Introduction

In teaching a mathematical course where the Navier[9]–Stokes[10,11] equation plays a role, one must mention the pioneering work of Jean LERAY[12,13] in the 1930s. Some of the problems that Jean LERAY left unanswered are still open today,[14] but some improvements were started by Olga LADYZHENSKAYA[15] [16], followed by a few others, like James SERRIN,[16] and my advisor, Jacques-Louis LIONS[17] [19], from whom I learnt the basic principles for the mathematical analysis of these equations in the late 1960s.

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9 Claude Louis Marie Henri NAVIER, French mathematician, 1785–1836. He worked in Paris, France.

10 Sir George Gabriel STOKES, Irish-born mathematician, 1819–1903. He held the Lucasian chair at Cambridge, England, UK.

11 Henry LUCAS, English clergyman, 1610–1663.

12 Jean LERAY, French mathematician, 1906–1998. He received the Wolf Prize in 1979. He held a chair (Théorie des équations différentielles et fonctionnelles) at Collège de France, Paris, France.

13 Ricardo WOLF, German-born (Cuban) diplomat and philanthropist, 1887–1981. The Wolf Foundation was established in 1976 with his wife, Francisca SUBIRANA-WOLF, 1900–1981, *to promote science and art for the benefit of mankind*.

14 Most problems are much too academic from the point of view of continuum mechanics, because the model used by Jean LERAY is too crude to be meaningful, and the difficulties of the open questions are merely of a technical mathematical nature. Also, Jean LERAY unfortunately called turbulent the weak solutions that he was seeking, and it must be stressed that turbulence is certainly not about regularity or lack of regularity of solutions, nor about letting time go to infinity either.

15 Olga Aleksandrovna LADYZHENSKAYA, Russian mathematician, 1922–2004. She worked at Russian Academy of Sciences, St Petersburg, Russia.

16 James B. SERRIN Jr., American mathematician, born in 1926. He works at University of Minnesota Twin Cities, Minneapolis, MN.

17 Jacques-Louis LIONS, French mathematician, 1928–2001. He received the Japan Prize in 1991. He held a chair (Analyse mathématique des systèmes et de leur contrôle) at Collège de France, Paris, France. I first had him as a teacher at

In the announcement of the course, I had mentioned that I would start by recalling some classical facts about the way to use functional analysis for solving partial differential equations of continuum mechanics, describing some fine properties of Sobolev[18] spaces which are useful, and studying in detail the spaces adapted to questions about incompressible fluids. I had stated then that the goal of the course was to describe some more recent mathematical models used in oceanography, and show how some of them may be solved, and that, of course, I would point out the known defects of these models.[19] I had mentioned that, for the oceanography part – of which I am no specialist – I would follow a book written by one of my collaborators, Roger LEWANDOWSKI[20] [18], who had learnt about some of these questions from recent lectures of Jacques-Louis LIONS. I mentioned that I was going to distribute notes, from a course on partial differential equations that I had taught a few years before, but as I had not written the part that I had taught on the Stokes equation and the Navier–Stokes equation at the time, I was going to make use of the lecture notes [23] from the graduate course that I had taught at University of Wisconsin, Madison WI, in 1974–1975, where I had added small technical improvements from what I had learnt. Finally, I had mentioned that I would write notes for the parts that I never covered in preceding courses.

I am not good at following plans. I started by reading about oceanography in a book by A. E. GILL[21] [15], and I began the course by describing some of the basic principles that I had learnt there. Then I did follow my plan of discussing questions of functional analysis, but I did not use any of the notes that I had written before. When I felt ready to start describing new models, Roger LEWANDOWSKI visited CARNEGIE[22] MELLON[23] University and gave a talk in the Center for Nonlinear Analysis seminar, and I realized that there were some questions concerning the models and some mathematical techniques which I had not described at all, and I changed my plans. I opted for describing the general techniques for nonlinear partial differential equations that I

École Polytechnique in 1966–1967, and I did research under his direction, until my thesis in 1971.

18 Sergei L'vovich SOBOLEV, Russian mathematician, 1908–1989. He worked in Novosibirsk, Russia, and there is now a SOBOLEV Institute of Mathematics of the Siberian branch of the Russian Academy of Sciences, Novosibirsk, Russia.

19 It seems to have become my trade mark among mathematicians, that I do not want to lie about the usefulness of models when some of their defects have already been pointed out. This is obviously the way that any scientist is supposed to behave, but in explaining why I have found myself so isolated and stubborn in maintaining that behavior, I have often invoked a question of religious training.

20 Roger LEWANDOWSKI, French mathematician, born in 1962. He works at Université de Rennes I, Rennes, France.

21 Adrian Edmund GILL, Australian-born meteorologist and oceanographer, 1937–1986. He worked in Cambridge, England, UK.

22 Andrew CARNEGIE, Scottish-born businessman and philanthropist, 1835–1919.

23 Andrew William MELLON, American financier and philanthropist, 1855–1937.

had developed, *homogenization*, *compensated compactness* and *H-measures*; there are obviously many important situations where they should be useful, and I found it more important to teach them than to analyze in detail some particular models for which I do not feel yet how good they are (which means that I suspect them to be quite wrong). Regularly, I was trying to explain why what I was teaching had some connection with questions about *fluids*.

It goes with my philosophy to *explain the origin of mathematical ideas* when I know about them, and as my ideas are often badly attributed, I like to mention *why and when I had introduced an idea*.

I have also tried to *encourage mathematicians to learn more about continuum mechanics and physics*, listening to the specialists and then trying to put these ideas into a sound mathematical framework. I hope that some of the discussions in these lecture notes will help in this direction.[24]

[This course mentions a few equations from continuum mechanics, and besides the Navier–Stokes equation I shall mention the Maxwell equation, the equation of linearized elasticity, and the wave equation, at least, but I did not always follow the classical notation used in texts of mechanics, writing $a, \mathbf{b}, \mathsf{C}$ for scalars, vectors and tensors, and using the notation $f_{,j}$ for denoting the partial derivative of f with respect to x_j. This course is intended for mathematicians, and even if many results are stated in an informal way, they correspond to theorems whose proofs usually involve functional analysis, and not just differential calculus and linear algebra, which are behind the notation used in mechanics.

It is then important to notice that partial differential equations are not written as pointwise equalities but in the sense of distributions, or more generally in some variational framework and that one deals with elements of function spaces, using operators and various types of convergence. Instead of the notation $\nabla a, \nabla.\mathbf{b}, \nabla \times \mathbf{b}$ used in mechanics, I write $grad\, a, div\, b, curl\, b$ (and I also recall sometimes the framework of differential forms), and I only use $\mathbf{b}$ for a vector-valued function b when the pointwise value is meant, in particular in integrands.

It may seem analogous to the remark known to mathematicians that "the function $f(x)$" is an abuse of language for saying "the function f whose elements in its domain of definition will often be denoted x", but there is something different here. The framework of functional analysis is not just a change of language, because it is crucial for understanding the point of view that I developed in the 1970s for relating what happens at a macroscopic level from the description at a microscopic/mesoscopic level, using convergences of weak type (and not just weak convergences), which is quite a different idea than the game of using ensemble averages, which destroys the physical meaning of the problems considered.]

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[24] I have gone further in the critical analysis of many principles of continuum mechanics, which I shall present as a different set of lecture notes, as an introduction to kinetic theory, taught in the Fall of 2001.

Detailed Description of Lectures

a.b refers to definition, lemma or theorem # b in lecture # a, while (a.b) refers to equation # b in lecture # a.

Lecture 1, Basic physical laws and units: The hypothesis of incompressibility and the speed of sound in water; salinity; units in the metric system; oceanography/meteorology; energy received from the Sun: the solar constant S; black-body radiation, Planck's law, surface temperature of the Sun; absorption, albedo, the greenhouse effect; convection of water induced by gravity and temperature, and salinity; how a greenhouse functions.
Lecture 2, Radiation balance of atmosphere: The observed percentages of energy in the radiation balance of the atmosphere; absorption and emission are frequency-dependent effects; the greenhouse with p layers (2.1)–(2.7); thermodynamics of air and water: lapse rate, relative humidity, latent heat; the Inter-Tropical Convergence Zone (ITCZ), the trade winds, cyclones and anti-cyclones.
Lecture 3, Conservations in ocean and atmosphere: The differences between atmosphere and ocean concerning heat storage; conservation of angular momentum, the trade winds, east–west dominant wings; conservation of salt; Eulerian and Lagrangian points of view; conservation of mass (3.1)–(3.3).
Lecture 4, Sobolev spaces I: Sobolev spaces $W^{1,p}(\Omega)$ (4.1)–(4.2); weak derivatives, theory of distributions; notation H^s for $p = 2$ and $\mathcal{H}$ for Hardy spaces; functions of $W^{1,p}(\Omega)$ have a trace on $\partial\Omega$ if it is smooth; integration by parts in $W^{1,1}(\Omega)$ (4.3); results from ordinary differential equations (4.4)–(4.8); conservation of mass (4.9)–(4.12); regularity of solutions of the Navier–Stokes and Euler equations, Riesz operators and singular integrals, Zygmund space, BMO, $\mathcal{H}^1$.
Lecture 5, Particles and continuum mechanics: Particles and continuum mechanics, distances between molecules; homogenization, microscopic/mesoscopic/macroscopic scales; "real" particles versus macroscopic particles as tools from numerical analysis; Radon measures (5.1), distributions (5.2)–(5.4);

momentum and conservation of mass (5.5)–(5.6); the homogenization problem related to oscillations in the velocity field.

Lecture 6, Conservation of mass and momentum: Euler equation (6.1); priority of Navier over Stokes and of Stokes over Riemann, Rankine and Hugoniot; similarity of the stationary Stokes equation and stationary linearized elasticity; kinetic theory, free transport equation and conservation of mass (6.2)–(6.4); transport equation with Lorentz force (6.5); Boltzmann's equation (6.6)–(6.7); Cauchy stress in kinetic theory (6.8); conservation of momentum (6.10); pressure on the boundary resulting from reflection of particles.

Lecture 7, Conservation of energy: Internal energy in kinetic theory (7.1); relation between internal energy and Cauchy stress in kinetic theory (7.2); heat flux in kinetic theory (7.3); conservation of energy (7.4); various origins of the internal energy; variation of thermodynamic entropy, H-theorem (7.5)–(7.6); local Maxwellian distribution (7.7); the parametrization of allowed collisions (7.8)–(7.9); the form of interaction term $Q(f, f)$ in Boltzmann's equation (7.10); the proof of (7.5): (7.11); letting the mean free path tend to 0; irreversibility, nonnegative character of solutions of Boltzmann's equation (7.12)–(7.13).

Lecture 8, One-dimensional wave equation: Longitudinal, transversal waves; approximating the longitudinal vibration of a string by small masses connected with springs (8.1)–(8.2); the limiting 1-dimensional wave equation (8.3)–(8.5); different scalings of string constants; time periodic solutions; linearization for the increase in length in 1-dimensional transversal waves and 2- or 3-dimensional problems; the linearized elasticity system (8.6)–(8.11); Cauchy's introduction of the stress tensor, by looking at the equilibrium of a small tetrahedron.

Lecture 9, Nonlinear effects, shocks: Beware of linearization; nonlinear string equation (9.1); Poisson's study of barotropic gas dynamics with $p = C\,\varrho^\gamma$ (9.2); what led Stokes to discover "Rankine–Hugoniot" conditions; Burgers's equation (9.3)–(9.5); characteristic curves and apparition of discontinuities (9.6)–(9.7); equations in the sense of distributions imply jump conditions (9.8)–(9.9); a two-parameter family of weak solutions for Burgers's equation with 0 initial datum (9.10); Lax's condition and Oleinik's condition for selecting admissible discontinuities; Hopf's derivation of Oleinik's condition using "entropies" (9.11)–(9.13), Lax's extension to systems; the equation for entropies of system (9.14) describing the nonlinear string equation (9.15)–(9.17); transonic flows.

Lecture 10, Sobolev spaces II: Description of functional spaces for the study of the Stokes and Navier–Stokes equations, boundedness of Ω, smoothness of $\partial\Omega$; $H^1(\Omega)$ (10.1), characteristic length, $H^1_0(\Omega)$; Poincaré's inequality (10.2); scaling, Poincaré's inequality does not hold for open sets containing arbitrary large balls (10.3)–(10.4); 10.1: Poincaré's inequality holds if Ω is included in a bounded strip (10.5), if $meas\,\Omega < \infty$ (10.11)–(10.12); Schwartz's convention for the Fourier transform (10.6), its action on derivation and multiplication (10.7); Plancherel's formula (10.8); Schwartz's extension of the Fourier transform to temperate distributions (10.9); the Fourier transform is an isometry on $L^2(R^N)$ (10.10); a sufficient condition for having Poincaré's inequality; the strain–stress constitutive relation in isotropic linearized elasticity (10.13).

Lecture 11, Linearized elasticity: Stationary linearized elasticity for isotropic materials (11.1)–(11.3); 11.1: Korn's inequality on $H_0^1(\Omega; R^N)$ (11.4), a proof using the Fourier transform (11.5), a proof by integration by parts (11.6)–(11.8); 11.2: Lax–Milgram lemma (11.9)–(11.10); variational formulation and approximation; the complex-valued case of the Lax–Milgram lemma; 11.3: a variant of the Lax–Milgram lemma (11.11); description of the plan for letting $\lambda \to +\infty$.

Lecture 12, Ellipticity conditions: Very strong ellipticity condition (12.1), the isotropic case; strong ellipticity condition (12.2) for stationary linearized elasticity, the isotropic case, the constant coefficients case with Dirichlet condition; the abstract framework for letting $\lambda \to +\infty$ in linearized elasticity (12.3)–(12.4), bounds for u^λ (12.5), variational form of the limit problem (12.6)–(12.7), strong convergence of u^λ (12.8)–(12.9); Lagrange multiplier; definition and characterization of $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$ (12.10)–(12.11); equations satisfied by u^λ and its limit u^∞ (12.12)–(12.14); De Rham's theorem and interpretation of (12.14); $grad\, S \in H^{-1}(\Omega; R^N)$ implies $S \in L^2(\Omega)$ if $\partial\Omega$ is smooth.

Lecture 13, Sobolev spaces III: $X(\Omega)$ (13.1); relation with Korn's inequality (13.2); 13.1: existence of the "pressure", and 13.2: existence of $u \in H_0^1(\Omega; R^N), div\, u = g$ whenever $\int_\Omega g\, dx = 0$, are equivalent if $\partial\Omega$ is smooth; proof based on regularity for a degenerate elliptic problem; 13.3: the equivalence lemma; applications of the equivalence lemma; 13.4: $X(R^N) = L^2(R^N)$ using the Fourier transform.

Lecture 14, Sobolev spaces IV: Approximation methods in $W^{1,p}(\Omega)$; truncation; properties of convolution in R^N (14.1)–(14.2); regularization by convolution (14.3); commutation of convolution and derivation (14.4), $C^\infty(R^N)$ is dense in $W^{1,p}(R^N)$; support of convolution product (14.5)–(14.6), $C^\infty(\overline{R_+^N})$ is dense in $W^{1,p}(R_+^N)$ for $\Omega = R^N$; localization, partition of unity, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ when Ω is bounded and $\partial\Omega$ is locally a continuous graph; extension from $W^{m,p}(R_+^N)$ to $W^{m,p}(R^N)$ (14.7)–(14.9); counter-example to the extension from $H^1(\Omega)$ to $H^1(R^2)$ for a plane domain with a cusp.

Lecture 15, Sobolev spaces V: $X(\Omega)$ is a local space; $C_c^\infty(\overline{R_+^N})$ is dense in $X(R_+^N)$; extension from $X(R_+^N)$ to $X(R^N)$ by transposition and construction of a restriction (15.1)–(15.3); the importance of regularity of $\partial\Omega$ for having $X(\Omega) = L^2(\Omega)$; 15.1: if $meas(\Omega) < \infty$, the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, by the Fourier transform; application to the convergence of $-\lambda\, div\, u^\lambda$ in $L^2(\Omega)$ to the "pressure", by the equivalence lemma.

Lecture 16, Sobolev embedding theorem: Differences between linearized elasticity and the Stokes equation for the evolution problems; variable viscosity, Poiseuille flows; stationary Navier–Stokes equation (16.1); 16.1: Sobolev embedding theorem, the original method of Sobolev and improvements using interpolation spaces, an inequality of Ladyzhenskaya (16.2) and a method of

Gagliardo and of Nirenberg (16.3)–(16.5); solving (16.1) as fixed point for Φ (16.6), estimates for Φ giving existence and uniqueness of a solution for small data and $N \leq 4$ (16.7)–(16.12), by the Banach fixed point theorem; solving (16.1) as fixed point for Ψ (16.13), estimates for Ψ (16.14)–(16.20); 16.2: existence of a fixed point for a contraction of a closed bounded nonempty convex set in a Hilbert space, monotone operators.

Lecture 17, Fixed point theorems: Existence of a solution of (16.1) for large data by the Schauder fixed point theorem for $N \leq 3$, by the Tykhonov fixed point theorem for $N = 4$; Faedo–Ritz–Galerkin method; existence of Faedo–Ritz–Galerkin approximations (17.1) by the Brouwer fixed point method applied to approximations Ψ_m (17.2), existence for large data for $N \leq 4$ by extraction of weakly converging subsequence and a compactness argument, valid for $N > 4$ in larger functional spaces; properties of the Brouwer topological degree; 17.1: nonexistence of tangent nonvanishing vector fields on S^{2N}; 17.2: nonexistence of a continuous retraction of a bounded open set of R^N onto its boundary; 17.3: Brouwer fixed point theorem.

Lecture 18, Brouwer's topological degree: $J_\varphi(u)$ (18.1); 18.1: the derivative of $J_\varphi(u)$ in the direction v is an integral on $\partial\Omega$ (18.2)–(18.3), vanishing if v vanish on $\partial\Omega$; 18.2: invariance by homotopy, $J_\varphi(u) = J_\varphi(w)$ if there is a homotopy from u to w avoiding $supp(\varphi)$ on $\partial\Omega$; 18.3: $J_\varphi(u)$ can be defined for $u \in C(\overline{\Omega}; R^N)$ avoiding $supp(\varphi)$ on $\partial\Omega$; 18.4: if $J_\varphi(u) \neq 0$ there exists $\mathbf{x} \in \Omega$ such that $\mathbf{u}(\mathbf{x}) \in supp(\varphi)$; proof of 18.1: (18.4)–(18.7); 18.5: definition of degree $deg(u; \Omega, \mathbf{p})$; 18.6: formula for degree if $\mathbf{u}(\mathbf{z}) = \mathbf{p}$ has a finite number of solutions where ∇u is invertible (18.8); Sard's lemma.

Lecture 19, Time-dependent solutions I: Spaces V, H for the Stokes or Navier–Stokes equations (19.1)–(19.2); semi-group theory; abstract ellipticity for $A \in \mathcal{L}(V, V')$ (19.3); 19.1: $u' + A\,u = f \in L^1(0,T;H) + L^2(0,T;V')$, $u(0) = u_0 \in H$ (19.4)–(19.5), by Faedo–Ritz–Galerkin (19.6); 19.2: properties of $W^{1,1}(0,T)$ and Gronwall's inequality; estimates for (19.6): (19.7)–(19.16); a variant of Gronwall's inequality (19.17)–(19.19), giving estimate (19.20).

Lecture 20, Time-dependent solutions II: Taking the limit in (19.6), (20.1)–(20.3), giving existence in 19.1; an identity for proving uniqueness in 19.1, (20.4); spaces $W_1(0,T)$ and $W(0,T)$ (20.5)–(20.8); properties of $W_1(0,T)$, for proving (20.4); problem with time derivative in Faedo–Ritz–Galerkin, and special choice for a basis; regularization effect when the initial datum is not in the right space; backward uniqueness in the case $A^T = A$, Agmon–Nirenberg result of log-convexity for $|u(t)|$.

Lecture 21, Time-dependent solutions III: Problem in the definition of H in (19.2); problem with the "pressure" in the nonstationary Stokes equation (21.1)–(21.7); 21.1: regularity in space when $A^T = A$, $u_0 \in V, f \in L^2(0,T;H)$, regularizing effect for $u_0 \in H, \sqrt{t}\,f \in L^2(0,T;H)$; problem of identifying H' with H; estimate for the "pressure" in the case $\Omega = R^N$ (21.8)–(21.11); avoiding cutting the transport operator into two terms (21.12)–(21.14); the nonlinear term (21.15) and its estimate in dimension 2, 3, 4 (21.16)–(21.17).

Lecture 22, Uniqueness in 2 dimensions: Cutting the transport term into two terms works for $N = 2$; 21.1: uniqueness for the abstract Navier–Stokes equation for $N = 2$ (21.1)–(21.6); a quasilinear diffusion equation (21.7), with the Artola uniqueness result (21.8)–(21.11).
Lecture 23, Traces: $H(div; \Omega)$ (23.1); space is local, $C^\infty(\overline{\Omega}; R^N)$ dense if $\partial\Omega$ smooth; formula defining the normal trace $u.\nu$ (23.2), in dual of traces of $H^1(\Omega)$ (23.3); interpretation in terms of differential forms, $H(curl; \Omega)$ (23.4); $H^s(R^N)$ (23.5); for $s > 1/2$, restriction on $x_N = 0$ is defined on $H^s(R^N)$, and the trace space is $H^{s-(1/2)}(R^{N-1})$ (23.6)–(23.10); 23.1: orthogonal of H in $L^2(\Omega; R^N)$ is the space $\{grad(p) \mid p \in H^1(\Omega)\}$, if injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact; 23.2: if $meas\,\Omega < \infty$ and $X(\Omega) = L^2(\Omega)$ then V is dense in H; discussion of $X(\Omega) = L^2(\Omega)$ if $\partial\Omega$ is smooth, and how to change the definitions of the spaces if the boundary is not smooth enough; Faedo–Ritz–Galerkin method for existence of Navier–Stokes equation for $N = 3$ (23.11)–(23.12); singular solutions of the stationary Stokes equation in corners (23.13)–(23.18).
Lecture 24, Using compactness: 24.1: J.-L. Lions's lemma (24.1); 24.2: u_n bounded in $L^p(0,T;E_1)$ and convergent in $L^p(0,T;E_3)$ imply u_n convergent in $L^p(0,T;E_2)$ if injection of E_1 into E_2 is compact (24.2); 24.3: u_n bounded in $L^{p_1}(0,T;E_1)$ and convergent in $L^{p_3}(0,T;E_3)$ gives u_n convergent in $L^{p_2}(0,T;E_2)$ if interpolation inequality holds; hypothesis of reflexivity; 24.4: u_n bounded in $L^p(0,T;E)$ and $||\tau_h u_n - u_n||_{L^p(0,T;E)} \le M\,|h|^\eta$ imply u_n bounded in $L^q(0,T;E)$; 24.5: u_n bounded in $L^p(0,T;E_1)$ and $||\tau_h u_n - u_n||_{L^p(0,T;E_3)} \le M\,|h|^\eta$ imply u_n compact in $L^p(0,T;E_2)$ if injection of E_1 into E_2 is compact; application to extracting subsequences from Faedo–Ritz–Galerkin approximation with special basis for the Navier–Stokes equation and $N \le 3$.
Lecture 25, Existence of smooth solutions: 25.1: If $N = 2$ and Ω smooth enough, $u_0 \in V$ and $f \in L^2((0,T) \times \Omega; R^2)$ then regularity of the linear case holds (25.1)–(25.2); can one improve bounds using interpolation inequalities; 25.2: if $N = 3$ and Ω smooth enough, $u_0 \in V$ and $f \in L^2((0,T) \times \Omega; R^3)$ then there exists $T_c \in (0,T]$ and a solution with the regularity of the linear case for $t \in (0,T_c)$ (25.3)–(25.4); 25.3: if $N = 3$ and Ω smooth enough, $|u_0|\,||u_0||$ small and $f = 0$ then a global solution with the regularity of the linear case exists for $t \in (0,\infty)$ (25.5)–(25.7); the case $f \ne 0$ (25.8); extending an idea of Foias for showing $u \in L^1(0,T;L^\infty(\Omega;R^3))$ for $N = 3$ (25.9)–(25.12).
Lecture 26, Semilinear models: Reynolds number, scaling of norms, the problems that norms give global information and not local information; a different approach shown on models of kinetic theory, the 2-dimensional Maxwell model (26.1), Broadwell model (26.2); using functional spaces with physical meaning; a special class of semilinear models (26.3)–(26.4) and why I had introduced it; 26.1: spaces $V_c \subset W_c$ and L^1 estimate in (x,t) for $u\,v$ (26.5)–(26.7); extension of the idea, compensated integrability.
Lecture 27, Size of singular sets: Leray's self-similar solutions (27.1); the question of estimating the Hausdorff dimension of singular sets; a bound for the

1/2 Hausdorff dimension in t (27.2); different scaling in $(\mathbf{x}, t)$ and the equation for "pressure" (27.3); maximal functions (27.4), Hedberg's program of proving local inequalities using maximal functions (27.5), application to pointwise estimates for the heat equation (27.6)–(27.10).
Lecture 28, Local estimates, compensated integrability: Hedberg's truncation method, a proof of F.-C. Liu's inequality using Hedberg's approach (28.1)–(28.2), a Hedberg type version of the Gagliardo–Nirenberg inequality (28.3); a result of compensated integrability improving Wente by estimates based on interpolation and Lorentz spaces.
Lecture 29, Coriolis force: Equations in a moving frame and Coriolis force (29.1)–(29.3); analogy, Lorentz force, incompressible fluid motion, nonlinearity as $\mathbf{u} \times curl(-\mathbf{u}) + grad(|\mathbf{u}|^2/2)$ (29.4)–(29.8), conservation of helicity.
Lecture 30, Equation for the vorticity: Equation for vorticity, for $N = 2$ and for $N = 3$ (30.1)–(30.6).
Lecture 31, Boundary conditions in linearized elasticity: Other boundary conditions for linearized elasticity, Neumann condition (31.1) and compatibility conditions (31.2)–(31.3); studying linearized rigid displacements (31.4); other type of boundary conditions; traction at the boundary for a Newtonian fluid (31.5)–(31.6).
Lecture 32, Turbulence, homogenization: Microstructures in turbulent flows; the defect of probabilistic postulates; homogenization.
Lecture 33, G-convergence and H-convergence: Weak convergence, linear partial differential equations in theory of distributions; conservation of mass using differential forms; G-convergence and H-convergence; exterior calculus, differential forms, exterior derivative, Poincaré lemma; weak convergence as a way to relate mesoscopic and macroscopic levels, analogy between proofs in H-convergence and the way some physical quantities are measured and other physical quantities are identified; 33.1: div-curl lemma, its relation with differential forms.
Lecture 34, One-dimensional homogenization, Young measures: 1-dimensional homogenization by div-curl lemma; the G-convergence and H-convergence approaches; effective coefficients cannot be computed in terms of Young measures in dimension $N \geq 2$, physicists' formulas are approximations; importance of both balance equations and constitutive relations; 34.1: Young measures.
Lecture 35, Nonlocal effects I: Turbulence as an homogenization problem for a first order transport operator (35.1); memory effects appearing by homogenization; a model problem with a memory effect in its effective equation (35.2)–(35.3), proof by the Laplace transform (35.4)–(35.9); irreversibility without probabilistic framework; a transport problem with a nonlocal effect in (x, t) in its effective equation (35.10)–(35.15).
Lecture 36, Nonlocal effects II: Frequency-dependent coefficients in Maxwell's equation (36.1), principle of causality, pseudo-differential operators; the model problem with time dependent coefficients (36.1)–(36.8), by a perturbation ex-

pansion approach; "analogies" with Feynman diagrams and Padé approximants.

Lecture 37, A model problem: A model problem with a term $\mathbf{u} \times curl(\mathbf{v}_n)$ added to the stationary Stokes equation (37.1)–(37.2), the derivation of the effective equation (37.3)–(37.14), by methods from H-convergence; an effective term corresponding to a dissipation quadratic in $\mathbf{u}$ and not in $grad\,\mathbf{u}$, which can be computed with H-measures.

Lecture 38, Compensated compactness I: The time dependent analog requires a variant of H-measures; 38.1: the quadratic theorem of compensated compactness (38.1)–(38.4); chronology of discoveries; correction for $U \otimes U$ written as the computation of a convex hull, a formula simplified by introduction of H-measures.

Lecture 39, Compensated compactness II: Constitutive relations (39.1), balance equations (39.2), question about how to treat nonlinear elasticity (39.3); H-measures can handle variable coefficients; how compensated compactness constrains Young measures (39.4); examples: compactness, convexity, monotonicity, Maxwell's equation; proof of necessary conditions.

Lecture 40, Differential forms: Maxwell's equation expressed with differential forms (40.1)–(40.6); 40.1: generalization of div-curl lemma for p-forms and q-forms; generalizations to Jacobians, special case of exact forms (40.7); 40.2: one cannot use the weak topology in the general div-curl lemma; other necessary conditions; how helicity appears in the framework of differential forms, analogy between Lorentz force and the equations for fluid flows.

Lecture 41, The compensated compactness method: 41.1: case when the characteristic set is the zero set of a nondegenerate quadratic form; the question of making the list of interesting quantities in nonlinear elasticity (41.1); wave equation (41.2), conservation of energy (41.3), where the energy goes, equipartition of energy; use of entropies for Burgers's equation for passing to the limit for weakly converging sequences (41.4)–(41.12), entropy condition (41.13) and Murat's lemma.

Lecture 42, H-measures and variants: Wigner transform, avoiding using one characteristic length, the hints for H-measures; definitions for H-measures (42.1)–(42.4); constructing the right "pseudo-differential" calculus (42.5)–(42.12); localization principle (42.13)–(42.14); small-amplitude homogenization (42.15)–(42.18); propagation equations for H-measures (42.19)–(42.27); the variant with one characteristic length, semi-classical measures of P. Gérard (42.28).

Biographical data: Basic biographical information for people whose name is associated with something mentioned in the lecture notes.