3.3 Linearization and Stability

Linearization of a nonlinear differential or difference equation of the form (2.27) about a given trajectory (in particular an equilibrium state) yields a linear system which, in general, will be time-varying. Since stability is a local property one might expect that the linearization provides sufficient information to determine whether or not the trajectory is stable. This is the idea behind an approach adopted by Liapunov which is now known as *Liapunov's indirect method.* In order to prepare the ground for the development of this method we need to consider stability problems for linear systems. Actually linear models are often used in areas of application and especially in control, so linear stability analysis is important in its own right. Moreover, as we shall see, the theory is well developed and yields a number of specific stability criteria which lead to computable tests. By using Liapunov's indirect method these linear stability tests can be applied to nonlinear systems as well.
Our stability analysis will be for both time-varying and time–invariant linear systems and we will also include methods related to Liapunov functions since they will enable us to prove the validity of the indirect method. In the last section of this chapter the stability analysis of linear systems will be continued with a derivation of the classical algebraic stability criteria for *time–invariant* linear systems.
In the first subsection we characterize the asymptotic and exponential stability of time–varying linear systems via their evolution operator Φ and associated Liapunov and Bohl coefficients. Whilst these results are quite satisfactory they suffer from the drawback that one needs to compute Φ in order to check them. In Subsection 3.3.2 time-invariant systems are considered and we show that the conditions are equivalent to constraints on the spectrum of A. We illustrate the results with some examples and also carry out an extended case study where the numerical stability of linear multi-step discretization methods, described in Section 2.5 is analyzed. In Subsection 3.3.4 we examine the possibility of using time–dependent quadratic forms as Liapunov functions for time-varying linear systems. Then these quadratic forms are used for nonlinear systems (2.27) to derive stability properties of a given trajectory from stability properties of the associated linearized model (Liapunov's indirect method). In fact we will see that the properties of asymptotic stability and of instability can be tested via the linearized model. However this is not possible for (marginal) stability, since the stability of a solution which is not asymptotically stable can be destroyed by arbitrary small perturbations of the system equation (see Subsection 3.3.2). Finally, in the last subsection we consider time-invariant systems and *time–invariant* quadratic Liapunov functions. For their construction a linear matrix equation, the *algebraic Liapunov equation*, must be solved. We analyze this equation in some detail and characterize the asymptotic stability and instability of a time-invariant linear system via the solutions of the associated algebraic Liapunov equation. This in turn allows us to conclude that an equilibrium point $\overline{x}$ of a nonlinear system is exponentially stable if and only if the spectrum $\sigma(A)$ of the matrix obtained from the linearization at $\overline{x}$ satisfies $\sigma(A) \in \mathbb{C}_-$ (resp. $\sigma(A) \in \mathbb{D}$). In addition, if $\sigma(A) \notin \overline{\mathbb{C}_-}$ (resp. $\sigma(A) \notin \overline{\mathbb{D}}$), then $\overline{x}$ is an unstable equilibrium point of the nonlinear system.

3.3.1 Stability Criteria for Time-Varying Linear Systems

In this subsection we analyze the stability of finite dimensional time-varying linear systems described by differential and difference equations. Recall that for these systems the (asymptotic) stability of the equilibrium solution at the origin is equivalent to the (asymptotic) stability of any other solution (see Subsection 3.1.2). Hence we may attribute the stability properties to the system itself instead of the solutions. We first express the stability properties of a linear system in terms of the associated evolution operator $\Phi(t, t_0)$. We then introduce Liapunov and Bohl exponents which measure the (uniform) exponential growth (or decrease) of the trajectories. The main theorem of the subsection is Theorem 3.3.15 where uniform exponential stability is shown to be equivalent to the Bohl exponent being negative and also to the existence of a uniform estimate on the L^p-norm of the evolution operator Φ.
We consider the following system equations

$$\dot{x}(t) = A(t)x(t), \quad t \in T \subset \mathbb{R} \tag{1a}$$
$$x(t+1) = A(t)x(t), \quad t \in T \subset \mathbb{Z} \tag{1b}$$

where the time domain T is either an interval in $\mathbb{R}$ or in $\mathbb{Z}$ which is unbounded to the right. By assumption $A(\cdot) \in PC(T; \mathbb{K}^{n\times n})$ in (1a) and $A(t) \in \mathbb{K}^{n\times n}$, $t \in T$ in (1b). In both cases $A(\cdot)$ generates an evolution operator $\Phi(\cdot, \cdot)$ (see Section 2.2), and the solution of (1) satisfying $x(t_0) = x^0$ is given by $x(t) = \Phi(t, t_0)x^0$, $t \in T_{t_0}$ for all $x^0 \in \mathbb{K}^n$. Linear systems of the form (1) induce a global flow $\mathfrak{F} = (T, X, \varphi)$ on $X = \mathbb{K}^n$ given by $\varphi(t; t_0, x^0) = \Phi(t, t_0)x^0$, $(t_0, x^0) \in T \times \mathbb{K}^n$, $t \in T_{t_0}$.
As in the previous section we provide $\mathbb{K}^n$ with the Euclidean norm and $\mathbb{K}^{n\times n}$ with the corresponding operator norm (spectral norm). The first two propositions are immediate consequences of Definitions 3.1.8, 3.1.9.

Proposition 3.3.1. *Let $(z^1, \ldots, z^n)$ be any basis of $\mathbb{K}^n$. Then the following statements are equivalent.*

(i) The system (1) *is stable at time t_0 (resp. uniformly stable).*

(ii) There exists a constant M which may depend on t_0 (resp. independent of t_0) such that $\|\Phi(t, t_0)\| \le M$ for all $t \in T_{t_0}$.

(iii) There exists a constant M which may depend on t_0 (resp. independent of t_0) such that $\|\Phi(t, t_0)z^i\| \le M$ for all $t \in T_{t_0}$, $i \in \underline{n}$.

Proof: (i) $\Rightarrow$ (ii). Suppose that (1) is stable at time t_0 (resp. uniformly stable). Then for $\varepsilon = 1$, there exists $\delta > 0$ depending on t_0 (resp. independent of t_0) such that

$$\|x^0\| \le \delta \quad \Rightarrow \quad \|\Phi(t, t_0)x^0\| \le 1\,, \quad t \in T_{t_0}\,.$$

Hence $\|\Phi(t, t_0)\| \le \delta^{-1}$ for all $t \in T_{t_0}$.
As (ii) $\Rightarrow$ (iii) is trivial it only remains to prove (iii) $\Rightarrow$ (i). Suppose (iii). Since there exist $a, b > 0$ such that

$$a \max_{i\in\underline{n}} |\xi_i| \le \|\sum_{i=1}^{n} \xi_i z^i\| \le b \max_{i\in\underline{n}} |\xi_i|, \quad \xi \in \mathbb{K}^n. \tag{2}$$

we have for all $x^0 = \sum_{i=1}^n \xi_i z^i \in \mathbb{K}^n$,

$$\|\Phi(t,t_0)x^0\| = \|\Phi(t,t_0)\sum_{i=1}^n \xi_i z^i\| \le \max_{i\in\underline{n}} |\xi_i| \sum_{i=1}^n \|\Phi(t,t_0)z^i\| \le a^{-1}nM\|x^0\|.$$

This proves (i). □

Proposition 3.3.2. *Let $(z^1,\dots,z^n)$ be any basis of $\mathbb{K}^n$. Then the following statements are equivalent.*

(i) The system (1) *is asymptotically stable at time t_0 (resp. uniformly asymptotically stable).*

(ii) The system (1) *is globally asymptotically stable at time t_0 (resp. globally uniformly asymptotically stable).*

(iii) $\|\Phi(t,t_0)\| \to 0$ as $t \to \infty$ (resp. uniformly in t_0).

(iv) For $i \in \underline{n}$, $\|\Phi(t,t_0)z^i\| \to 0$ as $t \to \infty$ (resp. uniformly in t_0).

Proof: (i) ⇒ (ii) follows directly from linearity and (ii) ⇒ (iv) and (iii) ⇒ (i) are trivial.
(iv) ⇒ (iii). Suppose (iv) holds, then for every $\varepsilon > 0$ there exists a time $\tau(\varepsilon)$ depending on t_0 (independent of t_0) such that $\|\Phi(t,t_0)z^i\| < \varepsilon$ for all $t \in T_{t_0+\tau(\varepsilon)}$, $i \in \underline{n}$. But then, for every $x^0 = \sum_{i=1}^n \xi_i z^i$, $\|x^0\| = 1$ we have $\max_{i\in\underline{n}} |\xi_i| \le a^{-1}$ where $a > 0$ satisfies (2), and thus

$$\|\Phi(t,t_0)x^0\| = \|\sum_{i=1}^n \xi_i \Phi(t,t_0)z^i\| \le a^{-1}n\varepsilon\,, \quad t \in T_{t_0+\tau(\varepsilon)}\,,$$

hence (iii) holds. □

Remark 3.3.3. In the discrete time case

$$\Phi(t,t_0) = A(t-1)A(t-2)\dots A(t_0)\,, \quad t \in T_{t_0}\,. \tag{3}$$

So if (1b) is (asymptotically) stable at time $t_0 \in T$ it will also be (asymptotically) stable at time $\tau \in T$ for all $\tau < t_0$. A similar statement also holds for $\tau \in T$, $\tau > t_0$ provided that $\det A(k) \neq 0$ for $k = t_0, \dots, \tau - 1$. Furthermore, taking norms in (3) we obtain

$$(\forall t \in T : \|A(t)\| \le \gamma\,) \quad \Rightarrow \quad \|\Phi(t,t_0)\| \le \gamma^{t-t_0}\,, \quad t_0 \in T\,, \quad t \in T_{t_0}\,. \tag{4}$$

Hence the zero state of (1b) will be uniformly stable if $\|A(t)\| \le 1$, $t \in T$ and it will be uniformly asymptotically stable if $\|A(t)\| \le \gamma < 1$ for all $t \in T$. These conditions, however, are far from being necessary. □

An estimate of the spectral norm $\|\Phi(t,t_0)\|$ for the continuous time case is provided by the next lemma.

Lemma 3.3.4. *If $\Phi(t,t_0)$ is the evolution operator of* (1a)*, then*

$$e^{-\int_{t_0}^t \|A(s)\|ds} \le \sigma_{\min}(\Phi(t,t_0)) = \|\Phi(t_0,t)\|^{-1}, \;\; \|\Phi(t,t_0)\| \le e^{\int_{t_0}^t \|A(s)\|ds}\,, \;\; t \ge t_0\,. \tag{5}$$

Proof: Since $\Phi(t_0, t) = \Phi(t, t_0)^{-1}$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(t, t_0) &= A(t)\Phi(t, t_0),\\ \frac{\partial}{\partial t}\Phi(t_0, t) &= -\Phi(t, t_0)^{-1}A(t)\Phi(t, t_0)\Phi(t, t_0)^{-1} = -\Phi(t, t_0)^{-1}A(t) = -\Phi(t_0, t)A(t)\end{aligned}$$

for a.e. $t > t_0$. Integrating yields

$$\Phi(t, t_0) - I = \int_{t_0}^{t} A(s)\Phi(s, t_0)ds, \qquad \Phi(t_0, t) - I = -\int_{t_0}^{t} \Phi(t_0, s)A(s)ds, \quad t \geq t_0.$$

Hence for $t \geq t_0$

$$\|\Phi(t, t_0)\| \leq 1 + \int_{t_0}^{t} \|A(s)\|\|\Phi(s, t_0)\|ds, \quad \|\Phi(t_0, t)\| \leq 1 + \int_{t_0}^{t} \|\Phi(t_0, s)\|\|A(s)\|ds.$$

By Gronwall's Lemma 2.1.18, we have

$$\|\Phi(t, t_0)\| \leq e^{\int_{t_0}^{t} \|A(s)\|ds}, \qquad \|\Phi(t_0, t)\| \leq e^{\int_{t_0}^{t} \|A(s)\|ds}, \quad t \geq t_0.$$

So the second inequality holds and the first is a consequence of $\sigma_{\min}(\Phi(t, t_0)) = \|\Phi(t_0, t)\|^{-1}$ for $t \geq t_0$. □

As a corollary of this lemma and Propositions 3.3.1, 3.3.2 we obtain

Corollary 3.3.5. *The continuous time system* (1a) *with time-domain $T = [t_0, \infty)$ is uniformly stable if $\int_{t_0}^{\infty} \|A(s)\|ds < \infty$. It is (asymptotically) stable at time $t_1 \in T$ if and only if it is (asymptotically) stable at time $t_0 \in T$.*

Proof: The first part is clear from the previous Proposition 3.3.1 and the above lemma. The second follows from the estimates

$$\|\Phi(t, t_0)\| \leq \|\Phi(t, t_1)\|\|\Phi(t_1, t_0)\|, \; \|\Phi(t, t_1)\| \leq \|\Phi(t, t_0)\|\|\Phi(t_0, t_1)\|, \; t \geq t_1 \geq t_0 \quad (6)$$

and Proposition 3.3.2 and (5). □

The following proposition shows that for periodic systems (1) the stability properties can be characterized via those of an associated time–invariant linear system. We will see in the next subsection that efficient stability tests are available for such systems.

Proposition 3.3.6. *Suppose the generators $A(\cdot)$ of* (1) *are periodic with period $\tau \in T$, $T = \mathbb{R}$ or $\mathbb{Z}$, $\tau > 0$: $A(t + \tau) = A(t)$, $t \in T$. Then* (1) *is uniformly stable (uniformly asymptotically stable) if and only if the time–invariant discrete time system*

$$\hat{x}(k + 1) = \Phi(\tau, 0)\hat{x}(k), \quad k \in \mathbb{N} \quad (7)$$

is stable (asymptotically stable) where Φ is the evolution operator generated by (1).

Proof: By periodicity $\Phi(t,t_0) = \Phi(t+\tau, t_0+\tau)$, $t \in T_{t_0}$, $t_0 \in T$. Hence if $t \in T_{t_0}$ and

$$t_0 = k_0\tau + t_0' , \qquad t = k\tau + t' , \;\; 0 \le t_0', \, t' < \tau, \quad k, \, k_0 \in \mathbb{N}, \tag{8}$$

then

$$\begin{aligned} \Phi(t,t_0) &= \Phi(t,k\tau)\Phi(k\tau,(k-1)\tau)\cdots\Phi((k_0+1)\tau,t_0) \\ &= \Phi(t',0)\Phi(\tau,0)^{k-k_0-1}\Phi(\tau,t_0') \,. \end{aligned} \tag{9}$$

By (3), (5) there exists $c > 0$ such that $\|\Phi(t',0)\|$, $\|\Phi(\tau,t_0')\| \le c$ for all t_0', $t' \in [0,\tau] \cap T$. Therefore (9) implies

$$\|\Phi(t,t_0)\| \le c^2 \|\Phi(\tau,0)^{k-k_0-1}\| \, , \quad t \in T_{t_0} \text{ as in (8)} \, .$$

Applying Proposition 3.3.1 (Proposition 3.3.2) we see that (1) is uniformly (asymptotically) stable if (7) has this property. The converse implication is obvious since $\Phi(\tau,0)^k = \Phi(k\tau,0)$. □

The next example shows that a system (1) may be unstable even though every time–invariant system $\dot{x}(t) = A(\tau)x(t)$ (resp. $x(t+1) = A(\tau)x(t)$) frozen at time $\tau \in T$ is asymptotically stable. It is also possible that every frozen system is unstable yet (1) is stable, see Ex. 7.

Example 3.3.7. Consider the two dimensional periodic system of period 2π, where

$$A(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} . \tag{10}$$

Then $\sigma(A(\tau)) = \{-1\}$, $\tau \in \mathbb{R}_+$ and we will see in the next subsection that time–invariant continuous time systems with spectrum in the open left half plane are asymptotically stable. However it is easily verified that the evolution operator generated by $A(\cdot)$ is such that

$$\Phi(t,0) = \begin{bmatrix} e^t(\cos t + \frac{1}{2}\sin t) & e^{-3t}(\cos t - \frac{1}{2}\sin t) \\ e^t(\sin t - \frac{1}{2}\cos t) & e^{-3t}(\sin t + \frac{1}{2}\cos t) \end{bmatrix} ,$$

which is clearly unbounded. □

Let us now turn to exponential stability. The linear system (1) is *(uniformly) exponentially stable* if there exist for every $t_0 \in T$ a constant $M > 0$, and a decay rate $\omega < 0$ which may depend upon t_0 (resp. independent of t_0), such that

$$\|\Phi(t,t_0)\| \le M e^{\omega(t-t_0)}, \quad t \in T_{t_0} \, . \tag{11}$$

The next theorem is rather surprising. A similar result does not hold for nonlinear systems.

Theorem 3.3.8. *The system* (1) *is uniformly exponentially stable if and only if it is uniformly asymptotically stable.*

Proof: The only if part follows immediately from (11) and Proposition 3.3.2. Conversely suppose that (1) is uniformly asymptotically stable. By Proposition 3.3.2 there exists $\tau \in T$ such that $\|\Phi(t+\tau,t)\| \le 1/2$ for all $t \in T$. Hence using the concatenation property of Φ

$$\|\Phi(t_0+k\tau,t_0)\| \le \|\Phi(t_0+k\tau,t_0+(k-1)\tau)\| \dots \|\Phi(t_0+\tau,t_0)\| \le 2^{-k}.$$

Now suppose $t_0+k\tau \le t < t_0+(k+1)\tau$, $t \in T_{t_0}$, $k \in \mathbb{N}$, then

$$\|\Phi(t,t_0)\| \le \|\Phi(t,t_0+k\tau)\|\|\Phi(t_0+k\tau,t_0)\| \le \|\Phi(t,t_0+k\tau)\|2^{-k}.$$

By Proposition 3.3.1 there exists $M' > 0$ such that $\|\Phi(t,t_0+k\tau)\| \le M'$ for all $t \ge t_0+k\tau,\ k \in \mathbb{N}$, and hence

$$\|\Phi(t,t_0)\| \le M' 2^{-[(t-t_0)/\tau-1]}, \qquad t \in T_{t_0},\ t_0 \in T.$$

Setting $M = 2M'$, $\omega = -(\ln 2)/\tau$ we obtain (11). □

In his doctoral thesis in 1892 Liapunov introduced characteristic numbers associated with the flow generated by the differential equation (1a). They are now known as *Liapunov exponents* and we will be particularly interested in the upper one which characterizes the supreme exponential growth rate of the system. Our definition is applicable to both continuous and discrete time systems (1).

Definition 3.3.9 (Liapunov exponents). If $\Phi(\cdot,\cdot)$ is the evolution operator of (1) and $t_0 \in T$, the upper and lower Liapunov exponents $\overline{\alpha}(\Phi)$, $\underline{\alpha}(\Phi)$ are defined by

$$\begin{aligned}\overline{\alpha}(\Phi) &= \inf\{\omega \in \mathbb{R};\ \exists M_\omega > 0\ \forall t \in T_{t_0}:\ \ \|\Phi(t,t_0)\| \le M_\omega e^{\omega(t-t_0)}\}\\ \underline{\alpha}(\Phi) &= \sup\{\omega \in \mathbb{R};\ \exists M_\omega > 0\ \forall\, t \in T_{t_0}\ \forall x \in \mathbb{K}^n:\ \ \|\Phi(t,t_0)x\| \ge M_\omega e^{\omega(t-t_0)}\|x\|\}\end{aligned}$$

(where we set $\inf \emptyset := \infty$, $\sup \emptyset := -\infty$).

It is easily seen that the two Liapunov exponents do not depend upon t_0 in the continuous time case. In the discrete time case this is also true if $\det A(t) \ne 0$ for all $t \in T$. But, if $\det A(t_1) = 0$ for some $t_1 \in T$ then $\det \Phi(t,t_0) = 0$ for all (t,t_0) with $t_0 \le t_1 \le t$. So by (3) $\overline{\alpha}(\Phi) = -\infty$ if we choose $t_0 \le t_1$ (as we will always do in this case). Therefore we need not indicate the dependency on t_0 in our notation of the Liapunov exponents.
While exponential stability can be characterized by $\overline{\alpha}(\Phi) < 0$ (see the next remark), *uniform* exponential stability can be characterized in terms of the upper *Bohl exponent* introduced by Bohl in 1913.

Definition 3.3.10 (Bohl exponents). If $\Phi(\cdot,\cdot)$ is the evolution operator generated via (1), the upper and lower Bohl exponents $\overline{\beta}(\Phi)$, $\underline{\beta}(\Phi)$ are defined by

$$\overline{\beta}(\Phi) = \inf\{\omega \in \mathbb{R};\ \exists M_\omega\ \forall t_0 \in T\ \forall t \in T_{t_0}:\ \|\Phi(t,t_0)\| \le M_\omega e^{\omega(t-t_0)}\},$$
$$\underline{\beta}(\Phi) = \sup\{\omega \in \mathbb{R};\ \exists M_\omega\ \forall t_0 \in T\ \forall t \in T_{t_0}\ \forall x \in \mathbb{K}^n:\ \|\Phi(t,t_0)x\| \ge M_\omega e^{\omega(t-t_0)}\|x\|\}.$$

Remark 3.3.11. (i) Clearly $\overline{\alpha}(\Phi) \le \overline{\beta}(\Phi)$ and $\underline{\beta}(\Phi) \le \underline{\alpha}(\Phi)$.

(ii) If $\|A(t)\| \le \gamma$, for all $t \in T$ and some $\gamma > 0$, it follows from (3) in the discrete time case that $\overline{\beta}(\Phi) \le \ln\gamma$, whereas in the continuous time case we have $\overline{\beta}(\Phi) \le \gamma$ and $\underline{\beta}(\Phi) \ge -\gamma$ by (5).

(iii) Suppose that $\overline{\alpha}(\Phi) < \infty$ (resp. $\overline{\beta}(\Phi) < \infty$), then given $\gamma > \overline{\alpha}(\Phi)$ (resp. $\gamma > \overline{\beta}(\Phi)$) there exists M depending on γ such that $\|\Phi(t,t_0)\| \le Me^{\gamma(t-t_0)}$, $t \in T_{t_0}$ for a given $t_0 \in T$ (resp. $\|\Phi(t,t_0)\| \le Me^{\gamma(t-t_0)}$, $t_0 \in T$, $t \in T_{t_0}$). So we conclude that the system (1) is exponentially stable at time t_0 (resp. uniformly exponentially stable) if and only if $\overline{\alpha}(\Phi) < 0$ (resp. $\overline{\beta}(\Phi) < 0$).

(iv) If $\tilde{\Phi}(t,t_0) = \Phi(t_0,t)^*$ denotes the evolution operator generated by $-A(t)^*$, then $\underline{\beta}(\Phi) = \overline{\beta}(\tilde{\Phi})$. □

In general the Bohl and Liapunov exponents are not the same as the following scalar example shows.

Example 3.3.12. (Perron). Consider the scalar system

$$\dot{x}(t) = a(t)x(t)\,, \quad \text{where} \quad a(t) = \sin\ln t + \cos\ln t, \;\; t > 0. \tag{12}$$

The corresponding evolution operator is

$$\Phi(t,t_0) = e^{t\sin\ln t - t_0 \sin\ln t_0}\,, \quad t \geq t_0 > 0$$

and $\|\Phi(t,1)\| \leq e^t$, $t \geq 1$ so $\overline{\alpha}(\Phi) \leq 1$.
For small $\varepsilon > 0$ let $\ln t_n = 2n\pi + \pi/4 + \varepsilon$, $\ln t_{0n} = 2n\pi + \pi/4$, then

$$t_n \sin\ln t_n - t_{0n}\sin\ln t_{0n} = e^{2n\pi+\pi/4}\big[e^{\varepsilon}\sin(\pi/4+\varepsilon) - \sin\pi/4\big]\,.$$

But for small ε, $e^{\varepsilon}\sin(\pi/4+\varepsilon) - \sin\pi/4 \approx (1+\varepsilon)(1+\varepsilon)/\sqrt{2} - 1/\sqrt{2} \approx \sqrt{2}\,\varepsilon$. Hence given any small $\delta > 0$ there exists $\varepsilon > 0$ such that $e^{\varepsilon}\sin(\pi/4+\varepsilon) - \sin\pi/4 \geq (\sqrt{2}-\delta)(e^{\varepsilon}-1)$. And so for this ε

$$t_n \sin\ln t_n - t_{0n}\sin\ln t_{0n} \geq (\sqrt{2}-\delta)(t_n - t_{0n})\,.$$

But then

$$|\Phi(t_n,t_{0n})| \geq e^{(\sqrt{2}-\delta)(t_n-t_{0n})}\,.$$

Since $t_n - t_{0n} \to \infty$ as $n \to \infty$, this shows $\overline{\beta}(\Phi) \geq \sqrt{2}$. Now $|a(t)| \leq \sqrt{2}$, $t > 0$. Hence $|\Phi(t,t_0)| = |e^{\int_{t_0}^t a(s)ds}| \leq e^{\sqrt{2}(t-t_0)}$, $t \geq t_0 > 0$ and so in fact $\overline{\beta}(\Phi) = \sqrt{2}$. □

Remark 3.3.13. In the continuous time case if $A(\cdot)$ generates $\Phi(\cdot,\cdot)$, then for any $\lambda \in \mathbb{C}$, $A(\cdot) + \lambda I_n$ generates $\Phi_\lambda(t,t_0) = e^{\lambda(t-t_0)}\Phi(t,t_0)$ and

$$\overline{\alpha}(\Phi_\lambda) = \overline{\alpha}(\Phi) + \operatorname{Re}\lambda, \qquad \overline{\beta}(\Phi_\lambda) = \overline{\beta}(\Phi) + \operatorname{Re}\lambda\,. \tag{13}$$

If $a(\cdot)$ is as in the above example and $-\sqrt{2} < \lambda < -1$ we see that $\overline{\alpha}(\Phi_\lambda) < 0$ and $\overline{\beta}(\Phi_\lambda) > 0$ so that all solutions of $\dot{x} = (a(t)+\lambda)x$ decrease exponentially although $\overline{\beta}(\Phi_\lambda) > 0$. □

It is easily verified (see Ex. 5) that for the upper upper Liapunov exponent we have

$$\overline{\alpha}(\Phi) = \limsup_{t\to\infty} \frac{\ln\|\Phi(t,0)\|}{t}\,. \tag{14}$$

The corresponding formula for the Bohl exponent is given in the next proposition.

Proposition 3.3.14. $\overline{\beta}(\Phi) < \infty$ *if and only if*

$$\sup_{t_0,t\in T,\, 0\leq t-t_0\leq 1} \|\Phi(t,t_0)\| < \infty\,, \tag{15}$$

and when this is the case

$$\overline{\beta}(\Phi) = \limsup_{t_0,\, t-t_0\to\infty} \frac{\ln\|\Phi(t,t_0)\|}{t-t_0}\,. \tag{16}$$

Proof: Suppose $\overline{\beta}(\Phi) < \infty$, then choosing $\gamma > \max\{\overline{\beta}(\Phi), 0\}$ there exists $M(\gamma) > 0$ such that

$$\|\Phi(t,t_0)\| \le M(\gamma)e^{\gamma(t-t_0)}\,, \quad t_0 \in T\,,\ t \in T_{t_0}\,. \tag{17}$$

Hence $\sup_{0 \le t-t_0 \le 1} \|\Phi(t,t_0)\| \le M(\gamma)e^{\gamma} < \infty$. Conversely suppose (15) holds so that $\|\Phi(\tau,\sigma)\| \le K$ for some $K \ge 1$ and all $\sigma, \tau \in T$, $0 \le \tau - \sigma \le 1$. Then for every $t_0 \in T$, $t \in T_{t_0}$ such that $t_0 + (n-1) \le t < t_0 + n$

$$\|\Phi(t,t_0)\| \le \|\Phi(t,t_0+n-1)\| \prod_{k=1}^{n-1} \|\Phi(t_0+k,t_0+k-1)\| \le K^n \le Ke^{(t-t_0)\ln K}. \tag{18}$$

So $\overline{\beta}(\Phi) \le \ln K$ and this concludes the proof of the equivalence statement.
To prove (16) we suppose $\overline{\beta}(\Phi) < \infty$. Then (17) holds for every $\gamma > \overline{\beta}(\Phi)$ and so

$$\mu = \limsup_{t_0, t-t_0 \to \infty} \frac{\ln\|\Phi(t,t_0)\|}{t-t_0} \;\le\; \limsup_{t_0, t-t_0 \to \infty} \frac{\ln M(\gamma)}{t-t_0} + \gamma = \gamma\,.$$

Hence $\mu \le \overline{\beta}(\Phi)$. Conversely, for every $\gamma > \mu$ there exists a time $t_\gamma \in T$ such that

$$\frac{\ln\|\Phi(t,t_0)\|}{t-t_0} \le \gamma\,, \quad \text{i.e.} \quad \|\Phi(t,t_0)\| \le e^{\gamma(t-t_0)}\,, \quad t_0 \in T_{t_\gamma}\,,\ t \in T_{t_0+t_\gamma}.$$

By (18)

$$K_\gamma := \sup\{\|\Phi(t,t_0)\|;\ t_0,\, t \in T,\ 0 \le t - t_0 \le t_\gamma\} \le Ke^{t_\gamma \ln K} < \infty. \tag{19}$$

So

$$\|\Phi(t,t_0)\| \le K_\gamma e^{|\gamma| t_\gamma} e^{\gamma(t-t_0)}, \quad t_0 \le t \le t_0 + t_\gamma. \tag{20}$$

Therefore

$$\|\Phi(t,t_0)\| \le Ne^{\gamma(t-t_0)}\,, \quad t_0 \in T_{t_\gamma}\,,\ t \in T_{t_0}\,,$$

where $N = \max\{1,\ K_\gamma e^{|\gamma| t_\gamma}\}$. But by (20) this same estimate is also valid for $0 \le t_0 \le t \le t_\gamma$. Finally if $t_0 \le t_\gamma < t$ we have

$$\|\Phi(t,t_0)\| \le \|\Phi(t,t_\gamma)\|\|\Phi(t_\gamma,t_0)\| \le Ne^{\gamma(t-t_\gamma)}Ne^{\gamma(t_\gamma-t_0)} = N^2 e^{\gamma(t-t_0)}$$

and so there exists M such that $\|\Phi(t,t_0)\| \le Me^{\gamma(t-t_0)}$, for all $t_0 \in T\,,\ t \in T_{t_0}$. Thus $\overline{\beta}(\Phi) \le \mu$ and (16) is proved. □

Note that in the discrete time case (15) holds if and only if $\sup_{t\in T} \|A(t)\| =: \gamma < \infty$ in which case $\overline{\alpha}(\Phi) \le \overline{\beta}(\Phi) \le \ln\gamma$.
The following theorem gives an alternative characterization for uniform exponential stability of (1). It is closely related to the Liapunov results which we will develop in Subsection 3.3.4.

Theorem 3.3.15. *Suppose the evolution operator* Φ *of* (1) *satisfies* $\overline{\beta}(\Phi) < \infty$ *then the following statements are equivalent.*

(i) The system (1) *is uniformly exponentially stable.*

(ii) $\overline{\beta}(\Phi) < 0$.

(iii) For any $p \in (0, \infty)$ *there exists a constant* c *independent of* $t_0 \in T$ *such that*

$$\int_{t_0}^{\infty} \|\Phi(t,t_0)\|^p dt \le c \quad \big(\textit{resp.} \quad \sum_{t=t_0}^{\infty} \|\Phi(t,t_0)\|^p \le c\big), \;\; t_0 \in T. \tag{21}$$

(iv) For any $p \in (0, \infty)$ *there exists a constant* c *independent of* $t_0 \in T$ *such that*

$$\int_{t_0}^{\infty} \|\Phi(t,t_0)x\|^p dt \le c\|x\|^p \,\big(\textit{resp.} \;\sum_{t=t_0}^{\infty} \|\Phi(t,t_0)x\|^p \le c\|x\|^p\big), \;\; x \in \mathbb{K}^n, \;\; t_0 \in T. \tag{22}$$

Proof: The proof is for the continuous time case. (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (iv) is clear.
(i) $\Rightarrow$ (iii): Suppose (i) then there exist constants $M > 0, \omega < 0$ independent of $t_0 \in T$ such that $\|\Phi(t,t_0)\| \le Me^{\omega(t-t_0)}$, for all $t_0 \in T$, $t \in T_{t_0}$. Hence (21) holds with $c = M^p/p(-\omega)$.
(iv) $\Rightarrow$ (i): Since $\overline{\beta}(\Phi) < \infty$ there exists $\overline{M}, \overline{\omega} > 0$ independent of t_0 such that $\|\Phi(t,t_0)\| \le \overline{M}e^{\overline{\omega}(t-t_0)}$, for all $t_0 \in T$, $t \in T_{t_0}$. So

$$\begin{aligned} \frac{1-e^{-p\overline{\omega}(t-t_0)}}{p\overline{\omega}}\|\Phi(t,t_0)x\|^p &= \int_{t_0}^{t} e^{-p\overline{\omega}(t-s)}\|\Phi(t,t_0)x\|^p ds \\ &\le \int_{t_0}^{t} e^{-p\overline{\omega}(t-s)}\|\Phi(t,s)\|^p\|\Phi(s,t_0)x\|^p ds \\ &\le \overline{M}^p \int_{t_0}^{t} \|\Phi(s,t_0)x\|^p ds \le \overline{M}^p c\|x\|^p, \;\; t \ge t_0\,. \end{aligned} \tag{23}$$

Hence there exists γ independent of $t_0 \in T$ such that $\|\Phi(t,t_0)\| \le \gamma$, $t \ge t_0$. But then for $t \ge t_0$, $t_0 \in T$

$$(t-t_0)\|\Phi(t,t_0)x\|^p = \int_{t_0}^{t} \|\Phi(t,t_0)x\|^p ds \le \int_{t_0}^{t} \|\Phi(t,s)\|^p\|\Phi(s,t_0)x\|^p ds \le \gamma^p c\|x\|^p.$$

So for $\tau = 2^p\gamma^p c$

$$\|\Phi(t_0+\tau, t_0)\| \le 1/2\,, \quad t_0 \in T\,. \tag{24}$$

Now suppose $t_0 + (n-1)\tau \le t < t_0 + n\tau$, then from (24)

$$\|\Phi(t,t_0)\| \le \|\Phi(t,t_0+(n-1)\tau)\| \prod_{k=1}^{n-1} \|\Phi(t_0+k\tau, t_0+(k-1)\tau)\| \le \frac{\gamma}{2^{n-1}} < 2\gamma e^{-(\ln 2)(t-t_0)/\tau}.$$

Hence $\overline{\beta}(\Phi) < -(\ln 2)/\tau$. The reader is asked to prove the discrete time case in Ex. 23. □

We now consider the effect of time–varying linear coordinate transformations of the form $\tilde{x}(t) = S(t)^{-1}x(t)$ on the system (1), where $S(\cdot) \in PC^1(T; \mathbf{Gl}_n(\mathbb{C}))$ (resp. $S(t) \in \mathbf{Gl}_n(\mathbb{C}), t \in T$). The associated similarity transformation converts the system (1) into

$$\dot{\tilde{x}}(t) = \tilde{A}(t)\tilde{x}(t), \;\; t \in T, \quad (\text{resp. } \tilde{x}(t+1) = \tilde{A}(t)\tilde{x}(t), \;\; t \in T) \tag{25}$$

where

$$\tilde{A}(t) = S(t)^{-1}A(t)S(t) - S(t)^{-1}\dot{S}(t),\ t \in T, \quad (\text{resp.} \ = S(t+1)^{-1}A(t)S(t),\ t \in T).$$

The evolution operator of the system (25) is

$$\tilde{\Phi}(t,s) = S(t)^{-1}\Phi(t,s)S(s), \quad t,s \in T. \tag{26}$$

In order that these transformations preserve stability properties additional assumptions must be imposed.

Definition 3.3.16 (Liapunov and Bohl transformation). A time-varying transformation $S(\cdot) \in PC^1(T; \mathbf{Gl}_n(\mathbb{C}))$ (resp. $S(t) \in \mathbf{Gl}_n(\mathbb{C})$, $t \in T$) is called a *Liapunov transformation* if $S(\cdot)$, $S(\cdot)^{-1}$ and $\dot{S}(\cdot)$ are bounded on T. It is called a *Bohl transformation* if

$$\inf\left\{\varepsilon \in \mathbb{R};\ \exists M_\varepsilon > 0\ \forall\, t,s \in T : \|S(t)^{-1}\|\,\|S(s)\| \le M_\varepsilon e^{\varepsilon|t-s|}\right\} = 0.$$

It is easily seen that the Liapunov transformations on T form a group with respect to pointwise multiplication, and this group of transformations preserves the properties of stability, instability and asymptotic stability. The next proposition shows that the property of exponential stability is invariant with respect to the larger group of Bohl transformations.

Proposition 3.3.17. *The Bohl exponent is invariant with respect to Bohl transformations.*

Proof: Let $\dot{\tilde{x}}(t) = \tilde{A}(t)\tilde{x}(t)$, (resp. $\tilde{x}(t+1) = \tilde{A}(t)\tilde{x}(t)$) be similar to (1) via a Bohl transformation $S(\cdot)$. Since the evolution operator of the transformed equation is given by (26), we have

$$\|\tilde{\Phi}(t,s)\| \le \|S(t)^{-1}\|\|\Phi(t,s)\|\|S(s)\|, \quad t,s \in T.$$

But by Definitions 3.3.10 and 3.3.16, for every $\varepsilon > 0$, there exists a constant M_ε such that

$$\|S(t)^{-1}\|\|S(s)\| \le M_\varepsilon e^{\varepsilon(t-s)}, \quad \|\Phi(t,s)\| \le M_\varepsilon e^{(\beta(\Phi)+\varepsilon)(t-s)}, \quad t \ge s \in T.$$

So $\beta(\tilde{\Phi}) \le \beta(\Phi)$. Using the fact that $S(\cdot)^{-1}$ is also a Bohl transformation we conclude that $\beta(\tilde{\Phi}) = \beta(\Phi)$. □

It is a simple exercise to show that every time-varying linear system (1) can be transformed into the trivial system $\dot{x} = 0$ by a time-varying coordinate transformation. In the context of stability theory it is interesting to know which time-varying systems can be transformed into time–invariant ones via Liapunov or Bohl transformations. According to a result of Liapunov this is always possible for periodic systems.

Proposition 3.3.18. *Suppose the generator $A(\cdot)$ of* (1) *is periodic with period $\tau > 0$, $\tau \in T$: $A(t+\tau) = A(t)$, $t \in T$, and $\det A(t) \neq 0, t \in T$ in the discrete time case. Then there exists a Liapunov transformation such that the transformed system* (25) *is time-invariant.*

Proof: Suppose $\Phi(\cdot,\cdot)$ is generated by $A(\cdot)$. Since $A(\cdot)$ is periodic, we have

$$\dot{\Phi}(t+\tau,0) = A(t)\Phi(t+\tau,0),\ t \in T\ \ (\text{resp. } \Phi(t+\tau+1,0) = A(t)\Phi(t+\tau,0),\ t \in T).$$

So there must exist a constant nonsingular matrix V such that $\Phi(t+\tau,0) = \Phi(t,0)V$. Choose $L \in \mathbb{C}^{n\times n}$ such that $e^L = V$ and set $S(t) = \Phi(t,0)e^{-tL/\tau}$, $t \in T$. Then

$$S(t+\tau) = \Phi(t+\tau,0)e^{-(tL/\tau)-L} = \Phi(t,0)e^L e^{-(tL/\tau)-L} = S(t).$$

Hence $S(\cdot)$ is periodic with period τ. In the continuous time case Φ is automatically invertible and in the discrete time case this is a consequence of the assumption that $\det A(t) \neq 0$, $t \in T$. It follows therefore that $S(t)$, $t \in T$ is invertible. Moreover

$$\dot{S}(t) = A(t)\Phi(t,0)e^{-tL/\tau} - \Phi(t,0)e^{-tL/\tau}\tau^{-1}L = A(t)S(t) - S(t)\tau^{-1}L,\ t \geq 0.$$

And in the discrete case

$$S(t+1) = \Phi(t+1,0)e^{-(t+1)L/\tau} = A(t)S(t)e^{-L/\tau},\ t \in T.$$

So the transformed system (25) is given by $\hat{A}(t) = \tau^{-1}L$, $t \in T$, (resp. $= e^{L/\tau}$). Clearly $S(\cdot) \in PC^1(T;\mathbf{Gl}_n(\mathbb{C}))$ (resp. $S(t) \in \mathbf{Gl}_n(\mathbb{C})$, $t \in T$) and the boundedness of $S(\cdot)$, $S(\cdot)^{-1}$, $\dot{S}(\cdot)$ is a consequence of periodicity. Hence S is a Liapunov transformation and this completes the proof. □

3.3.2 Time–Invariant Systems: Spectral Stability Criteria

We consider systems of the form

$$\dot{x}(t) = Ax(t), \quad t \in T, \qquad (\text{resp. } x(t+1) = Ax(t), \quad t \in T) \tag{27}$$

where $A \in \mathbb{K}^{n\times n}$ and $T = \mathbb{R}_+$ (resp. $T = \mathbb{N}$). The following result relates growth properties of the semigroup generated by the matrix $A \in \mathbb{K}^{n\times n}$ to the spectrum of A, $\sigma(A)$.

Lemma 3.3.19. *Given $A \in \mathbb{K}^{n\times n}$ and $\omega \in \mathbb{R}$. If*

$$\alpha(A) = \max\{\operatorname{Re}\lambda\,;\,\lambda \in \sigma(A)\} < \omega,\ (\textit{resp. } \varrho(A) = \max\{|\lambda|\,;\,\lambda \in \sigma(A)\} < e^{\omega}) \tag{28}$$

then there exists M, depending on ω such that

$$\|e^{At}\| \leq Me^{\omega t}, \quad t \in \mathbb{R}_+, \quad (\textit{resp. } \|A^t\| \leq Me^{\omega t}, \quad t \in \mathbb{N}). \tag{29}$$

Proof: The proof will be for the discrete time case. Since the spectral norm and the spectrum of a real linear operator do not change by complexification we may assume $\mathbb{K} = \mathbb{C}$. Let $(z^1,\ldots,z^n)$ be a basis of $\mathbb{C}^n$ consisting of generalized eigenvectors z^i of order m_i, corresponding to eigenvalues λ_i of A. Applying Proposition 3.3.1 to the time–invariant evolution operator $\Phi(t) = (A^t e^{-\omega t})$ we see that $(A^t e^{-\omega t})_{t\in\mathbb{N}}$ is bounded if and only if $(A^t e^{-\omega t} z^i)_{t\in\mathbb{N}}$ is bounded for all $i \in \underline{n}$. Now if $\varrho(A) = 0$, then

$A^t = 0$ for t sufficiently large and it follows from (2.2.55) that for $t \geq m_i - 1$ and $\varrho := \varrho(A) \neq 0$

$$\|A^t e^{-\omega t} z^i\| = e^{-\omega t} \| \sum_{\nu=0}^{m_i-1} \lambda_i^{t-\nu} \binom{t}{\nu} (A - \lambda_i I)^\nu z^i \| \leq \left[\varrho e^{-\omega}\right]^t \sum_{\nu=0}^{m_i-1} \binom{t}{\nu} \varrho^{-\nu} \|(A - \lambda_i I)^\nu z^i\|.$$

Since for every $\alpha \in (0,1)$ and every polynomial $p(t) \in \mathbb{K}[t]$ we have $\lim_{t\to\infty} \alpha^t p(t) = 0$, we see that $\|A^t e^{-\omega t} z^i\| \to 0$ as $t \to \infty$ and so there exists $M > 0$ such that $\|A^t e^{-\omega t}\| \leq M$ for all $t \in \mathbb{N}$. This proves (29). □

If $(\Phi(t))$ is the semigroup of operators generated by A (continuous or discrete time), then it is an easy consequence (see Ex. 1) of the above lemma that the Liapunov and Bohl exponents are equal. They are sometimes called the *growth rate* of Φ, denoted by $\omega(A)$ and are given by

$$\omega(A) = \overline{\beta}(\Phi) = \overline{\alpha}(\Phi) = \alpha(A), \quad \left(\omega(A) = \overline{\beta}(\Phi) = \overline{\alpha}(\Phi) = \ln \varrho(A)\right). \tag{30}$$

For time invariant systems stability and uniform stability properties are equivalent and hence as a consequence of Theorem 3.3.8 *asymptotic stability is equivalent to uniform exponential stability.* The following theorem derives necessary and sufficient conditions for the asymptotic stability of the system (27).

Theorem 3.3.20. *The system* (27) *is asymptotically (or, equivalently, exponentially) stable if and only if*

$$\operatorname{Re}\lambda < 0, \qquad (\textit{resp.}\ \ |\lambda| < 1), \quad \lambda \in \sigma(A). \tag{31}$$

Proof: The proof is for the discrete time case. If (31) holds then $\ln \varrho(A) < 0$ and so by Lemma 3.3.19 there exists $\omega < 0$, and M such that

$$\|A^t x_0\| \leq \|A^t\| \|x_0\| \leq M e^{\omega t} \|x_0\|, \qquad t \in \mathbb{N},\ x^0 \in \mathbb{K}^n.$$

This implies that (27) is exponentially stable. To prove necessity suppose there exists $\lambda \in \sigma(A)$ such that $|\lambda| \geq 1$ and let $z \in \mathbb{C}^n$ be a corresponding eigenvector. Then

$$\|A^t z\| = \|\lambda^t z\| \geq \|z\| \qquad t \in \mathbb{N}$$

and hence (27) is not asymptotically stable. □

Theorem 3.3.21. *The system* (27) *is stable if and only if both of the following conditions hold for all* $\lambda \in \sigma(A)$

(i) $\operatorname{Re}\lambda \leq 0$, *(resp.* $|\lambda| \leq 1$*).*

(ii) If $\operatorname{Re}\lambda = 0$ *(resp.* $|\lambda| = 1$*) then there exist* k_λ *linearly independent eigenvectors, where* k_λ *is the algebraic multiplicity of* λ*.*

Proof: The proof is for the discrete time case. By Proposition 3.3.1 the origin is stable if and only if all the (generalized) eigenmotions are bounded. This clearly implies $|\lambda| \le 1$ for all $\lambda \in \sigma(A)$. Now suppose there exists $\lambda \in \sigma(A)$ with $|\lambda| = 1$ and a generalized eigenvector z of order $m > 1$ then for $t \ge m-1$

$$A^t z = \lambda^t \sum_{\nu=0}^{m-1} \binom{t}{\nu} \lambda^{-\nu}(A-\lambda I)^\nu z \quad \text{and} \quad \|A^t z\| = \|\sum_{\nu=0}^{m-1} \binom{t}{\nu} \lambda^{-\nu}(A-\lambda I)^\nu z\|.$$

The RHS is a polynomial in t of degree $m-1 \ge 1$ and is therefore unbounded. Thus conditions (i), (ii) are necessary.
Conversely if (i) and (ii) hold there exist generalized eigenvectors of order $m > 1$ only for eigenvalues $\lambda \in \sigma(A)$ with $|\lambda| < 1$. We know already that these eigenmotions tend exponentially to the origin as $t \to \infty$. On the other hand if z is an eigenvector corresponding to $\lambda \in \sigma(A)$, then since $|\lambda| \le 1$

$$\|A^t z\| = \|\lambda^t z\| \le \|z\|.$$

Hence all generalized eigenmotions are bounded and (27) is stable. □

Figure 3.3.1 shows the stability regions for the eigenvalues in the continuous and discrete time case. They are denoted by $\mathbb{C}_-$ and $\mathbb{D}$ respectively.

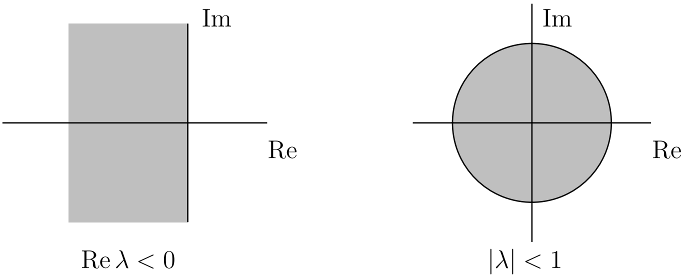

Figure 3.3.1: Stability regions for continuous and discrete time systems

As a consequence of Lemma 2.3.9 and Proposition 2.3.10 we have the following characterization of asymptotic stability in terms of properties of the solutions of the controlled systems

$$\dot{x}(t) = Ax(t) + Bu(t),\ t \in \mathbb{R}_+, \quad x(t+1) = Ax(t) + Bu(t),\ t \in \mathbb{N}. \tag{32}$$

We denote the solutions with $x(0) = x^0 \in \mathbb{K}^n$ by $\varphi(t; x^0, u(\cdot))$, $t \in \mathbb{R}_+$ (resp. $\mathbb{N}$).

Proposition 3.3.22. *The following are equivalent.*

(i) $\sigma(A) \subset \mathbb{C}_-$ *(resp.* $\sigma(A) \subset \mathbb{D}$*).*

(ii) For every $x^0 \in \mathbb{K}^n$, $e^{At}x^0 \to 0$ *(resp.* $A^t x^0 \to 0$*) as* $t \to \infty$.

(iii) If $u(\cdot) \in L^2(\mathbb{R}_+; \mathbb{K}^m)$ *(resp.* $\ell^2(\mathbb{N}; \mathbb{K}^m)$*), then* $\varphi(\cdot; x^0, u(\cdot)) \in L^2(\mathbb{R}_+; \mathbb{K}^n)$ *(resp.* $\ell^2(\mathbb{N}; \mathbb{K}^n)$*) for all* $x^0 \in \mathbb{K}^n$.

Proof: The proof is for the continuous time case. (i) implies (ii) by Theorem 3.3.20. Let $z \in \mathbb{C}^n$ be an eigenvector for the eigenvalue $\lambda \in \sigma(A)$, then $e^{At}z = e^{\lambda t}z$. Choosing $x^0 = z$ in the complex case and $x^0 = \operatorname{Re} z$ or $\operatorname{Im} z$ in the real case we see from (2.2.44) that if (ii) holds then necessarily $\operatorname{Re}\lambda < 0$. So (ii) implies (i). By

Proposition 2.3.10 with $C = I_n$ and $t_0 = 0$, (i) implies (iii). Now suppose $u(\cdot) \equiv 0$ and (iii) holds, then $x(\cdot) = e^{A\cdot}x^0 \in L^2(\mathbb{R}_+; \mathbb{K}^n)$ for all $x^0 \in \mathbb{K}^n$. But $\dot{x}(\cdot) = Ax(\cdot)$ and so $x(\cdot)$ is absolutely continuous and $\dot{x}(\cdot) \in L^2(\mathbb{R}_+; \mathbb{K}^n)$. Applying Lemma 2.3.9 with $p = n$, $y(\cdot) = x(\cdot)$ and $t_0 = 0$ we see that $x(t) = e^{At}x^0 \to 0$ as $t \to \infty$ for all $x^0 \in \mathbb{K}^n$. So (iii) implies (ii). □

Remark 3.3.23. In the previous section (Proposition 3.3.6) we showed that a periodic system with evolution operator Φ is stable (resp. uniformly asymptotically stable) if and only if the associated discrete time system with system matrix $\Phi(\tau, 0)$ is stable (resp. asymptotically stable). The eigenvalues of $\Phi(\tau, 0)$ are called the *characteristic multipliers* of (1). It follows from Theorems 3.3.20 and 3.3.21 that a periodic system (1) is stable (resp. asymptotically stable) if and only if its characteristic multipliers $\mu \in \sigma(\Phi(\tau, 0))$ satisfy conditions (i) (ii) of Theorem 3.3.21 (resp. (31)) in their discrete time versions. In contrast we have seen in Example 3.3.7 that, in general, the stability properties of a periodic time-varying system cannot be determined via the eigenvalues of $A(t)$. □

In the next two examples we illustrate the stability criteria by applying them to second order scalar systems.

Example 3.3.24. Consider the second order differential equation

$$\ddot{\xi}(t) + 2\alpha\dot{\xi}(t) + \beta\xi(t) = 0, \quad t > 0. \tag{33}$$

The matrix A of the corresponding state space system has eigenvalues $\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \beta}$. So

(a) if $\alpha > 0$, $\beta > 0$, the origin is *exponentially stable*;

(b) if $\alpha > 0$, $\beta = 0$, the origin is *marginally stable* (i.e. stable but not asymptotically stable);

(c) if $\alpha = 0$, $\beta > 0$, the origin is *marginally stable*;

(d) if $\alpha = 0$, $\beta = 0$, there is a generalized eigenvector for the zero eigenvalue and so the origin is *unstable*;

(e) if $\alpha < 0$ or $\beta < 0$, the origin is *unstable*.

The stability chart, i.e. the set of all parameter values $(\alpha, \beta) \in \mathbb{R}^2$ for which the system is asymptotically stable is given by the positive orthant $(0, \infty)^2$. □

Example 3.3.25. Using the approximations

$$\dot{\xi}(t) \;\approx\; \frac{\xi(t+\tau) - \xi(t)}{\tau}, \quad \ddot{\xi}(t) \;\approx\; \frac{\xi(t+2\tau) - 2\xi(t+\tau) + \xi(t)}{\tau^2}, \quad \tau > 0$$

the differential equation of the previous example gives rise to the difference equation

$$\xi(t+2\tau) - 2(\alpha\tau - 1)\xi(t+\tau) + (1 - 2\alpha\tau + \beta\tau^2)\xi(t) = 0, \qquad t \in \mathbb{N}\tau. \tag{34}$$

We will examine the stability properties of this discrete time system and compare the results with those obtained in the previous example. In order to do this we first obtain results for the general second order difference equation

$$\xi(t+2) + a_1\xi(t+1) + a_0\xi(t) = 0, \quad t \in \mathbb{N}. \tag{35}$$

The eigenvalues of the matrix $A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$ of the corresponding state space system are

$$\lambda_{1,2} = (1/2)[-a_1 \pm (a_1^2 - 4a_0)^{1/2}].$$

The parameter set (a_0, a_1) for which the system is stable must satisfy

$$\begin{array}{rll} -1 \le & (1/2)[-a_1 \pm (a_1^2 - 4a_0)^{1/2}] & \le 1 \text{ if } a_1^2 \ge 4a_0 \\ & (1/4)(a_1^2 + (4a_0 - a_1^2)) & \le 1 \text{ if } a_1^2 < 4a_0. \end{array}$$

The first condition is equivalent to $a_1^2 - 4a_0 \le (2+a_1)^2$ and $(a_1 - 2)^2 \ge a_1^2 - 4a_0$, i.e.

$$1 + a_1 + a_0 \ge 0 \text{ and } 1 - a_1 + a_0 \ge 0. \tag{36}$$

The second condition is equivalent to

$$1 - a_0 \ge 0 \text{ if } a_1^2 < 4a_0. \tag{37}$$

This leads to the stability chart for (35) shown on the LHS of Figure 3.3.2. The shaded

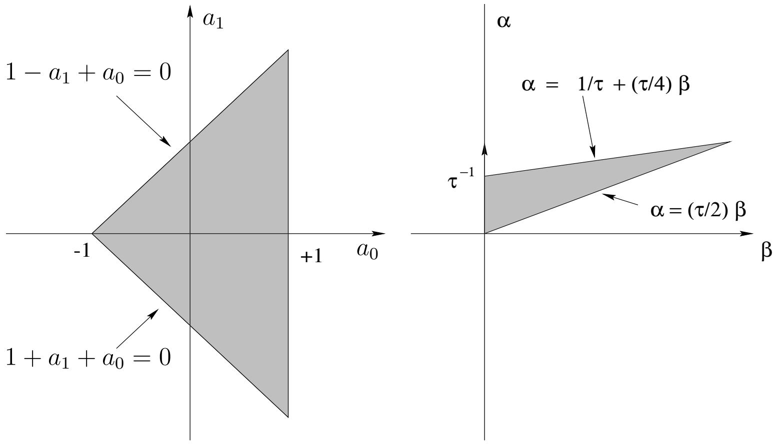

Figure 3.3.2: Stability charts for (35) and (34)

region inside the left triangle represents values of the parameters (a_0, a_1) for which $|\lambda_i| < 1$, $i = 1, 2$ and hence values for which the system is asymptotically stable. Now consider the boundary of the triangle. When $1 - a_1 + a_0 = 0$ (resp. $1 + a_1 + a_0 = 0$) then $\sigma(A) = \{-1, -a_0\}$ (resp. $\{+1, a_0\}$) and so if $a_0 < 1$ the system is marginally stable, and this is also the case if $a_0 = 1$, $|a_1| < 2$. However if $a_0 = 1$, $|a_1| = 2$ there are generalized eigenvectors of order 2 so the system is unstable.

Note that if $a_0 = a_1 = 0$ the system matrix A is nilpotent and so $A^{t+2} = 0$ for $t \in \mathbb{N}$. Thus any initial state is transferred to the origin in finite time. This can never occur in the differentiable case.

For the discretized differential equation (34), we have

$$a_0 = 1 - 2\alpha\tau + \tau^2\beta, \quad a_1 = 2\alpha\tau - 2.$$

Hence $1 + a_1 + a_0 = \tau^2\beta$, $1 - a_1 + a_0 = 4 - 4\alpha\tau + \tau^2\beta$, $1 - a_0 = 2\alpha\tau - \tau^2\beta$. So the discretized system will be asymptotically stable if $\beta > 0$, $2\alpha\tau - \tau^2\beta > 0$, $4 - 4\alpha\tau + \tau^2\beta > 0$. The stability chart is shown on the RHS of Figure 3.3.2. Note that for any $\beta > 0$, $\alpha > 0$ there exists τ sufficiently small such that the discretized system is asymptotically stable and as $\tau \to 0$ the shaded region fills up the positive orthant $(0, \infty)^2$. Thus the stability chart for the discretized system gradually approaches the stability region of the differentiable system as $\tau \to 0$. This is not the case for all discretization schemes as we will show in the next subsection. □

We conclude this subsection with a brief discussion of the relationship between spectral stability criteria for continuous and discrete time systems (27) (see Ex. 27). There is a well known rational map transforming the open left half plane $\mathbb{C}_-$ onto the open unit disk $\mathbb{D}$ and vice versa, the so-called Möbius map

$$m(\cdot) \; : \; \lambda \mapsto \frac{\lambda+1}{\lambda-1}, \quad \lambda \in \mathbb{C} \setminus \{1\} \tag{38}$$

with inverse $m^{-1}(\cdot) = m(\cdot)$. In particular this map sends 0 to -1, ∞ to 1, -1 to 0. The matrix version of this transformation

$$A \mapsto (A+I)(A-I)^{-1} \tag{39}$$

is well defined on $\{A \in \mathbb{K}^{n\times n};\ 1 \notin \sigma(A)\}$ and is known as the *Cayley transform.*

Proposition 3.3.26. *Given $A \in \mathbb{K}^{n\times n}$, $1 \notin \sigma(A)$, let $\hat{A} = (A+I)(A-I)^{-1}$, then $A = (\hat{A}-I)^{-1}(\hat{A}+I)$ and*

$$\sigma(\hat{A}) = \left\{(\lambda+1)(\lambda-1)^{-1},\ \lambda \in \sigma(A)\right\}. \tag{40}$$

Proof: Suppose $Ax = \lambda x$, $x \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, $x \neq 0$, then $(A-I)x = (\lambda-1)x$, $(\lambda-1)^{-1}x = (A-I)^{-1}x$ and $(A+I)x = (\lambda+1)x$. Hence

$$\hat{A}x = (\lambda+1)(\lambda-1)^{-1}x$$

and so $(\lambda+1)(\lambda-1)^{-1} \in \sigma(\hat{A})$, i.e. $m(\sigma(A)) \subset \sigma(\hat{A})$. Since $\hat{A}(A-I) = A+I$, we get $(\hat{A}-I)A = \hat{A}+I$ that is $A = (\hat{A}-I)^{-1}(\hat{A}+I)$. So applying the above argument to $\hat{A}$ instead of A and making use of $m^{-1}(\cdot) = m(\cdot)$ we obtain (40). □

As a result of the above proposition and Theorem 3.3.20 we see that the continuous time system $\dot{x} = Ax$ is asymptotically stable if and only if the discrete time system $x(t+1) = (A+I)(A-I)^{-1}x(t)$ is asymptotically stable. For further details and applications of the Cayley transform see Subsection 3.4.6 and Subsection 5.3.7.

3.3.3 Numerical Stability of Discretization Methods

In this subsection we examine the stability of linear multistep discretizations methods as described in Subsection 2.3.1 (see (35))

$$x^\tau(k+1) = x^\tau(k-p) + \tau[b_{-1}f_{k+1} + b_0 f_k + \ldots + b_\nu f_{k-\nu}], \quad p \in \mathbb{N}. \tag{41}$$

We apply this integration formula to the scalar differential equation,

$$\dot{x} = ax \tag{42}$$

(with $a \in \mathbb{R}$, $a\tau b_{-1} \neq 1$), i.e. we set $f_k = ax^\tau(k)$. Substitution in (41) yields the difference equation

$$x^\tau(k+1) = x^\tau(k-p) + a\tau[b_{-1}x^\tau(k+1) + b_0 x^\tau(k) + \ldots + b_\nu x^\tau(k-\nu)].$$

Let us first assume $\nu \geq p$, then introducing $x_1(k) = x^\tau(k-\nu)$, $x_2(k) = x^\tau(k-\nu+1), \ldots, x_{\nu+1}(k) = x^\tau(k)$, we obtain the matrix difference equation

$$x(k+1) = Ax(k)$$

where $x(k) = [x_1(k), \ldots, x_{\nu+1}(k)]^\top \in \mathbb{R}^{\nu+1}$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 1 & \ldots & \ldots & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & \ldots & \ldots & 1 \\ a\tau b_\nu \gamma^{-1} & \ldots & \ldots & \ldots & (1 + a\tau b_p)\gamma^{-1} & \ldots & a\tau b_0 \gamma^{-1} \end{bmatrix}, \ \gamma = 1 - a\tau b_{-1} \neq 0.$$

The characteristic equation of A, after multiplication by γ is

$$(1 - a\tau b_{-1})\lambda^{\nu+1} - \lambda^{\nu-p} - a\tau[b_0\lambda^\nu + \ldots + b_\nu] = 0. \tag{43}$$

If $\tau = 0$, the eigenvalues of A are 0 (with multiplicity $\nu - p$) and the $p+1$ distinct roots $\omega_1, \ldots, \omega_{p+1}$ of $z^{p+1} = 1$. It follows from Corollaries 4.2.4 and 4.2.3 of the next chapter that for small $\tau \geq 0$ the eigenvalues of A can be written in the form $\lambda_1(\tau), \ldots, \lambda_\nu(\tau)$ where the first $p+1$ eigenvalues with $\lambda_i(0) = \omega_i$ are analytic in τ

$$\lambda_i(\tau) = \omega_i + \alpha_i\tau + O(\tau^2), \quad i = 1, 2, \ldots, p+1 \tag{44}$$

and the remaining eigenvalues with $\lambda_i(0) = 0$, $i = p+2, \ldots, \nu+1$ are continuous in τ. Hence if $\tau \geq 0$ is sufficiently small, then

$$|\lambda_i(\tau)| < 1, \quad i = p+2, \ldots, \nu+1.$$

Substituting (44) in (43) and equating terms of order 1 in τ, yields

$$\alpha_i = \frac{a}{p+1}[b_{-1}\omega_i + b_0 + \ldots + b_\nu\omega_i^{-\nu}], \quad i = 1, \ldots, p+1. \tag{45}$$

Hence

$$\lambda_i^{p+1}(\tau) = \omega_i^{p+1} + a\tau[b_{-1}\omega_i^{p+1} + b_0\omega_i^p + \ldots + b_\nu\omega_i^{p-\nu}] + O(\tau^2)$$

and since $\omega_i^{p+1} = 1$,

$$|\lambda_i^{p+1}(\tau)|^2 = 1 + 2\tau a \operatorname{Re}[b_{-1} + b_0\omega_i^p + \ldots + b_\nu\omega_i^{p-\nu}] + O(\tau^2).$$

We say that a particular discretization method is *stable* if, on application to an asymptotically stable scalar differential equation (42), the resulting discrete time system is also asymptotically stable for sufficiently small τ. Thus the discretization method (41) is stable if and only if all the roots of (43) lie in $\mathbb{D}$, for every $a < 0$ and τ sufficiently small $0 < \tau \leq \delta(a)$. So a sufficient condition is

$$\operatorname{Re}[b_{-1} + b_0\omega_i^p + \ldots + b_\nu\omega_i^{p-\nu}] > 0, \quad i = 1, \ldots, p+1, \tag{46}$$

and a necessary condition is

$$\mathrm{Re}[b_{-1} + b_0\omega_i^p + \ldots + b_\nu \omega_i^{p-\nu}] \geq 0, \quad i = 1, \ldots, p+1.$$

In the case of equality higher order approximation of $\lambda_i(\tau)$ must be considered to determine whether or not the discretization method (41) is stable.
The case $p > \nu$ leads to the same conclusions and is slightly easier to analyze since when $\tau = 0$ there are no roots at 0. Now let us apply the results to some of the integration schemes introduced in Section 2.5.

Example 3.3.27. (Euler's method). In this case (see (2.5.25)), $p = 0$, $\nu = 0$, $b_{-1} = 0$, $b_0 = 1$, so (46) holds and Euler's method is stable. □

Example 3.3.28. (Runge-Kutta method). This is a single step method with $p = 0$, $\nu = 0$, $b_{-1} = 0$, $b_0 = 1 + (1/2)\tau a + (1/6)\tau^2 a^2 + (1/24)\tau^3 a^3$ (see Example 2.5.12), so (46) holds and the Runge-Kutta method is stable. □

Example 3.3.29. (Midpoint method). For this method (see Example 2.5.11) $p = 1$, $\nu = 0$, $b_{-1} = 0$, $b_0 = 2$. So $\omega_1 = +1$, $\omega_2 = -1$ and

$$\mathrm{Re}[b_0\omega_1] > 0 \text{ but } \mathrm{Re}[b_0\omega_2] < 0.$$

Hence the midpoint method is unstable. □

Example 3.3.30. (Adams-Bashforth methods). For these methods (see Example 2.5.13), $p = 0$, $b_{-1} = 0$ and some typical values of b_i, $i = 0, \ldots, \nu$ are given in Table 2.5.11. The stability condition (46) is $b_0 + b_1 + \ldots + b_\nu > 0$. Note that the values of b_i given in Table 2.5.11 all have the property that $\sum_{i=0}^{\nu} b_i = 1$, so the Adams-Bashforth methods are stable. □

Example 3.3.31. (Milne's method). The implicit corrector of Milne's method (see Example 2.5.14) applied to the scalar differential equation (42), yields

$$x^\tau(k+1) = x^\tau(k-1) + (a\tau/3)[x^\tau(k+1) + 4x^\tau(k) + x^\tau(k-1)].$$

Hence $p = 1$, $\nu = 1$, $b_{-1} = 1/3$, $b_0 = 4/3$, $b_1 = 1/3$. So $\omega_1 = 1$, $\omega_2 = -1$ and substitution in the left hand side of (46) gives

$$\mathrm{Re}[b_{-1} + b_0\omega_1 + b_1] = 2 > 0, \quad \text{but} \quad \mathrm{Re}[b_{-1} + b_0\omega_2 + b_1] = -2/3 < 0.$$

So the corrector of Milne's method is unstable and this will be the case for the predictor-corrector algorithm as well. □

These results seem to contradict some of the convergence properties of the above schemes as described in Section 2.5. However, the important thing to remember is that convergence is defined relative to a *finite* interval $[0, b]$ whereas stability is a requirement on the asymptotic behaviour as $t \to \infty$. We illustrate this distinction by the following example.

Example 3.3.32. (Instability of the midpoint rule). Applying the midpoint rule to the scalar differential equation (42) yields the difference equation

$$x^\tau(k+1) = x^\tau(k-1) + 2\tau a x^\tau(k), \quad k \in \mathbb{N} \tag{47}$$

Assume $a < 0$ so that (42) is asymptotically stable. The eigenvalues of the second order system (47) are given by

$$\lambda_1(\tau) = a\tau + \sqrt{1 + a^2\tau^2} = 1 + a\tau + O(\tau^2), \quad \lambda_2(\tau) = a\tau - \sqrt{1 + a^2\tau^2} = -1 + a\tau + O(\tau^2).$$

Every solution of (47) can be represented in the form

$$x^\tau(k) = c_1\lambda_1^k(\tau) + c_2\lambda_2^k(\tau). \tag{48}$$

If $\tau = t/k$, with $t > 0$ fixed arbitrarily, then the first eigenmotion

$$x^\tau(k) = \lambda_1(\tau)^k x_0 = (1 + at/k + O(t^2/k^2)\,)^k x_0 \;\rightarrow\; e^{at}x_0 \quad \text{as} \quad k \to \infty.$$

Hence, on any compact interval $[0, b]$, this eigenmotion of (47) generated by the initial conditions $x^\tau(0) = x_0$, $x^\tau(1) = \lambda_1(\tau)x_0$ yields a uniform approximation of the eigenmotion $e^{at}x_0$ of (42) generated by $x(0) = x_0$. Moreover the eigenmotion $x^\tau(k) = \lambda_1(\tau)^k x_0$ tends to 0 for $k \to \infty$, as does $e^{at}x_0$. However, the discretization (47) has a second eigenmotion

$$x^\tau(k) = \lambda_2(\tau)^k x_0 = (-1 + a\tau + O(\tau^2)\,)^k, \quad k \in \mathbb{N}$$

which is an *unbounded* oscillation. This eigenmotion of (47) is called *spurious* or *parasitic* since it does not correspond to a solution of the differential equation (42). Any deviation from the initial conditions $x^\tau(0) = x_0$, $x^\tau(1) = \lambda_1(\tau)x_0$ or any rounding error will excite this spurious eigenmotion and then, for any given $\tau > 0$, this eigenmotion will completely dominate the true solution after some time. This is illustrated in Table 3.3.3 where we apply the midpoint rule to

$$\dot{x}(t) = -x(t), \quad x(0) = 1. \tag{49}$$

We started the algorithm with the exact value $x^\tau(0) = 1$ and an order 2 approximation of

t	TRUE SOL'N	ERRORS
$k\tau$	$x(k\tau)$	
0.0	1.00000000	0.0000000
0.5	0.60653066	-0.0000003
1.0	0.36787944	-0.0000006
1.5	0.22313016	-0.0000011
2.0	0.13533528	-0.0000019
2.5	0.08208500	-0.0000031

t	TRUE SOL'N	ERRORS
$k\tau$	$x(k\tau)$	
5.0	0.00673795	-0.0000371
7.5	0.00055308	-0.0004520
10.0	0.00004540	-0.0055066
12.5	0.00000373	-0.0670841
15.0	0.00000031	-0.8172517
17.5	0.00000003	-9.9561596

Table 3.3.3: Errors of the midpoint rule applied to (49) with $\tau = 0.001$

the corresponding value at time τ : $x^\tau(1) = 1 - \tau$. Note that we have a good approximation of the true solution for t in the range $[0, 2.5]$ because of the relatively small stepsize $\tau = 0.001$. However, for values $t \geq 15$ the parasitic oscillations excited by the initial errors and by rounding errors become so strong that any correlation between the true and the "approximate" solutions is lost. □

The previous example illustrates the general problem. Suppose we apply a linear multistep method of the form (41) to an initial value problem $\dot{x} = Ax$, $x(0) = x_0$. If the differential system is n-dimensional and $\nu \geq 1$ or $p \geq 1$ then the dimension of the state space X of the corresponding discrete time system $x(t+1) = A_\tau x(t)$ is higher, namely

$$\dim X = n \cdot (\max\{\nu, p\} + 1). \tag{50}$$

Only n of the $n \cdot (\max\{\nu, p\} + 1)$ eigenvalues of A_τ (counting multiplicities) approximate eigenvalues of A, all the others correspond to parasitic eigenmotions of the discrete time system introduced by the multistep method. Thus a crucial question is whether or not these parasitic eigenmotions are tending to zero with an appropriate decay rate as $t \to \infty$. *Numerically unstable* integration methods of the form (41) generate unstable discrete time systems (27) when applied to certain asymptotically stable differentiable systems (27). Although these integration methods may be very efficient for the solution of initial value problems *on fixed compact intervals* they are not suitable for the approximation of differentiable *systems* by discrete time *systems* (see Section 2.5).

3.3.4 Liapunov Functions for Time-Varying Linear Systems

In this subsection we return to time-varying linear systems of the form (1). The time domain T is either an interval in $\mathbb{R}$ unbounded to the right or an interval in $\mathbb{Z}$ unbounded to the right. For linear systems it is natural to choose (time–varying) quadratic forms $x \mapsto V(t, x) = \langle x, P(t)x \rangle$, $t \in T$, as possible candidates for Liapunov functions. Here we characterize stability properties of (1) in terms of these quadratic Liapunov functions. In contrast to the previous section (where we assumed Liapunov functions to be given) we develop a systematic construction procedure. At the end of the subsection we will see that quadratic Liapunov functions provide a tool for deriving stability properties of a given nonlinear system trajectory from stability properties of the associated linearized model. Thus we will use Liapunov's direct method in order to prove the validity of Liapunov's indirect method.

Throughout the subsection we assume $P(t)$, $t \in T$ is symmetric if $\mathbb{K} = \mathbb{R}$ and Hermitian if $\mathbb{K} = \mathbb{C}$. Moreover in the continuous time case we suppose that $P(\cdot) : T \mapsto \mathcal{H}_n(\mathbb{K})$ is continuous and piecewise continuously differentiable, i.e. $P(\cdot) \in PC^1(T; \mathcal{H}_n(\mathbb{K}))$[1]. We do not assume $P(\cdot)$ to be continuously differentiable since our construction process will only yield piecewise continuously differentiable $P(\cdot)$ if $A(\cdot) \in PC(T; \mathbb{K}^{n \times n})$ has jump points.

Now consider

$$V(t, x) = \langle x, P(t)x \rangle \qquad (t, x) \in T \times \mathbb{K}^n \tag{51}$$

as a candidate for a Liapunov function for the linear system (1). In the continuous time case the derivative of V along the flow of (1a) is defined by

$$\begin{aligned} \dot{V}(t, x) &= \langle x, \dot{P}(t)x \rangle + \langle A(t)x, P(t)x \rangle + \langle x, P(t)A(t)x \rangle \\ &= \langle x, (\dot{P}(t) + A(t)^* P(t) + P(t)A(t))x \rangle, \quad (t, x) \in T \times \mathbb{K}^n \end{aligned} \tag{52}$$

[1] This means that the derivative $\dot{P}(t)$ exists for all $t \in T \setminus S$ where $S \subset T$ is a subset without accumulation point in $\mathbb{R}$ and the limit $\lim_{t \downarrow s} \dot{P}(t)$ exists at every $s \in S$. Extending $\dot{P}(\cdot)$ by $\dot{P}(s) = \lim_{t \downarrow s} \dot{P}(t)$ to all of T, we obtain a piecewise continuous and right continuous matrix function $\dot{P}(\cdot) : T \to \mathcal{H}_n(\mathbb{K})$.

where $\dot{P}(t)$ is defined for all $t \in T$ as in the footnote. In the discrete time case

$$\begin{aligned} \dot{V}(t,x) &= \langle A(t)x, P(t+1)A(t)x\rangle - \langle x, P(t)x\rangle \\ &= \langle x, (A(t)^*P(t+1)A(t) - P(t))x\rangle\,, \quad (t,x) \in T \times \mathbb{K}^n\,. \end{aligned} \tag{53}$$

Suppose we define a matrix $Q(t) \in \mathbb{K}^{n\times n}$, $t \in T$ by

$$\dot{P}(t) + A(t)^*P(t) + P(t)A(t) + Q(t) = 0\,, \quad t \in T \tag{54a}$$

$$A(t)^*P(t+1)A(t) - P(t) + Q(t) = 0\,, \quad t \in T\,. \tag{54b}$$

Then in the continuous time case $Q(\cdot) \in PC(T; \mathcal{H}_n(\mathbb{K}))$ and in the discrete time case $Q(\cdot) = (Q(t))_{t\in T} \in \mathcal{H}_n(\mathbb{K})^T$, i.e. $Q(\cdot)$ is a sequence in $\mathcal{H}_n(\mathbb{K})$ defined on T. In both cases

$$\dot{V}(t,x) = -\langle x, Q(t)x\rangle\,, \quad (t,x) \in T \times \mathbb{K}^n\,. \tag{55}$$

As a counterpart of Theorem 3.2.17 for quadratic Liapunov functions we have

Theorem 3.3.33. *Suppose that $P(\cdot) \in PC^1(T; \mathcal{H}_n(\mathbb{K}))$ and $Q(\cdot) \in PC(T; \mathcal{H}_n(\mathbb{K}))$ (resp. $P(\cdot)$, $Q(\cdot) \in \mathcal{H}_n(\mathbb{K})^T$) satisfy* (54). *If $\alpha_1, \alpha_2, \alpha_3 > 0$, then*

(i) $\forall t \in T : P(t) \succeq \alpha_1 I_n,\ Q(t) \succeq 0 \quad \Rightarrow \quad$ *stability of* (1) *at any time* $t_0 \in T$.

(ii) $\forall t \in T : \alpha_2 I_n \succeq P(t) \succeq \alpha_1 I_n\,,\ Q(t) \succeq 0 \quad \Rightarrow \quad$ *uniform stability of* (1) *on* T.

(iii) $\forall t \in T : \alpha_2 I_n \succeq P(t) \succeq \alpha_1 I_n\,,\ Q(t) \succeq \alpha_3 I_n \Rightarrow$ *uniform asymptotic stability of* (1).

Proof: In the discrete time case the theorem is a specialization of Theorem 3.2.17 using (51) as a Liapunov function. However, for the continuous time case, V will not, in general be a Liapunov function in the sense of Definition 3.2.16 since V may not be continuously differentiable on $T \times \mathbb{K}^n$. But $t \mapsto V(t, x(t)) = \langle x(t), P(t)x(t)\rangle$ is continuous and piecewise continuously differentiable for trajectories $x(\cdot)$ of (1). By (52), for all $t \in T$ where $x(\cdot)$ and $P(\cdot)$ are both differentiable, we have

$$\frac{dV}{dt}(t, x(t)) = \langle \dot{x}(t), P(t)x(t)\rangle + \langle x(t), \dot{P}(t)x(t)\rangle + \langle x(t), P(t)\dot{x}(t)\rangle = \dot{V}(t, x(t)). \tag{56}$$

Hence if the premises in (i) are satisfied, V is a generalized Liapunov function for (1) and then (i) follows from Theorem 3.2.7. In a similar way (ii) and (iii) follow since V is bounded in the sense of (5) (and strictly decreasing along the flow of (1)) if the premises in (ii) (resp. (iii)) hold. □

For quadratic functions the instability Theorem 3.2.22 specializes to the following result.

Theorem 3.3.34. *Suppose that $P(\cdot) \in PC^1(T; \mathcal{H}_n(\mathbb{K}))$ and $Q(\cdot) \in PC(T; \mathcal{H}_n(\mathbb{K}))$ (resp. $P(\cdot), Q(\cdot) \in \mathcal{H}_n(\mathbb{K})^T$) satisfy* (54). *If there exists $(t_0, x^0) \in T \times \mathbb{K}^n$ and positive constants α_3, α_2 such that $\langle x^0, P(t_0)x^0\rangle < 0$ and for all $t \in T_{t_0}$, $x \in \mathbb{K}^n$*

$$\langle x, P(t)x\rangle < 0 \quad \Rightarrow \quad \langle x, Q(t)x\rangle \geq \alpha_3 \|x\|^2 \quad \textit{and} \quad |\langle x, P(t)x\rangle| \leq \alpha_2 \|x\|^2$$

then (1) *is unstable at time* $t_0 \in T$.

The proof is set as Ex. 9.

Example 3.3.35. The damped Mathieu equation is of the form

$$\ddot{y} + 2\zeta\dot{y} + (a - 2r\cos 2t)y = 0\,, \quad t \geq 0$$

where $\zeta > 0$, $a > 0$, $r \in \mathbb{R}$ are constants. If $x_1 = y$, $x_2 = \dot{y}$ we obtain the state space system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a + 2r\cos 2t & -2\zeta \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} =: A(t)x(t)\,, \quad t \geq 0\,. \tag{57}$$

Consider the matrix function

$$P(t) = \begin{bmatrix} 2\rho\zeta^2 + a - 2r\cos 2t & \rho\zeta \\ \rho\zeta & 1 \end{bmatrix}, \quad t \geq 0$$

where ρ is constant. A straight forward calculation yields

$$\dot{P}(t) + A(t)^*P(t) + P(t)A(t) = \begin{bmatrix} -2\zeta\rho(a - 2r\cos 2t) + 4r\sin 2t & 0 \\ 0 & -\zeta(4 - 2\rho) \end{bmatrix}.$$

So $Q(t)$ as defined by (54a) is

$$Q(t) = \begin{bmatrix} 2\zeta\rho(a - 2r\cos 2t) - 4r\sin 2t & 0 \\ 0 & \zeta(4 - 2\rho) \end{bmatrix}, \quad t \geq 0\,.$$

There exist positive constants α_1, α_2, α_3 such that $\alpha_2 I_2 \succeq P(t) \succeq \alpha_1 I_2$, $Q(t) \succeq \alpha_3 I_2$, $t \geq 0$

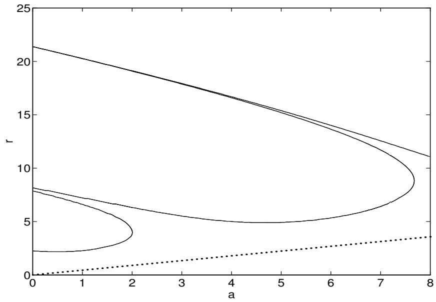

Figure 3.3.4: Stability domain for the Mathieu equation

provided

$$0 < \rho < 2\,, \quad 2\rho\zeta^2 + a - 2r\cos 2t > \rho^2\zeta^2\,, \quad 2\rho\zeta(a - 2r\cos 2t) - 4r\sin 2t > 0\,, \quad t \geq 0\,.$$

And these inequalities will hold if the following time–invariant inequalities are satisfied:

$$0 < \rho < 2\,, \quad \rho(2 - \rho)\zeta^2 + a > 2|r|\,, \quad \rho\zeta a > 2|r|(1 + \rho^2\zeta^2)^{1/2}\,. \tag{58}$$

Now suppose $\zeta a > |r|\,(1 + 4\zeta^2)^{1/2}$ then by choosing ρ close to 2 it can be shown that (58) holds and hence by Theorem 3.3.33 the time-varying system (57) is uniformly exponentially stable. For $\zeta = 1$ the stability domain determined by this inequality is shown in Figure 3.3.4 (below the dotted line) together with the actual stability boundaries (below the continuous lines). □

In order to prepare the ground for Liapunov's indirect method we now seek a partial converse to statement (iii) in Theorem 3.3.33. For this the following lemma is useful.

Lemma 3.3.36. *Suppose that $A(\cdot)$ generates a uniformly exponentially stable evolution operator $\Phi(\cdot,\cdot)$. Given a bounded $Q(\cdot) \in PC(T;\mathcal{H}_n(\mathbb{K}))$ (resp. bounded $Q(\cdot) \in \mathcal{H}_n(\mathbb{K})^T$), the only bounded $P(\cdot) \in PC^1(T;\mathcal{H}_n(\mathbb{K}))$ (resp. bounded $P(\cdot) \in \mathcal{H}_n(\mathbb{K})^T$) which solves* (54) *is*

$$P(t) = \int_t^\infty \Phi(s,t)^* Q(s)\Phi(s,t)ds\,, \quad t \in T, \tag{59a}$$

$$P(t) = \sum_{s=t}^{\infty} \Phi(s,t)^* Q(s)\Phi(s,t)\,, \qquad t \in T\,. \tag{59b}$$

Proof: The proof is for the continuous time case. Suppose $P(\cdot) \in PC^1(T;\mathcal{H}_n(\mathbb{K}))$ is a bounded solution of (54a) then

$$\begin{aligned} \frac{\partial}{\partial s}\left(\Phi(s,t)^* P(s)\Phi(s,t)\right) &= \Phi(s,t)^*\Big[\dot{P}(s) + A(s)^*P(s) + P(s)A(s)\Big]\Phi(s,t) \\ &= -\Phi(s,t)^*Q(s)\Phi(s,t)\,, \quad a.e.\ s > t,\ t \in T. \end{aligned} \tag{60}$$

By assumption there exist constants $p, q, M > 0$ and $\omega < 0$ such that $\|P(s)\| \le p$ and

$$\|Q(s)\| \le q\,, \quad \|\Phi(s,t)\| \le Me^{\omega(s-t)}\,, \quad s \ge t,\ t \in T. \tag{61}$$

So we may integrate (60) on $[t,\infty)$, $t \in T$ to obtain (59a). It remains to show that $P(\cdot)$ defined by (59a) is a bounded solution of (54a) on T. Since (61) holds, $P(\cdot)$ is well defined by (59a), Hermitian and bounded on T (see (62a)). Now just as in the proof of Lemma 3.3.4, we have

$$\frac{\partial \Phi(s,t)}{\partial t} = -\Phi(s,t)A(t) \text{ for a.e. } s > t.$$

Differentiating the integral in (59a)

$$\begin{aligned} \dot{P}(t) &= -Q(t) - A(t)^* \int_t^\infty \Phi(s,t)^*Q(s)\Phi(s,t)ds - \int_t^\infty \Phi(s,t)^*Q(s)\Phi(s,t)A(t)ds \\ &= -Q(t) - A(t)^*P(t) - P(t)A(t)\,, \quad a.e.\ t \in T. \end{aligned}$$

This shows that $P(\cdot) \in PC^1(T;\mathcal{H}_n(\mathbb{K}))$ and solves (54a). □

Note that if (61) holds and $P(\cdot)$ is given by (59) then $P(\cdot)$ is bounded by

$$\|P(t)\| \;\le\; M^2q\int_t^\infty e^{2\omega(s-t)}ds = M^2q/(-2\omega)\,, \quad t \in T\,, \tag{62a}$$

$$\|P(t)\| \;\le\; M^2q\sum_{s=t}^{\infty} e^{2\omega(s-t)} = M^2q/(1-e^{2\omega})\,, \quad t \in T\,. \tag{62b}$$

$Q(t)$ needs not necessarily be positive definite in order to conclude uniform asymptotic stability of (1) via the Liapunov function (51). Suppose that $Q(t) = C(t)^*C(t)$, where $C(\cdot) \in PC(T; \mathbb{K}^{p\times n})$ (resp. $C(\cdot) \in (\mathbb{K}^{p\times n})^T$). We will need an extra assumption which is expressed in terms of the matrices

$$\mathcal{Q}(t, t_0) = \int_{t_0}^{t} \Phi(s, t_0)^* C(s)^* C(s) \Phi(s, t_0) ds, \quad t \in T_{t_0},\ t_0 \in T \tag{63a}$$

$$\mathcal{Q}(t, t_0) = \sum_{s=t_0}^{t-1} \Phi(s, t_0)^* C(s)^* C(s) \Phi(s, t_0), \quad t \in T_{t_0},\ t_0 \in T. \tag{63b}$$

We will say that $(A(\cdot), C(\cdot))$ is *uniformly observable* on T (see Volume II) if there exist constants $\tau > 0, c > 0$ such that

$$\mathcal{Q}(t_0 + \tau, t_0) \succeq cI_n, \quad t_0 \in T. \tag{64}$$

Clearly in this case $\mathcal{Q}(t_1, t_0) \succeq cI_n$ for all $t_0, t_1 \in T$ such that $t_1 - t_0 \geq \tau$. In the next theorem we will show that condition (64) (instead of $Q(t) \succeq \alpha_3 I_n$, see Theorem 3.3.33 (iii)) suffices to obtain uniform asymptotic stability of (1) via the Liapunov function (51).

Remark 3.3.37. If $Q(t) = C(t)^* C(t)$ satisfies

$$Q(t) \succeq \alpha_3 I_n,\ t \in T, \tag{65}$$

then in the discrete time case $(A(\cdot), C(\cdot))$ is *uniformly observable* and we may choose the observability time $\tau = 1$. This need not be the case for continuous time systems (see Ex. 22). However, if $\underline{\beta}(A) > -\infty$ and (65) holds, then $(A(\cdot), C(\cdot))$ is *uniformly observable* and the observability time τ can be made arbitrarily small (with $c > 0$ chosen appropriately). In fact, we have, by assumption, the existence of $\varepsilon > 0, \omega < 0$ such that

$$\Phi(t, t_0)^* \Phi(t, t_0) \succeq \varepsilon e^{\omega(t - t_0)} I_n, \quad t \geq t_0,\ t_0 \in T.$$

Hence it follows from (65) that for all $\tau > 0$,

$$\mathcal{Q}(t_0 + \tau, t_0) \succeq \alpha_3 \int_{t_0}^{t_0+\tau} \Phi(s, t_0)^* \Phi(s, t_0) ds \succeq c_\tau I_n, \quad t_0 \in T,$$

where $c_\tau = \varepsilon \alpha_3 (1 - e^{\omega\tau})/(-\omega)$. □

Theorem 3.3.38. *Suppose that $Q(t) = C(t)^* C(t)$ for a bounded $C(\cdot) \in PC(T; \mathbb{K}^{p\times n})$ (resp. bounded $C(\cdot) \in (\mathbb{K}^{p\times n})^T$) and $(A(\cdot), C(\cdot))$ is uniformly observable on T. Then the following are equivalent.*

(i) $A(\cdot)$ generates a uniformly exponentially stable evolution operator.

(ii) There exists a solution $P(\cdot) \in PC^1(T; \mathcal{H}_n(\mathbb{K}))$ (resp. $P(\cdot) \in \mathcal{H}_n(\mathbb{K})^T$) of (54) such that $\alpha_2 I_n \succeq P(t) \succeq \alpha_1 I_n$, $t \in T$, for some $\alpha_1, \alpha_2 > 0$.

(iii) There exists a bounded positive definite solution $P(\cdot) \in PC^1(T; \mathcal{H}_n(\mathbb{K}))$ (resp. bounded positive definite solution $P(\cdot) \in \mathcal{H}_n(\mathbb{K})^T$) of (54).

Proof: The proof is for the continuous time case.
(i) ⇒ (ii) Suppose that (1a) is uniformly exponentially stable, then by Lemma 3.3.36

$$P(t) = \int_t^\infty \Phi(s,t)^* C(s)^* C(s) \Phi(s,t) ds \quad \succeq 0 \tag{66}$$

satisfies (54a) and hence satisfies (60). An upper bound for $P(t)$ is given by (62a). (60) implies that for $t_1 \geq t_0$, $t_0 \in T$

$$0 \preceq \Phi(t_1,t_0)^* P(t_1) \Phi(t_1,t_0) = P(t_0) - \int_{t_0}^{t_1} \Phi(s,t_0)^* C(s)^* C(s) \Phi(s,t_0) ds\,. \tag{67}$$

By the observability assumption there exists $\tau > 0$ satisfying (64) where $\mathcal{Q}(t,t_0)$ is defined by (63a). Hence (ii) follows from

$$P(t_0) \succeq \int_{t_0}^{t_1} \Phi(s,t_0)^* C(s)^* C(s) \Phi(s,t_0) ds \succeq cI_n\,, \quad t_1 \geq t_0 + \tau,\ \ t_0 \in T. \tag{68}$$

(ii) ⇒ (iii) is trivial.
(iii) ⇒ (i) Suppose (iii) and let $v(t) = \langle x(t), P(t)x(t)\rangle$ where $x(t) = \Phi(t,t_0)x^0$, $(t_0,x^0) \in T \times \mathbb{K}^n$. Then $\dot v(t) = -\langle x(t), C(t)^* C(t) x(t)\rangle$ (see (56) and (55)) and integrating from t_0 to $t_1 > t_0$ yields (67) and hence (68). So $\alpha_2 I_n \succeq P(t) \succeq cI_n$, $t \in T$ for some $\alpha_2 > 0$ and $c > 0$ as in (64). Using again the assumption of uniform observability, we have

$$v(t+\tau) - v(t) = -\langle x(t), \mathcal{Q}(t+\tau,t)x(t)\rangle \leq -c\|x(t)\|^2 \leq -(c/\alpha_2)v(t)\,,\ \ t \geq t_0,\ t_0 \in T.$$

Setting $\tilde c = c/\alpha_2$, we obtain $v(t+\tau) \leq (1-\tilde c)v(t)$, $t \geq t_0$ and hence $\tilde c < 1$ and

$$0 \leq v(t+k\tau) \leq (1-\tilde c)^k v(t)\,, \qquad k \in \mathbb{N}\,,\ \ t \geq t_0\,,\ t_0 \in T.$$

For every $t = t_0 + k\tau + r$ where $k \in \mathbb{N}$, $r \in [0,\tau)$ we get

$$v(t) \leq (1-\tilde c)^k v(t_0+r) \leq (1-\tilde c)^{(t-t_0-\tau)/\tau} \langle x(t_0+r), P(t_0+r)x(t_0+r)\rangle\,.$$

By Theorem 3.3.33 the system (1a) is uniformly stable on T and hence there exists $M > 0$ such that $\|\Phi(t,t_0)x^0\| \leq M\|x^0\|$, $t > t_0$, $t_0 \in T$. So

$$\langle \Phi(t,t_0)x^0, P(t)\Phi(t,t_0)x^0\rangle = v(t) \leq M^2\alpha_2(1-\tilde c)^{-1}(1-\tilde c)^{(t-t_0)/\tau}\|x^0\|^2\,.$$

Now using the lower bound of $P(t)$ and choosing $\omega = (2\tau)^{-1}\ln(1-\tilde c) < 0$, we conclude

$$\|\Phi(t,t_0)\| \leq M([\tilde c(1-\tilde c)])^{-1/2} e^{\omega(t-t_0)}\,, \quad t \geq t_0,\ \ t_0 \in T.$$

Hence (1) is uniformly exponentially stable on T. □

We now derive a necessary and sufficient *instability* criterion in terms of quadratic Liapunov functions.

Theorem 3.3.39. *Let $t_0 \in T$ and suppose that $Q(t) = C(t)^*C(t)$, for a bounded $C(\cdot) \in PC(T_{t_0};\mathbb{K}^{p\times n})$ (resp. bounded $C(\cdot) \in (\mathbb{K}^{p\times n})^{T_{t_0}}$), $(A(\cdot),C(\cdot))$ is uniformly observable on T_{t_0} and there exists a bounded $P(\cdot) \in PC^1(T_{t_0};\mathcal{H}_n(\mathbb{K}))$ (resp. bounded $P(\cdot) \in \mathcal{H}_n(\mathbb{K})^{T_{t_0}}$) which solves (54). Then the following are equivalent* *(i)* (1) *is not exponentially stable at time* t_0*.*

(ii) (1) *is unstable at* t_0*.*

(iii) There exist $\tau > 0$, $\omega > 0$, $M_\omega > 0$ *and* $x^0 \in \mathbb{K}^n$, $x^0 \neq 0$ *such that*

$$\|\Phi(t_0 + k\tau, t_0)x^0\| \geq M_\omega e^{k\tau\omega}\|x^0\|, \quad k \in \mathbb{N}.$$

(iv) There exist $x^0 \in \mathbb{K}^n$ *such that* $\langle x^0, P(t_0)x^0\rangle < 0$*.*

Moreover, when this is the case the Bohl exponent $\overline{\beta}(\Phi) > 0$*.*

Proof: The proof is for the continuous time case.
(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is obvious.
(i) $\Rightarrow$ (iv): Suppose $\langle x, P(t)x\rangle \geq 0$ for all $(t, x) \in T_{t_0} \times \mathbb{K}^n$ then by Theorem 3.3.38 $A(\cdot)$ generates an exponentially stable evolution operator at time t_0.
(iv) $\Rightarrow$ (iii): Suppose that $x^0 \in \mathbb{K}^n$ is such that $\langle x^0, P(t_0)x^0\rangle < 0$. Setting $t_1 = t_0 + k\tau$, $k \in \mathbb{N}$ in (67) (where τ satisfies the uniform observability condition (64)) we obtain from (67)

$$\begin{aligned}\langle x^0, P(t_0)x^0\rangle &- \langle \Phi(t_0 + k\tau, t_0)x^0, P(t_0 + k\tau)\Phi(t_0 + k\tau, t_0)x^0\rangle \\ &= \int_{t_0}^{t_0+k\tau} \|C(s)\Phi(s, t_0)x^0\|^2 ds = \sum_{j=1}^{k} \int_{t_0+(j-1)\tau}^{t_0+j\tau} \|C(s)\Phi(s, t_0)x^0\|^2 ds \\ &= \sum_{j=1}^{k} \int_{t_0+(j-1)\tau}^{t_0+j\tau} \|C(s)\Phi(s, t_0 + (j-1)\tau)\Phi(t_0 + (j-1)\tau, t_0)x^0\|^2 ds \\ &\geq c\sum_{j=1}^{k} \|\Phi(t_0 + (j-1)\tau, t_0)x^0\|^2 \,. \end{aligned} \tag{69}$$

Assume $\langle x^0, P(t_0)x^0\rangle = -\alpha\|x^0\|^2$, $\alpha > 0$, $\|P(t)\| \leq \alpha_2$ for all $t \in T_{t_0}$ and set $r_k = \sum_{j=1}^{k} \|\Phi(t_0 + (j-1)\tau, t_0)x^0\|^2$, then $r_1 = \|x^0\|^2$ and using (69)

$$cr_k + \alpha\|x^0\|^2 \leq -\langle \Phi(t_0+k\tau, t_0)x^0, P(t_0+k\tau)\Phi(t_0+k\tau, t_0)x^0\rangle \leq \alpha_2\|\Phi(t_0+k\tau, t_0)x^0\|^2.$$

Hence

$$\alpha_2(r_{k+1} - r_k) = \alpha_2\|\Phi(t_0 + k\tau, t_0)x^0\|^2 \geq cr_k + \alpha\|x^0\|^2 \quad , \quad r_1 = \|x^0\|^2 \,. \tag{70}$$

So $r_{k+1} \geq (1 + c/\alpha_2)r_k + \alpha/\alpha_2\|x^0\|^2$. From which it is easy to see that

$$r_k \geq \left[(1 + c/\alpha_2)^{k-1}(1 + \alpha/c) - \alpha/c\right] \|x^0\|^2 \,.$$

Inserting this inequality in (70) we obtain

$$\alpha_2\|\Phi(t_0 + k\tau, t_0)x^0\|^2 \geq (1 + c/\alpha_2)^{k-1}(\alpha + c)\|x^0\|^2 = \alpha_2 M^2 e^{2k\omega\tau}\|x^0\|^2 \tag{71}$$

where $M = ((\alpha+c)/(\alpha_2+c))^{1/2}$, $\omega = (2\tau)^{-1}\ln(1+c/\alpha_2) > 0$. This proves (iii) and $\overline{\beta}(\Phi) \geq \omega > 0$. □

Remark 3.3.40. (i) The system (1) is not uniformly exponentially stable if and only if there exists a $t_0 \in T$ such that one of the conditions (ii)-(iv) is satisfied.
(ii) Since the in the continuous time case the Liapunov exponent $\overline{\alpha}(\Phi)$ is independent of t_0 we must have $\overline{\alpha}(\Phi) > 0$. The same conclusion also holds for the discrete time case provided $\det A(t) \neq 0$ for all $t \in T$ (this will be the case if $\underline{\beta}(\Phi) > -\infty$).
(iii) In comparison with the Stability Theorem 3.3.38 the Instability Theorem 3.3.39 is rather unsatisfactory since it assumes the existence of a bounded solution of (54). On the other hand Theorem 3.3.39 is surprising in that it shows that under its assumptions marginal stability cannot occur. More precisely we obtain as a consequence of Theorem 3.3.39 and Theorem 3.3.38: If $\overline{\beta}(\Phi) = 0$ and $Q(t) = C(t)^*C(t)$, where $C(\cdot) \in PC(T;\mathbb{K}^{p\times n})$ is bounded (resp. $C(\cdot) \in (\mathbb{K}^{p\times n})^T$ is bounded) then either there is no bounded $P(\cdot) \in PC^1(T_{t_0};\mathcal{H}_n(\mathbb{K}))$ (resp. $P(\cdot) \in \mathcal{H}_n(\mathbb{K})^{T_{t_0}}$) solving (54), for any $t_0 \in T$, or $(A(\cdot),C(\cdot))$ is not uniformly observable. □

We now describe *Liapunov's indirect method* of stability analysis which proceeds via linearization. Consider the nonlinear equations (2.27), namely

$$\dot{x}(t) = f(t,x(t)), \quad t \in T \tag{72a}$$
$$x(t+1) = f(t,x(t)), \quad t \in T. \tag{72b}$$

Let $\overline{x} \in X \subset \mathbb{K}^n$ be an equilibrium point of (72) and assume that f satisfies the conditions (A1), (A2) in Subsection 3.2.2. In addition we also require that

$$f(t,x) = A(t)(x-\overline{x}) + h(t,x-\overline{x}) \ , (t,x) \in T \times X \,. \tag{73}$$

where $A(\cdot) \in PC(T;\mathbb{K}^{n\times n})$ (resp. $A(\cdot) \in (\mathbb{K}^{n\times n})^T$) and for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\|h(t,x-\overline{x})\| \le \varepsilon\|x-\overline{x}\|, \quad (t,x) \in T \times B(\overline{x},\delta)\,. \tag{74}$$

Theorem 3.3.41 (Liapunov's indirect method). *Suppose that f satisfies* (73), (74), *$A(\cdot)$ generates a uniformly exponentially stable evolution operator and $(A(\cdot),I)$ is uniformly observable. Then the equilibrium point $\overline{x}$ is uniformly exponentially stable for the nonlinear system* (72)*. More precisely for $x^0 \in B(\overline{x},r)$, $r > 0$ sufficiently small, the solutions $\varphi(t;t_0,x^0)$ of* (72) *have infinite life time and there exist constants $M > 0, \omega < 0$ such that for all $t_0 \in T$*

$$x^0 \in B(\overline{x},r) \Longrightarrow \forall t \in T_{t_0} :\ \|\varphi(t;t_0,x^0)-\overline{x}\| \le Me^{\omega(t-t_0)}\|x^0-\overline{x}\|. \tag{75}$$

Proof: The proof is for the discrete time case. Let $Q(t) \equiv I_n$, then by Theorem 3.3.38 there exists a solution $P(t) \in \mathcal{H}_n(\mathbb{K})$, $t \in T$ of (54b) with $\alpha_2 I_n \succeq P(t) \succeq \alpha_1 I_n$, $t \in T$, for some $\alpha_1,\alpha_2 > 0$. Consider $V(t,x) = \langle x-\overline{x}, P(t)(x-\overline{x})\rangle$, $(t,x) \in T \times X$. Setting $\Delta x = x - \overline{x}$ we obtain by (73) for every $(t,x) \in T \times X$

$$\begin{aligned}
\dot{V}(t,x) &= V(t+1,f(t,x)) - V(t,x)\\
&= \langle A(t)\Delta x + h(t,\Delta x), P(t+1)(A(t)\Delta x + h(t,\Delta x))\rangle - \langle \Delta x, P(t)\Delta x\rangle\\
&= -\|\Delta x\|^2 + 2\,\mathrm{Re}\langle h(t,\Delta x), P(t+1)A(t)\Delta x\rangle + \langle h(\Delta x,t), P(t+1)h(t,\Delta x)\rangle.
\end{aligned}$$

So

$$\dot{V}(t,x) \le -\|\Delta x\|^2 + 2\|h(t,\Delta x)\|\,\|P(t+1)\|\,\|A(t)\|\,\|\Delta x\| + \|P(t+1)\|\,\|h(t,\Delta x)\|^2.$$

In the discrete time case $\|A(\cdot)\|$ is bounded on T by the uniform stability assumption, so we may choose $\varepsilon > 0$ sufficiently small to obtain

$$1 - 2\alpha_2\|A(t)\|\varepsilon - \alpha_2\varepsilon^2 \ge 1/2, \quad t \in T.$$

By (74), there exists a $\delta > 0$, such that $B(\overline{x},\delta) \subset X$ and

$$\|h(t,\Delta x)\| \le \varepsilon\|\Delta x\| \quad \text{for} \quad (t,x) \in T \times B(\overline{x},\delta).$$

Hence

$$\dot{V}(t,x) \le -(1/2)\|\Delta x\|^2 \quad \text{for} \quad (t,x) \in T \times B(\overline{x},\delta).$$

Setting $D = B(\overline{x},\delta)$ we may apply Corollary 3.2.20 with $p = 2$ to conclude that the equilibrium point $\overline{x}$ is uniformly exponentially stable and there exist constants $M > 0, \omega < 0$ such that (75) holds. □

Remark 3.3.42. (i) The formulation of Theorem 3.3.41 in terms of uniform exponential stability is essential and cannot be replaced by for example asymptotic stability (see *Bellman* (1953) [44] for a counter example).

(ii) By Remark 3.3.37 $(A(\cdot), I)$ is necessarily uniformly observable in the discrete time case and this will also hold in the continuous time case if $\underline{\beta}(\Phi) > -\infty$. □

In order to apply the previous theorems to the stability analysis of a non-constant *trajectory* of the nonlinear system (72) we proceed as described in Subsection 2.1.4. Suppose that $\tilde{x} : T_{t_0} \to X$ is a trajectory of (72) and that $(t, \tilde{x}(t) + x) \in X$ for all $t \in T_{t_0}$, $x \in B(0,\rho)$, $\rho > 0$. Let

$$g(t,x) = f(t,\tilde{x}(t)+x) - f(t,\tilde{x}(t)), \quad (t,x) \in T_{t_0} \times B(0,\rho).$$

Then $\overline{x} = 0$ is an equilibrium point of (72) with f replaced by g and $\overline{x} = 0$ is exponentially stable for this system if and only if $\tilde{x}(\cdot)$ is exponentially stable for (72). Now g will satisfy (73), (74) with $T = T_{t_0}$ (so that we may apply Theorem 3.3.41 to the equilibrium point $\overline{x} = 0$ of the system described by g) if and only if f is uniformly differentiable along the trajectory $\tilde{x}(\cdot)$ in the following sense

$$f(t,x) = f(t,\tilde{x}(t)) + A(t)(x - \tilde{x}(t)) + h(t, x - \tilde{x}(t)), \quad (t,x) \in T_{t_0} \times X\,. \tag{76}$$

where $A(\cdot) \in PC(T_{t_0};\mathbb{K}^{n\times n})$ (resp. $A(\cdot) \in (\mathbb{K}^{n\times n})^{T_{t_0}}$) and for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\|h(t, x - \tilde{x}(t))\| \le \varepsilon\|x - \tilde{x}(t)\|, \quad (t,z) \in T_{t_0} \times B(\tilde{x}(t),\delta)\,. \tag{77}$$

This condition will be satisfied if, for example, f is twice continuously differentiable with respect to x in an ε-neighbourhood of the integral curve $\{(t,\tilde{x}(t)); t \in T_{t_0}\}$, and its second derivative is bounded on this neighbourhood. This follows from the fact that

$$\|f(t,x) - f(t,\tilde{x}(t)) - f'(t,\tilde{x}(t))(x - \tilde{x}(t))\| \le (1/2)\|f''(t,\tilde{x}(t) + \theta(t)(x - \tilde{x}(t)))\|\|x - \tilde{x}(t)\|^2$$

where f' and f'' denote the first and the second derivative of f with respect to x and $\theta(t) \in [0,1]$.
In the following instability theorem we use a quadratic Liapunov function which is associated with a perturbation of the linearization (1) of (72). This widens the applicability of the theorem.

Theorem 3.3.43. *Assume that f satisfies (73), (74) and $t_0 \in T$. For some $r \geq 0$ let $A_r(t) = A(t) - rI_n$ (resp. $r \geq 1$, $A_r(t) = r^{-1}A(t)$), $t \in T$ and suppose the following hold*

(i) $(A_r(\cdot), I_n)$ is uniformly observable on T_{t_0}.

(ii) For $Q(t) \equiv I_n$ there exists a bounded $P_r(\cdot) \in PC^1(T_{t_0}; \mathcal{H}_n(\mathbb{K}))$ (resp. bounded $P_r(\cdot) \in \mathcal{H}_n(\mathbb{K})^{T_{t_0}}$) which solves (54) with $A(t)$ replaced by $A_r(t)$ on T_{t_0}.

Then, if $\dot{x}(t) = A_r(t)x(t)$ (resp. $x(t+1) = A_r(t)x(t)$) is unstable at time t_0, the equilibrium point $\overline{x}$ of the nonlinear system (72) will also be unstable at time t_0.

Proof: The proof is for the continuous time case. Suppose $\dot{x}(t) = A_r(t)x(t)$ is unstable at time t_0. By applying Theorem 3.3.39 with $Q(t) \equiv I_n$ we see that there exists $\tilde{x} \in X$, $\|\tilde{x}\| = 1$ such that $\langle \tilde{x}, P_r(t_0)\tilde{x}\rangle < 0$. Let $V(t,x) = \langle x - \overline{x}, P_r(t)(x - \overline{x})\rangle$ for $(t,x) \in T \times X$. Setting $\Delta x = x - \overline{x}$, the derivative of V along the flow of (72a) is given by

$$\begin{aligned}\dot{V}(t,x) &= \langle \Delta x, \dot{P}_r(t)\Delta x\rangle + \langle A(t)\Delta x + h(t,\Delta x), P_r(t)\Delta x\rangle + \langle \Delta x, P_r(t)(A(t)\Delta x + h(t,\Delta x))\rangle \\ &= -\|\Delta x\|^2 + 2r\langle \Delta x, P_r(t)\Delta x\rangle + 2\,\mathrm{Re}\langle h(t,\Delta x), P_r(t)\Delta x\rangle, \quad (t,x) \in T \times X.\end{aligned}$$

So

$$\dot{V}(t,x) \leq -\|\Delta x\|^2 + 2rV(t,x) + 2\|h(t,\Delta x)\|\,\|P_r(t)\|\,\|\Delta x\|.$$

By assumption there exists $\alpha_2 > 0$ such that $\|P_r(t)\| \leq \alpha_2$ for all $t \in T_{t_0}$. Choose $\varepsilon > 0$ such that $4\varepsilon\alpha_2 < 1$. By (74), there exists a $\delta > 0$, such that $B(\overline{x},\delta) \subset X$ and

$$\|h(t,\Delta x)\| \leq \varepsilon\|\Delta x\| \quad \text{for} \quad (t,x) \in T \times B(\overline{x},\delta).$$

Hence

$$\dot{V}(t,x) - 2rV(t,x) \leq -(1/2)\|\Delta x\|^2 \quad \text{for} \quad (t,x) \in T \times B(\overline{x},\delta). \tag{78}$$

First suppose that $r = 0$. Setting $D = B(\overline{x},\delta)$ and choosing $x^0 = \overline{x} + \rho\tilde{x}$ for any $\rho \in (0,\delta)$ we have an $x^0 \in B(\overline{x},\rho)$ with $V(t_0,x^0) < 0$. Moreover since $\dot{V}(t,x) \leq -(1/2)\|\Delta x\|^2$ and $|V(t,x)| \leq \alpha_2\|\Delta x\|^2$, for all $(t,x) \in T_{t_0} \times B(\overline{x},\delta)$, we may apply the Instability Theorem 3.2.22 to conclude that $\overline{x}$ is unstable at t_0 for (72a) .
Now suppose that $r > 0$ and assume by way of contradiction that $\overline{x}$ is stable for (72a) at time t_0. Then there exists a $\tilde{\delta} \in (0,\delta)$ such that

$$\|x - \overline{x}\| < \tilde{\delta} \Longrightarrow \|\varphi(t;t_0,x) - \overline{x}\| < \delta, \quad t \geq t_0.$$

For every $\rho \in (0,\tilde{\delta})$ we again choose $x^0 = \overline{x} + \rho\tilde{x}$, then $\|x^0 - \overline{x}\| = \rho$ and $V(t_0,x^0) < 0$. By (78) we have

$$\frac{d}{dt}\left[e^{-2r(t-t_0)}V(t,\varphi(t;t_0,x^0))\right] \leq -(1/2)e^{-2r(t-t_0)}\|\varphi(t;t_0,x^0) - \overline{x}\|^2 \leq 0, \quad t \geq t_0.$$

Hence, for $t \geq t_0$

$$V(t, \varphi(t; t_0, x^0) \leq e^{2r(t-t_0)} V(t_0, x^0) = -e^{2r(t-t_0)} |\langle \tilde{x}, P_r(t_0)\tilde{x}\rangle| \|x^0 - \overline{x}\|^2.$$

This contradicts the fact that $V(t,x)$ is bounded on $T \times B(\overline{x}, \delta)$. So $\overline{x}$ is unstable for (72a) at time t_0. □

One might think that if the linear system is unstable at time t_0, there will not exist any bounded solution $P_r(t)$ on T_{t_0} which solves (54) with $A(t)$ replaced by $A_r(t)$ and $Q(t) \equiv I_n$. In part this is suggested by the solution formulas (59) for the uniformly exponentially stable case. Note that even if the right hand side of this formula is well defined we cannot use the resulting $P_r(t)$ in applying the theorem since it is positive definite for all $t \in T_{t_0}$. Let us denote by $\Phi_r(\cdot,\cdot)$ the evolution operator generated by $A_r(\cdot)$, then integrating the equation (60) with the above replacements from t_0 to t, the unique solution with initial value $P_r(t_0)$ is

$$P_r(t) = \Phi_r(t_0,t)^* P_r(t_0)\Phi_r(t_0,t) - \int_{t_0}^{t} \Phi_r(s,t)^*\Phi_r(s,t)ds, \quad t \geq t_0 \,. \tag{79}$$

So in order to apply the theorem we have to seek a non-positive definite $P_r(t_0)$ for which the $P_r(\cdot)$ defined by (79) is bounded on T_{t_0}. Similar considerations apply in the discrete time case, see Ex. 14. We illustrate the continuous time case for the case $r = 0$ in the following simple example, see also Ex. 15.

Example 3.3.44. Consider the scalar system $\dot{x}(t) = a(t)x(t)$, $t \in \mathbb{R}$. First let us assume that $a(t) \equiv a > 0$, $t \in \mathbb{R}$, then the solution given by (59a) is not defined. However by (79) we have $p(t) = e^{-2a(t-t_0)}(p(t_0) + 1/2a) - 1/2a$, $t \geq t_0$. Choosing $p(t_0) = -1/2a$ we conclude from the above theorem that any nonlinear system satisfying (73), (74) for which $\dot{x}(t) = ax(t)$ is the linearization will be unstable at any time t_0.
Now suppose $a(t) = t$, $t \in \mathbb{R}$, then $\Phi(t,t_0) = e^{(t^2-t_0^2)/2}$ and so again the solution given by (59a) is not defined. But from (79) we get

$$p(t) = e^{t_0^2 - t^2} p(t_0) - \int_{t_0}^{t} e^{s^2-t^2} ds, \quad t \geq t_0.$$

Hence for $t \geq t_0 \geq 1/2$

$$e^{t_0^2-t^2}p(t_0) \geq p(t) \geq e^{t_0^2-t^2}p(t_0) - \int_{t_0}^{t} 2se^{s^2-t^2}ds = e^{t_0^2-t^2}p(t_0) - (1 - e^{t_0^2-t^2}).$$

So $p(t)$ is bounded for $t \geq t_0 \geq 1/2$ and since we may choose $p(1/2) < 0$ we conclude from the above theorem that any nonlinear system satisfying (73), (74) for which $\dot{x}(t) = tx(t)$ is the linearization will be unstable at any time $t_0 \geq 1/2$. □

3.3.5 Liapunov Functions for Time-Invariant Linear Systems

In this subsection we specialize the results of the previous one to the time–invariant case. The dynamic Liapunov equations (54) then reduce to linear matrix equations for which effective solution procedures are available. This enables us to construct Liapunov functions in an efficient way. Moreover we obtain a more satisfactory

instability criterion (Theorem 3.3.49 (iv)), and the application of Liapunov's indirect method is simplified by the fact that it only requires the differentiability of the right hand side of the nonlinear system at the equilibrium point in question.
In a time–invariant setting it is natural to assume that Q is constant and to require *time–invariant* solutions of (54). Then the dynamic Liapunov equations become static and take the form

$$A^*P + PA + Q = 0 \tag{80a}$$

$$A^*PA - P + Q = 0\,. \tag{80b}$$

In contrast to the dynamic Liapunov equations it is not clear whether these linear matrix equations have solutions. For the case where $\sigma(A) \subset \mathbb{C}_-$ (resp. $\sigma(A) \subset \mathbb{D}$) a solution could be constructed as in Lemma 3.3.36. However this would presuppose asymptotic stability. We need to prove the existence of solutions under more general conditions and with a view to later applications we do this by characterizing the eigenvalues of the Liapunov maps for a generalized version of the equations (80).

Proposition 3.3.45. *Suppose $A \in \mathbb{K}^{n\times n}$, $A_1 \in \mathbb{K}^{n_1\times n_1}$ and let $\mathbf{L}$ (resp. $\mathbf{L}^D$) be the associated generalized Liapunov operator*

$$\mathbf{L} : \mathbb{K}^{n_1\times n} \to \mathbb{K}^{n_1\times n}, \quad X \to \mathbf{L}(X) = A_1X + XA \tag{81a}$$

$$\mathbf{L}^D : \mathbb{K}^{n_1\times n} \to \mathbb{K}^{n_1\times n}, \quad X \to \mathbf{L}^D(X) = A_1XA - X. \tag{81b}$$

Then

$$\sigma(\mathbf{L}) = \{\mu_1 + \mu;\ \mu_1 \in \sigma(A_1),\ \mu \in \sigma(A)\} \tag{82a}$$

$$\sigma(\mathbf{L}^D) = \{\mu_1\mu - 1;\ \mu_1 \in \sigma(A_1),\ \mu \in \sigma(A)\}. \tag{82b}$$

In particular, $\mathbf{L}$, $\mathbf{L}^D : \mathbb{K}^{n_1\times n} \to \mathbb{K}^{n_1\times n}$ is a linear isomorphism if and only if

$$\mu_1 + \mu \neq 0 \quad (\textit{resp. } \mu_1\mu \neq 1), \qquad \mu_1 \in \sigma(A_1),\ \mu \in \sigma(A). \tag{83}$$

Proof: The proof is for the continuous time case. Suppose that $A_1x^1 = \mu_1x^1, x^1 \in \mathbb{C}^{n_1}, x^1 \neq 0$ and $xA = \mu x$, $x^\top \in \mathbb{C}^n, x \neq 0$. Then for $X = x^1x$, we have

$$\mathbf{L}(X) = A_1X + XA = A_1x^1x + x^1xA = (\mu_1 + \mu)x^1x = (\mu_1 + \mu)X.$$

Hence $\mu_1 + \mu \in \sigma(\mathbf{L})$, i.e. the inclusion $\subset$ in (82a). To prove the converse we transform A to Schur form. In Section 4.5 we will show that for $A \in \mathbb{K}^{n\times n}$, there exists a unitary matrix $U \in \mathbf{U}_n(\mathbb{C})$ such that U^*AU is in upper triangular complex Schur form, namely

$$U^*AU = S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ 0 & s_{22} & \cdots & s_{2n} \\ . & 0 & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & s_{nn} \end{bmatrix} \tag{84}$$

where each diagonal element is an eigenvalue of A. Now suppose that λ is an eigenvalue of $\mathbf{L}$ with eigenvector $X \in \mathbb{K}^{n_1\times n}, X \neq 0$. Then $A_1X + XA = \lambda X$ and multiplying on the right by U, we obtain

$$A_1XU + XUU^*AU = A_1XU + XUS = \lambda XU.$$

Defining $Z = [z^1\, z^2\, \ldots z^n] := XU$, $z^j \in \mathbb{C}^{n_1}, j \in \underline{n}$, then $Z \neq 0$ and

$$[A_1 + s_{jj} I_{n_1}] z^j = \lambda z^j - \sum_{i=1}^{j-1} s_{ij} z^i, \quad j \in \underline{n}\,. \tag{85}$$

Since $Z \neq 0$ there exists $j \in \underline{n}$, such that $z^j \neq 0$ and $z^k = 0, k < j$. But then from (85) $[A_1 + s_{jj} I_{n_1}] z^j = \lambda z^j$ and hence $\lambda - s_{jj} \in \sigma(A_1)$ and this completes the proof of (82). Since the linear map $\mathbf{L}$ is a vector space isomorphism if and only if $0 \notin \sigma(\mathbf{L})$ the second assertion follows. □

As a direct consequence of Proposition 3.3.45 the *generalized Liapunov equations*

$$A_1 P + PA + Q = 0\,, \quad (resp. \quad A_1 PA - P + Q = 0). \tag{86}$$

have unique solutions $P \in \mathbb{K}^{n_1 \times n}$ for every $Q \in \mathbb{K}^{n_1 \times n}$ if and only if condition (83) holds. In the present context the particular case where $A_1 = A^*$ is of special interest. If $Q = Q^*$ is Hermitian and P is a solution of (86) then P^* is also a solution of (86). Hence if (86) has a unique solution then the solution is necessarily Hermitian. This leads us to introduce the following *Liapunov operator* on the real vector space $\mathcal{H}_n(\mathbb{K})$ of Hermitian $n \times n$ matrices

$$\mathbf{L}_A : \mathcal{H}_n(\mathbb{K}) \to \mathcal{H}_n(\mathbb{K}), \quad X \mapsto A^* X + XA \quad (resp.\, \mathbf{L}_A^D : X \mapsto A^* XA - X). \tag{87}$$

As an immediate consequence of Proposition 3.3.45 we obtain

Corollary 3.3.46. *Suppose $A \in \mathbb{K}^{n \times n}$. The Liapunov operator $\mathbf{L}_A$ (resp. $\mathbf{L}_A^D$) is a linear bijection from $\mathcal{H}_n(\mathbb{K})$ onto itself if and only if*

$$\lambda + \overline{\mu} \neq 0, \quad (resp. \quad \lambda\, \overline{\mu} \neq 1) \quad \lambda, \mu \in \sigma(A). \tag{88}$$

In this (and only in this) case the algebraic Liapunov equation (80) *has a unique (Hermitian) solution for every $Q \in \mathcal{H}_n(\mathbb{K})$.*

If $\sigma(A) \subset \mathbb{C}_-$ (resp. $\sigma(A) \subset \mathbb{D}$), then by Lemma 3.3.36 we know that the solution of (80) is given by

$$P = \int_t^\infty e^{A^*(s-t)} Q e^{A(s-t)} ds \; = \int_0^\infty e^{A^* \rho} Q e^{A\rho} d\rho\,, \tag{89a}$$

$$P = \sum_{s=t}^\infty A^{*(s-t)} Q A^{s-t} \; = \sum_{\rho=0}^\infty A^{*\rho} Q A^\rho\,. \tag{89b}$$

Clearly, if $Q \succ 0$ then P defined by (89) is positive definite. Therefore

Corollary 3.3.47. *Suppose $A \in \mathbb{K}^{n \times n}$ and $\sigma(A) \subset \mathbb{C}_-$ (resp. $\sigma(A) \subset \mathbb{D}$). Then the Liapunov operator $\mathbf{L}_A\, (resp.\, \mathbf{L}_A^D) : \mathcal{H}_n(\mathbb{K}) \to \mathcal{H}_n(\mathbb{K})$ is invertible and $-\mathbf{L}_A^{-1}$ (resp. $-(\mathbf{L}_A^D)^{-1}$) is a positive operator from the vector space $\mathcal{H}_n(\mathbb{K})$ ordered by $\succeq$ into itself, i.e.*

$$Q \succ 0 \Rightarrow P = -\mathbf{L}_A^{-1}(Q) \succ 0, \quad (resp..-(\mathbf{L}_A^D)^{-1}(Q) \succ 0, \tag{90}$$

Remark 3.3.48. As a consequence of the next theorem the converse of Corollary 3.3.47 is also true. Hence $-\mathbf{L}_A^{-1}$, (*resp.* $-(\mathbf{L}_A^D)^{-1}$) is a positive operator on $\mathcal{H}_n(\mathbb{K})$ if and only if the associated system $\dot{x} = Ax$ (resp. $x(t+1) = Ax(t)$) is asymptotically stable. This observation shows that there is a close relationship between the stability theory of time–invariant linear systems and the theory of positive operators. □

For time-invariant systems the matrix $\mathcal{Q}(t, t_0)$ defined in (63) takes the form

$$\mathcal{Q}(t, t_0) = \int_{t_0}^{t} e^{A^*(s-t_0)} C^* C e^{A(s-t_0)} ds \; = \int_0^{t-t_0} e^{A^*\rho} C^* C e^{A\rho} d\rho \,, \tag{91a}$$

$$\mathcal{Q}(t, t_0) = \sum_{s=t_0}^{t-1} A^{*(s-t_0)} C^* C A^{s-t_0} \; = \sum_{\rho=0}^{t-t_0-1} A^{*\rho} C^* C A^{\rho} \,. \tag{91b}$$

Hence the pair (A, C) is uniformly observable if and only if there exists $c > 0, \tau > 0$, such that $\mathcal{Q}(\tau, 0) \geq cI_n$. And it is not difficult to show (cf. Volume II) that this will be the case if and only if (A, C) is *observable* in the sense that

$$\bigcap_{i=1}^{n} \ker\, CA^{i-1} = \{0\}. \tag{92}$$

As a consequence of these observations the results developed in the previous subsection take a simpler form.

Theorem 3.3.49. *Suppose $Q = C^*C$, where $C \in \mathbb{K}^{p\times n}$.*

(i) If (A, C) is observable, then (27) *is asymptotically stable if and only if there exists a solution P of* (80) *with $P \succ 0$.*

(ii) If there exists a solution P of (80) *with $P \succeq 0$ and* $\ker P \neq \{0\}$*, then (A, C) is not observable.*

(iii) If there exists a solution P of (80) *with $P \succ 0$, then the time-invariant system* (27) *is stable, and if in fact it is asymptotically stable then (A, C) is necessarily observable.*

(iv) Suppose (A, C) is observable and there exists a solution $P \in \mathcal{H}_n(\mathbb{K})$ of (80)*. Then there exists $x^0 \in \mathbb{K}^n$ with $\langle x^0, Px^0\rangle < 0$ if and only if* $\operatorname{Re}\lambda > 0$ *(resp. $|\lambda| > 1$) for some $\lambda \in \sigma(A)$.*

Proof: The proof is for the continuous time case.
(i) The "if" statement follows from Theorem 3.3.38. Conversely, suppose that (27) is asymptotically stable. Then P defined by (89a) solves (80a) and is positive definite by the observability of (A, C).
(ii) If $P \succeq 0$ is a solution of (80a) then it follows from (67) that

$$e^{A^*t} P e^{At} = P - \int_0^t e^{A^*\rho} C^* C e^{A\rho} d\rho, \quad t \geq 0.$$

Now suppose $x \in \ker P$, $x \neq 0$ then $-\int_0^t \|Ce^{A\rho}x\|^2 d\rho = \langle x, e^{A^*t} P e^{At} x\rangle \geq 0$ and hence $\langle x, \mathcal{Q}(t, 0)x\rangle = \int_0^t \|Ce^{A\rho}x\|^2 d\rho = 0$ for all $t \geq 0$. Thus (A, C) is not observable.

(iii) Suppose that $P \succ 0$ solves (80a) then (27) is stable by Theorem 3.3.33. If additionally $\sigma(A) \subset \mathbb{C}_-$ then $\int_0^\infty e^{A^*\rho} C^* C e^{A\rho} d\rho = P \succ 0$ and so (A, C) is observable.
(iv) follows from Theorem 3.3.39 and Theorem 3.3.38, see Remark 3.3.40. □

Remark 3.3.50. (i) If $Q \succ 0$ then rank $C = n$ and (92) is automatically satisfied.

(ii) For higher dimensions it is a nontrivial task to solve the linear matrix equation (80). In the next chapter we will describe an algorithm based on the reduction of A to Schur form. An alternative is to make an inspired choice of a $P = P^* \succ 0$ and compute Q from (80). If $Q \succ 0$ ($\succeq 0$) then (27) is asymptotically stable (stable) whereas if this is not the case no conclusion can be drawn.

(iii) If $P \succ 0$ solves (80) with $Q = Q^* \succ 0$ then $V(x) = \langle x, Px\rangle = \langle x, x\rangle_P = \|x\|_P^2$ satisfies

$$\dot{V}(x) = 2\,\mathrm{Re}\langle x, Ax\rangle_P < 0 \quad (\dot{V}(x) = \|Ax\|_P^2 - \|x\|_P^2 < 0), \quad x \in \mathbb{K}^n,\ x \neq 0. \tag{93}$$

So the flow is contracting with respect to the induced norm $\|\cdot\|_P$, i.e. the distance from the origin measured by this norm is continually decreasing along the trajectory.

(iv) If the spectral abscissa $\alpha(A) = 0$ (resp. $\varrho(A) = 1$) and $Q \succ 0$, then there is no solution of (80), see Remark 3.3.40.

(v) Given $r \in \mathbb{R}$ (resp. $r > 0$), $Q = Q^* \succ 0$, let us assume that (88) holds for the matrix $(A - rI)$ (resp. $r^{-1}A$). Then from the above theorem we have that $\mathrm{Re}\,\lambda > r$ (resp. $|\lambda| > r$) for some $\lambda \in \sigma(A)$ if and only if the solution $P_r \in \mathcal{H}_n(\mathbb{K})$ of following equation (94)

$$P(A - rI) + (A - rI)^* P + Q = 0 \tag{94a}$$
$$r^{-2}\, A^* P A - P + Q = 0. \tag{94b}$$

satisfies $\langle x^0, P_r x^0\rangle < 0$ for some $x^0 \in \mathbb{K}^n$. Moreover (88) will hold for all but a finite number of values of r. □

Example 3.3.51. We again consider the linear oscillator studied in Example 3.3.24. For different parameter combinations ($\alpha \in \mathbb{R}, \beta \geq 0$) we will determine the stability properties via the use of Liapunov functions. The Liapunov equation (80a) takes the following form

$$\begin{bmatrix} 0 & 1 \\ -\beta & -2\alpha \end{bmatrix}^* \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\beta & -2\alpha \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} = 0$$

i.e.

$$-2\beta p_2 + q_1 = 0,\ p_1 - 2\alpha p_2 - \beta p_3 + q_2 = 0,\ 2(p_2 - 2\alpha p_3) + q_3 = 0.$$

1. case: $\alpha \neq 0$, $\beta > 0$. In this case we choose $Q = I_2$ and obtain the solution

$$p_1 = \frac{\alpha}{\beta} + \frac{1}{4\alpha}(1 + \beta), \quad p_2 = \frac{1}{2\beta}, \quad p_3 = \frac{1 + \beta}{4\alpha\beta}.$$

Since $p_1 p_3 - p_2^2 > 0$, $P \succ 0$ if and only if $p_1 > 0$ (or $p_3 > 0$). $\langle x, Px\rangle_{\mathbb{R}^2} < 0$ for some $x \in \mathbb{R}^2$ (in fact $-P \succ 0$) if and only if $p_1 < 0$ (or $p_3 < 0$). Thus by Theorem 3.3.49 the system is *asymptotically stable* if $\alpha > 0$, $\beta > 0$ and it is *unstable* if $\alpha < 0$, $\beta > 0$.
If $\alpha = 0$ or $\beta = 0$ there are no solutions of the Liapunov equation when $Q = I_2$, so we examine the modified Liapunov equation (94a).
2. case: $\alpha < 0$, $\beta = 0$. The solution of (94a) with $Q = I_2$ is

$$P_r = \frac{1}{4r(\alpha + r)(2\alpha + r)} \begin{bmatrix} 2(\alpha + r)(2\alpha + r) & 2\alpha + r \\ 2\alpha + r & 2r^2 + 2\alpha r + 1 \end{bmatrix}.$$

Now $2(\alpha+r)(2\alpha+r)(2r^2+2\alpha r+1)-(2\alpha+r)^2 = r(2\alpha+8\alpha^3)+O(r^2)$, so for $\alpha<0$, $r>0$ sufficiently small there exists $x\in\mathbb{R}^2$ such that $\langle x, P_r x\rangle < 0$. Hence by Remark 3.3.50 A has an eigenvalue with $\operatorname{Re}\lambda > r$ for small $r>0$. So there is a $\lambda\in\sigma(A)$ with $\operatorname{Re}\lambda>0$ and hence the system is *unstable.*

If $\alpha\geq 0$, $\beta=0$ it is easily verified from the above formula that $P_r \succ 0$ for $r>0$ and $\langle x^0, P_r x^0\rangle < 0$ for some $x^0\in\mathbb{K}^n$ when $r<0$ is near $r=0$. Hence there is a $\lambda\in\sigma(A)$ with $\operatorname{Re}\lambda=0$. A similar analysis can be carried out for the case $\alpha=0,\ \beta>0$. But as in the case where $\beta=0,\ \alpha\geq 0$ no stability or instability result is obtained (only the existence of $\lambda\in\sigma(A)$ with $\operatorname{Re}\lambda=0$). Thus stability results for these remaining cases cannot be obtained with the choice of $Q=I_2$, even if we use the modified Liapunov equation (94a). In order to proceed using quadratic Liapunov functions we need to make an inspired choice for P (or equivalently Q). The total energy of the oscillator is $\frac{1}{2}(\beta x_1^2+x_2^2)$, so let us consider $P=\begin{bmatrix}\frac{1}{2}\beta & 0\\ 0 & \frac{1}{2}\end{bmatrix}$, with the associated

$$Q=-A^*P-PA=\begin{bmatrix}0 & 0\\ 0 & 2\alpha\end{bmatrix}=\begin{bmatrix}0 & \sqrt{2\alpha}\end{bmatrix}\begin{bmatrix}0\\ \sqrt{2\alpha}\end{bmatrix}=:C^*C. \tag{95}$$

The parameter values we still have to analyze are $\alpha\geq 0$, $\beta=0$ and $\alpha=0$, $\beta\neq 0$, however for these values there is more than one solution of (80a) with Q given by (95).

3. case: $\alpha=0,\ \beta>0$. In this case there are many solutions of (80a)

$$P(\gamma,\delta)=\begin{bmatrix}\beta\delta & \gamma\\ -\gamma & \delta\end{bmatrix},\qquad \gamma,\,\delta\in\mathbb{R}.$$

which are, in general, non-symmetric. However, $P(0,1)\in\mathcal{H}_2(\mathbb{R})$ is positive definite and so the system is *stable* by Corollary 3.3.46.

4. case: $\beta=0,\ \alpha>0$. In this case there are many symmetric solutions of (80a) with Q given by (95)

$$P(\gamma)=\begin{bmatrix}2\alpha\gamma & \gamma\\ \gamma & \frac{1}{2}+\frac{\gamma}{2\alpha}\end{bmatrix},\qquad \gamma\in\mathbb{R}.$$

Since $P(1)\succ 0$ for $\alpha>0$ we conclude from Theorem 3.3.49 the system is *stable.*

5. case: $\alpha=0,\ \beta=0$. In this case there are again many symmetric solutions of (80a)

$$P(\delta)=\begin{bmatrix}0 & 0\\ 0 & \delta\end{bmatrix},\qquad \delta\in\mathbb{R}$$

and the pair (A,C) is unobservable since $C=0$. We know from Example 3.3.24 that the system is (marginally) unstable in the present case but we cannot infer this result from Theorem 3.3.49.

This example illustrates the usual situation when Liapunov equations are used. Stability or instability can be deduced for most of the parameter values by the choice of $Q=I_n$. However certain combinations of the parameters (associated with the case $\operatorname{Re}\lambda=0$, $\lambda\in\sigma(A)$) require a more subtle analysis. □

We now turn to the time-invariant version of Liapunov's indirect method. Consider the nonlinear equations

$$\dot{x}(t) = f(x(t)),\qquad t\in\mathbb{R}, \tag{96a}$$
$$x(t+1) = f(x(t)),\qquad t\in\mathbb{Z} \tag{96b}$$

where f is Lipschitz continuous (resp. continuous) on an open subset $X \subset \mathbb{K}^n$, $\overline{x} \in X, f(\overline{x}) = 0$ (resp. $f(\overline{x}) = \overline{x}$). In addition, suppose that f is differentiable at $\overline{x}$ and $f'(\overline{x}) = A$, i.e.

$$f(x) = A(x - \overline{x}) + h(x - \overline{x}), \qquad x \in X. \tag{97}$$

and for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\|h(x - \overline{x})\| \leq \varepsilon \|x - \overline{x}\|, \qquad x \in B(\overline{x}, \delta). \tag{98}$$

Theorem 3.3.52. *Assume that* (97), (98) *hold for the nonlinear system* (96). *Then*

(i) if $\operatorname{Re}\lambda < 0$ *(*$|\lambda| < 1$*) for all* $\lambda \in \sigma(A)$*, the equilibrium state* $\overline{x}$ *is exponentially stable with respect to the nonlinear system* (96).

(ii) If $\operatorname{Re}\lambda > 0$ *(*$|\lambda| > 1$*) for some* $\lambda \in \sigma(A)$ *then the equilibrium state* $\overline{x}$ *is unstable with respect to the nonlinear system* (96).

Proof: Since (A, I_n) is uniformly observable (i) is an immediate consequence of Theorem 3.3.20 and Theorem 3.3.41. We prove (ii) for the continuous time case leaving the proof for the discrete time case to the reader (Ex. 23). Suppose $\operatorname{Re}\lambda_0 > 0$ for some $\lambda_0 \in \sigma(A)$ and choose $r \in (0, \operatorname{Re}\lambda_0)$ such that (88) holds for $A_r = A - rI_n$. Then (A_r, I_n) is uniformly observable and there exists a solution $P_r \in \mathcal{H}_n(\mathbb{K})$ of (94a). Moreover $\dot{x} = A_r x$ is unstable and so by Theorem 3.3.43 we must have that $\overline{x}$ is unstable for the nonlinear system (96a). □

As an immediate consequence of Theorem 3.3.52 we know that if the equilibrium point $\overline{x}$ of the nonlinear system (96) is unstable then there exists $\lambda \in \sigma(A)$ such that $\operatorname{Re}\lambda \geq 0\,(|\lambda| \geq 1)$, but the linearized system is not necessarily unstable. Conversely if the equilibrium point $\overline{x}$ of the nonlinear system is (asymptotically) stable then necessarily $\operatorname{Re}\lambda \leq 0\,(|\lambda| \leq 1)$, $\lambda \in \sigma(A)$, but we cannot infer that the linearization is (asymptotically) stable. In contrast in the case of *exponential* stability there is a tighter relationship between the behaviour of a nonlinear system near an equilibrium point and its linearization. In order to express this relationship in a succinct way we need the following definition. The solution of (96) with initial state x^0 will be denoted by $\varphi(t; x^0)$, $t \in T(x^0)$.

Definition 3.3.53. Let $r > 0$ be such that $B(\overline{x}, r) \subset X$. The infimum of all $\omega \in \mathbb{R}$ for which there exists $M_\omega \geq 1$ such that

$$x^0 \in B(\overline{x}, r) \Longrightarrow \forall t \in T(x^0) : \|\varphi(t; x^0) - \overline{x}\| \leq M_\omega e^{\omega t} \|x^0 - \overline{x}\| \tag{99}$$

is called the *(upper) growth rate* of the nonlinear system (96) with initial state in $B(\overline{x}, r)$ and is denoted by $\omega(f, \overline{x}, r)$. $\omega(f, \overline{x}) := \lim_{r \searrow 0} \omega(f, \overline{x}, r)$ is said to be the *(upper) growth rate* of (96) *at the equilibrium state* $\overline{x}$.

It follows from the definition that $0 < r_1 < r_2$ implies $\omega(f, \overline{x}, r_1) \leq \omega(f, \overline{x}, r_2)$ and therefore $\omega(f, \overline{x}) = \inf_{r>0} \omega(f, \overline{x}, r)$. By definition $\omega(f, \overline{x}, r) = \infty$ if there does not exist an $M_\omega \geq 1$, $\omega \in \mathbb{R}$ such that (99) holds.

Example 3.3.54. Let $f(x) = Ax, x \in \mathbb{K}^n$ where $A \in \mathbb{K}^{n\times n}$ is given and $\overline{x} = 0$. Then $\omega(f, 0, r) = \omega(f, 0) = \omega(A)$ for all $r > 0$ where $\omega(A)$ equals the upper Liapunov (or Bohl) coefficient of the semigroup $\Phi(t) = e^{At}$ generated by A, see (30). Hence Definition 3.3.53 generalizes the concept of growth rate as introduced in Subsection 3.3.2 for time-invariant linear systems. □

Theorem 3.3.55. *Assume* (97), (98) *hold for the nonlinear system* (96)*. Then the equilibrium point* $\overline{x}$ *is exponentially stable if and only if the linearization at* $\overline{x}$ *is exponentially stable. In this case* $\omega(f, \overline{x}) = \omega(A)$.

Proof: The proof is for the continuous time case, the proof for the discrete time case is set as Ex. 23. Assume $\omega(A) < 0$ and $\beta \in (0, -\omega(A))$. Given $\varepsilon > 0$ choose $\delta > 0$ such that (98) holds and consider the time-varying nonlinear equation

$$\dot{z}(t) = (A + \beta I_n)z(t) + \tilde{h}(t, z(t)), \quad \tilde{h}(t, z) = e^{\beta t}h(e^{-\beta t}z),\ z \in B(0, \delta),\ t \geq 0. \quad (100)$$

Now

$$\|z\| < \delta \Longrightarrow \|\tilde{h}(t, z)\| = e^{\beta t}\|h(e^{-\beta t}z)\| \leq e^{\beta t}\varepsilon\|e^{-\beta t}z\| = \varepsilon\|z\|, \quad t \geq 0. \quad (101)$$

Hence $\tilde{h}(\cdot, \cdot)$ has the property (74) for the pair (ε, δ). Moreover $\sigma(A+\beta I_n) \subset \mathbb{C}_-$ and $(A+\beta I_n, I_n)$ is uniformly observable. So we may apply Theorem 3.3.41 to conclude that there exist positive constants $\tilde{\delta}, \tilde{\varepsilon}, \tilde{M}$ such that

$$\|z(0)\| < \tilde{\delta} \Longrightarrow \|z(t)\| \leq \tilde{M}e^{-\tilde{\varepsilon}t}\|z(0)\|, \quad t \geq 0.$$

Let $x^0 = \overline{x} + z(0)$ and $\varphi(t; x^0) - \overline{x} = e^{-\beta t}z(t)$, $t \geq 0$, then

$$\|x^0 - \overline{x}\| < \tilde{\delta} \Longrightarrow \|\varphi(t; x^0) - \overline{x}\| \leq \tilde{M}e^{-(\beta+\tilde{\varepsilon})t}\|x^0 - \overline{x}\|, \quad t \geq 0.$$

Moreover for $t > 0$, we have

$$\begin{aligned} \dot{\varphi}(t; x^0) &= -\beta e^{-\beta t}z(t) + e^{-\beta t}[(A + \beta I_n)z(t) + e^{\beta t}h(e^{-\beta t}z(t))] \\ &= A(\varphi(t; x^0) - \overline{x}) + h(\varphi(t; x^0) - \overline{x}). \end{aligned}$$

So $\varphi(t; x^0)$ is the solution of (96a) with initial state x^0. We see, therefore, that $\overline{x}$ is exponentially stable for the system (96a) and its growth rate at $\overline{x}$, $\omega(f, \overline{x}) \leq \omega(A)$. Conversely, assume that $\overline{x}$ is exponentially stable for (96a). Then given $\varepsilon > 0$ and $\omega \in (\omega(f, \overline{x}), 0)$, there exists positive constants δ, M such that (98) holds and

$$\|x^0 - \overline{x}\| < \delta \Longrightarrow \|\varphi(t; x^0) - \overline{x}\| \leq Me^{\omega t}\|x^0 - \overline{x}\|, \quad t \geq 0.$$

Choose $\beta \in (0, -\omega)$ such that $\lambda + \overline{\mu} + 2\beta \neq 0$ for all $\lambda, \mu \in \sigma(A)$ and set $z(t) = e^{\beta t}(\varphi(t; x^0) - \overline{x})$, $t \geq 0$. Then $z(\cdot)$ satisfies (100) with initial state $x^0 - \overline{x}$ and by (101) $\tilde{h}(\cdot, \cdot)$ has the property (74) for the pair (ε, δ). Now if $\lambda_\beta \in \sigma(A + \beta I_n)$, then $\lambda_\beta = \lambda + \beta$ for some $\lambda \in \sigma(A)$. So by the restriction on the choice of β we see that $\lambda_\beta + \overline{\mu}_\beta \neq 0$ for all $\lambda_\beta, \mu_\beta \in \sigma(A + \beta I)$ and hence there exists a solution P of the algebraic Liapunov equation (80a) with A replaced by $A+\beta I_n$ and $Q = I_n$. Finally, since $(A+\beta I_n, I_n)$ is observable, we see that all the conditions of Theorem 3.3.43 for the equation given by (100) are satisfied. But $\|z(t)\| \leq Me^{(\omega+\beta)t}\|z(0)\|, t \geq 0$ and so the the equilibrium point 0 of (100) is exponentially stable. Therefore $A + \beta I_n$ cannot be unstable and $\operatorname{Re}\lambda \leq -\beta$ for all $\lambda \in \sigma(A)$. Thus $\omega(A) \leq \omega(f, \overline{x})$ and this completes the proof. □

Liapunov's indirect method provides a very simple way of determining whether or not an equilibrium state is stable since it relates the nonlinear flow to that of the linearized flow. However it is important to stress that stability or instability of an equilibrium state is a local property and from a practical point of view may give misleading information. For example an equilibrium state may be asymptotically stable but its basin of attraction may be so small that from a practical standpoint one should think of it as being unstable. Similar considerations apply to unstable equilibrium points. Although the construction of Liapunov functions for nonlinear systems may be difficult, the great advantage of Liapunov's direct method is that it provides information about the basin of attraction.

Example 3.3.56. Consider the nonlinear oscillator

$$\ddot{y} + h(y, \dot{y})\dot{y} + g(y) = 0$$

where $g(0) = 0$. Setting $x = [x_1, x_2]^\top = [y, \dot{y}]^\top$, we get the corresponding state space system

$$\dot{x} = \begin{bmatrix} x_2 \\ -g(x_1) - h(x_1, x_2)x_2 \end{bmatrix} := f(x).$$

Since

$$\frac{\partial f}{\partial x}(0) = \begin{bmatrix} 0 & 1 \\ -g'(0) & -h(0,0) \end{bmatrix},$$

the origin will be exponentially stable if and only if $g'(0) > 0$ *and* $h(0,0) > 0$. It will be unstable if either $g'(0) < 0$ *or* $h(0,0) < 0$. □

Example 3.3.57. Let us analyze the stability of an oscillator with nonlinear friction described by the following equation

$$\ddot{\xi} + (2\alpha + \dot{\xi}^2)\dot{\xi} + \beta\xi = 0.$$

Mechanical systems with this equation of motion are used to regulate the angular position ξ of a gyrating mass. The corresponding state space system is

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta & -2\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ x_2^3 \end{bmatrix}, \tag{102}$$

which has one equilibrium state at $(0,0)$. The linearization about this equilibrium state is $\dot{x} = \begin{bmatrix} 0 & 1 \\ -\beta & -2\alpha \end{bmatrix} x$ and we have analyzed the stability of this system in Example 3.3.24. Using these results and Theorem 3.3.52, we are able to conclude that the origin is exponentially stable if $\alpha > 0$, $\beta > 0$ and it is unstable if $\alpha < 0$. If $\alpha = 0$, $\beta > 0$ the origin of the linearized system is a centre and since it is only marginally stable we cannot apply Theorem 3.3.52. In order to obtain information about this case and the basin of attraction when $\alpha \geq 0$, consider the function

$$V(x) = (1/2)(\beta x_1^2 + x_2^2), \quad x \in \mathbb{R}^2.$$

This function associates with any state x, the corresponding total energy of the system. Then $\lim_{\|x\| \to \infty} V(x) = \infty$ and $\dot{V}(x) = -(2\alpha + x_2^2)x_2^2$. The largest invariant subset in $\{x \in \mathbb{R}^2 : \dot{V}(x) = 0\} = \{(x_1, 0); x_1 \in \mathbb{R}\}$ for (102) is $\{(0,0)\}$ when $\alpha \geq 0$, $\beta > 0$. So by Corollary 3.2.29 the origin is asymptotically stable even when $\alpha = 0$, $\beta > 0$. Moreover since every sublevel set $V(x) < \rho$ is bounded the asymptotic stability is global. □

Example 3.3.58. The discrete time system

$$x_1(t+1) = \alpha x_1(t) + x_2^2(t), \quad x_2(t+1) = x_1(t) + \beta x_2(t) \tag{103}$$

has two equilibrium points $\overline{x}^1 = (0,0)$ and $\overline{x}^2 = ((1-\alpha)(1-\beta)^2, (1-\alpha)(1-\beta))$.
The linearized system about $(0,0)$ is given by the matrix $\begin{bmatrix} \alpha & 0 \\ 1 & \beta \end{bmatrix}$ which has eigenvalues α, β. So the equilibrium state $(0,0)$ will be exponentially stable if $|\alpha| < 1$ and $|\beta| < 1$. It will be unstable if $|\alpha|$ or $|\beta|$ is greater than one.
The linearized system about the second equilibrium state is given by the matrix

$$\begin{bmatrix} \alpha & 2(1-\alpha)(1-\beta) \\ 1 & \beta \end{bmatrix}.$$

The characteristic equation is $(\lambda - \alpha)(\lambda - \beta) = 2(1-\alpha)(1-\beta)$. The shaded region in Figure 3.3.5 corresponds to those values of α, β for which this equilibrium state is exponentially stable. The boundaries $\alpha = 1$, $\beta = 1$ are obtained when $\lambda = +1$, the

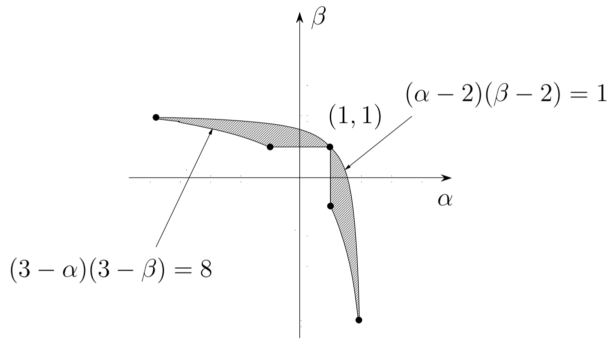

Figure 3.3.5: Stability chart for $\overline{x}^2$ with respect to (103)

boundary $(3-\alpha)(3-\beta) = 8$ is obtained when $\lambda = -1$ and the other part of the boundary is determined by setting $\lambda = e^{\imath\theta}$ with $\cos\theta = (\alpha+\beta)/2$. Note that the first equilibrium point is unstable if the second one is exponentially stable. □

3.3.6 Exercises

1. Prove that the growth rate $\omega(A) = \inf\{\omega \in \mathbb{R};\, \exists M > 0 : \ \|\Phi(t)\| \le M e^{\omega t}\}$ of a continuous (resp. discrete) time semigroup Φ with generator $A \in \mathbb{C}^{n\times n}$ is equal to the spectral abscissa $\alpha(A) = \max_{\lambda\in\sigma(A)} \operatorname{Re}\lambda$ (resp. the logarithm of the spectral radius $\ln \varrho(A) = \ln \max_{\lambda\in\sigma(A)} |\lambda|$).

2. If $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$ show that

$$e^{At} = e^{-t}\begin{bmatrix} \cos t + \sin t & \sin t \\ -2\sin t & \cos t - \sin t \end{bmatrix}, \quad A^t = (\sqrt{2})^t \begin{bmatrix} \cos\frac{\pi}{4}t - \sin\frac{\pi}{4}t & \sin\frac{\pi}{4}t \\ -2\sin\frac{\pi}{4}t & \sin\frac{\pi}{4}t + \cos\frac{\pi}{4}t \end{bmatrix}.$$

Determine

$$\lim_{t\to\infty} \frac{\ln\|e^{At}\|}{t}, \quad \text{and} \quad \lim_{t\to\infty} \frac{\ln\|A^t\|}{t}.$$

3. Find the continuous time evolution operator generated by $a(t) = t \sin t$ on $T = \mathbb{R}_+$. Show that the upper Liapunov exponent is $+1$, whereas the upper Bohl exponent is not finite.

4. Show the upper Liapunov exponent for a continuous time system (1) is finite if $\sup_{t\in\mathbb{R}_+} \int_t^{t+1} \|A(s)\| ds < \infty$.

5. Show that the Liapunov exponent of (1) is given by

$$\overline{\alpha}(\Phi) = \limsup_{t\to\infty} \frac{\ln\|\Phi(t,0)\|}{t} \quad \text{(where } \ln\|\Phi(t,0)\| = -\infty \text{ if } \|\Phi(t,0)\| = 0).$$

6. Prove Theorem 3.3.15 in the discrete time case.

7. Consider

$$A(t) = \begin{bmatrix} -11/2 + (15/2)\sin 12t & (15/2)\cos 12t \\ (15/2)\cos 12t & -11/2 - (15/2)\sin 12t \end{bmatrix}.$$

Show that the system $\dot{x} = A(t)x$ is exponentially stable even though $\sigma(A(t)) = (2, -13)$ for all $t \geq 0$.

8. Prove that every scalar system $\dot{x}(t) = a(t)x(t)$, $t \in \mathbb{R}_+$ (resp. $x(t+1) = a(t)x(t)$, $t \in \mathbb{N}$) which has a finite upper Bohl exponent β, can be transformed via a Bohl transformation $\theta(t) = e^{\int_0^t (a(s)-\beta)ds}$ (resp. $\theta(t) = e^{-\beta t}\prod_{s=0}^{t-1} a(s)$) into the time invariant system $\dot{x}(t) = \beta x(t)$, (resp. $x(t+1) = e^{\beta}x(t)$).

9. Consider the scalar system $\dot{x}(t) = (4t\sin t - 2t)x(t)$, $t > t_0$, $x(t_0) = x_0$, $t_0 \in \mathbb{R}_+$. Prove that the solution is

$$\Phi(t,t_0)x_0 = x_0 \exp(4\sin t - 4t\cos t - t^2 - 4\sin t_0 + 4t_0\cos t_0 + t_0^2).$$

Hence show that the origin is asymptotically stable at any time $t_0 \in \mathbb{R}_+$. Prove that $\Phi((2n+1)\pi, 2n\pi) = \exp((4n+1)\pi(4-\pi))$ for $n \in \mathbb{N}$ and so the origin is not uniformly asymptotically stable.

10. If $a(t) = -(1+t)^{-1}$, $q(t) = 2(1+t)^{-1} - 3(1+t)^{-2}$ for $t \in \mathbb{R}_+$ show that a solution of the Liapunov equation (54a) is $p(t) = 1-(1+t)^{-1}$. What conclusion can be drawn about the stability properties of the evolution operator generated by $a(\cdot)$ on $\mathbb{R}_+$.

11. Let $A(t) = \begin{bmatrix} -2+\cos t & -\sin t \\ -\sin t & -2-\cos t \end{bmatrix}$, $t \in \mathbb{R}_+$, choose $P(t) \equiv I_2$ and compute $Q(t)$ such that the Liapunov equation (54a) is satisfied. Use this to prove that the evolution operator generated by $A(\cdot)$ is uniformly exponentially stable.

12. Let $A(t) = \begin{bmatrix} 0 & 1 \\ -a(t) & -1/2 \end{bmatrix}$, $t \in \mathbb{N}$, choose $P(t) = \begin{bmatrix} a(t)^2 + 1/4 & 0 \\ 0 & 1 \end{bmatrix}$ and compute $Q(t)$ such that the discrete time Liapunov equation (54b) is satisfied. Hence show that the evolution operator generated by $A(\cdot)$ is uniformly exponentially stable if $|a(t)| < 1/2, t \in \mathbb{N}$.

13. Prove the Instability Theorem 3.3.34

14. Show that the unique solution of (54b) on T_{t_0} with initial state $P(t_0)$ is

$$P(t) = \Phi(t_0,t)^* P(t_0)\Phi(t_0,t) - \sum_{s=t_0}^{t-1} \Phi(s,t)^* Q(s)\Phi(s,t), \quad t \geq t_0.$$

15. Suppose $A(t) = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}$, $t \in \mathbb{R}$. Show that for $Q(t) = I_2$ there is not a bounded solution of (54a). However for every $r > 0$ if $A_r(t) = A(t) - rI_2$ there are bounded solutions of (54a) on $T_{1/2+r}$ when $A(t)$ is replaced by $A_r(t)$.

16. Determine conditions on a, b, ρ for which the equilibrium state is asymptotically stable for Goodwin's model of supply and demand considered in Example 1.2.1.

17. Determine whether or not the following matrices correspond to asymptotically stable systems in the continuous time and discrete time cases

$$\text{(a)} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 0 & 1 \\ -\frac{1}{8} & -\frac{1}{2} \end{bmatrix}.$$

Verify your conclusions by solving the Liapunov equations with $Q = I_2$.

18. Find the linearized equations of motion about the equilibrium states in Ex. 2.5, 2.6. Determine whether or not these systems are asymptotically stable.

19. Show that the system $\dot{x} = \alpha x^3$ is asymptotically stable if $\alpha < 0$ and unstable if $\alpha > 0$. Note that the linearized system about the origin is marginally stable for all $\alpha \in \mathbb{R}$. This example shows that no conclusions for the stability with respect to the nonlinear system can be drawn from this fact.

20. Consider the discrete time system with matrix

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_0 \end{bmatrix}.$$

Determine the values of a_0, a_1 for which there is not a unique solution of the Liapunov equation (80b). Solve the Liapunov equation when $Q = I_2$ and hence determine those values of a_0, a_1 for which the system is asymptotically stable, marginally stable or unstable. Solve the modified Liapunov equation (94b). What further conclusions can be drawn? Compare your results with those given in Example 3.3.25.

21. Suppose that the system $\dot{x} = Ax$ is asymptotically stable, where $A \in \mathbb{R}^{n\times n}$. For a step size $\tau > 0$ consider the following discretizations

$$\text{(a)} \quad \frac{x^\tau(t+1) - x^\tau(t)}{\tau} = Ax^\tau(t), \quad \text{(b)} \quad \frac{x^\tau(t+1) - x^\tau(t)}{\tau} = Ax^\tau(t+1), \quad t \in \mathbb{N}.$$

Show that the system in (b) is necessarily asymptotically stable but the system in (a) need not be asymptotically stable.

22. If $\Phi(\cdot,\cdot)$ is generated by $a(t) = -t$, $t \in \mathbb{R}_+$ show that for every $\tau > 0$, $\int_{t_0}^{t_0+\tau} \Phi(s,t_0)^2 ds \leq (2t_0)^{-1}$, $t_0 \geq 0$. This example shows that the pair $(a(\cdot), 1)$ is not uniformly observable.

23. Prove Theorems 3.3.43, 3.3.52 and 3.3.55 for the discrete time case.

24. Consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -2x_1 - 3x_2 - h(x_1 + x_2)$$

where $h : \mathbb{R} \to \mathbb{R}$ is continuous $h(0) = 0$ and $xh(x) > 0$ for all $x \neq 0$.

(i) Find a matrix $P \in \mathbb{R}^{n\times n}$ such that

$$PA + A^\top P + 4I = 0 \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

(ii) Use the function $V(x) = x^\top Px + \int_0^{x_1+x_2} h(s)\,ds$ to show that the origin is asymptotically stable.

25. Show that for the Lur'e problem (2.58) the function V (2.60) has the property

$$\dot V(x,\sigma) = -\langle x, Qx\rangle - rf^2(\sigma) + 2f(\sigma)\langle Pb + (1/2)c, x\rangle$$

where $PA + A^\top P + Q = 0$. Hence prove that if $\sigma(A) \subset \mathbb{C}_-$, $f(0) = 0$, $\sigma \neq 0 \Rightarrow \sigma f(\sigma) > 0$ and

$$r > \langle Pb + (1/2)c, Q^{-1}(Pb + (1/2)c)\rangle,$$

then the origin $x = 0$, $\sigma = 0$ is asymptotically stable.

26. Newton's method for solving the equation $F(x) = 0$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, is the iterative scheme

$$x(t+1) = x(t) - [F'(x(t))]^{-1}F(x(t)) \quad t \in \mathbb{N}$$

where the inverse is assumed to exist.
This method is used to solve the scalar equation $e^{-x} - x = 0$, so that

$$x(t+1) = x(t) + \frac{e^{-x(t)} - x(t)}{e^{-x(t)} + 1}, \quad t \in \mathbb{N}. \tag{104}$$

If $\overline{x}$ is the required unique solution find the linearized equations about $\overline{x}$ and show that $\overline{x}$ is an exponentially stable equilibrium state of (104). Use the function $V(x) = |x - \overline{x}|$ to obtain an estimate for the basin of attraction of this equilibrium state.

27. The second order differential system $\ddot\xi + \alpha\dot\xi + \beta\xi = 0$ is asymptotically stable if and only if $\alpha > 0$, $\beta > 0$. Use the Cayley transform to obtain necessary and sufficient conditions for the second order difference equation

$$\xi(t+2) + a\,\xi(t+1) + b\,\xi(t) = 0, \quad t \in \mathbb{N}$$

to be asymptotically stable.

28. Case study: A model for a continuous flow stirred tank reactor is given by

$$\begin{aligned} \dot T &= a(T_0 - T) + bkCe^{-\alpha/T} \\ \dot C &= a(C_0 - C) - kCe^{-\alpha/T} \end{aligned}$$

where C_0, T_0 are the concentration and temperature of the reactant in the influent and C, T are the concentration and temperature of the reactant in the effluent. a, b, α, k are positive constants.

(i) Show that all equilibrium states (C_e, T_e) satisfy

$$1 + \frac{a}{k}e^{\alpha/T_e} = \frac{bC_0}{T_e - T_0}, \qquad \frac{C_e - C_0}{T_e - T_0} = -\frac{1}{b}.$$

(ii) Linearize the equations about an equilibrium state and hence show that an equilibrium state is stable if

$$\frac{C_0}{T_e - T_0} > \frac{\alpha C_e}{T_e^2}.$$

(iii) If $T_0 = 300$, $C_0 = 10$, $a = 2^{-9}$, $b = 30$, $k = 0.5$, $\alpha = 3600 \ln 2$, find three equilibrium states and determine whether or not they are stable. Are there any other equilibrium states?

(iv) Use a computer to obtain a phase portrait of the system around the three equilibrium points.

3.3.7 Notes and References

Many of the results for time-varying linear systems can be found in *Daleckii and Krein* (1974) [118]. The notion of Bohl exponent is due to *Bohl* (1913) [65]. The proof of Theorem 3.3.15 is given in [118] and was proved for the case $p = 2$ in [120]. Our proof is based on that of [115]. Many of the books quoted in Section 3.2 contain results for time-varying systems. For further results on *time-varying* Liapunov transformations, see *Gantmacher* (1959 Vol. 2) [183].

The result that the growth rate of a strongly continuous semigroup is $\sup_{\lambda\in\sigma(A)} \text{Re}\,\lambda$ is known as the *spectrum determined growth condition.* It holds for a large class of strongly continuous semigroups on infinite dimensional Banach spaces. However it is not true in general, see *Zabczyk* (1975) [540] for a counterexample with $\sup_{\lambda\in\sigma(A)} = 0$ yet $\|S(t)\| = e^t$. For a discussion of numerical stability of discretization methods see for example *Stoer and Bulirsch* (1978) [485] and the references in Section 4.5.

The quadratic Liapunov function for linear systems was introduced in Liapunov's original work and many of the results in Subsection 3.3.4 and Subsection 3.3.5 can be found there. A good account can also be found in *Barbashin* (Translation 1970) [33]. Extensions of Liapunov's result which relate the inertia $i(A)$ to the inertia $i(P)$ where $PA + A^*P + Q = 0$, are called *inertia theorems* see *Carlson and Schneider* (1963) [91], *Wimmer* (1975) [531], *Glover* (1984) [188] and *Datta* (1999) [123].

Generalizations of Liapunov's Theorem 3.3.33 to infinite dimensional systems, continuous or discrete time have been obtained by *Datko* (1970) [119] and *Zabczyk* (1974) [539].

In the late 60's determining stability domains via Liapunov functions (or otherwise) was much in vogue and there have been many such attempts for the Mathieu equation. For example *Narenda and Taylor* (1973) [387] obtained the stability domain $\pi a\zeta/2 > |q|$, $a \gg \zeta^2$, $\zeta \ll 1$.

The linearization result in Subsection 3.3.5 is essentially due to Liapunov and the fact that an equilibrium point is exponentially stable if and only if the linearization at the equilibrium point is exponentially stable can be found in *Zabczyk* (1992) [541].