Error Analysis for H^1 Based Wavelet Interpolations

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Summary. We rigorously study the error bound for the H^1 wavelet interpolation problem, which aims to recover missing wavelet coefficients based on minimizing the H^1 norm in physical space. Our analysis shows that the interpolation error is bounded by the second order of the *local* sizes of the interpolation regions in the wavelet domain.

1 Introduction

In this paper, we investigate the theoretical error estimates for variational wavelet interpolation models.

The wavelet interpolation problem is to calculate unknown wavelet coefficients from given coefficients. It is similar to the standard function interpolations except the interpolation regions are defined in the wavelet domain. This is because many images are represented and stored by their wavelet coefficients due to the new image compression standard JPEG2000. The wavelet interpolation is one of the essential problems of image processing and closely related to many tasks such as image compression, restoration, zooming, inpainting, and error concealment, even though the term “interpolation” does not appear very often in those applications. For instance, wavelet inpainting and error concealment are to fill in (interpolate) damaged wavelet coefficients in given regions in the wavelet domain. Wavelet zooming is to predict (extrapolate) wavelet coefficients on a finer scale from a given coarser scale coefficients.

A major difference between wavelet interpolations and the standard function interpolations is that the applications of wavelet interpolations often impose regularity requirements of the interpolated images in the pixel domain, rather than the wavelet domain. For example, natural images (not including textures) are often viewed as piecewise smooth functions in the pixel domain.

This makes the wavelet interpolations more challenging as one usually cannot directly use wavelet coefficients to ensure the required regularity in the pixel domain. To overcome the difficulty, it seems natural that one can use optimization frameworks, such as variational principles, to combine the pixel domain regularity requirements together with the popular wavelet representations to accomplish wavelet interpolations.

A different reason for using variational based wavelet interpolations is from the recent success of partial differential equation (PDE) techniques in image processing, such as anisotropic diffusion for image denoising [25], total variation (TV) restoration [26], Mumford-Shah and related active contour segmentation [23, 10], PDE or TV image inpainting [1, 8, 7], and many more that we do not list here. Very often these PDE techniques are derived from variational principles to ensure the regularity requirements in the pixel domain, which also motive the study of variational wavelet interpolation problems.

Many variational or PDE based wavelet models have been proposed. For instance, Laplace equations, derived from H^1 semi-norm, has been used for wavelet error concealment [24], TV based models are used for compression [5, 12], noise removal [19], post-processing to remove Gibbs' oscillations [16], zooming [22], wavelet thresholding [11], wavelet inpainting [9], l^1 norm optimization for sparse signal recovery [3, 4], anisotropic wavelet filters for denoising [14], variational image decomposition [27]. These studies have demonstrated promising results, which show clear advantages of the combinations of wavelet and variational PDE strategies over the traditional methods.

Despite of the remarkable results obtained in combining variational PDE's with wavelets, the theoretical understandings for those models remain limited, specially for the nonlinear TV based models. Most of the existing studies are focused on the existence and uniqueness (or non-uniqueness) of the solutions of the variational wavelet models. A few recent investigations have been conducted to address the recover properties, including the well-known results reported in [3], in which a probabilistic theory for the exact recovery conditions of sparse signals based on random frequency samples has been developed. In [4], the authors have also studied the reconstruction error in probability sense for the random sampling model based on l^1 minimization of the Fourier frequencies for functions with certain power-law decaying frequencies.

To quantify the interpolation ability of those variational wavelet interpolation models, it is highly desirable to obtain rigorous error estimates, similar to the error bounds for the standard variational image inpainting problems as studied by Chan-Kang in [6] in which the recovery error is bounded by the square of local width of the inpainting region in the pixel domain if H^1 minimization is employed. However, this error analysis for variational wavelet interpolation models often faces different difficulties. For instance, the missing wavelet coefficients in wavelet space could have global influence in physical space, and the imposed regularity (smoothness) requirements are in physical space while the interpolations are performed in the wavelet space. Therefore, how to precisely estimate the regularity requirements in the wavelet space becomes the key to carry out the analysis. This might be very challenging, specially for the nonlinear TV models in which one cannot characterize the TV semi-norm by size properties on wavelet coefficients [20]. For these reasons, such error estimates are still lacking for most of the variational wavelet interpolation models. This paper is our first attempt in gaining an understanding of those models from the error estimate perspective. We investigate the error bound for the H^1 wavelet interpolation model. Similar to the results in [6], our analysis shows that the error bound depends quadratically on the local size of the interpolation regions in wavelet domain. The ultimate goal of our current study is to develop a general strategy and theory to study error estimates for general variational PDE based wavelet models in image processing. We hope the results obtained in this paper can shed some lights for the general theory.

The rest of the paper is arranged as following: in the next section, we present the general variational wavelet interpolation models. The error estimate is given in Section 3.

2 Variational Wavelet Interpolation Models

In this section, we give the variational models of wavelet interpolations, which have been used in many applications. To better illustrate the analysis and simplify the discussion, we restrict ourselves to the one dimensional models. The results can be extended to higher dimensions with appropriate modifications.

We shall start with a brief review of continuous wavelet transforms to introduce notations that will be useful in this paper. Detailed wavelet theory can be found in many texts, such as [15, 28, 21, 17, 13]. A continuous wavelet transform is based on a selected real function $\psi(x) \in L^2(\mathbb{R})$, called wavelet function, satisfying,

$$C_\psi = 2\pi \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} \, d\omega < +\infty, \tag{1}$$

where $\hat{\psi}$ is the Fourier Transform of ψ. For requirements on how to select ψ, we refer to [15]. A family of wavelet functions is constructed by dilation and translations of $\psi(x)$ in the following format,

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi(\frac{x-b}{a}), \tag{2}$$

where $b \in \mathbb{R}$ is the translation variable and $a > 0$ the dilation variable. We denote as $a \in \mathbb{R}^+$, and $V = \mathbb{R}^+ \times \mathbb{R}$. In the wavelet literature, different dilation values of a often refer to the different resolutions or scales.

Let $z(x)$ be any function in $L^2(\mathbb{R})$, its continuous wavelet transform is defined by

$$\beta(a,b) = \int_{-\infty}^{+\infty} z(x)\psi_{a,b}(x) \, dx. \tag{3}$$

Similar to the Fourier transform, the wavelet transform is perfectly invertible, and the inverse wavelet transform is given by

$$z(x) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\beta(a,b)}{a^2} \psi_{a,b}(x)\, dadb. \tag{4}$$

The continuous wavelet transform (3) provides a very redundant description of the function $z(x)$. For this reason, discrete wavelet transforms have been used more often in practice. To obtain the discrete wavelet transforms, one samples the continuous wavelet transform (3) at selected dyadic points. For example, a traditional (and also the most popular) selection takes $a_j = 2^j$ and $b_k = 2^j k$, where j, k are integers. This means that discrete wavelet coefficients are defined by

$$\beta_{j,k} = \beta(a_j, b_k) = \int_{-\infty}^{+\infty} z(x) \psi_{a_j,b_k}(x)\, dx, \tag{5}$$

and its reconstruction formula (discrete inverse wavelet transform) is given by

$$z(x) = \sum_{j,k} \beta_{j,k} \psi_{j,k}(x) = \sum_{j,k} \beta_{j,k} 2^{-\frac{j}{2}} \psi(2^{-j}x - k). \tag{6}$$

In the discrete wavelet representation (6), the wavelet functions $\psi_{a_j,b_k}(x)$ often form an orthonormal basis of L^2 space.

Wavelet transforms have been widely used in many applications, the most remarkable ones are in image processing such as compression, zooming, inpainting. A common challenge in those applications is that partial information of the discrete wavelet transforms $\beta(a_j, b_k)$ is not available for either deliberate (image compression) or involuntary (error concealment) reasons. For instance, the wavelet inpainting and error concealment consider problems that partial wavelet transforms are damaged or lost in the transmission or storage, and image compression algorithms record only selective, usually the significant, wavelet coefficients. Therefore, to restore the original images, one wants to recover the lost information based on the known coefficients. In image zooming or super-resolution, one wants to extend the information, which is only defined on a coarse grid, to a finer grid.

To solve these problems, one needs to interpolate the unavailable information from the known coefficients. To be mathematical precise, we describe the wavelet interpolation problem as following.

Let $z(x)$ be the original function having forward and inverse wavelet transforms defined by (5) and (6) respectively. If $I \subset V$ is a subset in which the discrete wavelet coefficients are not available, we denote

$$\alpha(a_j, b_k) = \begin{cases} \text{unknown} & \text{if}(a_j, b_k) \in I \\ \beta(a_j, b_k) & \text{if}(a_j, b_k) \in I^c \end{cases},$$

where I^c is the complement of I in V, as the wavelet transform for the to-be recovered function $u(x)$. The wavelet interpolation problem is to approximate the original function $z(x)$ by reconstructing $u(x)$ or $\alpha(a_j, b_k)$ on I from $\beta(a_j, b_k)$ on I^c.

Many different approaches have been proposed to achieve this goal. In this paper, we consider one strategy that uses variational principles in the optimization framework to help controlling the regularity of the interpolation. Let $F(\alpha)$ be an energy functional associated with $u(x)$. The variational wavelet interpolation problem is posed in the following form:

$$\min_{\alpha(a_j,b_k),(a_j,b_k)\in I} F(\alpha), \text{ subject to } \alpha(a_j, b_k) = \beta(a_j, b_k), \text{for } (a_j, b_k) \in I^c \quad (7)$$

Different energy functionals $F(\alpha)$ have been proposed. For example, the l^1 norm of the coefficients $\|\alpha\|_1$ has been used to recover sparse signals [3, 4]. The H^1 semi-norm $\|\nabla_x u\|_2^2$ is used in the error concealment algorithm [24]. The popular TV semi-norm $\|\nabla_x u\|_1$ has been used by different groups to wavelet inpainting [9], thresholding [11], compression [12], zooming [18, 22], and restoration [2, 16, 19]. Many of these models have achieved remarkable success in their applications. However, theoretical understandings are still limited, especially for the models using H^1 or TV norms. Most of the existing analysis is related to the existence and non-uniqueness of the minimizers. And it does not provide quantitative understandings on why the models work well. In this paper, we investigate the error estimate for the missing information recovery and hope to explain the observations being made in these applications.

3 Recovery Bound for the H^1 Model

In this paper, we focus on the H^1 variational wavelet interpolation model, which uses

$$F(\alpha) = \int |\nabla_x u(x, \alpha)|^2 dx, \quad (8)$$

in the wavelet interpolation model (7).

To simplify the analysis, we assume that the functions $u(x)$ and $z(x)$ are defined on an infinite domain with compact supports, which can be achieved by extending to the outside of the given finite regions to zero values smoothly. Under this assumption, the boundary treatment becomes trivial and we omit it in this paper.

We shall start the analysis by decomposing of the interpolation subset I into simple connected regions for each resolution, which become simple subintervals in one dimension. Given the structure of the space V, one can easily write

$$V = \bigcup_{a\in\mathbb{R}^+} \{(c, d) \in V | c = a\} = \bigcup_{a\in\mathbb{R}^+} V_a.$$

Subspaces V_a correspond to different resolutions or scales for different dilation values of a in the wavelet space.

For a given resolution with fixed value of a, we define

$$I_a = I \bigcap V_a,$$

which is the restriction of I onto the subspace V_a. It is easy to see that I_a is the subset to be interpolated on the resolution a. This leads to

$$I = \bigcup_a I_a,$$

which simply states that the interpolation subset I can be decomposed into subsets I_a on different resolutions a. It is worth to remind that a is taken as discrete values $a_j = 2^j$ in the discrete wavelet interpolation problem. In the one dimensional case, it is obvious that I_{a_j} is just a measurable subset of $\mathbb{R}$. One can further divide it into disjoint subintervals

$$I_{a_j} = \bigcup_m I_{a_j,m} = \bigcup_m (b^1_{a_j,m}, b^2_{a_j,m}),$$

with

$$I_{a_j,m} \bigcap I_{a_j,n} = \phi, \quad \text{for} \quad m \neq n,$$

and ϕ is the empty set. In other words, $I_{a_j,m} = (b^1_{a_j,m}, b^2_{a_j,m})$ is a simple connected subregion to be interpolated on the resolution a_j. The wavelet coefficients at two ending points $\alpha(a_j, b^1_{a_j,m}), \alpha(a_j, b^2_{a_j,m})$ are known to be $\beta(a_j, b^1_{a_j,m})$ and $\beta(a_j, b^2_{a_j,m})$ respectively. We call the width of the subinterval $|I_{a_j,m}| = |b^2_{a_j,m} - b^1_{a_j,m}|$ the local size of the interpolation region. We denote

$$\epsilon = \inf_{a_j} \max_m |I_{a_j,m}|,$$

which is the largest width of all subinterval, or the maximum value of the local sizes of the interpolation regions.

Theorem 1. *Assume $u(x)$ is a minimizer of (8). If the wavelet function $\psi(x)$ is in C^2 and $\frac{d^2\psi(x)}{dx^2} \in L^2$, then the continuous wavelet transform $\alpha(a,b)$ of $u(x)$ is C^2 with respect to b, and satisfies*

$$\begin{cases} -\Delta_b \alpha(a_j, b_k) = 0, & \textit{for all sample points } (a_j, b_k) \in I_{a_j,m} \\ \alpha(a_j, b^1_{a_j,m}) = \beta(a_j, b^1_{a_j,m}), & \alpha(a_j, b^2_{a_j,m}) = \beta(a_j, b^2_{a_j,m}), \end{cases} \tag{9}$$

and

$$|\Delta_b \alpha(a_j, b)| \leq a_j^{-1} \|z\|_{H^1} \|\psi\|_{H^1}, \tag{10}$$

where $\Delta_b = \frac{\partial^2}{\partial b^2}$ is the Laplace operator with respect to b for each fixed resolution a_j, and $\|\cdot\|_{H^1}$ is the standard H^1 semi norm.

Proof From the definition

$$\alpha(a,b) = \int u(x)\psi_{a,b}(x)dx,$$

we have

$$\Delta_b\alpha(a,b) = \int u(x)\Delta_b\psi_{a,b}(x)dx.$$

Using the dilation and translation structure (2) of $\psi_{a,b}(x)$, we observe

$$\nabla_x\psi_{a,b}(x) = -\nabla_b\psi_{a,b}(x), \quad \text{and} \quad \Delta_x\psi_{a,b}(x) = \Delta_b\psi_{a,b}(x).$$

These lead to

$$\Delta_b\alpha(a,b) = \int u(x)\Delta_x\psi_{a,b}(x)dx = \int \Delta_x u(x)\psi_{a,b}(x)dx < \infty, \tag{11}$$

which is continuous with respect to b.

Let us denote γ_{a_j,b_k} a unit vector taking the only nonzero value at a sample point (a_j,b_k).We consider the partial directional derivative of $(\partial_\alpha F)(\gamma_{a_j,b_k})$ defined by

$$\begin{aligned}
(\partial_\alpha F)(\gamma_{a_j,b_k}) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}(F(\alpha+\epsilon\gamma_{a_j,b_k}) - F(\alpha)) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int(|(\nabla_x(u(x,\alpha+\epsilon\gamma_{a_j,b_k}))|^2 - |\nabla_x u(x,\alpha)|^2)dx \\
&= \int 2\nabla_x u(x)\nabla_x\psi_{a_j,b_k}(x)dx \\
&= -2\int \Delta_x u(x)\psi_{a_j,b_k}(x)dx.
\end{aligned}$$

It is known from calculus of variation that the minimizer of (8) must satisfy

$$(\partial_\alpha F)(\gamma_{a_j,b_k}) = 0,$$

which implies

$$\int \Delta_x u(x)\psi_{a_j,b_k}(x)dx = 0, \quad \text{for any sample point } (a_j,b_k) \in I_{a_j,m}.$$

This is the Euler-Lagrange equation for the variational problem in wavelet space. From this equation and (11), we get (9).

We also have

$$
\begin{aligned}
|\Delta_b \alpha(a_j, b)| &= |\int u(x) \Delta_b \psi_{a_j,b}(x) dx| \\
&= |\int u(x) \Delta_x \psi_{a_j,b}(x) dx| \\
&= |-\int \nabla_x u(x) \nabla_x \psi_{a_j,b}(x) dx| \\
&\leq \left(\int |\nabla_x u(x)|^2 dx \right)^{\frac{1}{2}} \left(\int |\nabla_x \psi_{a_j,b}(x)|^2 dx \right)^{\frac{1}{2}} \\
&= a_j^{-1} \|u(x)\|_{H^1} \|\psi\|_{H^1}.
\end{aligned}
$$

Since $u(x)$ is a minimizer, we must have $\|u(x)\|_{H^1} \leq \|z(x)\|_{H^1}$ which completes the estimate (10) and the proof the theorem.

Theorem 2. *If the wavelet function $\psi(x)$ is in C^2 and $\frac{d^2\psi(x)}{dx^2} \in L^2$, then the discrete wavelet transform $\alpha(a_j, b_k)$ of the minimizer $u(x)$ of (8) satisfies*

$$
|\alpha(a_j, b_k) - \beta(a_j, b_k)| \leq 2a_j^{-1} \epsilon^2 \|z\|_{H^1} \|\psi\|_{H^1}. \tag{12}
$$

Proof For each fixed resolution a_j, we define

$$
g(b) = \alpha(a_j, b) - \beta(a_j, b),
$$

which is C^2 with respect to b. Let us consider this function on the interpolation interval $I_{a_j,m}$. The interpolation problem ensures that $g(b)$ vanishes at two ending points of $I_{a_j,m}$ because $\alpha(a_j, b)$ and $\beta(a_j, b)$ take the same values, i.e.

$$
g(b^1_{a_j,m}) = g(b^2_{a_j,m}) = 0.
$$

Given any one point $b \in I_{a_j,m}$, we have Taylor expansions,

$$
g(b^1_{a_j,m}) = g(b) + g'(b)(b^1_{a_j,m} - b) + \frac{1}{2} g''(\xi_1)(b^1_{a_j,m} - b)^2,
$$

and

$$
g(b^2_{a_j,m}) = g(b) + g'(b)(b^2_{a_j,m} - b) + \frac{1}{2} g''(\xi_2)(b^2_{a_j,m} - b)^2,
$$

where ξ_1 and ξ_2 are two points in $I_{a_j,m}$. Thus

$$
\begin{aligned}
g(b) &= g(b) - \frac{b^2_{a_j,m} - b}{b^2_{a_j,m} - b^1_{a_j,m}} g(b^1_{a_j,m}) - \frac{b - b^1_{a_j,m}}{b^2_{a_j,m} - b^1_{a_j,m}} g(b^2_{a_j,m}) \\
&= \frac{1}{2}(g''(\xi_1)(b^1_{a_j,m} - b)^2 + g''(\xi_2)(b^2_{a_j,m} - b)^2) \\
&\leq \max_{\xi \in I_{a_j,m}} |g''(\xi)| \epsilon^2
\end{aligned}
$$

Similar to the proof of (10), we obtain

$$\begin{aligned}|g''(b)| &= |\Delta_b(\alpha(a_j, b) - \beta(a_j, b))| \\ &= |\int (u(x) - z(x))\Delta_b \psi_{a_j,b}(x)dx| \\ &= |\int (u(x) - z(x))\Delta_x \psi_{a_j,b}(x)dx| \\ &= |-\int \nabla_x(u(x) - z(x))\nabla_x \psi_{a_j,b}(x)dx| \\ &\leq 2a_j^{-1}\|z(x)\|_{H^1}\|\psi(x)\|_{H^1},\end{aligned}$$

which completes the proof.

We remark that for multi-dimensional wavelet interpolation problems, Theorems 1 and 2 still hold with the understanding that $I_{a,m}$ becomes multi-dimensional regions. We will not address them in detail in this paper.

4 A Numerical Example

The estimate obtained in Section 3 shows that the approximation error for the H^1 wavelet interpolation model is bounded quadratically by the local size of the interpolation regions. In this section, we compute the H^1 wavelet interpolations of a simple function

$$z(x) = sin(4\pi x) \quad x \in (0, 1).$$

To illustrate the quadratic rate, we arbitrarily select l consecutive low frequency coefficients to be interpolated. We note that the doubled number l corresponds to the doubled size ϵ of the local interpolation region. We measure the maximum approximation error in the coefficients defined by

$$\text{EIC} = \max_k |\alpha_{a_j,b_k} - \beta_{a_j,b_k}|.$$

And the error rate is calculated by

$$\text{rate} = \log_2(\frac{\text{EIC}(2l)}{\text{EIC}(l)}).$$

The error and its rate for different number l are shown in Table 1. It clearly demonstrates that the error rate is close to 2 if the interpolation region is in the low frequencies. We also remark that our numerical experiments show that if the interpolation regions do not contain low frequencies, the error is much smaller than the quadratic estimate, which suggests that the rate may be improved if no low frequency coefficient is interpolated.

Conclusion and future work: The analysis shows that the recovery property of H^1 wavelet interpolation model is bounded quadratically by the local, not global, sizes of the interpolation regions in the wavelet domain,

l	EIC	rate
2	0.00045	1.76
4	0.00151	1.84
8	0.00541	1.89
16	0.02014	1.91
32	0.07565	1.71
64	0.24725	

Table 1. The maximum error in the coefficients for the H^1 wavelet interpolation model. The error rate indicate that error is bounded quadratically by the consecutive number of coefficients to be interpolated.

which is similar to the results for the pixel domain image inpainting problems reported in [6]. It explains that good restorations can be achieved if the local interpolation regions are small even if their total size is large. For instance, if the interpolation regions are randomly distributed as small disjoint regions in the wavelet domain, good interpolation computations are achieved even the total size of the interpolation regions is significant. On the contrary, if there is one large region to be interpolated, the error would be large in this region. This error bound is also consistent with many computations such as these reported in [24] and [9].

The results reported here are for H^1 based wavelet interpolation model. However it is well known that H^1 based model often over smooths edges in images. TV or other nonlinear energy based models can preserve the discontinuities better. The recovery bounds for those models are beyond the scope of this paper and we will not address them here.

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