^2, \quad S_5 = B_2^2, \quad T_5 = 2A_2 \cdot B_2;$$

$$Q_6 = A_3^2, \quad S_6 = B_3^2, \quad T_6 = 2A_3 \cdot B_3;$$

$$Q_7 = A_1 \cdot A_2, \quad S_7 = B_1 \cdot B_2, \quad T_7 = A_2 \cdot B_1 + A_1 \cdot B_2;$$

$$J_1 = \iint\limits_{\Delta\Gamma_{jk}} \frac{(x')^2}{R_1^3} d\Gamma, \quad J_2 = \iint\limits_{\Delta\Gamma_{jk}} \frac{(y')^2}{R_1^3} d\Gamma, \quad J_3 = \iint\limits_{\Delta\Gamma_{jk}} \frac{x'y'}{R_1^3} d\Gamma,$$

$$J_4 = \iint\limits_{\Delta\Gamma_{jk}} \frac{x'}{R_1^2} d\Gamma, \quad J_5 = \iint\limits_{\Delta\Gamma_{jk}} \frac{y'}{R_1^2} d\Gamma.$$

Now the integrals J_1, J_2,..., J_5 after the transition to the polar coordinates are obtained in quadratures

$$\begin{aligned} &J_1 = D\left(\sin\varphi_2 - \sin\varphi_1\right), \quad J_2 = I_1 - J_1, \quad J_3 = -D\left(\cos\varphi_2 - \cos\varphi_1\right), \\ &J_4 = D\left[A_1\left(\varphi_2 - \varphi_1\right) + B_1 \ln\left|\frac{\cos\varphi_1}{\cos\varphi_2}\right|\right], \quad J_5 = D\left[A_2\left(\varphi_2 - \varphi_1\right) + B_2 \ln\left|\frac{\cos\varphi_1}{\cos\varphi_2}\right|\right]. \end{aligned} \tag{2.26}$$

Considering the coordinates X_i, Y_i, Z_i, (i= 1, 2, 3) of the apices of the triangular subelement $\Delta\Gamma_{jk}$ to be known, for the sake of completeness we give the formulae to determine the A$_i$, B_i (i= 1, 2, 3), and D values:

$$A_1 = \frac{X_* - X_1}{D}, \quad A_2 = \frac{Y_* - Y_1}{D}, \quad A_3 = \frac{Z_* - Z_1}{D},$$
$$B_1 = \frac{X_3 - X_2}{\hat{R}}, \quad B_2 = \frac{Y_3 - Y_2}{\hat{R}}, \quad B_3 = \frac{Z_3 - Z_2}{\hat{R}},$$
$$\hat{R} = \sqrt{(X_3 - X_2)^2 + (Y_3 - Y_2)^2 + (Z_3 - Z_2)^2},$$
$$D = \sqrt{(X_* - X_1)^2 + (Y_* - Y_1)^2 + (Z_* - Z_1)^2},$$
$$X_* = X_2 + (X_2 - X_3) \cdot t, \quad Y_* = Y_2 + (Y_2 - Y_3) \cdot t, \quad Z_* = Z_2 + (Z_2 - Z_3) \cdot t,$$
$$t = ((X_2 - X_1)(X_3 - X_2) + (Y_2 - Y_1)(Y_3 - Y_2) + (Z_2 - Z_1)(Z_3 - Z_2)) / \hat{R}^2.$$

The presented expressions Eqs. (2.23), (2.26) for the improper integrals over a flat triangular domain with a singularity in one of its apices enable the accuracy of calculation of the diagonal coefficients of the canonical equation matrix to be essentially increased at a simultaneous decrease of the computation time (in comparison with only numerical integration being used).

In the analytical integration we mostly followed the approach suggested by Cruse [12] who was among the first to obtain analytical expressions for diagonal coefficients of the influence matrix for a flat triangular boundary element. In addition to [12], we have also obtained analytical expressions for special (limiting) integrals, arising at the application of boundary elements on the half-space surface. As one should expect, the final equations (2.23), (2.26) with the accuracy to identity transformations, are in agreement with the results of [12]. Besides, note that the expressions obtained here have the advantages of giving directly the formulae with the known coordinates of the boundary-element nodes what is convenient for practical applications.

In a series of numerical comparison experiments it has been found that the presented analytical-and-numerical integration method appeared comparable in efficiency with the known methods (in view of speed at the given calculation accuracy). In our case the natural increase of the computation time for the integration of the Mindlin's solution is caused by the fact the latter being more complicated than the Kelvin's solution (the presence of additional eighteen deformation cores) and is to a great extent justified by the condition of vanishing of stress at the half-space surface being automatically satisfied.

In view of the comparison performed we would give a brief account of other existing approaches to the formation of influence matrices in the direct boundary-element method for spatial problems of theory of elasticity [5, 6, 35–38]. In all of the known studies the results are obtained in closed form only in the case when the external normal does not change its direction, i.e. for flat boundary elements being used.

A rather descriptive numerical-and-analytical method of calculation of the matrix of an algebraic analogue of the system of boundary integral equations was suggested by Yakimchuk and Kvitka [38]. In the boundary-element plane a local polar coordinate system was introduced. The surface integrals were reduced to iterated integrals for which the integration over the polar radius was performed analytically using a software for computation of indefinite integrals (*Analytic* language). Numerical integration over the angular variable, using the quadrature formulae, is recommended. The procedure of formation of the influence matrix, suggested in [38] which its authors call semianalytical, does not seem to have visible advantages and has not been further developed or spread for the solution of spatial problems of theory of elasticity in the studies performed by other groups. Since we have performed efficient numerical integration of the fundamental Mindlin's solutions over the optimal quadrature formulae without a transition to consideration of iterated integrals, the semianalytical method, proposed in [38], in case being applied to a half-space, will only create additional difficulties and will obviously be inefficient.

Expressions of a rather cumbersome structure at the analytical calculation of integrals from the Kelvin's solution with density functions in the form of algebraic polynomials are given in [36]. The formulae have a sufficiently general form and are applicable both in the cases the collocation point (pole) belonging to the integration domain ($K \in \varDelta\Gamma$) and being located outside it ($K \notin \varDelta\Gamma$). Parallel translation and rotation of the Cartesian coordinate system axes are used for the transition to the plane of the boundary element $\Delta\Gamma$. Later, in [37], Roytfarb et al. have also obtained in a closed analytical form the expressions for the coefficients of the Kelvin influence matrix in a special case (with respect to [37]) of piecewise constant approximation of the sought densities and using a local polar coordinate system linked to a side of a polygon, arbitrarily oriented in space. In spite of the obvious efficiency of the methods developed in [36, 37], they possess certain inconveniences in the practical application of the obtained results for solution of problems for an elastic half-space. Namely, the formulae for the primitives contain a great number of transcendental functions, the reliable calculation of which is known to require double-precision computations and, hence, additional computation time. In the case under consideration, when part of the terms $(U_{ij})^c$ in the Mindlin's solution is always subject to numerical integration, the use of analytical transformation to obtain all the coefficients of the Kelvin influence matrix (the total number of integrals is $3m \times 3m$ where m is the number of the boundary elements on the contact surface) is absolutely unjustified. As has been shown by intentional numerical experiments, this increases the computation time by factor of 1.5/2 without a noticeable increase of the calculation accuracy.

In [5, 6, 35], the earlier approach of [36] is developed, using the method of analytical integration of the Mindlin's solutions over triangular flat elements. The practical applicability of the proposed method was restricted by the presence of primitives only for the cases when the flat integration domain was parallel to the axes of the global coordinate system, in which the expressions for the integrands were written. Such approach leaves beyond consideration a great class of problems, important for applications, when at the approximation of contact surfaces boundary elements with different angles of inclination with respect to the coordinate axes arise (Sect. 3.4). Unfortunately, until now we failed to obtain primitives in double integrals from the additional terms of the Mindlin's solution for flat, arbitrarily oriented boundary elements even using such modern powerful software for analytical transformations as *Matcad, Maple, Mathematica, Derive* etc.

2.4 Punch in the Shape of a Rotation Body, Deepened into an Elastic Half-Space

If the contact surface of a punch is a rotation surface and its loading and the boundary conditions possess axial symmetry, the spatial problem of theory of elasticity is essentially simplified. In this case it is quite natural to use more precise and efficient procedures of numerical solution, taking into account the symmetry of the problem. This will save the computation resources and revise the calculation formulae, finally resulting in a more rational design solutions.

In a cylindrical coordinate system (r, φ, z), for which the Oz axis is combined with the punch axis, all the parameters of the stress-strained state are independent of the angular coordinate ϕ and, due to such azimuthal symmetry, the contact problem becomes two-dimensional. The simplest axisymmetric punches are a sphere, a cylinder, a cone (including a frustum of a cone). More complicated axisymmetric structures are presented in Sect. 3.4.

Contact problems with an axial symmetry for punches, deepened into an elastic half-space, can be divided in two groups. The first one corresponds to the forced

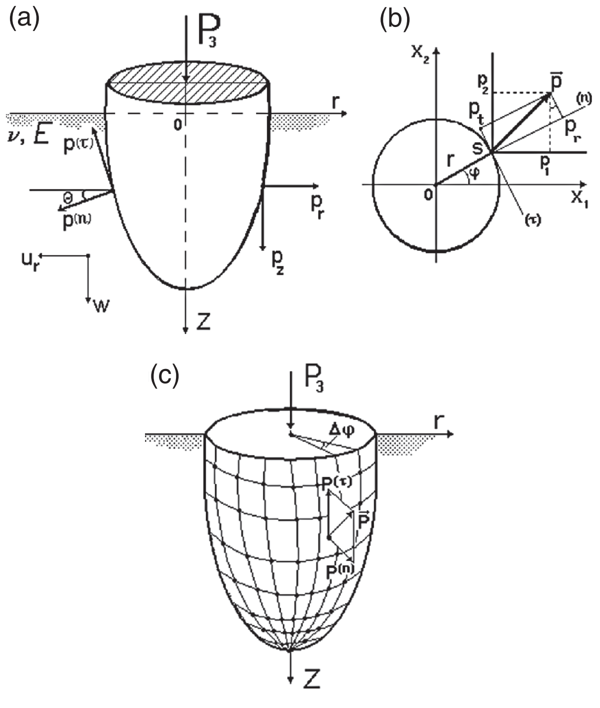

Fig. 2.6 Axially symmetrical contact problem for a rigid punch, deepened into an elastic half-space: (**a**) calculation scheme; (**b**) contact stress in the horizontal plane; (**c**) cyclic (over the angular coordinate) discretization of the contact surface and representation of the contact stress vector on the boundary elements

loading of the punch along the symmetry axis and is characterized by action of tangential and normal stresses on the contact surface as well as radial and axial displacements in the stressed base (Fig. 2.6a). The second group corresponds to the punch torsion under a torque, collinear to the symmetry axis, and is characterized by the action of a pair of tangential contact stresses and solely tangential displacements in the elastic half-space, i.e. the stress distribution, inversely symmetrical with respect to the axis (Fig. 2.7a).

For axisymmetric problems, being one of the most important classes of spatial problems of theory of elasticity, there are efficient methods of solution; a great number of forms of general solution are known [31]. The problem is eventually two-dimensional, for its solution well developed means of theory of analytic and *p*-analytic functions can be used [3, 26]. According to [31], the studies of axisymmetric stress-strained state of bodies of finite size is one of the most extensively developed fields of theory of elasticity, the better results have been achieved only for the flat problem. Nevertheless, still no analytical solutions of contact problems with axial symmetry for punches of even the simplest shape, deepened into an elastic half-space, are available.

Below we present the integral equations for contact problems with axial symmetry and construct efficient boundary-element algorithms of their solution, suitable

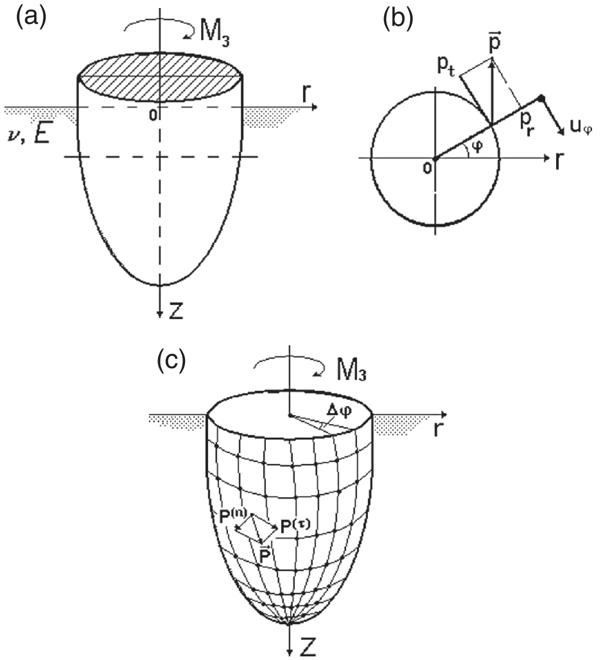

Fig. 2.7 Torsion of an elastic half-space by a rigid deepened punch in the shape of a rotation body: (**a**) calculation scheme; (**b**) stress-strained state of the horizontal plane; (**c**) cyclic (over the angular coordinate) discretization of the contact surface and representation of the contact stress vector on the boundary elements

for punches in the shape of rotation bodies without any restrictions on the meridional cross-section shape.

2.4.1 Axisymmetric Contact Problem

At central loading by an axial force $P_3 = P_z$ a deepened absolutely rigid punch in the shape of a rotation body will be displaced only vertically. The stressed state of an elastic half-space is characterized by radial p_r and vertical p_z components of the contact stress vector (there is no tangential stress due to the axial symmetry) which depend only on the vertical coordinate (Fig. 2.6a, b).

The system of equations of the spatial contact problem for a deepened axisymmetric punch, written using the theorem of reciprocity of works for the basic and the auxiliary states (see Sect. 2.1), is given by

$$\begin{cases} \iint\limits_{\Gamma} \left[p_r(N) U_r^{(1)}(K,N) + p_z(N) W^{(1)}(K,N) \right] d\Gamma - \Delta_3 \iint\limits_{\Gamma} p_z^{(1)}(K,N) d\Gamma = 0, \\ \iint\limits_{\Gamma} \left[p_r(N) U_r^{(3)}(K,N) + p_z(N) W^{(3)}(K,N) \right] d\Gamma - \Delta_3 \iint\limits_{\Gamma} p_z^{(3)}(K,N) d\Gamma = \frac{1}{2} \Delta_3 \end{cases} \tag{2.27}$$

where Δ_3 is vertical displacement of the punch, $p_r(N)$, $p_z(N)$ are the projections of the contact stress vector in the point N on the cylindrical coordinate system axes, $U_r^{(k)}(K,N)$, $W^{(k)}(K,N)$ are displacements of points of the elastic half-space, determined from the following formulae

$$\begin{cases} U_r^{(k)}(K,N) = U_{1\,k}(K,N) \cdot \cos\varphi + U_{2\,k}(K,N) \cdot \sin\varphi, \\ W^{(k)}(K,N) = U_{3\,k}(K,N), \quad k = 1,3 \end{cases} \tag{2.28}$$

$U_{ij}(K,N)$, $i,j = \overline{1,3}$ is the fundamental Mindlin's solution, written in the global Cartesian coordinate system, Γ is the punch contact surface, points $N \in \Gamma$ and $K \in \Gamma$.

The obtained integral equations (2.27) are essentially simplified if one takes into account the equilibrium equations of an elastic body in the shape of the deepened punch in the auxiliary state under the action of unit concentrated forces (see Sect. 2.1):

$$\iint\limits_{\Gamma} p_z^{(1)}(K,N) d\Gamma = 0, \qquad \iint\limits_{\Gamma} p_z^{(3)}(K,N) d\Gamma = \frac{1}{2}. \tag{2.29}$$

After substitution of Eq. (2.29) into Eq. (2.27), one obtains integral equations of the axisymmetric contact problem for a rigid punch, deepened into an elastic half-space in the shape of a rotation body, under an axial load:

$$\begin{cases} \iint\limits_{\Gamma} \left[p_r(N)U_r^{(1)}(K,N) + p_z(N)W^{(1)}(K,N)\right] d\Gamma = 0, \\ \iint\limits_{\Gamma} \left[p_r(N)U_r^{(3)}(K,N) + p_z(N)W^{(3)}(K,N)\right] d\Gamma = \Delta_3, \end{cases} \tag{2.30}$$

Boundary integral equations for the axisymmetric problem of Eq. (2.30) can be as well obtained in a formal way using the general equations of the spatial contact problem (2.9) in a special case when the punch does not undergo any rotations ($\psi_1 = \psi_2 = \psi_3 = 0$) and displacements across the symmetry axis ($\Delta_1 = \Delta_2 = 0$). Then the integral equation system (2.9) is given by

$$\begin{cases} \iint\limits_{\Gamma} \left[p_1(N)U_{11}(K,N) + p_2(N)U_{12}(K,N) + p_3(N)U_{13}(K,N)\right] d\Gamma = 0, \\ \iint\limits_{\Gamma} \left[p_1(N)U_{21}(K,N) + p_2(N)U_{22}(K,N) + p_3(N)U_{23}(K,N)\right] d\Gamma = 0, \\ \iint\limits_{\Gamma} \left[p_1(N)U_{31}(K,N) + p_2(N)U_{32}(K,N) + p_3(N)U_{33}(K,N)\right] d\Gamma = \Delta_3. \end{cases} \tag{2.31}$$

In the plane, orthogonal to the punch symmetry axis, for each contact point S we introduce a local coordinate system whose axes are directed tangentially (t) and normally (n) to the cross-section contour (Fig. 2.6b). Then, using the formulae for transformation of the displacement and stress vector components at the axis rotation

$$\begin{cases} p_1 = p_r \cdot \cos\varphi - p_t \cdot \sin\varphi, \\ p_2 = p_r \cdot \sin\varphi - p_t \cdot \cos\varphi, \end{cases} \quad \begin{cases} U_1 = U_r \cdot \cos\varphi - U_t \cdot \sin\varphi, \\ U_2 = U_r \cdot \sin\varphi - U_t \cdot \cos\varphi, \end{cases} \tag{2.32}$$

the system (2.31) can be presented in projections on the cylindrical coordinate system axes

$$\begin{cases} \iint\limits_{\Gamma} \left[p_r(N)U_r^{(1)}(K,N) + p_t(N)U_t^{(1)}(K,N) + p_3(N)U_3^{(1)}(K,N)\right] d\Gamma = 0, \\ \iint\limits_{\Gamma} \left[p_r(N)U_r^{(2)}(K,N) + p_t(N)U_t^{(2)}(K,N) + p_3(N)U_3^{(2)}(K,N)\right] d\Gamma = 0, \\ \iint\limits_{\Gamma} \left[p_r(N)U_r^{(3)}(K,N) + p_t(N)U_t^{(3)}(K,N) + p_3(N)U_3^{(3)}(K,N)\right] d\Gamma = \Delta_3. \end{cases} \tag{2.33}$$

Due to the symmetry at the axial loading the tangential components of the contact stress vector will be zero ($p_t = 0$) what considerably simplifies the system (2.33). Taking into account the fact that the first and the second equations of Eq. (2.33) are linearly dependent (i.e. one of them is a consequence of the other), the integral equation system of the axisymmetric contact problem takes the above form of Eq. (2.30).

Equation (2.30) should be complemented with an integral equation of equilibrium

$$\iint_{\Gamma} p_z(N)\, d\Gamma = P_3 \tag{2.34}$$

where P_3 is the resultant of the external forces, applied to the punch in the direction of the z axis.

Note that other five integral equations of equilibrium in the system (2.10) are fulfilled identically since at the axial loading $P_1 = P_2 = M_1 = M_2 = M_3 = 0$, p_r and p_z are independent of the angular coordinate φ, and after the transition form double integrals to iterated integrals each of the terms will contain zero factors

$$\int_0^{2\pi} \cos\varphi d\varphi = 0, \int_0^{2\pi} \sin\varphi d\varphi = 0.$$

Thus, the axisymmetric problem of theory of elasticity, consisting in the determination of the contact forces p_r, p_z and vertical displacements Δ_3, is reduced to the solution of the integral equation system (2.30) under the integral condition of Eq. (2.34) being fulfilled. Having found the solution, one can easily calculate the normal $p^{(n)}$ and tangential $p^{(\tau)}$ contact stresses based on the known relations

$$\begin{cases} p^{(n)} = p_r \cdot \cos\theta + p_z \cdot \sin\theta, \\ p^{(\tau)} = -p_r \cdot \sin\theta + p_z \cdot \cos\theta \end{cases} \tag{2.35}$$

where θ is the angle between the external normal to the contact surface and the horizontal plane (Fig. 2.6a).

For an approximate solution of the axisymmetric contact problem under consideration we use the direct boundary-element method combined with the piecewise constant approximation of the contact stress function what will enable the integral equations (2.30) and (2.34) to be reduced to a system of linear algebraic equations.

The most convenient way is to divide the contact surface of the punch in the shape of a rotation body into flat boundary elements whose nodes are formed by intersection of the "geographical" system of coordinate lines. For this purpose we build Q planes, containing the symmetry axis, turned by equal angles $\Delta\varphi = 2\pi/Q$. As a result, on the punch surface Q meridional zones will be formed. Then we build $M' = M+1$ horizontal planes, not necessarily equidistant. Consequently, the surface of the deepened part of the punch will be divided into $M \times Q$ boundary elements, among which there will be Q triangular and $(M-1) \times Q$ quadrangular elements

(Fig. 2.6c). We can say that we use cyclic discretization of the rotation surface over the angular coordinate since at a rotation around the symmetry axis by an angle, multiple of $\Delta\varphi = 2\pi/Q$, there will be a coincidence of all the boundary-element nodes with their initial positions. Note that the variation of the distance between the horizontal planes enables one, taking into account the curvature of the punch generatrix, to perform discretization uniformly with the required vertical condensation.

After the discretization of the surface of contact between the deepened part of the punch and the elastic half-space, the formed system of linear algebraic equations of the boundary-element method is given by

$$\begin{cases} \sum\limits_{n=1}^{N} [p_r(N_n) \iint\limits_{\Delta\Gamma_n} d\Gamma + p_z(N_n) \iint\limits_{\Delta\Gamma_n} W^{(1)}(K_i,N)d\Gamma] = 0, \\ \sum\limits_{n=1}^{N} [p_r(N_n) \iint\limits_{\Delta\Gamma_n} U_r^{(3)}(K_i,N)d\Gamma + p_z(N_n) \iint\limits_{\Delta\Gamma_n} W^{(3)}(K_i,N)d\Gamma] = \Delta_3, \\ \sum\limits_{n=1}^{N} p_z(N_n)\Delta s_n = P_3, \quad i = 1,2,...,N. \end{cases} \tag{2.36}$$

Here the following notations are used: $N = M \times Q$ is the total number of the bound elements on the punch contact surface; $p_r(N_n)$, $p_z(N_n)$ are the averaged values of the radial and vertical contact stress, respectively, within the n-th boundary element; K_i are the collocation points (the gravity centres of the boundary elements); ΔS_n is the area of the n-th boundary element.

We further reduce the system (2.36), assuming the above discretization of the contact surface between the punch and the base to be regular, cyclic over the angular coordinate. The latter condition enables one to increase essentially the dimensionality of the algebraic analogue in comparison with the system of Eqs. (2.10) and (2.9) for the general spatial contact problem. If the punch generatrix was divided by the horizontal planes into M sections, and over the angular coordinate into Q meridional zones (being determined by equal dihedral angles $\Delta\varphi = 2\pi/Q$), then the system (2.36) with the account of the cyclicity requirement is given by

$$\begin{cases} \sum\limits_{m=1}^{M} \left[p_r(N_m) \sum\limits_{q=1}^{Q} \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_r^{(1)}(K_i,N)d\Gamma + p_z(N_m) \sum\limits_{q=1}^{Q} \iint\limits_{\Delta\Gamma_{M(q-1)+m}} W^{(1)}(K_i,N)d\Gamma \right] = 0, \\ \sum\limits_{m=1}^{M} \left[p_r(N_m) \sum\limits_{q=1}^{Q} \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_r^{(3)}(K_i,N)d\Gamma + p_z(N_m) \sum\limits_{q=1}^{Q} \iint\limits_{\Delta\Gamma_{M(q-1)+m}} W^{(3)}(K_i,N)d\Gamma \right] = \Delta_3, \\ \sum\limits_{m=1}^{M} p_z(N_m)\Delta s_m = P_3/Q, \quad i = 1,2,...,M. \end{cases} \tag{2.37}$$

Here $M = N/Q$ is the number of boundary elements in a single meridional zone of the punch contact surface.

For the sake of convenient realization of numerical algorithms the algebraic analogue Eq. (2.37) of the integral equation system of the axisymmetric contact problem is presented in the matrix form:

$$\mathbf{A} \cdot Z = B \tag{2.38}$$

where $A = \begin{pmatrix} D_{2M\times 2M} & C_{2M\times 1} \\ T_{1\times 2M} & 0 \end{pmatrix}$ is a square block matrix of the order $(2M + 1)$,

$$D_{2M\times 2M} = \begin{pmatrix} a_{11}^{(1)} & b_{11}^{(1)} & a_{21}^{(1)} & b_{21}^{(1)} & \ldots & a_{M1}^{(1)} & b_{M1}^{(1)} \\ a_{11}^{(3)} & b_{11}^{(3)} & a_{21}^{(3)} & b_{21}^{(3)} & \ldots & a_{M1}^{(3)} & b_{M1}^{(3)} \\ \ldots & \ldots & \ldots & \ldots & & \ldots & \ldots \\ a_{1M}^{(1)} & b_{1M}^{(1)} & a_{2M}^{(1)} & b_{2M}^{(1)} & \ldots & a_{MM}^{(1)} & b_{MM}^{(1)} \\ a_{1M}^{(3)} & b_{1M}^{(3)} & a_{2M}^{(3)} & b_{2M}^{(3)} & \ldots & a_{MM}^{(3)} & b_{MM}^{(3)} \end{pmatrix} \text{ is the influence matrix,}$$

$$C_{2M\times 1} = -\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad T_{1\times 2M} = (0,\Delta s_1;0,\Delta s_2;\ldots;0,\Delta s_M)\,;$$

Z and **B** are column vectors of the size $(2M + 1)$:

$$\mathbf{Z} = (p_r(N_1), p_z(N_1); \quad p_r(N_2), p_z(N_2);\ldots p_r(N_M), p_z(N_M);\Delta_3)^T,$$

$$\mathbf{B} = (0,0;0,0;\ldots;P_3)^T;$$

ΔS_i $(i = 1, 2, \ldots M)$ are the areas of flat triangles or quadrangles, dividing the meridional zone into the boundary elements whose numbering is determined in accordance with the vertical coordinate variation;

$$a_{im}^{(k)} = \sum_{q=1}^{Q} \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_r^{(k)}(K_i, N)d\Gamma =$$

$$= \sum_{q=1}^{Q} \cos\varphi_q \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_{1k}(K_i, N)d\Gamma + \sum_{q=1}^{Q} \sin\varphi_q \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_{2k}(K_i, N)d\Gamma;$$

$$b_{im}^{(k)} = \sum_{q=1}^{Q} \iint\limits_{\Delta\Gamma_{M(q-1)+m}} W^{(k)}(K_i, N)d\Gamma = \sum_{q=1}^{Q} \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_{3k}(K_i, N)d\Gamma$$

are the coefficients of the influence matrix $\mathbf{D}_{2M\times 2M}$, determined by the numerical-and-analytical integration method, described above in Sect. 2.3, $\varphi_q = (2q-1)\pi/Q$, $k = 1, 3; 4$, $m = 1, 2,\ldots, M$.

It is seen that the dimensionality of the algebraic analogue of the boundary contact problem equals $(2M+1)$ with respect to the values of Δ_3 and $p_r(N_m)$, $p_z(N_m)$ $(m = 1, 2,\ldots M)$ where M is the boundary element number along the punch generatrix.

2.4.2 Torsion of an Axisymmetric Punch in an Elastic Half-Space

Consider a punch in the shape of a rotation body, deepened into an elastic half-space, under the action of a sole torque $M_3 \neq 0$ $(P_1 = P_2 = P_3 = M_1 = M_2 = 0)$. Then the punch displacement will be determined only by the angle $\psi_3 \neq 0$ $(\psi_1 = \psi_2 = \Delta_1 = \Delta_2 = \Delta_3 = 0)$, characterizing the punch rotation around the Oz axis (Fig. 2.7a). The equation system (2.9) in the special case under consideration is given by

$$\begin{cases} \iint\limits_{\Gamma} \left[p_1(N)U_{11}(K,N) + p_2(N)U_{12}(K,N) + p_3(N)U_{13}(K,N)\right] d\Gamma = \eta \cdot \psi_3, \\ \iint\limits_{\Gamma} \left[p_1(N)U_{21}(K,N) + p_2(N)U_{22}(K,N) + p_3(N)U_{23}(K,N)\right] d\Gamma = -\xi \cdot \psi_3, \\ \iint\limits_{\Gamma} \left[p_1(N)U_{31}(K,N) + p_2(N)U_{32}(K,N) + p_3(N)U_{33}(K,N)\right] d\Gamma = 0. \end{cases} \tag{2.39}$$

Similarly to the case of the axisymmetric problem (see Sect. 2.4.1), using the formulae (2.32) for the transformation of the vector components at the axis rotation, the system (2.39) is presented in projections onto the cylindrical coordinate system axes

$$\begin{cases} \iint\limits_{\Gamma} \left[p_r(N)U_r^{(1)}(K,N) + p_t(N)U_t^{(1)}(K,N) + p_z(N)U_z^{(1)}(K,N)\right] d\Gamma = \eta \cdot \psi_3, \\ \iint\limits_{\Gamma} \left[p_r(N)U_r^{(2)}(K,N) + p_t(N)U_t^{(2)}(K,N) + p_z(N)U_z^{(2)}(K,N)\right] d\Gamma = -\xi \cdot \psi_3, \\ \iint\limits_{\Gamma} \left[p_r(N)U_r^{(3)}(K,N) + p_t(N)U_t^{(3)}(K,N) + p_z(N)U_z^{(3)}(K,N)\right] d\Gamma = 0. \end{cases} \tag{2.40}$$

Since, according to the problem formulation, the punch does not undergo axial forces, there will arise no vertical stress on the contact surface, i.e. $p_z = 0$. Besides, the system (2.40) will be additionally simplified due to the fact that its first and second equations are linearly dependent (one is the consequence of the other). As a result, the system of boundary integral equations for the contact problem of an elastic half-space torsion due to the axial rotation of a deepened punch will be given by

$$\begin{cases} \iint\limits_{\Gamma} \left[p_r(N) U_r^{(1)}(K,N) + p_t(N) U_t^{(1)}(K,N) \right] d\Gamma = \eta \cdot \psi_3, \\ \iint\limits_{\Gamma} \left[p_r(N) U_r^{(3)}(K,N) + p_t(N) U_t^{(3)}(K,N) \right] d\Gamma = 0, \end{cases} \tag{2.41}$$

where

$$\begin{cases} U_r^{(k)}(K,N) = U_{1\,k}(K,N) \cdot \cos\varphi + U_{2\,k}(K,N) \cdot \sin\varphi, \\ U_t^{(k)}(K,N) = -U_{1\,k}(K,N) \cdot \sin\varphi + U_{2\,k}(K,N) \cdot \cos\varphi, k = 1,3. \end{cases} \tag{2.42}$$

The equation system (2.41) becomes closed if it is complemented by the integral equilibrium equation

$$\iint\limits_{\Gamma} \left[x_1 p_2(N) - x_2 p_1(N) \right] d\Gamma = M_3. \tag{2.43}$$

Evidently, the other five equilibrium equations of the system (2.10) will be identically fulfilled due to the symmetry of the problem and independence of the contact stress on the angular coordinate, similarly to the case of the axial loading (Sect. 2.4.1).

The equilibrium equation (2.43), similarly to the boundary integral equations (2.41), can be written in terms of the radial and tangential projections of the stress vector. Taking into account that $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, and p_1 and p_2 are expressed in terms of p_r and p_t using Eq. (2.32), the equilibrium equation (2.43) is given by

$$\iint\limits_{\Gamma} p_t \cdot r d\Gamma = M_3 \tag{2.44}$$

where $r = \sqrt{x_1^2 + x_2^2} = r(z)$ is the radial coordinate of the contact surface points.

Thus, the inverse symmetrical problem of theory of elasticity, consisting in the determination of the contact forces p_r and p_t and rotation angles ψ_3, is reduced to the solution of the integral equation system (2.41), the integral condition of Eq. (2.44) being fulfilled. Note that the found solution will simultaneously determine the normal $p^{(n)} = p_r$ and tangential $p^{(\tau)} = p_t$ contact stresses (Fig. 2.7c).

The approximate solution of the inverse symmetrical contact problem will be obtained similarly to the above considered (Sect. 2.4.1) axisymmetric problem, using the direct boundary-element method in combination with the piecewise constant approximation of the contact stress function. Assuming the discretization with the cyclic symmetry to be performed (Fig. 2.7c) and omitting cumbersome intermediate calculations, the integral equations (2.41) and (2.44) can be readily reduced to the following system of linear algebraic equations of the direct boundary-element method:

$$\begin{cases} \sum\limits_{m=1}^{M}\left[p_r(N_m)\sum\limits_{q=1}^{Q}\iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_r^{(1)}(K_i,N)d\Gamma + p_t(N_m)\sum\limits_{q=1}^{Q}\iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_t^{(1)}(K_i,N)d\Gamma\right] = \eta_i\cdot\psi_3, \\ \sum\limits_{m=1}^{M}\left[p_r(N_m)\sum\limits_{q=1}^{Q}\iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_r^{(3)}(K_i,N)d\Gamma + p_t(N_m)\sum\limits_{q=1}^{Q}\iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_t^{(3)}(K_i,N)d\Gamma\right] = 0, \\ \sum\limits_{m=1}^{M} p_t(N_m)r_m\Delta s_m = M_3/Q, \quad i=1,2,...,M. \end{cases} \tag{2.45}$$

Here $p_r(N_m)$ and $p_t(N_m)$ are the averaged values of the radial and tangential contact stress, respectively, within the m-th boundary element, the rest of notations being the same as those in Sect. 2.4.1.

In the matrix form the algebraic analogue (2.45) of the integral equation system is given by

$$\mathbf{A}\cdot Z = B \tag{2.46}$$

where $\mathrm{A} = \begin{pmatrix} \mathrm{F}_{2M\times 2M} & \mathrm{G}_{2M\times 1} \\ \mathrm{H}_{1\times 2M} & 0 \end{pmatrix}$ is a square block matrix of the order $(2M+1)$,

$$\mathrm{F}_{2M\times 2M} = \begin{pmatrix} g_{11}^{(1)} & h_{11}^{(1)} & g_{21}^{(1)} & h_{21}^{(1)} & \cdots & g_{M1}^{(1)} & h_{M1}^{(1)} \\ g_{11}^{(3)} & h_{11}^{(3)} & g_{21}^{(3)} & h_{21}^{(3)} & \cdots & g_{M1}^{(3)} & h_{M1}^{(3)} \\ \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots \\ g_{1M}^{(1)} & h_{1M}^{(1)} & g_{2M}^{(1)} & h_{2M}^{(1)} & \cdots & g_{MM}^{(1)} & h_{MM}^{(1)} \\ g_{1M}^{(3)} & h_{1M}^{(3)} & g_{2M}^{(3)} & h_{2M}^{(3)} & \cdots & g_{MM}^{(3)} & h_{MM}^{(3)} \end{pmatrix} \text{ is the influence matrix,}$$

$$\mathrm{G}_{2M\times 1} = -\begin{pmatrix} \eta_1 \\ 0 \\ \eta_2 \\ 0 \\ \vdots \\ \eta_M \\ 0 \end{pmatrix}, \mathrm{H}_{1\times 2M} = (0,r_1\Delta s_1;0,r_2\Delta s_2;...;0,r_M\Delta s_M)\,;$$

Z and **B** are column vectors of the size $(2M + 1)$:

$$\mathrm{Z} = (p_r(N_1), p_t(N_1); p_r(N_2),p_t(N_2);...p_r(N_M),p_t(N_M);\psi_3)^T ,$$

$$\mathrm{B} = (0,0;0,0;...;M_3)^T ;$$

$$g_{im}^{(k)} = \sum_{q=1}^{Q} \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_r^{(k)}(K_i, N)\, d\Gamma =$$

$$= \sum_{q=1}^{Q} \cos\varphi_q \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_{1\,k}(K_i, N)\, d\Gamma + \sum_{q=1}^{Q} \sin\varphi_q \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_{2\,k}(K_i, N)\, d\Gamma;$$

$$h_{im}^{(k)} = \sum_{q=1}^{Q} \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_t^{(k)}(K_i, N)\, d\Gamma =$$

$$= -\sum_{q=1}^{Q} \sin\varphi_q \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_{1\,k}(K_i, N)\, d\Gamma + \sum_{q=1}^{Q} \cos\varphi_q \iint\limits_{\Delta\Gamma_{M(q-1)+m}} U_{2\,k}(K_i, N)\, d\Gamma$$

are the coefficients of the influence matrix $\mathbf{F}_{2M\times 2M}$, $\varphi_q = (2q-1)\pi/Q$, $k = 1, 3$; $i, m = 1, 2,\ldots,M$.

Evidently, the dimensionality and the structure of the matrix representation of the algebraic analogue of the boundary contact problem in terms of ψ_3 and $p_r(N_m)$, $p_z(N_m)$ $(m = 1, 2,\ldots M)$ values are the same as in the above axisymmetric case.

Thus, we have considered the integral equations of the contact problems for deepened punches with axial symmetry and have constructed an efficient method of their solution ob the base of direct boundary-element formulation. The results of the formulations presented in Sects. 2.4.1 and 2.4.2 enable numerical solutions of the class of contact problems of theory of elasticity, important for practical application, to be effectively constructed. Attention should be paid to the following main advantages of the elaborated algorithms, increasing the efficiency of application of the boundary-element method in engineering.

As noted above, spatial contact problems of theory of elasticity with axial symmetry are two-dimensional, since, due to the independence of the geometrical shape of the punch and the boundary conditions on the angular coordinate, the characteristics of the stress-strained state of the base will be determined only by virtue of the radial r and vertical z coordinates. The application of the boundary-element method with a special cyclic discretization of the contact surface additionally reduces the geometrical dimensionality of the problem: the contact stresses are to be determined only for the points of the broken line, approximating the punch generatrix. Consequently the application of the boundary-element method, enabling the possibility of further reduction of dimensionality, reduces the axisymmetric contact problem to a one-dimensional one.

The time, required for the solution of the contact problems with axial symmetry, will be essentially shortened in comparison with the problems in general spatial formulation, due to the two reasons. First, the dimensionality of the resolving systems of algebraic equations is reduced, and for these systems, in case Gauss elimination method being used, the solution time is proportional to N^3 (N is the number of the equations). Besides, due to the angular (cyclic) symmetry the time for the computation of the influence matrix coefficients will be reduced by factor of $3Q/2$. In practice, in our calculations of contact problems with axial symmetry, based on the proposed reduced formulation, with the number of the boundary elements of about 400, the total computation time was reduced in average by factor of 20 in comparison with that required in case of application of the spatial scheme of the most general way. It should be also mentioned that by increasing the number of the boundary elements along the angular coordinate one can increase the accuracy of the numerical solution of contact problems with axial symmetry without increasing the dimensionality of the system of resolving algebraic equations, increasing only the computation time for the formation of the influence matrix, i.e. without extending the computer RAM resources.

Finally we note that in a great many of studies, devoted to the solution of axisymmetric problems of theory of elasticity for the finite-size bodies, using the Kelvin's solution, the implementation of the boundary-element method implies a procedure of analytical integration over the angular coordinate (See, e.g., the references in [7, 10, 17]). This results in complete elliptical integrals of the first and second order, which, in turn, for the sake of convenience of further numerical calculations, are presented in the form of an expansion over polynomials [1]. A number of authors note that in the axisymmetric case the integral cores have a rather complicated form and the related calculations are cumbersome. In [29], devoted to the axisymmetric contact problem for a rigid deepened punch, integration of the Mindlin's solution over the angular coordinate is performed numerically. Evidently, such an algorithm can be efficient only for the punches of cylindrical shape when the radial coordinate of the contact surface points remains constant. The approach developed here seems more convenient since it enables the solutions for both axisymmetric problems and problems of general spatial formulation to be obtained, based on the same computation algorithm of formation of influence matrices, without loss in accuracy.

2.5 Contact Problems for Rigid Punches Located on the Elastic Base Surface

Considering a spatial contact problem for a rigid punch and an elastic base, we analyze a limiting case when the punch is not deepened, i.e. is located on the elastic base surface (Fig. 2.8). We also assume the punch bottom to be flat; then the contact domain F will be a part of the half-space surface. Hence, in the boundary integral equations of the general spatial contact problem (2.9), (2.10) one should imply $z = \zeta = 0$, and the fundamental Mindlin's solution to be transformed into a

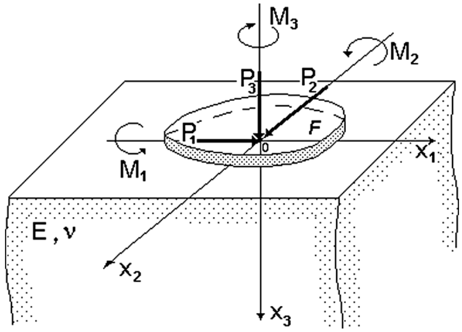

Fig. 2.8 Calculation scheme for the spatial contact problem for a non-deepened rigid punch with a flat bottom, located at the surface of the elastic base

combined Boussinesq-Cerruti solution. The account of the above statements enables the boundary integral equations of the spatial problem of theory of elasticity for a flat rigid punch, contacting an elastic half-space on its surface, to be written in the following form:

$$\begin{cases} \iint\limits_F \left[p_1(N)U^{(1)}(K,N) + p_2(N)V^{(1)}(K,N) + p_3(N)W^{(1)}(K,N)\right] ds = \Delta_1 + \eta \cdot \psi_3, \\ \iint\limits_F \left[p_1(N)U^{(2)}(K,N) + p_2(N)V^{(2)}(K,N) + p_3(N)W^{(2)}(K,N)\right] ds = \Delta_2 - \xi \cdot \psi_3, \\ \iint\limits_F \left[p_1(N)U^{(3)}(K,N) + p_2(N)V^{(3)}(K,N) + p_3(N)W^{(3)}(K,N)\right] ds = \Delta_3 - \eta \cdot \psi_1 + \xi \cdot \psi_2. \end{cases} \tag{2.47}$$

To make the system (2.47) closed we take into account six static equilibrium equations whose form is more simple than in Eqs. (2.10):

$$\begin{cases} \iint\limits_F p_1(N)ds = P_1, \iint\limits_F p_2(N)ds = P_2, \iint\limits_F p_3(N)ds = P_3 \\ \iint\limits_F p_3(N)x_2 ds = M_1, \iint\limits_F p_3(N)x_1 ds = -M_2, \iint\limits_F \left[p_2(N)x_1 - p_1(N)x_2\right] ds = M_3. \end{cases} \tag{2.48}$$

In Eqs. (2.47), (2.48) the following notations are assumed: $p_1(N) = p_x(x, y)$, $p_2(N) = p_y(x, y)$, $p_3(N) = p_z(x, y)$ are the sought contact stress functions, acting in a flat domain F; $U^{(k)}$, $V^{(k)}$, and $W^{(k)}$ ($k = 1, 2, 3$) are the components of the combined Boussinesq-Cerruti solution for the displacements of the half-space surface due to unit concentrated forces, acting in the direction of the coordinate axes $x_k (k = 1, 2, 3)$.

In comparison with the initial system of Eqs. (2.9), (2.10) for the general spatial contact problem for a deepened punch, the integral equation system (2.47) and (2.48) is simpler. First, the contact pressure functions have to be determined in a flat domain and, hence, both the discretization of the contact domain and the interpolation of the discrete numerical results at the analysis of the contact problem solution will be easier than for a curved surface. Second, the integration procedure at the formation of the influence matrix of the resolving system of algebraic equations of the boundary-element method will require less time than in the case of the full Mindlin's solution. Nevertheless, the total computation time for the solution of the spatial contact problem for a non-deepened punch will be of the same order as for the solution of the contact problem in the most general spatial formulation.

Then we consider two important special cases of spatial loading of non-deepened punches, for which the integral equation system (2.47), (2.48) is essentially simplified, following separately the force balance at contact deformation in the vertical and horizontal planes, respectively. The first case corresponds to the punch indentation by a vertical force and pull-out torques, acting with respect to the coordinate axes in the punch base plane; in the contact domain only vertical (normal) stress exists. In the second case, the punch, linked to the half-space, undergoes a torque; only horizontal (tangential) stresses act on the contact surface.

2.5.1 *Indentation of a Punch with a Flat Smooth Base into an Elastic Half-Space*

Let the punch base be smooth, i.e. in the contact domain F tangential stress is zero. In this case the external load system will not include horizontal forces P_1, P_2 and torque M_3 leading to the tangential stress in the contact domain. The punch will be indented into the base by a vertical force P_3 and torques M_1, M_2, rotating the punch around the axes Ox_1 and Ox_2. Besides, since no friction between the punch bottom and the base surface is assumed, then only one of the three displacement vector components will be varied, namely the vertical one. Thus, in the system (2.47), (2.48) one should imply $\Delta_1 = \Delta_2 = 0$, $\psi_3 = 0$, $P_1 = P_2 = M_3 = 0$, $p_1 = p_2 = 0$. Then the interaction of the punch with the base will be characterized by the function of vertical stress (contact pressure) $p_3 = p(x, y)$ and vertical displacement of the points of the flat punch bottom $W = \Delta_3 - \eta \cdot \psi_1 + \xi \cdot \psi_2$. The above assumptions will reduce the integral equation system of the spatial contact problem for a smooth punch with a flat base to a single equation

$$\iint_F p_3(N)W^{(3)}(K,N) = \Delta_3 - \eta \cdot \psi_1 + \xi \cdot \psi_2. \tag{2.49}$$

In the equilibrium equation system (2.48) three conditions, corresponding to the horizontal force balance, are fulfilled identically, and the other three are given by

$$\begin{cases} \iint\limits_F p_3(N)ds = P_3, \\ \iint\limits_F p_3(N)x_2 ds = M_1, -\iint\limits_F p_3(N)x_1 ds = M_2. \end{cases} \tag{2.50}$$

The system (2.49), (2.50) can be written in the form, corresponding to the notations, established in the literature

$$\iint\limits_p (\xi,\eta)\omega(x,y,\xi,\eta)d\xi d\eta = W_c + \psi_x \cdot (x - x_c) + \psi_y \cdot (y - y_c), \tag{2.51}$$

$$\begin{aligned} &\iint\limits_F p(x,y)dxdy = P, \iint\limits_F p(x,y)ydxdy = P \cdot y_c - M_x, \\ &\iint\limits_F p(x,y)xdxdy = P \cdot x_c + M_y \end{aligned} \tag{2.52}$$

where

F is the domain of the punch contact with the elastic base;
$p(x,y) = p_3(x,y,0)$ is the sought contact pressure function,
$W(x,y) = W_c + \psi_x \cdot (x - x_c) + \psi_y \cdot (y - y_c)$ is the displacement of the point $N(x,y)$ of the contact surface of the punch and the elastic base,
W_c is the vertical displacement of the punch centre (x_c, y_c) (point of application of the external forces and torques),
ψ_x and ψ_y are the punch slopes with respect to the coordinate axes,
$\omega(x,y,\xi,\eta) = W^{(3)}(K,N) = (1-\nu^2)/\pi E\sqrt{(x-\xi)^2 + (y-\eta)^2}$ is the influence function (Boussinesq solution),
P, M_x, M_y are the external vertical force and tilting moments.

Equation (2.51) expresses the fact that the punch displacement $W(N)$ is numerically equal to the sum of works from the contact forces $p\,(x, y)$ in the basic state on the vertical displacements of the base surface

$$\omega\,(x,y,\xi,\eta) = \frac{(1-\nu^2)}{\pi E} \cdot \frac{1}{R}, \quad R = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

of the auxiliary state, resulting from the action of a unit vertical concentrated force.

Thus, the spatial contact problem on the indentation of a rigid punch with a smooth flat bottom into an elastic half-space is reduced to the finding from a 2-D integral equation (2.51) of the parameters W_c, ψ_x and ψ_y, determining the punch position, and the contact pressure function $p(\xi,\eta)$ over its bottom, satisfying the equilibrium conditions (2.52).

While characterizing the integral equation (2.51), note that, in spite of its simple form, the general studies of the corresponding mixed problem of theory of elasticity are far from being finished [4, 15, 30]. A closed solution of the integral equation (2.51), except for its axisymmetric case, has been found only for elliptical and ring-shaped contact domains. Construction of approximate solutions of the integral equation (2.51) for contact domains of a rather general shape is a serious computational problem. Our overview of the main solution methods for the spatial contact problem for flat complex-shaped punches, interacting with elastic, spatially nonhomogeneous bases, is presented in Sect. 4.1.

We obtain a finite-measure algebraic analogue of the integral equation system (2.51), (2.52) of the contact problem under consideration using the boundary element method. For this purpose first we approximate the contact surface by boundary elements of, in general, polygonal shape. This can be done using practically any of the known considerable amount of mesh generators for arbitrary flat domains, used in finite-difference, finite- and boundary-element methods and having their own specific features. The proposed and applied here algorithms of automatic mesh of flat surfaces of a rather general shape are considered in Sect. 3.3. They are rather economical (require small computation time in comparison with the numerical solution of the problem, for which the mesh is built) and are capable of controlling the mesh geometry using easily treated "free" parameters (condensation degree, maximal and minimal boundary element size etc.).

In the simplest version of the boundary-element method the points of application of the unit concentrated forces are located in the gravity centres of the boundary elements and a piecewise contact approximation of the contact pressures $p(\xi,\eta) = \text{const}$ within each boundary element is assumed. Then, instead of Eqs. (2.51) and (2.52), a linear algebraic equation system with $(m+3)$ unknowns is obtained:

$$\begin{cases} p_1\delta_{i1} + p_2\delta_{i2} + ... + p_m\delta_{im} - W_c - \psi_x \cdot (x_c - x_i) - \psi_y \cdot (y_c - y_i) = 0,\ i = \overline{1,m}, \\ p_1\Delta s_1 + p_2\Delta s_2 + ... + p_m\Delta s_m = P, \\ p_1\Delta s_1 x_1 + p_2\Delta s_2 x_2 + ... + p_m\Delta s_m x_m = P \cdot x_c + M_y, \\ p_1\Delta s_1 y_1 + p_2\Delta s_2 y_2 + ... + p_m\Delta s_m y_m = P \cdot y_c - M_x, \end{cases} \tag{2.53}$$

Here $p_i(\xi_i,\ \eta_i)$ are the sought contact stresses for the boundary elements (i= 1, . . ., m),

$$\delta_{ij} = \iint\limits_{F_j} \omega\,(x,y,\xi,\eta)\,d\xi\,d\eta \tag{2.54}$$

is the vertical displacement of the base surface in the point (x_i, y_i) coinciding with the gravity centre of the i-th element, due to a unit load, uniformly distributed over the domain F_j of the j-th element, ΔS_i (i = 1, 2,. . .m) are the areas of the flat boundary elements whose combination approximates the contact domain F.

For numerical solution of the system (2.53) one can take the advantage of its matrix form

$$\mathbf{A} \cdot Z = B \tag{2.55}$$

where

$\mathbf{A} = \begin{pmatrix} \mathbf{D}_{m\times m} & \mathbf{C}_{m\times 3} \\ \mathbf{T}_{3\times m} & 0 \end{pmatrix}$ – is a block matrix of the size $(m+3)\times(m+3)$,

$\mathbf{D} = \left\| \delta_{ij} \right\|$ is the influence matrix, $i, j = 1,\ldots m$; **C** and **T** are rectangular matrices

$$\mathbf{C} = -\begin{pmatrix} 1 & x_1 - x_c & y_1 - y_c \\ 1 & x_2 - x_c & y_2 - y_c \\ \ldots & \ldots & \ldots \\ 1 & x_m - x_c & y_m - y_c \end{pmatrix}, \mathbf{T} = \begin{pmatrix} \Delta s_1 & \Delta s_2 & \ldots & \Delta s_m \\ \Delta s_1 \cdot x_1 & \Delta s_2 \cdot x_2 & \ldots & \Delta s_m \cdot x_m \\ \Delta s_1 \cdot y_1 & \Delta s_2 \cdot y_2 & \ldots & \Delta s_m \cdot y_m \end{pmatrix};$$

Z and **B** are column matrices of the size $(m+3)$:
$\mathrm{Z} = \left(p_1, p_2 \ldots p_m;\, W_c, \psi_x, \psi_y\right)^T$, $\mathrm{B} = \left(0{,}0{,}\ldots 0;\, P;\, P \cdot x_c - M_y; P \cdot y_c + M_x\right)^T$.

The discretization of the contact domain (the punch bottom) will result in the location of all the boundary elements in the same plane (the half-space surface $z=0$). Therefore, in order to increase the computation algorithm efficiency and accuracy we use in Eq. (2.54) the same procedure of analytical calculation of both singular ($i=j$) and regular ($i \neq j$) surface integrals over an arbitrary boundary element with a polygonal contour. This is achieved on the base of algebraic assembling (according to the choice of sign while moving along a closed circuit) of singular integrals over triangles with a singularity in the concentrated force application point. For example, for a triangular boundary element with the apices *A, B,* and *C* and an arbitrary point $K(\xi, \eta)$, located outside (Fig. 2.9a) and inside (Fig. 2.9b) the integration domain the same sum

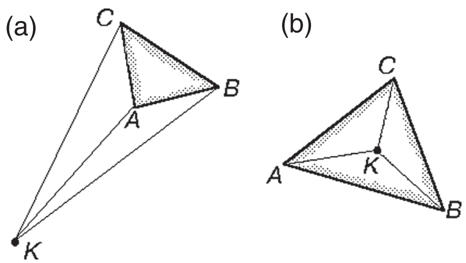

Fig. 2.9 Geometrical scheme of the integration domain in case the point *K* of application of a unit concentrated force being located (**a**) outside and (**b**) inside the boundary element

$$\int_{ABC} = \int_{KAB} + \int_{KBC} + \int_{KCA} .$$

is used, corresponding to the positive (anti-clockwise) direction of moving along the circuit.

The exact calculation of integrals of the form

$$\int\int \omega(x, y, \xi, \eta) d\xi d\eta,$$

over a triangular domain $\Delta F = M_1 M_2 M_3$, when the integrand has a singularity in the first apex, is reduced to the above obtained integral I_1 (Sect. 2.3). According to Eq. (2.23), in a general case, when the triangle is arbitrarily oriented in the elastic half-space, the latter integral can be found by a combination of logarithmic and trigonometric functions. In our case, when the punch is located on the half-space surface ($Z_1 = Z_2 = Z_3 = 0$), Eq. (2.23) remains unchanged, and the expressions for its parameters are considerably simplified, having the form

$$D = \sqrt{(X_* - X_1)^2 + (Y_* - Y_1)^2}, \quad \hat{R} = \sqrt{(X_3 - X_2)^2 + (Y_3 - Y_2)^2},$$
$$X_* = X_2 - (X_3 - X_2) \cdot p, \quad Y_* = Y_2 - (Y_3 - Y_2) \cdot p,$$
$$p = ((X_2 - X_1)(X_3 - X_2) + (Y_2 - Y_1)(Y_3 - Y_2)) / \hat{R}^2.$$

The simplest form of the system (2.53) is achieved for a circular punch at central ($x_c = 0$, $y_c = 0$) loading (axisymmetric contact problem). Due to the axial symmetry ($M_x = M_y = 0$), ($\psi_x = \psi_y = 0$), hence instead of Eq. (2.53) one obtains

$$\begin{cases} p_1\delta_{i1} + p_2\delta_{i2} + ... + p_m\delta_{im} - W_c = 0, \; i = \overline{1,m}, \\ p_1\Delta s_1 + p_2\Delta s_2 + ... + p_m\Delta s_m = P. \end{cases} \tag{2.56}$$

The reduction of the system (2.56) is performed similarly to the case of the axisymmetric problem of a deepened punch indentation (Sect. 2.4.1), assuming cyclic discretization of the circular (or the ring-shaped) contact domain. A typical example of such discretization by means of a regular ($\Delta\varphi$ = const) grid of boundary elements whose nodes are obtained by intersection of rays and concentric circles in a polar coordinate system, is shown in Fig. 2.10. Afterwards the system (2.56) is transformed in the following way. In each line of this system there are several terms which, due to the cyclic symmetry, contain the same contact force values, corresponding to the same boundary element number along the radius (or the number of the ring-shaped layer). Combine all such terms in the line, corresponding to each ring-shaped layer. If L is the number of the boundary elements along the punch radius, then the algebraic analogue of the integral equation system on the contact problem for central indentation of a circular (or ring-shaped) punch with a flat smooth bottom takes its most simple form

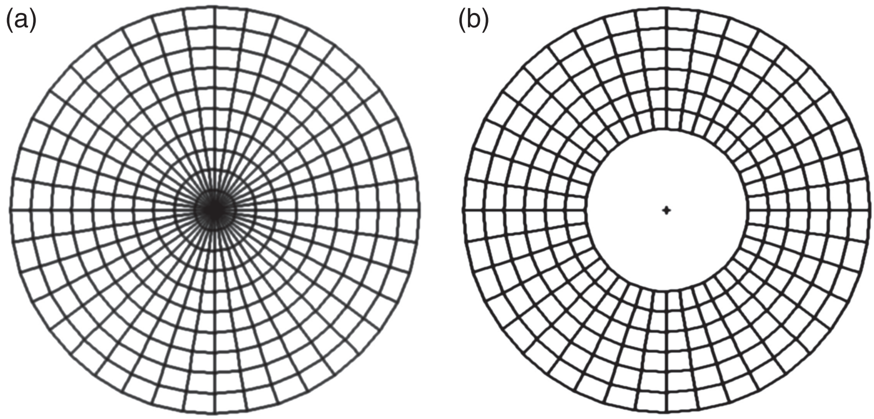

Fig. 2.10 Cyclically symmetrical discretization of circular (**a**) and ring-shaped (**b**) contact domains

$$\sum_{j=1}^{L} A_{ij} \cdot p_j - W_c = 0,\ i = \overline{1,L},\ \sum_{j=1}^{L} p_j \cdot \Delta s_j = \frac{P \cdot L}{m} \tag{2.57}$$

Here the values A_{ij} are found using the influence coefficients δ_{ij} ($i = 1,\ldots, L$; $j = 1,\ldots, m$) from the formula

$$A_{ij} = \sum_{k=1}^{m/L} \delta_{i,j+L(k-1)}.$$

It is evident that the system (2.57) can be also easily obtained from the system (2.37) as a degenerate case of a deepened punch with a smooth bottom $W^{(3)}(K,N) = \omega(x, y, \xi, \eta), \rho_\tau = 0$.

2.5.2 *Torsion of an Elastic Half-Space by a Rigid Punch*

Let the system of the external load on the punch include only a torque M_3, resulting in the formation of tangential stresses in the contact domain. Neither other torques M_1, M_2, nor both vertical P_3 and horizontal P_1, P_2 external forces are assumed. In this case the punch will rotate with the elastic base, and its position is characterized only by the rotation angle ψ_3 around the Oz axis. In the system (2.47) let $\Delta_1 = \Delta_2 = \Delta_3 = 0$, ($P_1 = P_2 = M_1 = M_2 = 0$). Due to the absence of vertical loads and tilting moments it can be reasonably assumed that the torsion does not affect the pressure distribution below the punch bottom, hence $p_3 = 0$ [19]. Contrary to the above case of the punch indentation (Sect. 2.5.1), now the contact interaction of the punch with the base will be characterized by two functions of contact tangential stress $p_1 = p_x(x, y)$ and $p_2 = p_y(x, y)$. The integral equations (2.47) of the spatial contact problem of torsion for the case of a flat-bottom punch are reduced to the following system:

$$\begin{cases} \iint\limits_F \left[p_1(N)U^{(1)}(K,N) + p_2(N)V^{(1)}(K,N)\right] ds = \eta \cdot \psi_3, \\ \iint\limits_F \left[p_1(N)U^{(2)}(K,N) + p_2(N)V^{(2)}(K,N)\right] ds = -\xi \cdot \psi_3. \end{cases} \tag{2.58}$$

As follows from Eq. (2.48), the system (2.58) is made closed by a single (the others are fulfilled identically) equilibrium equation, given by

$$\iint\limits_\Gamma \left[p_2(N)x_1 - p_1(N)x_2\right] ds = M_3. \tag{2.59}$$

In Eqs. (2.58) and (2.59) $p_1(N) = p_x(x, y)$ and $p_2(N) = p_y(x, y)$ are the sought functions of tangential stresses acting in the contact domain F;$U^{(k)}$ and $V^{(k)}$ ($k = 1, 2$) are the components of the Cerruti solution for the half-space surface displacements under unit concentrated horizontal forces in the direction of the Ox and Oy axes, respectively.

Having introduced commonly used notations, we present the system (2.58), (2.59) in a more convenient form for the subsequent analysis:

$$\begin{cases} \iint\limits_F \left[p_1(\xi,\eta)\Omega_1(x,y,\xi,\eta) + p_2(\xi,\eta)\Omega_2(x,y,\xi,\eta)\right] d\xi d\eta = y \cdot \psi, \\ \iint\limits_F \left[p_1(\xi,\eta)\Omega_2(x,y,\xi,\eta) + p_2(\xi,\eta)\Omega_3(x,y,\xi,\eta)\right] d\xi d\eta = -x \cdot \psi, \end{cases} \tag{2.60}$$

$$\iint\limits_\Gamma \left[p_2(x,y)x - p_1(x,y)y\right] dxdy = M \tag{2.61}$$

where $p_1(x, y)$ and $p_2(x, y)$ are the sought contact tangential stress functions, $\Omega_i(x, y, \xi, \eta)$, ($i = 1, 2, 3$) are the Cerruti displacement functions:

$$\Omega_1(x,y,\xi,\eta) = U^{(1)}(K,N) = \frac{1+\nu}{\pi E}\left[\nu\frac{x_1^2}{R^3} + (1-\nu)\frac{1}{R}\right],$$

$$\Omega_2(x,y,\xi,\eta) = U^{(2)}(K,N) = V^{(1)}(K,N) = \frac{\nu(1+\nu)}{\pi E}\frac{x_1 y_1}{R^3},$$

$$\Omega_3(x,y,\xi,\eta) = V^{(2)}(K,N) = \frac{1+\nu}{\pi E}\left[\nu\frac{y_1^2}{R^3} + (1-\nu)\frac{1}{R}\right];$$

$x_1 = x - \xi$, $y_1 = y - \eta$, $R = \sqrt{x_1^2 + x_2^2}$; M is the external torque rotating the punch in the horizontal plane by an angle ψ.

Thus, the spatial contact problem on surface torsion of an elastic base by a rigid flat-bottomed punch at full coupling is reduced to the following: from the two-dimensional integral equation system (2.60) one should find the angle ψ and two functions $p_1(x, y)$ and $p_2(x, y)$, satisfying the integral equilibrium condition (2.61).

The exact solutions of the mixed problem of theory of elasticity (2.60) and (2.61) are known only for circular and elliptical punches [19]. Approximate solutions of contact problems on torsion of elastic bases of various types have been obtained by a number of authors [2, 8, 11, 13, 23, 25, 27, 28, 33], but in most cases for circular and ring-shaped punches. For a rectangular domain, the punch rotation angle due to a given torque was determined by Mozhevitinov [22]; however, the contact problem was not solved, but a linear distribution of tangential stresses, depending on the contact point distance from the rotation axis, was suggested. As will be shown below, the numerical solution of the integral equation system (2.60), (2.61) for punches with complex-shaped bottom can be efficiently performed using the boundary-element method in its direct formulation.

If the discretization of the contact domain F is performed by any of the available methods, then a finite-measure algebraic analogue of the system (2.60), (2.61) can be readily obtained in the approximation of piecewise constant functions of contact tangential stresses on the boundary elements similarly to how it has been made in Sect. 2.4.2. We write the resolving solutions of the torsion contact problem under consideration in a discrete form:

$$\begin{cases} \sum\limits_{q=1}^{M} [p_1(Q_q) \iint\limits_{\Delta F_n} \Omega_1(P_i, Q)dF + p_2(Q_q) \iint\limits_{\Delta F_n} \Omega_2(P_i, Q)dF] = y_i \cdot \psi, \\ \sum\limits_{q=1}^{M} [p_1(Q_q) \iint\limits_{\Delta F_n} \Omega_2(P_i, Q)dF + p_2(Q_q) \iint\limits_{\Delta F_n} \Omega_3(P_i, Q)dF] = -x_i \cdot \psi, \\ \sum\limits_{q=1}^{M} \left[p_2(P_q)x_q - p_1(P_q)y_q\right] \Delta s_q = M, \quad q = 1,2,...,m. \end{cases} \tag{2.62}$$

Here the following notations are used: $p_1(Q_q)$, $p_2(Q_q)$ are the averaged values of the corresponding radial and vertical contact stress within the q-th boundary element, P_i are the collocation points (the boundary element gravity centres), ΔS_q is the q-th boundary element area, m is the total number of the boundary elements.

In the matrix notation the algebraic analogue Eq. (2.62) of the integral equation system is given by

$$\mathbf{A} \cdot \mathbf{Z} = \mathbf{B} \tag{2.63}$$

where $\mathbf{A} = \begin{pmatrix} \mathbf{F}_{2m\times 2m} & \mathbf{G}_{2m\times 1} \\ \mathbf{H}_{1\times 2m} & 0 \end{pmatrix}$ is a square block matrix of the order $(2m + 1)$,

$$\mathbf{F}_{2m\times 2m} = \left(\begin{array}{cc|cc|cc|cc} f_{11} & g_{11} & f_{21} & g_{21} & \cdot & \cdot & f_{m1} & g_{m1} \\ g_{11} & h_{11} & g_{21} & h_{21} & \cdot & \cdot & g_{m1} & h_{m1} \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline f_{1m} & g_{1m} & f_{2m} & g_{2m} & \cdot & \cdot & f_{mm} & g_{mm} \\ g_{1m} & h_{1m} & g_{2m} & h_{2m} & \cdot & \cdot & g_{mm} & h_{mm} \end{array}\right)$$ is the influence matrix

$$\mathbf{G}_{2m\times 1} = -\left(y_1, -x_1, y_2, -x_2, \ldots, y_m, -x_m\right)^T,$$
$$\psi = \pi q_0/(4G),\ \tau(r) = q_0 r(a^2 - r^2)^{-1/2},\ (q_0 = 3M/4\pi a^3).$$

Z and **B** are column vectors of the size ($2M$+1):

$$\mathbf{Z} = (p_1(P_1), p_2(P_1); p_1(P_2), p_2(P_2); \ldots p_1(P_m), p_2(P_m); \psi)^T,$$
$$\mathbf{B} = (0,0;0,0;\ldots;M)^T;$$

$$f_{iq} = \iint\limits_{\Delta F_q} \Omega_1(P_i, Q)dF = \iint\limits_{\Delta F_q} \Omega_1(x_i, y_i, \xi, \eta)d\xi d\eta,$$

$$g_{iq} = \iint\limits_{\Delta F_q} \Omega_2(P_i, Q)dF = \iint\limits_{\Delta F_q} \Omega_2(x_i, y_i, \xi, \eta)d\xi d\eta,$$

$$h_{iq} = \iint\limits_{\Delta F_q} \Omega_3(P_i, Q)dF = f\Omega_3(x_i, y_i, \xi, \eta)d\xi d\eta$$

are the coefficients of the influence matrix $\mathbf{F}_{2m\times 2m}$, $i, q = 1, 2, \ldots, m$.

It is evident that the structure of the given matrix representation of the algebraic analogue of the boundary contact problem with respect to the values of ψ_3 and $p_1(P_q)$, $p_2(P_q)$ ($q = 1, 2, \ldots m$) is similar to the case of the half-space torsion by a deepened punch in the shape of a rotation body considered above in Sect. 2.4.2 with the only difference that the blocks in the influence matrix $\mathbf{F}_{2m\times 2m}$ for a flat-bottomed punch are symmetrical. The coefficients f_{iq}, g_{iq}, h_{iq} of the influence matrix for the Cerruti solution used here are determined analytically using the primitives for the integrals I_1, I_4, I_5, I_7, obtained in Sect. 2.3 in a closed form. Here, similarly to the case of the punch indentation (Sect. 2.5.1), for the calculation of both singular and regular integrals the same method is used, based on the algebraic assembling of integrals over triangles, resting on the sides of flat polygon-type boundary elements. The formulae of the primitives (2.26) for I_1, I_4, I_5, I_7 remain unchanged, and their parameters are determined from simplified 2-D formulae due to the zero values of the node z-coordinates for all the boundary elements.

We finish the consideration of a half-space torsion by a rigid flat-bottomed punch by a boundary-element formulation of the problem for the circular (or ring-shaped) contact domain.

First we perform cyclic discretization into boundary elements (Fig. 2.10). Then we take into account the axial symmetry of the problem, for which only tangential stresses are possible. Similarly to the above case, we locate the points of application of unit concentrated forces in the boundary-element gravity centres and assume tangential forces $p_t = \tau$, acting in the direction, normal to the radius, to be constant within each boundary element. The algebraic analogue of the integral equation system of the contact problem of a rigid rotation under the torque of a circular (or ring-shaped) punch, bound to the half-space, will be obtained from the system (2.45) as a limiting case of the deepened punch in the shape of a rotation body ($U_t^{(1)}(P,Q) = \tilde{\Omega}(x,y,\xi,\eta)\, p_r = p_z = 0$) in the form

$$\begin{cases} \sum\limits_{j=1}^{L} B_{ij} \cdot \tau_j - y_i \cdot \psi = 0,\ i = \overline{1,L} \\ \sum\limits_{j=1}^{L} \tau_j \cdot r_j \cdot \Delta s_j = \frac{M \cdot L}{m} \end{cases} \tag{2.64}$$

where

$$B_{ij} = \sum_{k=1}^{m/L} \lambda_{i,j+L(k-1)};\ \lambda_{qn} = \iint\limits_{\Delta F_n} \tilde{\Omega}(P_q,Q)dF = \iint\limits_{\Delta F_n} \tilde{\Omega}(x_q,y_q,\xi,\eta)d\xi d\eta;$$

$$\tilde{\Omega}(x,y,\xi,\eta) = -\Omega_1(x,y,\xi,\eta) \cdot \cos\varphi + \Omega_2(x,y,\xi,\eta) \cdot \sin\varphi;$$

$n = 1,\ldots,m$; m is the total number of boundary elements in the contact domain, $q = 1,\ldots,L$; L is the number of boundary elements along the radial direction.

Note that the problem of torsion of an elastic half-space by a round punch of the radius a has an exact solution [19], enabling the rigid rotation ψ of the punch and the distribution of tangential forces $\tau(r)$ below the punch to be obtained:

$$\psi = \pi q_0/(4G),\ \tau(r) = q_0 r(a^2 - r^2)^{-1/2} (q_0 = 3M/4\pi a^3).$$

An approximate analytical solution of the problem of an elastic half-space torsion due to a ring-shaped punch rotation was obtained in [8].

In the algebraic analogues (2.53), (2.62), and (2.64) the influence matrix coefficients are presented for the classic contact model of an elastic homogeneous half-space. However, all the formulations of spatial contact problems for flat-bottomed punches considered remain valid also for non-classic bases, for which the influence functions exist or can be obtained. The algorithm of solution of such contact problems remains the same, except for the calculation of the influence matrix coefficients. The literature analysis shows that the great majority of the influence functions for non-classic elastic bases can be given by

$$\Lambda_i(x,y,\xi,\eta) = \Lambda_0(x,y,\xi,\eta) + \Lambda_*(x,y,\xi,\eta)$$

where $\Lambda_0(x, y, \xi, \eta)$ is the influence function for a homogeneous elastic half-space (a combined Boussinesq-Cerruti solution), and $\Lambda_*(x, y, \xi, \eta)$ is an additional (non-classic) component, taking into account nonhomogeneity, anisotropy, lamination and other mechanical characteristics of the bases. The additional components of the influence functions $\Lambda_*(x, y, \xi, \eta)$ are regular and do not introduce any principal difficulties in numerical integration. Here, similarly to the case of the Mindlin's solution integration, a numerical-and-analytical procedure, described in Sect. 2.3 and consisting in analytical integration of the components of the influence functions Λ_0 for the elastic half-space and numerical integration of the non-classic component Λ_*, appears to be efficient.

Concerning the above boundary-element formulations of contact problems for a round (or ring-shaped) punch (2.57) and (2.64), it should be noted that in spite of the existing corresponding exact solutions, these algebraic analogues are of great importance for the general development of the proposed version of the numerical boundary-element method. First, they serve as a convenient tool for numerical algorithm testing. Second, these formulations remain unchanged (except for the technical procedure of the influence matrix determination) for various spatial contact models being used and, hence, are a universal tool for the studies of spatial contact interaction of the simplest type (a centrally loaded round punch). The latter case is important for the identification of the parameters of the existing and constantly elaborated influence functions. The formulations of Eqs. (2.57) are (2.64) are undoubtedly helpful as well due to the fact they are valid for practically any important contact problem with axial symmetry, namely for a flat ring-shaped punch. The numerical solution of this problem using the proposed boundary-element algorithms requires, in comparison with the round punch problem, only an obvious slight modification at the contact domain discretization under the condition of the cyclic symmetry being preserved (Fig. 2.10b). Not so many solutions have been obtained for the ring-shaped punch problem (much less than for the round punch, see Sect. 4.1); however, the practical interest to it is rather high [18, 32].

In spite of the simplicity and convenience of the given boundary-element formulation of the contact problems with axial symmetry (2.57) and (2.64), one should take into account that they remain valid only for the base models, for which the influence functions are symmetrically-differential, i.e. when $\Lambda(x,y,\xi,\eta) = \Lambda(x-\xi,y-\eta)$. Otherwise, for the numerical solution of contact problems for round and ring-shaped punches the general boundary-element algorithms of Eqs. (2.53) and (2.62) should be used.

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