2

Dissipativity and Passivity

This chapter introduces the concepts of passive and dissipative systems which lay the foundation of the developments described in this book. Most of the definitions follow Willems [138, 139], Byrnes *et al.* [24] and Sepulchre *et al.* [110] with possibly different notations. The implication of passivity is discussed in terms of the input-output behavior and stability of the process system.

2.1 Concept of Passive Systems

Much of the discussion presented in this chapter is related to system stability. Therefore, we start with a brief review of the stability of nonlinear systems. Consider a nonlinear system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\left(x, u\right), \tag{2.1}$$

where $x \in X \subset \mathbb{R}^n$ and $u \in U \subset \mathbb{R}^m$ are the state and input vector variables, respectively. The stability of this system is concerned with its *free* dynamics when the input variable $u = 0$. Assume that

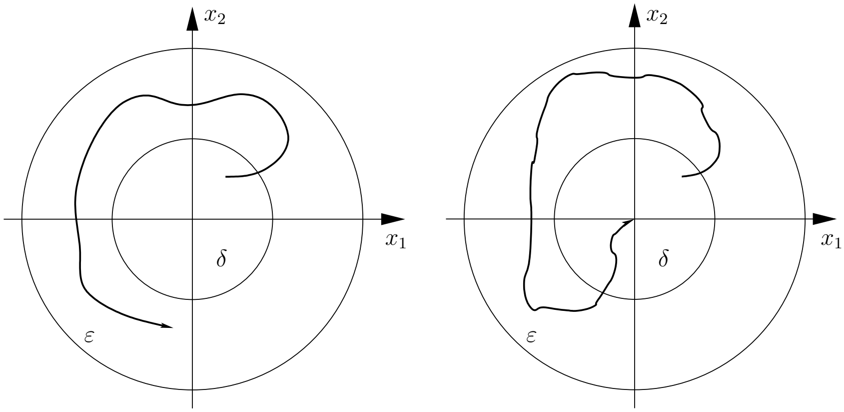
$$f^*\left(x\right) = f\left(x, 0\right), \tag{2.2}$$

where the components of the n dimensional vector $f^*\left(x\right)$ are local Lipschitz functions of x, *i.e.*, $f^*\left(x\right)$ satisfies the following Lipschitz condition:

$$\left\|f^*\left(x_1\right) - f^*\left(x_2\right)\right\| \leq L\left\|x_1 - x_2\right\| \tag{2.3}$$

for all x_1, x_2 in a neighbourhood of x_0, where L is a positive constant and $\|\cdot\|$ is the Euclidean norm (*i.e.*, $\|x\| = \sqrt{x^T x}$). The Lipschitz condition guarantees that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f^*\left(x\right) \tag{2.4}$$



(a) Stable (b) Asymptotically stable

Fig. 2.1. Lyapunov stability

has a unique solution with the initial condition $x(0) = x_0$. A point $x^* \in X$ is called an equilibrium point of (2.4) if $f^*(x^*) = 0$. The equilibrium point $x = 0$ is *stable* if for each $\varepsilon > 0$, $\delta = \delta(\varepsilon) > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| < \varepsilon$ for all $t \geq 0$ (as shown in Figure 2.1a for $X \subset \mathbb{R}^2$). This equilibrium point is said to be *asymptotically stable* (AS) if it is stable and δ can be chosen such that $\|x(0)\| < \delta$ implies that $x(t)$ approaches the origin as t tends to infinity (as shown in Figure 2.1b). When the origin is asymptotically stable, the *region of attraction* is defined as the set of initial points $x(0)$ such that the solution of (2.4) approaches the origin as t tends to infinity. If the region of attraction is the entire state-space X, then the origin is *globally asymptotically stable* (GAS) [66]. Unlike linear systems, a nonlinear system may have multiple equilibrium points, of which some are stable and some are unstable. A sufficient condition for the stability of an equilibrium point is given by the Lyapunov stability criterion, which can be used to determine the stability of an equilibrium point without solving the state equation. Let $V(x)$ be a continuously differentiable (also denoted as C^1) scalar function defined in X that contains the origin. A function $V(x)$ is said to be *positive definite* if

$$V(0) = 0 \quad \text{and} \quad V(x) > 0, \quad \forall\, x \neq 0. \tag{2.5}$$

It is said to be *positive semidefinite* if

$$V(x) \geq 0, \quad \forall\, x. \tag{2.6}$$

Similarly, a function $V(x)$ is said to be *negative definite* if $V(0) = 0$ and $V(x) < 0$ for $x \neq 0$ and is said to be *negative semidefinite* if $V(x) \leq 0$ for all x.

Theorem 2.1 (Lyapunov stability criterion [67]). *Let $x = 0$ be an equilibrium point of a system described by (2.4). Function f^* is locally Lipschitz* *and X contains the origin. The origin is stable if there exists a C^1 positive definite function $V(x) : X \to \mathbb{R}$ such that $\frac{\mathrm{d}V(x)}{\mathrm{d}t}$ is negative semidefinite and it is asymptotically stable if $\frac{\mathrm{d}V(x)}{\mathrm{d}t}$ is negative definite, where $\frac{\mathrm{d}V(x)}{\mathrm{d}t}$ is the derivative along the trajectory of (2.4), i.e.,*

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} = \frac{\partial V(x)}{\partial x} f^*(x). \tag{2.7}$$

The function $V(x)$ in the above theorem, if it exists, is called a *Lyapunov function.*

A stronger type of stability is called *exponential stability*, which is defined as follows:

Definition 2.2 (Exponential stability [67]). *A system is globally exponentially stable (GES) if and only if there exists a Lyapunov function $V(x)$ such that*

$$\rho_1|x|^2 \le V(x) \le \rho_2|x|^2, \tag{2.8}$$

and with zero input,

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \le -\rho_3|x|^2, \tag{2.9}$$

where $\rho_i > 0$, $i = 1, 2, 3$ are suitable scalar constants. If these conditions hold, it follows that there exists some constant $\rho \ge 0$ such that with $x(0) = x_0$,

$$|x(t)| \le \rho|x_0|e^{-\rho_3 t/2} \quad \forall\ t \ge 0. \tag{2.10}$$

If the above condition is valid for x only in a neighbourhood of $x = 0$, the system is locally exponentially stable (LES).

First introduced by Popov [95], the concept of passive systems originally arose in the context of electrical circuit theory. A network consisting of only passive components, *e.g.*, inductors, resistors and capacitors, does not generate any energy and therefore is stable (*e.g.*, [6, 49]). In the early 1970s, Willems [138, 139] developed a systematic framework for dissipative systems, including passive systems, by introducing the notation of a storage function and a supply rate. Passivity, dissipativity and relevant stability conditions are cornerstones of modern control theory. In this section, an introduction to passive systems is presented through a very simple example of a gravity tank, followed by rigorous definitions.

Example 2.3 (Gravity tank). Consider the gravity under flow tank system illustrated in Figure 2.2. Assume that the input is the inlet volumetric flow rate $u = F_i(t)$, the state variable is the liquid level $x(t)$ and the output variable is the liquid pressure $y = p(t) = \rho g x(t)$. (Liquid pressure measurement is often used in level control.) Suppose that the outlet is flowing under the influence of gravity, *i.e.*,

$$F_o(t) = C_v\sqrt{x(t)}, \tag{2.11}$$

where C_v denotes the valve coefficient and F_o is the mass flow rate. The mass balance is given by

$$\rho A \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \rho F_i(t) - \rho F_o(t) = \rho F_i(t) - \rho C_v \sqrt{x(t)}, \tag{2.12}$$

leading to

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= -\frac{C_v}{A}\sqrt{x(t)} + \frac{1}{A}F_i(t), \\ y(t) &= p(t) = \rho g x(t), \end{aligned} \tag{2.13}$$

where A is the cross-sectional area of the tank and ρ is the density of the liquid. Denote the mass in the tank as m. Half of the potential energy stored in the tank is given by the following equation:

$$S(t) = S(x(t)) = \frac{1}{2}m(t)gx(t) = \frac{1}{2}[\rho A x(t)]gx(t) = \frac{1}{2}A\rho g x^2(t). \tag{2.14}$$

The inlet flow into the system increases the potential energy in the tank. The increment of potential energy per unit time can be represented by a function of the input and output:

$$w(t) = y(t)u(t) = \rho g F_i(t)x(t). \tag{2.15}$$

The rate of change of the potential energy is given by taking the derivative along the trajectory of $x(t)$:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \frac{\partial S}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = A\rho g x(t)\left[\frac{1}{A}\left(F_i(t) - C_v\sqrt{x(t)}\right)\right] \tag{2.16}$$

$$= -C_v \rho g x(t)\sqrt{x(t)} + \rho g F_i(t)x(t) \tag{2.17}$$

$$< w(t). \tag{2.18}$$

Note that in the range of definition of x, the first term of (2.17) is always negative. Therefore the rate of change of the stored energy in the tank is less than that supplied to it by the inlet flow rate (represented by $w(t)$). As such, the tank system "dissipates" its potential energy through both the inlet flow (F_i) and the liquid pressure p, which is a function of both the input and output. This is called a *dissipative system*. Because the potential energy $S(t)$ is a positive definite function of the state variable $x(t)$, it can be treated as a Lyapunov function. When $F_i(t) = 0$,

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} < 0, \quad \forall\, x \neq 0. \tag{2.19}$$

Therefore, the equilibrium $x = 0$ is asymptotically stable (AS). If the outlet valve is completely shut off (*i.e.*, $C_v = 0$), then the energy flow into the tank is totally stored. In this case, this process becomes *lossless* and the equilibrium $x = 0$ is stable.

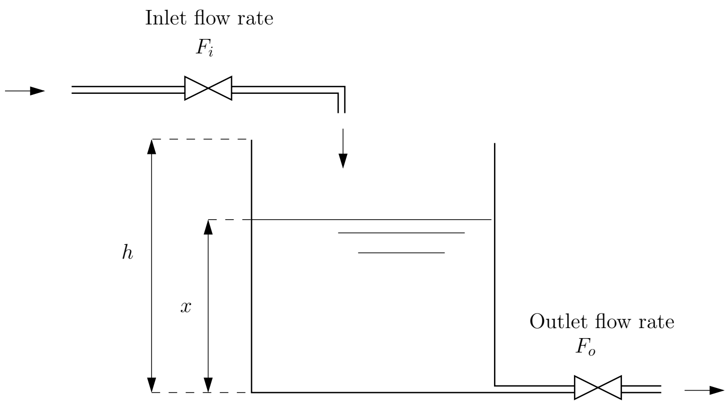


Fig. 2.2. A gravity tank system

Comparing (2.17) with (2.19), it can be seen that by introducing the energy function, (2.17) gives the stability of a free system (with zero input) and also how its input and output affect the state variable. If we generalize the energy function to any nonnegative function of the states, then we can define a class of nonlinear processes. Consider the following nonlinear system:

$$H : \begin{cases} \dot{x} &= f\,(x, u) \\ y &= h(x, u), \end{cases} \tag{2.20}$$

where $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ and $y \in Y \subset \mathbb{R}^m$ are the state, input and output variables, respectively, and X, U and Y are state, input and output spaces, respectively. The representation $x(t) = \phi(t, t_0, x_0, u)$ is used to denote the state at time t reached from the initial state x_0 at t_0.

Definition 2.4 (Supply rate [138]). *The supply rate $w(t) = w(u(t), y(t))$ is a real valued function defined on $U \times Y$, such that for any $u\,(t) \in U$ and $x_0 \in X$ and $y(t) = h(\phi(t, t_0, x_0, u))$, $w(t)$ satisfies*

$$\int_{t_0}^{t_1} |w(t)|\, dt < \infty \tag{2.21}$$

for all $t_1 \geq t_0 \geq 0$.

Definition 2.5 (Dissipative systems [138]). *System H with supply rate $w(t)$ is said to be dissipative if there exists a nonnegative real function $S(x)$: $X \rightarrow \mathbb{R}^+$, called the storage function, such that, for all $t_1 \geq t_0 \geq 0$, $x_0 \in X$ and $u \in U$,*

$$S(x_1) - S(x_0) \leq \int_{t_0}^{t_1} w(t)dt, \tag{2.22}$$

where $x_1 = \phi(t_1, t_0, x_0, u)$ and $\mathbb{R}^+$ is a set of nonnegative real numbers.

The above condition states that a system is dissipative if the increase in its energy (storage function) during the interval (t_0, t_1) is no greater than the energy supplied (via the supply rate) to it. If the storage function is differentiable, *i.e.*, it is C^1, then we can write (2.22) as

$$\frac{\mathrm{d}S\left(x\left(t\right)\right)}{\mathrm{d}t} \leq w(t). \tag{2.23}$$

The interpretation is that the rate of increase of energy is no greater than the input power.

According to the above definition, a storage function has to be positive semidefinite. The next definition describes the notion of available storage, the largest amount of energy that can be extracted from the system given the initial condition $x\left(0\right) = x$:

Definition 2.6 (Available storage [138]). *The available storage, S_a of a system H with supply rate w, is the function $S_a : X \rightarrow \mathbb{R}^+$ defined by:*

$$S_a\left(x\right) \triangleq \sup_{\substack{x(0)=x \\ u(t)\in U \\ t_1>0}} \left\{ -\int_0^{t_1} w\left(u\left(t\right), y\left(t\right)\right) dt \right\}. \tag{2.24}$$

The available storage is nonnegative, since $S_a\left(x\right)$ is the supremum over a set of values including the zero element. The available storage function plays an important role in dissipative/passive systems. If a system is dissipative, the available storage function $S_a(x)$ is finite for each $x \in X$. Moreover, any possible storage function $S(x)$ satisfies

$$0 \leqslant S_a(x) \leqslant S(x) \tag{2.25}$$

for each $x \in X$. If S_a is a continuous (C^0) function, then S_a itself is a possible storage function. Conversely, if $S_a\left(x\right)$ is finite for every $x \in X$, then the system is dissipative with respect to the supply rate $w(t)$.

The supply rate can be any function defined on the input and output space that satisfies (2.21). When a bilinear supply rate is adopted, passive systems can be defined as:

Definition 2.7 (Passive systems [24]). *A system is said to be passive if it is dissipative with respect to the following supply rate:*

$$w\left(u\left(t\right), y\left(t\right)\right) = u^T\left(t\right) y\left(t\right), \tag{2.26}$$

and the storage function $S\left(x\right)$ satisfies $S(0) = 0$.

Two extreme cases of passive systems are lossless and state strictly passive systems:

Definition 2.8 (Lossless systems [24]). *A passive system H with storage function $S(x)$ is said to be lossless if for all $t_1 \geq t_0 \geq 0$, $x_0 \in X$ and $u \in U$,*

$$S(x) - S(x_0) = \int_{t_0}^{t_1} y^T(t)\, u(t)\, dt. \tag{2.27}$$

Definition 2.9 (State strictly passive systems [24]). *A passive system H with storage function $S(x)$ is said to be state strictly passive if there exists a positive definite function $V : X \to \mathbb{R}^+$ such that for all $t_1 \geq t_0 \geq 0$, $x_0 \in X$ and $u \in U$,*

$$S(x) - S(x_0) = \int_{t_0}^{t_1} y^T(t)\, u(t)\, dt - \int_{t_0}^{t_1} V(x(t))\, dt. \tag{2.28}$$

This definition is referred to as *strict passivity* in [24]. Here we define it as *state strict passivity* to discriminate it from other types of strict passivity discussed later in this book, such as strict input passivity and strict output passivity.

In the tank system example, the storage function is the total potential energy stored in the tank system, given by (2.14). The supply rate given by (2.15) is the inner product of the input and output. Therefore, the tank system is state strictly passive when the outlet valve is open and is lossless when the outlet valve is closed. Storage functions are not limited to physical energies. Any nonnegative real functions defined on state variables can be understood as a type of *abstract energy*, like the Lyapunov functions. They are potential candidates for the storage functions. For example, for the tank system, if we choose the output as the liquid level x, then the supply rate $w(t) = F_i(t)\, x(t)$. With the storage function $S(x) = \frac{1}{2}Ax^2$, it is obvious that

$$\frac{\mathrm{d}S(x)}{\mathrm{d}t} = -C_v x(t) \sqrt{x(t)} + F_i(t)\, x(t) < w(t), \tag{2.29}$$

which shows that the process is passive (more precisely, state strictly passive). In this case, the physical meanings of the storage function and supply rate are not explicit, making it more difficult to determine its passivity directly from our understanding of the mass and energy balance. However, the process possesses all the useful properties of passive systems that we are going to discuss in the next section.

2.2 Properties of Passive Systems

2.2.1 Stability of Passive Systems

In the tank example, we can see that the concept of passivity implies stability if a positive definite storage function is used. Because the storage function is only required to be positive semidefinite in Definition 2.5, stability is not always ensured by passivity. For example, if a system has two states $x = [x_1, x_2]^T$ and the storage function is positive semidefinite, *e.g.*, $S(x) = \frac{1}{2}x_1^2$, then passivity with this storage function does not imply the stability of x_2. In this case, additional conditions on zero-state detectability and observability are required:

Definition 2.10 (Zero-state observability and detectability [24]). *A system as given in (2.20) is zero-state observable (ZSO) if for any $x \in X$,*

$$y(t) = h(\phi(t, t_0, x, 0)) = 0, \quad \forall\, t \geq t_0 \geq 0 \ \text{ implies } x = 0, \tag{2.30}$$

and the system is locally ZSO if there exists a neighbourhood X_n of 0, such that for all $x \in X_n$, (2.30) holds. The system is zero-state detectable (ZSD) if for any $x \in X$,

$$y(t) = h(\phi(t, t_0, x, 0)) = 0, \quad \forall\, t \geq t_0 \geq 0 \ \text{ implies } \lim_{t \to \infty} \phi(t, t_0, x, 0) = 0, \tag{2.31}$$

and the system is locally ZSD if there exists a neighbourhood X_n of 0, such that for all $x \in X_n$, (2.31) holds.

With the definition of zero-state detectability (ZSD), the link between passivity and Lyapunov stability can be established:

Theorem 2.11 (Passivity and stability [110]). *Let a system H (as represented in (2.20)) be passive with a C^1 storage function $S(x)$ and $h(x, u)$ be C^1 in u for all x. Then the following properties hold:*

1. *If $S(x)$ is positive definite, then the equilibrium $x = 0$ of H with $u = 0$ is Lyapunov stable.*
2. *If H is ZSD, then the equilibrium $x = 0$ of H with $u = 0$ is Lyapunov stable.*
3. *If in addition to either Condition 1 or Condition 2, $S(x)$ is radially unbounded (i.e., $S(x) \to \infty$ as $\|x\| \to \infty$), then the equilibrium $x = 0$ in the above conditions is globally stable (GS).*

It can be also found that if system H is state strictly passive with a positive definite storage function, then the equilibrium $x = 0$ with $u = 0$ is asymptotically stable. The boundedness of the storage function implies the boundedness of the state variables. However, passivity tells more than just stability. It relates the input and output to the storage function and thus defines a set of useful input-output properties, which are explained in the next section.

2.2.2 Kalman–Yacubovich–Popov Property

One of the most important properties of passive systems is related to the following definition:

Definition 2.12 (Kalman–Yacubovich–Popov property [24]). *Consider a control affine system without throughput (as a special case of the system in (2.20)):*

$$H : \begin{cases} \dot{x} & = f(x) + g(x)\,u \\ y & = h(x), \end{cases} \tag{2.32}$$

where $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ and $y \in Y \subset \mathbb{R}^m$. It is said to have the Kalman–Yacubovitch–Popov (KYP) property if there exists a C^1 nonnegative function $S(x) : X \to \mathbb{R}^+$, with $S(0) = 0$ such that

$$L_f S(x) = \frac{\partial S(x)}{\partial x} f(x) \leq 0, \tag{2.33}$$

$$L_g S(x) = \frac{\partial S(x)}{\partial x} g(x) = h^T(x), \tag{2.34}$$

for each $x \in X$.

The term $L_f S(x) = \frac{\partial S(x)}{\partial x} f(x)$ is called the *Lie derivative*, which is defined as follows:

Definition 2.13 (Lie derivative). *Given a C^1 nonlinear scalar function $S(x) : \mathbb{R}^n \to \mathbb{R}$ and a vector function:*

$$f(x) = [f_1(x), f_2(x), \cdots, f_n(x)]^T \in \mathbb{R}^n \to \mathbb{R}^n, \tag{2.35}$$

on a common domain $X \subset \mathbb{R}^n$. The derivative of $S(x)$ along f is defined as

$$L_f S(x) = \frac{\partial S(x)}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial S(x)}{\partial x_i} f_i(x). \tag{2.36}$$

The repeated Lie derivative is defined as

$$L_f^k S(x) = \frac{\partial \left(L_f^{k-1} S(x) \right)}{\partial x} f(x), \tag{2.37}$$

with $L_f^0 S(x) = S(x)$.

Proposition 2.14 ([57]). *A system H which has the KYP property is passive, with a storage function $S(x)$. Conversely, a passive system having a C^1 storage function has the KYP property.*

For the tank system in Example 2.3, $f(x) = -\frac{C_v}{A}\sqrt{x}$, $g(x) = \frac{1}{A}$ and $h(x) = \rho g x(t)$. With the storage function defined in (2.14), it is easy to verify that the tank system has the KYP property. Because the liquid level $x(t) \geq 0$,

$$L_f S(x) = -\rho g C_v x(t) \sqrt{x(t)} \leq 0, \tag{2.38}$$

$$L_g S(x) = A \rho g x(t) \frac{1}{A} = \rho g x(t) = y(t). \tag{2.39}$$

For a linear time invariant (LTI) system, there exists a quadratic storage function $S(x) = x^T Px$ (with a positive definite matrix P), leading to the following linear version of the KYP condition:

Proposition 2.15 ([139]). *Consider a stable LTI system:*[1]

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{2.40}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$. This system is passive if and only if there exist matrices $P, L \in \mathbb{R}^{n\times n}, Q \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ with $P > 0$, $L > 0$ (positive definite) such that

$$\begin{aligned} A^T P + PA &= -Q^T Q - L, \\ B^T P - C &= -W^T Q, \\ W^T W &= D + D^T. \end{aligned} \tag{2.41}$$

For systems with relative degree 0 (*i.e.*, $D \neq 0$), the above condition can be represented using a linear matrix inequality (LMI), which is often referred to as the positive-real lemma:

Lemma 2.16 (Positive-real Lemma [21]). *A stable LTI system given in (2.40) with $D \neq 0$ is passive if and only if there exists a positive definite matrix P such that:*

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D - D^T \end{bmatrix} < 0. \tag{2.42}$$

When $D = 0$, the above condition is reduced to

$$A^T P + PA < 0, \tag{2.43}$$

$$B^T P = C. \tag{2.44}$$

Equations (2.43) and (2.44) are the linear versions of (2.33) and (2.34), respectively.

2.2.3 Input-Output Property

Obviously, while (2.33) is related to the stability, (2.34) defines an input-output property. The input-output property of passive systems is called *positive realness*:

[1] In this book, the linear system given in (2.40) is said to be *stable* if $\mathrm{Re}[\lambda_i(A)] < 0$, $\forall\ i = 1, \ldots, n$. The system is actually *asymptotically stable* according to Section 2.1.

Definition 2.17 (Positive real systems [24]). *A system is said to be positive real if for all* $t_1 \geq t_0 \geq 0$, $u \in U$,

$$\int_{t_0}^{t_1} y^T(t)u(t)dt \geqslant 0, \tag{2.45}$$

whenever $x(t_0) = 0$.

The initial condition of the state variable $x_0 = x(t_0) = 0$ (consequently $S(x_0) = 0$) is assumed because positive realness is only an input-output property, which says nothing about the states. Clearly, passive systems are positive real. To tell whether a positive real system is passive, we need an additional *reachability* condition:

Definition 2.18 (Reachability and controllability [138]). *The state-space of a dynamic system* H *(as in (2.20)) is said to be reachable from* x_{-1} *if for any* $x \in X$*, there exists a* $t_{-1} \leq 0$ *and* $u \in U$ *such that*

$$x = \phi(0, t_{-1}, x_{-1}, u). \tag{2.46}$$

It is said to be controllable to x_1 *if for any* $x \in X$*, there exists a* $t_1 \geq 0$ *and* $u \in U$ *such that*

$$x_1 = \phi(t_1, 0, x, u). \tag{2.47}$$

A positive real system is passive if any state is reachable from the origin and S_a is at least continuous (C^0). A thorough treatment of passive systems from the perspective of input-output systems can be found in [32]. In the case of linear systems, *positive realness* and *passivity* are synonyms, provided that the system is detectable.

The input-output relationship is often more conveniently represented by system operators. The system operator is a mapping defined on signal spaces. For example, system H with the input and output signals $u(t)$ and $y(t)$ can be understood as a mapping from u to y (with certain initial conditions on the state variable $x(t)$). In this case, $H : u(t) \longmapsto y(t)$ is a system operator and the system output can be represented as

$$y(t) = Hu(t). \tag{2.48}$$

The mapping from $y(t)$ to $u(t)$ is referred to as the inverse of H, denoted as H^{-1}. For a vector signal function on time $f(t) = [f_1(t), \ldots, f_m(t)]^T$, where $f_i(t)$ are scalar functions and $t \geq 0$, the "size" of the signal can be quantified by using norms. Here we introduce the so-called 2-norm:

Definition 2.19 (2-norm of a signal). *The 2-norm of a vector time-domain signal* $f(t) \in \mathbb{R}^m$ *is defined as*

$$\|f\|_2 \triangleq \left[\sum_{i=1}^{m} \int_0^\infty f_i^2(t)\, dt\right]^{\frac{1}{2}} = \sqrt{\int_0^\infty f^T(t)\, f(t)\, dt}. \tag{2.49}$$

Define the inner product as follows:

$$\langle f, g \rangle \triangleq \int_0^\infty f(t)^T g(t)\, dt. \tag{2.50}$$

Then (2.49) can be written as

$$\|f\|_2 = \sqrt{\langle f, f \rangle}. \tag{2.51}$$

The set of vector functions of $f : \mathbb{R}^+ \to \mathbb{R}^m$ which have a bounded 2-norm, *i.e.*

$$\|f\|_2 < \infty, \tag{2.52}$$

are called the $\mathcal{L}_2^m$ space (the superscript indicates the dimension). This is a Hilbert space (a linear space with inner product). The $\mathcal{L}_2^m$ space can be extended to allow functions that are unbounded, when $t \to \infty$, by introducing the truncation operator:

Definition 2.20 (Truncation operator [130]). *Let $f : \mathbb{R}^+ \to \mathbb{R}^m$. Then for each $T \geq 0$, the function $f_T(t)$ is defined by*

$$f_T(t) = \begin{cases} f(t), & 0 \leq t < T \\ 0, & t \geq T, \end{cases} \tag{2.53}$$

and is called the truncation of f to the interval $[0, T]$.

The space that consists of all functions f such that $f_T(t) \in \mathcal{L}_2^m$ is called the extension of $\mathcal{L}_2^m$, denoted as $\mathcal{L}_{2e}^m$. Now the definition of input-output stability can be given as follows:

Definition 2.21 (Input-output stability [130]). *Let $H : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^p$. System H is said to be $\mathcal{L}_2$ stable if $Hu \in \mathcal{L}_2^p$ for any $u \in \mathcal{L}_{2e}^m$.*

The mapping H is said to have finite $\mathcal{L}_2$ gain if there exist finite constants γ and b such that for all $T \geq 0$,

$$\|(Hu)_T\|_2 \leq \gamma \|u_T\|_2 + b, \ \forall\, u \in \mathcal{L}_{2e}^m. \tag{2.54}$$

Using the above definition, we can define passivity from the perspective of the input-output property:

Definition 2.22 ([130]). *Let $H : u \in \mathcal{L}_{2e}^m \mapsto y \in \mathcal{L}_{2e}^m$. Then system H is passive if there exists some constant β such that*

$$\langle Hu, u \rangle_T = \langle y, u \rangle_T \geq \beta, \ \forall\, u \in \mathcal{L}_{2e}^m, \ \forall\, T \geq 0. \tag{2.55}$$

The above inequality is equivalent to the positive real condition given in (2.45), with the assumption $t_0 = 0$. The introduction of constant β is due to the fact that $x(t_0) = 0$ is not assumed in (2.55). One possible case is $\beta = S(x(0))$, where $S(x)$ is the storage function.

Because both (2.45) and (2.55) are symmetrical in terms of u and y, the following proposition is obvious:

Proposition 2.23 ([116]). *Consider a positive real system H which maps u to y. Its inverse (denoted H^{-1}) which maps y to u is also positive real if it exists.*

For stable linear systems, the above input-output property can be defined on the transfer functions by introducing positive real transfer functions.

Definition 2.24 (Positive real transfer function [139]). *A transfer function $G(s)$ is positive real if*

- *$G(s)$ is analytic in $\mathrm{Re}(s) > 0$;*
- *$G(j\omega) + G^*(j\omega) \geq 0$ for any frequency ω that $j\omega$ is not a pole of $G(s)$. If there are poles $p_1, p_2, \ldots, p_q$ of $G(s)$ on the imaginary axis, they are nonrepeated and the residue matrix at the poles $\lim_{s \to p_i} (s - p_i)G(s)$ $(i = 1, \ldots, q)$ is Hermitian and positive semidefinite.*

Transfer function $G(s)$ is said to strictly positive real (SPR) if

- *$G(s)$ is analytic in $\mathrm{Re}(s) \geq 0$;*
- *$G(j\omega) + G^*(j\omega) > 0 \;\forall\; \omega \in (-\infty, +\infty)$.*

Furthermore $G(s)$ is said to be extended strictly positive real (ESPR) if it is SPR and $G(j\infty) + G^*(j\infty) > 0$ [123].

Here, $G^*(j\omega)$ is the complex conjugate transpose of $G(j\omega)$.

Theorem 2.25 ([139]). *A linear system as given in (2.40) is passive (or strictly passive) if and only if its transfer function $G(s) := C(sI - A)^{-1}B + D$ is positive real (or strictly positive real).*

The above theorem (together with Definition 2.24) forms an input-output version of the positive-real lemma in the frequency domain. The above theorem is often used as the definition of linear passive systems. According to Theorem 2.25, $G_1(s) = \frac{1}{s+1}$ is a strictly passive system and $G_2(s) = \frac{1}{s}$ is a passive system. It is worth pointing out that *any* PID controller

$$K(s) = k_c \left[1 + \frac{1}{\tau_I s} + \tau_D s \right], \quad k_c > 0, \tag{2.56}$$

is passive. So is any multiloop PID controller.

2.2.4 Phase-related Properties

The above input-output property implies another interesting characteristic of passive systems – they are *phase bounded*. This is very obvious for SISO passive systems, because the condition $G(j\omega) + G^*(j\omega) \geq 0$ is then reduced to $\mathrm{Re}(G(j\omega)) \geq 0$, which means that the real part of their frequency response is always nonnegative. This is what the term “positive real” originally referred to. Clearly, the phase shift of a stable SISO passive system in response to a sinusoidal input is always within $[-90°, 90°]$ and the phase shift of a SISO strictly passive system is always within $(-90°, 90°)$.

The above statement is also true for multi-input multi-output (MIMO) linear systems. Here we adopt the following phase definition for MIMO systems given by Postlethwaite *et al.*:

Definition 2.26 (Phase of MIMO LTI systems [96]). *Consider an MIMO LTI system with a transfer function $G(s) \in \mathbb{C}^{m\times m}$. Perform the polar decomposition on its frequency response:*

$$\begin{aligned} G(j\omega) &= X(j\omega)\Lambda(j\omega)V^*(j\omega) \\ &= [X(j\omega)V^*(j\omega)]\,[V(j\omega)\Lambda(j\omega)V^*(j\omega)] = U(j\omega)H(j\omega), \end{aligned} \tag{2.57}$$

where $\Lambda(j\omega)$ is an $m \times m$ diagonal, real and nonnegative matrix; $X(j\omega)$ and $V(j\omega)$ are unitary matrices. $U(j\omega) = X(j\omega)V^*(j\omega)$ is also a unitary matrix and $H = V(j\omega)\Lambda(j\omega)V^*(j\omega)$ is a Hermitian matrix. The phase of the system at frequency ω is defined as the principal arguments of the eigenvalues of $U(j\omega)$.

Theorem 2.27 (Phase condition for MIMO LTI strictly passive systems [13]). *Consider an MIMO LTI system with a transfer function $G(s) \in \mathbb{C}^{m\times m}$. If the system is strictly passive, then its phase shift lies in the open interval $(-90°, 90°)$ for any real ω.*

The proof of the above theorem is given in Section B.1. If the frequency response of a stable linear system has a phase shift within $[-90°, 90°]$ for all frequencies, this system also satisfies both of the following conditions:

1. it is minimum phase;
2. the difference between the degree of the denominator polynomial and the degree of the numerator polynomial (*i.e.*, the relative degree) is less than 2.

This can be illustrated by a simple SISO case. Consider a stable and minimum phase transfer function $G(s) = \frac{p(s)}{q(s)}$ with $G(0) > 0$, where the numerator polynomial $p(s)$ is of mth order and the denominator polynomial $q(s)$ is of nth order. Because $G(s)$ has only left half plane (LHP) zeros and poles at frequency $\omega = \infty$, the phase shift will be $90°(n-m)$. Therefore, for the system to be phase bounded by $[-90°, 0°]$ at all frequencies, it must satisfy $n - m < 2$. (A positive phase shift will occur when $G(0) < 0$.)

Phase is not defined for nonlinear systems. However, the above phase-related conditions can be extended to nonlinear systems. The relative degree can be understood as the number of times one has to differentiate the output to have the input explicitly appearing. Therefore, we can define the relative degree for nonlinear systems as follows:

Definition 2.28 (Relative degree [61]). *A SISO control affine nonlinear system*

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x), \end{aligned} \tag{2.58}$$

is said to have relative degree r at point x_0 if

1. $L_g L_f^k h(x) = 0$ *for all x in a neighbourhood of x_0 and all $k < r - 1$;*
2. $L_g L_f^{r-1} h(x_0) \neq 0$,
   where $L_f^k h(x)$ is the kth order Lie derivative of h along f.
   A multivariable nonlinear control affine system as in the following equation:

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{q} g_j(x) u_j, \\ y_i &= h_i(x), \quad i = 1, \ldots, p, \end{aligned} \tag{2.59}$$

has a vector relative degree given by $\{r_1, r_2, \cdots, r_p\}$ at a point x_0 if

1. $L_{g_i} L_f^k h_i(x) = 0, \quad i = 1, \ldots, p,\ k = 0, \ldots, r_i - 2$ *for all x in a neighbourhood of x_0.*
2. *The characteristic matrix $C(x)$, given by*

$$C(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & L_{g_2} L_f^{r_1-1} h_1(x) & \cdots & L_{g_p} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & L_{g_2} L_f^{r_2-1} h_2(x) & \cdots & L_{g_p} L_f^{r_2-1} h_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_p-1} h_p(x) & L_{g_2} L_f^{r_p-1} h_p(x) & \cdots & L_{g_p} L_f^{r_p-1} h_p(x) \end{bmatrix}_{p \times p}$$

is nonsingular at x_0. The total relative degree is defined as $r = \sum_{i=1}^{p} r_i$.

For the linear SISO system $\dot{x} = Ax + Bu$, $y = Cx$, the relative degree is equal to the difference between the degree of the denominator polynomial and the degree of the numerator polynomial of the transfer function $H(s) = C(sI - A)^{-1}B$ of the system. To extend the concept of minimum phase systems to nonlinear systems, we need to look at the zero dynamics:

Definition 2.29 (Zero dynamics). *Consider the system in (2.32) with the constraint $y = 0$, i.e.,*

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ 0 &= h(x). \end{aligned} \tag{2.60}$$

The constrained system (2.60) is called the zero-output dynamics, or briefly, the zero dynamics.

If the matrix $L_g h(0) \triangleq \frac{\partial h(x)}{\partial x} g(x)\Big|_{x=0}$ of the system in (2.32) is nonsingular and the distribution spanned by the vector fields $g_1(x), \cdots, g_m(x)$ is involutive in a neighbourhood of $x = 0$, then there exists new local coordinates (z, y) under which the system can be represented as the so-called normal form:

$$\begin{aligned} \dot{z} &= q(z, y), \\ \dot{y} &= b(z, y) + a(z, y) u. \end{aligned} \tag{2.61}$$

The zero dynamics of system (2.32) are given by

$$\dot{z} = q(z, 0). \tag{2.62}$$

Denote $q(z, 0)$ by $f_0(z)$. Then, the function $q(z, y)$ can be expressed in the form

$$q(z, y) = f_0(z) + p(z, y) y, \tag{2.63}$$

where $p(z, y)$ is a smooth function (see [24]).

Definition 2.30 (Minimum phase nonlinear systems [24]). *Consider the system in (2.32). Suppose that $L_g h(0)$ is nonsingular. Then the system is said to be:*

1. *minimum phase if its zero dynamics are asymptotically stable in a neighbourhood of $z = 0$;*
2. *weakly minimum phase if there exists a positive differentiable function $W(z)$ with $W(0) = 0$, such that*
$$\frac{\partial W(z)}{\partial z} f_0(z) \leq 0 \tag{2.64}$$
in a neighbourhood of $z = 0$.

Similarly, we can define globally minimum phase and globally weakly minimum phase if the normal form and minimum phase are global. Now we are in the position to study the phase-related properties of nonlinear passive systems.

Theorem 2.31 ([24]). *Consider system H given in (2.32). Assume that* rank $\{L_g h(x)\}$ *is constant in a neighbourhood of $x = 0$. If system H is passive with a C^2 storage function $S(x)$ which is positive definite, then*

1. *$L_g h(0)$ is nonsingular and H has relative degree $\{1, \cdots, 1\}$.*
2. *The zero dynamics of H exist locally at $x = 0$, and H is weakly minimum phase.*

Because system H in consideration does not have a feedthrough term, its relative degree could not be below $\{1, \cdots, 1\}$. A passive SISO nonlinear system has a relative degree of 1 or 0 (if there is a feedthrough term). The above theorem shows that nonlinear passive systems have phase-related input-output properties similar to those their linear counterparts possess. These properties imply output feedback stability conditions which will be discussed in the next section.

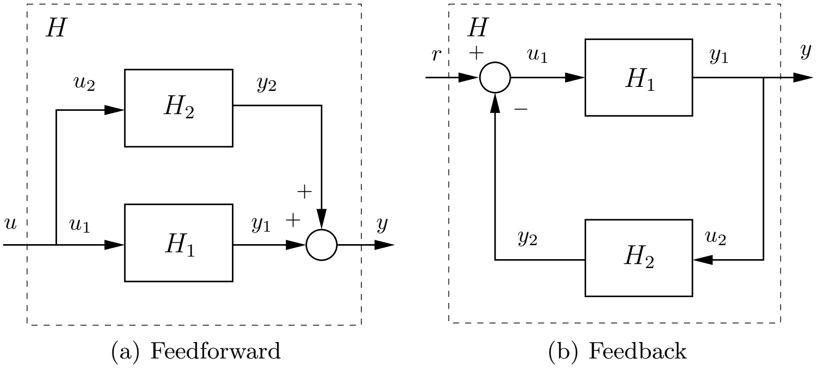


Fig. 2.3. Interconnections of passive systems

2.3 Interconnection of Passive Systems

The phase-related properties of passive systems imply important output feedback stability conditions, which can be used to determine the stability of networks of interconnected systems. A passive system is very easy to control via output feedback. For example, a linear passive system (*e.g.*, $G(s) = \frac{1}{s}$) can be stabilized by *any* proportional only controller with a positive gain. Similarly, we have the following stability condition for nonlinear systems:

Theorem 2.32. *For a nonlinear passive system H given in (2.32), a proportional only output feedback control law $u = -ky$ asymptotically stabilizes the equilibrium $x = 0$ for* any $k > 0$*, provided that H is ZSD.*

Proof. Assume that H is passive with storage function $S(x)$. For $u = -y$, the time derivative of S satisfies

$$\dot{S}(x) \leq -ky^T y < 0, \quad \forall\, y \neq 0. \tag{2.65}$$

The bounded solution of $\dot{x} = f(x, -y)$ is confined in $\{x|\, h(x) = 0\}$. If H is ZSD, then $x \to 0$.

The output feedback stability condition is not limited to static feedback:

Theorem 2.33 (Interconnections of passive systems). *Suppose that systems H_1 and H_2 are passive (as shown in Figure 2.3). Then the two systems, one obtained by the parallel interconnection, and the other obtained by feedback interconnection, are both passive. If systems H_1 and H_2 are ZSD and their respective storage functions $S_1(x_1)$ and $S_2(x_2)$ are C^1, then the equilibrium $(x_1, x_2) = (0, 0)$ of both interconnections is stable.*

Proof. Passivity: Because H_1 and H_2 are passive, there exist two positive semidefinite storage functions $S_1(x_1)$ and $S_2(x_2)$ such that

$$S_i(x_i(t_1)) - S_i(x_i(t_0)) \leq \int_{t_0}^{t_1} u_i^T y_i dt, \quad i = 1,\ 2, \tag{2.66}$$

where x_1, x_2 are the state variables of H_1 and H_2, respectively. Define $x = \left[x_1^T, x_2^T\right]^T$ and $S(x) = S_1(x_1) + S_2(x_2)$. Note that $S(x)$ is positive semidefinite and

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} \left(u_1^T y_1 + u_2^T y_2\right) dt. \tag{2.67}$$

For the parallel interconnection, $u = u_1 = u_2$ and $y = y_1 + y_2$. Therefore,

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} u^T y dt. \tag{2.68}$$

For the feedback case, $u_2 = y_1$ and $u_1 = r - y_2$:

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} r^T y_1 dt. \tag{2.69}$$

Therefore, both interconnections are passive.
If systems H_1 and H_2 are ZSD, the equilibrium $(x_1, x_2) = (0, 0)$ of both interconnections is Lyapunov stable, according to Theorem 2.11.

The above conditions can be extended to partial parallel and feedback connections:

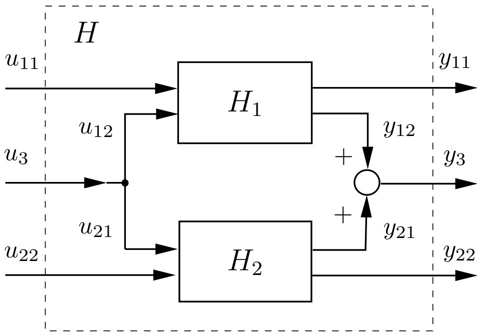
Proposition 2.34 (Partial interconnection of passive systems). *Consider systems $H_1 : u_1 \longmapsto y_1$ and $H_2 : u_2 \longmapsto y_2$, where $u_1 = \left[u_{11}^T, u_{12}^T\right]^T$, $u_2 = \left[u_{21}^T, u_{22}^T\right]^T$, $y_1 = \left[y_{11}^T, y_{12}^T\right]^T$, $y_2 = \left[y_{21}^T, y_{22}^T\right]^T$. If systems H_1 and H_2 are passive, then the two systems, one obtained by partial parallel interconnection, and the other obtained by partial feedback interconnection (as shown in Figure 2.4), are both passive. If systems H_1 and H_2 are ZSD and their respective storage functions $S_1(x_1)$ and $S_2(x_2)$ are C^1, then the equilibrium $(x_1, x_2) = (0, 0)$ of both interconnections is stable.*

Proof. Similar to the proof of Theorem 2.33, because H_1 and H_2 are passive, there exist two positive semidefinite storage functions $S_1(x_1)$ and $S_2(x_2)$ such that

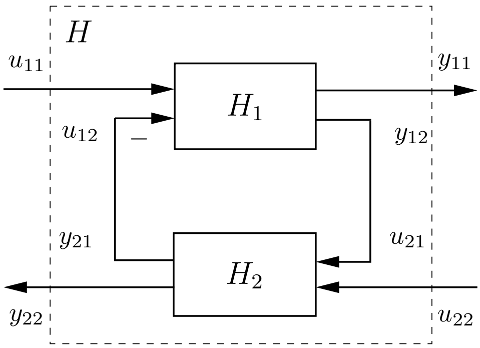
$$S_i(x_i(t_1)) - S_i(x_i(t_0)) \leq \int_{t_0}^{t_1} u_i^T y_i dt,\ i = 1, 2, \tag{2.70}$$

where x_1, x_2 are the state variables of H_1 and H_2, respectively. Define $x = \left[x_1^T, x_2^T\right]^T$ and $S(x) = S_1(x_1) + S_2(x_2)$. Note that $S(x)$ is positive semidefinite and

$$\begin{aligned} S(x(t_1)) - S(x(t_0)) &\leq \int_{t_0}^{t_1} \left(u_1^T y_1 + u_2^T y_2\right) dt \\ &= \int_{t_0}^{t_1} \left(u_{11}^T y_{11} + u_{12}^T y_{12} + u_{21}^T y_{21} + u_{22}^T y_{22}\right) dt. \end{aligned} \tag{2.71}$$



(a) Parallel connection



(b) Feedback connection

Fig. 2.4. Partial interconnected systems

For the partial parallel interconnection, define $u_3 = u_{12} = u_{21}$, $y_3 = y_{12} + y_{21}$. The overall system inputs and outputs are $u = \left[u_{11}^T, u_3^T, u_{22}^T\right]^T$ and $y = \left[y_{11}^T, y_3^T, y_{22}^T\right]^T$, respectively. Therefore,

$$\begin{aligned} S\left(x\left(t_1\right)\right) - S\left(x\left(t_0\right)\right) &\leq \int_{t_0}^{t_1} \left(u_{11}^T y_{11} + u_3^T y_3 + u_{22}^T y_{22}\right) dt \\ &= \int_{t_0}^{t_1} y^T u dt. \end{aligned} \tag{2.72}$$

For the feedback case, $u_{12} = -y_{21}$ and $u_{21} = y_{12}$. The overall system inputs and outputs are $u = \left[u_{11}^T, u_{22}^T\right]^T$ and $y = \left[y_{11}^T, y_{22}^T\right]^T$, respectively. Then

$$\begin{aligned} S\left(x\left(t_1\right)\right) - S\left(x\left(t_0\right)\right) &\leq \int_{t_0}^{t_1} \left(u_{11}^T y_{11} - y_{21}^T y_{12} + y_{12}^T y_{21} + u_{22}^T y_{22}\right) dt \\ &= \int_{t_0}^{t_1} \left(u_{11}^T y_{11} + u_{22}^T y_{22}\right) dt = \int_{t_0}^{t_1} y^T u dt. \end{aligned} \tag{2.73}$$

Therefore, both interconnections are passive. If systems H_1 and H_2 are ZSD, from Theorem 2.11, the equilibrium $(x_1, x_2) = (0, 0)$ of both interconnections is Lyapunov stable.

As a result, if a process is passive, it can be stabilized at the equilibrium point $(x = 0)$ by any passive controller, even if it is highly nonlinear and/or highly coupled. For example, the gravity tank can be stabilized by *any* PID controller with a positive controller gain. The controller gain can be arbitrarily large to reduce the response time without causing instability. This motivates stability analysis and control design based on passivity. The above stability condition can be further extended by introducing the notion of a passivity index.

2.4 Passivity Indices

2.4.1 Excess and Shortage of Passivity

To extend the passivity-based stability conditions to more general cases for both passive and nonpassive systems, we need to define the passivity indices that quantify the degree of passivity. The passivity indices can be defined in terms of an excess or shortage of passivity.

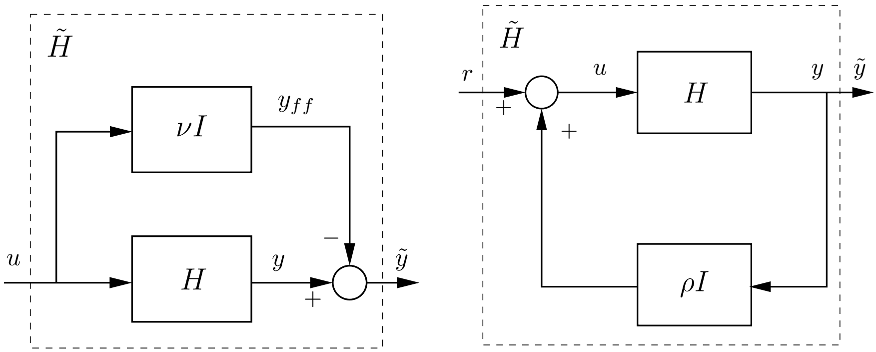
Let system H, as given in (2.32), be passive with a C^1 storage function $S(x)$. Consider a static feedfoward $y_{ff} = -\nu u$ $(\nu > 0)$ such that the overall system $\tilde{H}$ has the output $\tilde{y} = y - \nu u$ (as shown in Figure 2.5a). Because the feedforward is static, its state-space is void. Therefore, the storage function of the overall system remains $S(x)$. If $\tilde{H}$ is also passive, then,

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} u^T \tilde{y} dt = \int_{t_0}^{t_1} \left(u^T y - \nu u^T u\right) dt. \tag{2.74}$$

This is equivalent to the condition that H is dissipative with respect to the supply rate $w(u, y) = u^T y - \nu u^T u$. In this case, system H is said to have excessive input feedforward passivity of ν, denoted as IFP(ν). The feedforward system $-\nu I$ is not passive because $\int_{t_0}^{t_1} u^T y_{ff} dt = \int_{t_0}^{t_1} -\nu u^T u dt < 0$, violating the positive real condition. From this example, it can be seen that the excess of passivity in H can compensate for the shortage of passivity in the feedforward system. Similarly, if H is nonpassive, but it is dissipative with respect to the supply rate $w(u, y) = u^T y + \nu u^T u$ $(\nu > 0)$, then system $H + \nu I$ is passive. In this case, H lacks input feedforward passivity, denoted as IFP($-\nu$).

Another situation is the negative feedback interconnection (as shown in Figure 2.5b). Let $\tilde{H}$ be the closed-loop system of H with a positive feedback ρI $(\rho > 0)$. Assume that $\tilde{H}$ is passive with a C^1 storage function $S(x)$, then,

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} r^T y dt = \int_{t_0}^{t_1} \left(u^T y - \rho y^T y\right) dt. \tag{2.75}$$



(a) Input feedforward passivity (b) Output feedback passivity

Fig. 2.5. Excess and shortage of passivity

This is equivalent to the dissipativity of system H with respect to the supply rate $w(u,y) = u^T y - \rho y^T y$. In this case, H is said to have excessive output feedback passivity of ρ, denoted as OFP(ρ). If H is not passive, but it is dissipative with respect to the supply rate $w(u,y) = u^T y + \rho y^T y$ ($\rho > 0$), then system H can be rendered passive by a *negative* feedback ρI. In this case, H is said to lack output feedback passivity, denoted as OFP($-\rho$). Mathematically,

Definition 2.35 (Excess/shortage of passivity [110]). *Let $H : u \longmapsto y$. System H is said to be:*

1. *Input feedforward passive (IFP) if it is dissipative with respect to supply rate $w(u,y) = u^T y - \nu u^T u$ for some $\nu \in \mathbb{R}$, denoted as IFP(ν).*
2. *Output feedback passive (OFP) if it is dissipative with respect to supply rate $w(u,y) = u^T y - \rho y^T y$ for some $\rho \in \mathbb{R}$, denoted as OFP(ρ).*

In this book, a positive value of ν or ρ means that the system has an excess of passivity. In this case, the process is said to be *strictly input passive* or *strictly output passive*, respectively. Clearly, if a system is IFP(ν) or OFP(ρ), then it is also IFP($\nu - \varepsilon$), or OFP($\rho - \varepsilon$) $\forall\, \varepsilon > 0$.

The IFP and OFP can also be defined on the input-output version of passivity:

Definition 2.36 ([130]). *Let $H : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^m$. System H is strictly input passive if there exist β and $\delta > 0$ such that*

$$\langle Hu, u \rangle_T \geq \delta \left\| u_T \right\|_2^2 + \beta, \ \forall\ u \in \mathcal{L}_{2e}^m, \ T \geq 0. \tag{2.76}$$

H is strictly output passive if there exist β and $\varepsilon > 0$ such that

$$\langle Hu, u \rangle_T \geq \varepsilon \left\| (Hu)_T \right\|_2^2 + \beta, \ \forall\ u \in \mathcal{L}_{2e}^m, \ T \geq 0. \tag{2.77}$$

A strictly output passive system has a finite $\mathcal{L}_2$ gain [130]. Furthermore, a system that has excessive OFP with a C^1 storage function has a stable equilibrium $x = 0$ when $u = 0$, provided that the system is ZSD. This can be seen from the following:

$$\begin{aligned} \dot{S} &\leq u^T y - \rho y^T y = u^T y - \rho h^T(x) h(x) \\ &< -\rho h^T(x) h(x) < 0, \quad \forall\ h(x) \neq 0 \text{ and } u = 0. \end{aligned} \tag{2.78}$$

Following a proof similar to Theorem 2.32, $x \to 0$ when $t \to \infty$.

IFP and OFP systems have the following scaling property:

Proposition 2.37 (IFP/OFP Scaling [110]). *For systems H and αH, where α is a constant, the following statements are true:*

1. *If H is OFP(ρ), then αH is OFP$\left(\frac{1}{\alpha}\rho\right)$.*
2. *If H is IFP(ν), then αH is IFP($\alpha\nu$).*

Note that the strict passivity definition for linear systems given in Theorem 2.25 is the IFP plus the stability condition, not the linear version of state strict passivity for nonlinear systems. More precisely, a linear system is *strictly passive* if it is *stable* and IFP(ν), $\nu > 0$.

Example 2.38. To illustrate the definition of IFP and OFP, let us consider a linear integrating system:

$$H : \begin{cases} \dot{x} &= u \\ y &= x. \end{cases} \tag{2.79}$$

This system is lossless (passive but not strictly passive). By definition, system H with a positive feedforward ν:

$$H_1 : \begin{cases} \dot{x} &= u \\ y &= x + \nu u \end{cases} \tag{2.80}$$

will have excessive IFP of ν. This can be seen by using a storage function $S(x) = \frac{1}{2}x^2$:

$$\dot{S} = xu = yu - \nu u^2. \tag{2.81}$$

From an input-output point of view, $H(s) = 1/s$ is passive, and

$$H_1(s) = H(s) + \nu = (\nu s + 1)/s \tag{2.82}$$

has excessive IFP of ν. According to Theorem 2.25, $H_1(s)$ is not strictly passive because it is not stable.

Similarly, $H(s)$ with a negative feedback of ρ ($\rho > 0$),

$$H_2(s) = \frac{\frac{1}{s}}{1 + \rho\frac{1}{s}} = \frac{1}{s + \rho}, \tag{2.83}$$

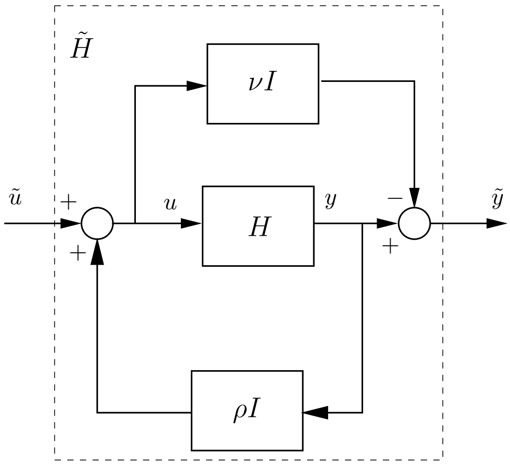


Fig. 2.6. Simultaneous IFP and OFP

will be OFP(ρ) and also strictly passive. Linear strictly output passive systems may not be strictly passive (due to the fact that strict passivity for linear systems requires strict IFP). For example,

$$H_3\left(s\right) = \frac{s}{s+1} \tag{2.84}$$

is OFP(1), but it is not strictly passive because $H_3\left(0\right) + H_3^*\left(0\right) = 0$.

More general supply rates can be used to define simultaneous IFP and OFP. Consider a system H with both input feedforward νI and output feedback ρI, as shown in Figure 2.6. If the overall system $\tilde{H}$ is passive, then system H is dissipative with respect to the supply rate:

$$w\left(u, y\right) = \left(1 + \rho\nu\right) y^T u - \nu u^T u - \rho y^T y. \tag{2.85}$$

In the above discussion, the feedforward and feedback are assumed to be static and decentralized. A more general case is when they are arbitrary nonlinear multivariable (thus vector) functions, *e.g.*

$$w\left(u, y\right) = y^T u - \nu^T\left(u\right) u - \rho^T\left(y\right) y, \tag{2.86}$$

where $v\left(u\right) = \left[v_1\left(u\right), \cdots v_m, \left(u\right)\right]^T$ and $\rho\left(u\right) = \left[\rho_1\left(u\right), \cdots, \rho_m\left(u\right)\right]^T$.

Another generalization of the supply rate was given by Hill and Moylan [57]:

$$w\left(u\left(t\right), y\left(t\right)\right) = y^T\left(t\right) Q y\left(t\right) + 2u^T\left(t\right) S y\left(t\right) + u^T\left(t\right) R u\left(t\right), \tag{2.87}$$

where $Q, R, S \in \mathbb{R}^{m\times m}$ are constant weighting matrices, with Q and R symmetrical. This corresponds to multivariable but linear and static feedforward and feedback required to render the process system passive.

2.4.2 Passivity Indices for Linear Systems

For a stable linear system with a transfer function $G(s)$, the IFP index, denoted as $\nu(G(s))$, can be calculated based on the KYP lemma. If $G(s)$ has excessive IFP, then there exists a largest $\nu > 0$ such that the process with the feedforward $-\nu I$ is positive real, *i.e.*,

$$G(j\omega) - \nu I + [G(j\omega) - \nu I]^* > 0, \quad \forall\, \omega. \tag{2.88}$$

Therefore, we can have the following definition:

Definition 2.39. *The input feedforward passivity index for a stable linear system $G(s)$ is defined as*[2]

$$\nu(G(s)) \triangleq \frac{1}{2} \min_{\omega\in\mathbb{R}} \underline{\lambda}(G(j\omega) + G^*(j\omega)), \tag{2.89}$$

where $\underline{\lambda}$ denotes the minimum eigenvalue.

If ν is negative, then the minimum feedforward required to render the process passive is νI. The above definition also gives a numerical approach for calculating the IFP index. For linear systems, it is possible to define a tighter IFP index conveniently by employing a frequency-dependent passivity index:

Definition 2.40 ([11]). *The input feedforward passivity index for a stable linear system $G(s)$ at frequency ω is given by*

$$\nu_F(G(s), \omega) \triangleq \frac{1}{2} \underline{\lambda}(G(j\omega) + G^*(j\omega)). \tag{2.90}$$

By using the above definition, we can specify the condition that a dynamic feedforward $G_{ff}(s)$ needs to satisfy so that $G(s) + G_{ff}(s)$ is passive. For a stable process $G(s)$, a stable $G_{ff}(s)$ should be chosen such that

$$\nu_F(G_{ff}(s), \omega) + \nu_F(G(s), \omega) > 0 \;\; \forall\, \omega \in \mathbb{R}. \tag{2.91}$$

It is more difficult to calculate the OFP index numerically because it involves feedback loops. If a process $G(s)$ is minimum phase (therefore, $G^{-1}(s)$ exists and is stable. $G(s)$ does not need to be stable), then with a positive feedback of ρI, the closed-loop system is

$$G_{cl}(s) = G(s)[I - \rho G(s)]^{-1} = \left[G(s)^{-1} - \rho I\right]^{-1}. \tag{2.92}$$

According to Proposition 2.23, $G_{cl}(s)$ is passive if and only if

$$G_{cl}^{-1}(s) = G(s)^{-1} - \rho I \tag{2.93}$$

is passive. Therefore, the OFP index of $G(s)$ is the IFP index of $G^{-1}(s)$. We can have the following definition:

[2] This definition is similar to the passivity index proposed in [135], except that in [135] a positive value of ν implies that the system lacks passivity.

Definition 2.41. *The output feedback passivity index for a minimum phase linear system $G(s)$ is defined as*

$$\rho(G(s)) \triangleq \frac{1}{2} \min_{\omega \in \mathbb{R}} \underline{\lambda}\left(G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^*\right). \tag{2.94}$$

The OFP index at frequency ω is given by

$$\rho_F(G(s), \omega) \triangleq \frac{1}{2} \underline{\lambda}\left(G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^*\right). \tag{2.95}$$

For processes that are nonminimum phase and unstable, we need both feedback and feedforward to render the process passive. In this case, the IFP and OFP indices are dependent. Special passivity indices need to be defined so that they can be conveniently computed and used in system analysis and control design. We will introduce these indices in other chapters.

2.5 Passivation

To render a process passive via either feedback or feedforward is called *passivation*. This is possible if the process lacks either IFP or OFP. Because passive systems are stable and easy to control, passivation is often a useful step in control design. For example, we may passivate a process and then stabilize the passivated system with a (strictly) passive controller (*e.g.*, a static output feedback controller given in Theorem 2.32).

2.5.1 Input Feedforward Passivation

Many stable processes can be passivated by a static feedforward. For example, a linear system $G_1(s) = \frac{1-s}{s^3+s^2+s+1}$ can be passivated by a static unit feedforward because $G(s) = G_1(s) + 1 = \frac{s^3+s^2+2}{s^3+s^2+s+1}$ is minimum phase, has a relative degree of 0 and is positive real.

Consider a control affine process H as in (2.32). Assume that the process has a globally stable equilibrium at $x = 0$ with a Lyapunov function $V(x)$. Use $V(x)$ as a storage function, then

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} = \frac{\partial V(x)}{\partial x} f(x) + \frac{\partial V(x)}{\partial x} g(x) u \le \frac{\partial V(x)}{\partial x} g(x) u. \tag{2.96}$$

As shown in Figure 2.5a with the feedforward νI, $\tilde{y} = h(x) + \nu u$. Then $\tilde{y}^T u = h^T(x) u + \nu u^T u$. As long as there exists a ν such that

$$\nu u^T u > \left[\frac{\partial V(x)}{\partial x} g(x) - h^T(x)\right] u, \tag{2.97}$$

$\dot{V}(x) \le \tilde{y}^T u$.

This result can be generalized to dynamic feedforward systems. Any stable control affine process (of which a Lyapunov function can be found) can be passivated with a feedforward dynamic system. As shown in Figure 2.3a, assume that a system

$$H_1 : \begin{cases} \dot{x} &= f_1(x) + g_1(x) u_1 \\ y_1 &= h_1(x), \end{cases} \tag{2.98}$$

is nonpassive but has a globally stable equilibrium point $x = 0$ with a Lyapunov function $V(x)$. A feedforward system H_2 can be designed to passivate H_1. One way to design such a feedfoward passivater is to assume that the passivated system H has the same state equation as that of H_1 and find an appropriate output function $y(t) = h(x)$ such that H is passive. According to the KYP lemma (Proposition 2.14), if we use $V(x)$ as a storage function, then, the condition $L_f V(x) = \frac{\partial V(x)}{\partial x} f_1(x) \leq 0$ is always satisfied. If we choose $h(x) = \left[\frac{\partial V(x)}{\partial x} g_1(x)\right]^T$, then H is passive. The feedforward system H_2 can be obtained by subtracting y from y_1:

$$H_2 : \begin{cases} \dot{x} &= f_1(x) + g_1(x) u_2 \\ y_2 &= \left[\frac{\partial V(x)}{\partial x} g_1(x)\right]^T - h_1(x). \end{cases} \tag{2.99}$$

Such a feedforward will stabilize the zero dynamics of H_1 (so that H is made weakly minimum phase) and reduce its relative degree to no greater than $\{1, \cdots, 1\}$.

For linear systems, the feedforward system can be easily obtained using the linear version of the KYP lemma. Detailed discussion will be given in later chapters. However, it is not possible to passivate an unstable process with feedforward because the feedforward does not affect the free dynamics of the process (when $u = 0$). Such systems can only be passivated via feedback.

2.5.2 Output Feedback Passivation

Passivation of unstable processes is a topic which attracted much interest because it can be an effective approach to stabilization of nonlinear processes. Most research work is concerned with passivation by state feedback. A thorough development of this topic can be found in [24]. A control affine system given in (2.32) is said to be *feedback passive* (or *feedback equivalent to a passive system*) if there exists a state feedback transformation [24]:

$$u = \alpha(x) + \beta(x) v, \tag{2.100}$$

with invertible $\beta(x)$ such that the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(x) + g(x)\beta(x) v, \\ y &= h(x), \end{aligned} \tag{2.101}$$

is passive. The condition for feedback passivity is given in the following theorem:

Theorem 2.42 (State feedback passivity [24]). *Consider the control affine system in (2.32). Assume* $\operatorname{rank}\left(L_g h(x)|_{x=0}\right) = m$ *(where* m *is the number of outputs). Then this system is feedback passive with a* C^2 *positive definite storage function* $S(x)$ *if and only of it has relative degree* $\{1, \cdots 1, \}$ *at* $x = 0$ *and is weakly minimum phase.*

Clearly, the above condition says that we cannot render a nonminimum phase system or a system with a relative degree larger than 1 passive via feedback, because a passive system needs to be weakly minimum phase and have a relative degree no greater than 1, but the relative degree and the zero dynamics *cannot* be altered by feedback [69]. In this case, passivation is only possible via feedforward.

For the output feedback case, an additional condition is required:

Theorem 2.43 (Output feedback passivity).

1. *Necessary condition: If the system in (2.32) can be rendered passive with a* C^2 *storage function* $S(x)$*, then it has relative degree* $\{1, \cdots, 1\}$ *at* $x = 0$ *and is weakly minimum phase, and* $L_g h(x)|_{x=0}$ *is symmetrical and positive definite.*
2. *Sufficient condition: The system in (2.32) can be rendered locally passive with a* C^2 *positive definite storage function* $S(x)$ *by an output feedback if its Jacobian linearization at* $x = 0$ *is minimum phase and* $\left.\frac{\partial h(x)}{\partial x} g(x)\right|_{x=0}$ *is symmetrical and positive definite.*

To get some intuition from the above conditions, let us look at the case of linear systems. For a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{2.102}$$

$$L_g h(x) = \frac{\partial h(x)}{\partial x} g(x) = CB. \tag{2.103}$$

Theorem 2.42 says the linear system is feedback passive if it (1) has a relative degree of 1 (due to the assumption $D = 0$, the relative degree cannot be 0) (2) is weakly minimum phase (it may have zeros in the LHP and on the imaginary axis) and (3) $\operatorname{rank}(CB) = m$. Note that if CB is nonsingular, then the linear system has a relative degree of 1. Therefore, Condition (3) implies Condition (1) for linear systems.

Clearly, any *state feedback* cannot change any of the above conditions, because with a state feedback, $u = r - Kx$ (r is an exogenous input such as reference), the closed-loop system will be

$$\begin{aligned}\dot{x} &= (A - BK)\,x + Br,\\ y &= Cx.\end{aligned} \tag{2.104}$$

For example, systems like $G_1(s) = \frac{1-s}{s+1}$ cannot be passivated by any state feedback controllers.

For *output feedback* passivity, CB must also be (1) symmetrical and (2) positive definite. The first condition implies that the input and output of a process need to be properly paired. The second condition imposes the limitation on the sign of the steady-state gain. For example, a linear system $G_2(s) = \frac{-1}{-s+1}$ can be stabilized by a negative feedback controller $K(s)$ with negative steady-state gain, but cannot be made positive real because the closed-loop system $\frac{G_2(s)}{1+G_2(s)K(s)}$ will have a negative steady-state gain.

2.6 Passivity Theorem

We have shown the stability condition of passive systems in feedback in Theorem 2.33. By using the notions of strict input passivity and strict output passivity, asymptotic stability conditions for interconnected passive systems can be derived. These conditions are called the Passivity Theorem. The simplest version of the Passivity Theorem is as follows:

Theorem 2.44 (Passivity Theorem [110]). *Assume that systems H_1 and H_2 are ZSD and dissipative with C^1 storage functions $S_1(x_1)$ and $S_2(x_2)$. Then the equilibrium $(x_1, x_2) = (0, 0)$ of their feedback connection (as shown in Figure 2.7a) with $r \equiv 0$ is asymptotically stable (AS) if*

1. *H_1 and H_2 are strictly output passive; or,*
2. *H_1 and H_2 are strictly input passive; or,*
3. *H_1 is GAS and strictly input passive and H_2 is passive.*

If storage functions $S_1(x_1)$ and $S_2(x_2)$ are radially unbounded, then the feedback connection is globally asymptotically stable (GAS).

Proof. The proof of the above theorem can be found in [110]. Here we provide a simplified version of the proof to clarify the intent. The storage function for the closed-loop is chosen as $S(x_1, x_2) = S_1(x_1) + S_2(x_2)$.

1. Since H_1 and H_2 are strictly output passive, there exist $\rho_1, \rho_2 > 0$ such that

$$\dot{S}_1(x_1) \le y_1^T u_1 - \rho_1 y_1^T y_1, \tag{2.105}$$

$$\dot{S}_2(x_2) \le y_2^T u_2 - \rho_2 y_2^T y_2. \tag{2.106}$$

Then,

$$\dot{S}(x_1, x_2) \le y_1^T u_1 - \rho_1 y_1^T y_1 + y_2^T u_2 - \rho_2 y_2^T y_2. \tag{2.107}$$

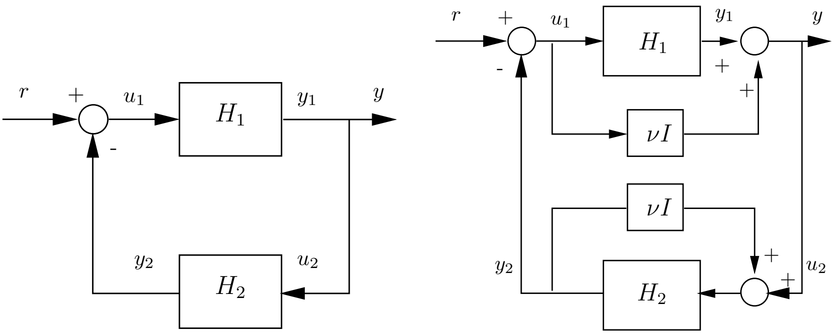


Fig. 2.7. Extended passivity condition

Because $u_1 = -y_2, u_2 = y_1$,

$$\begin{aligned} \dot{S}\left(x_1, x_2\right) &\leq -y_1^T y_2 - \rho_1 y_1^T y_1 + y_2^T y_1 - \rho_2 y_2^T y_2 \\ &= -\rho_1 y_1^T y_1 - \rho_2 y_2^T y_2 < 0, \quad \forall\, y_1, y_2 \neq 0. \end{aligned} \tag{2.108}$$

The bounded solution of (x_1, x_2) is confined in $\{(x_1, x_2)|\,(y_1, y_2) = (0, 0)\}$. Because H_1 and H_2 are ZSD, $(x_1, x_2) \to (0, 0)$.

2. Since H_1 and H_2 are strictly input passive, there exist $\nu_1, \nu_2 > 0$ such that

$$\dot{S}_1\left(x_1\right) \leq y_1^T u_1 - \nu_1 u_1^T u_1, \tag{2.109}$$

$$\dot{S}_2\left(x_2\right) \leq y_2^T u_2 - \nu_2 u_2^T u_2. \tag{2.110}$$

Then

$$\begin{aligned} \dot{S}\left(x_1, x_2\right) &\leq y_1^T u_1 - \nu_1 u_1^T u_1 + y_2^T u_2 - \nu_2 u_2^T u_2 \\ &\leq -\nu_1 y_2^T y_2 - \nu_2 y_1^T y_1 < 0, \quad \forall\, y_1, y_2 \neq 0. \end{aligned} \tag{2.111}$$

Similar to Part 1, $(x_1, x_2) \to (0, 0)$.

3. In this case, there exists a $\nu_1 > 0$ such that

$$\dot{S}_1\left(x_1\right) \leq y_1^T u_1 - \nu_1 u_1^T u_1, \tag{2.112}$$

$$\dot{S}_2\left(x_2\right) \leq y_2^T u_2, \tag{2.113}$$

$$\begin{aligned} \dot{S}\left(x_1, x_2\right) &\leq y_1^T u_1 - \nu_1 u_1^T u_1 + y_2^T u_2 \\ &= -\nu_1 y_2^T y_2 < 0, \quad \forall\, y_2 \neq 0. \end{aligned} \tag{2.114}$$

Because $\dot{S}$ is bounded only by $y_2^T y_2$, the bounded solution of (x_1, x_2) is confined in $\{(x_1, x_2)|\, y_2 = 0\}$ and $u_1 = 0$. Because H_1 is GAS and H_2 is ZSD, $(x_1, x_2) \to (0, 0)$.

If storage functions $S_1(x_1)$ and $S_2(x_2)$ are radially unbounded, then *all* the above results hold globally.

The input-output version of the Passivity Theorem can be presented as follows:

Theorem 2.45 ([130]). *Consider the closed-loop system shown in Figure 2.7a with* $H_1,\ H_2 : \mathcal{L}_{2e}^m \rightarrow \mathcal{L}_{2e}^m$*. Assume that for any* $r \in \mathcal{L}_2^m$ *there are solutions* $u_1,\ u_2 \in \mathcal{L}_{2e}^m$*. If*

1. *H_1 is passive and H_2 is strictly input passive; or,*
2. *H_1 is strictly output passive and H_2 is passive,*

then, $u_2 = y_1 = H_1(u_1) \in \mathcal{L}_2^m$*, i.e., the closed-loop system from r to y_1 is $\mathcal{L}_2$ stable.*

Furthermore, if the input-output stability of systems H_1 and/or H_2 is assumed, we have

Theorem 2.46 ([130]). *Consider the closed-loop system shown in Figure 2.7a with* $H_1,\ H_2 : \mathcal{L}_{2e}^m \rightarrow \mathcal{L}_{2e}^m$*. Assume for any* $r \in \mathcal{L}_2^m$ *that there are solutions* $u_1,\ u_2 \in \mathcal{L}_{2e}^m$*. If*

1. *H_1 is passive and H_2 is strictly input passive and $\mathcal{L}_2$ stable; or*
2. *Both H_1 and H_2 are strictly output passive,*

then, $y_1, y_2 \in \mathcal{L}_2^m$*, i.e., both of the closed-loop systems from r to y_1 and from r to y_2 are $\mathcal{L}_2$ stable.*

For linear systems, Condition 1 of the above theorem simply means:

Proposition 2.47 (Passivity theorem for linear systems). *Consider two LTI systems H_1 and H_2 in negative feedback configuration, as shown in Figure 2.7a. The closed-loop system is asymptotically stable if H_1 is strictly passive and H_2 is passive.*

This can be clearly seen from the example of two SISO systems H_1 and H_2. In this case, the phase shifts of H_1 and H_2 lie within $(-90°, 90°)$ and $[-90°, 90°]$, respectively. Therefore, the total phase shift of the open-loop system never reaches $-180°$, producing no critical frequency in the open-loop Bode diagram. According to the Nyquist-Bode stability condition, the closed-loop system is stable *regardless* of the amplitude ratio of $H_1(j\omega) H_2(j\omega)$. The system has infinite gain margin.

By using the concepts of excess and shortage of passivity, we can extend the above results further to general (possibly nonpassive) systems. Assume that system H_1 in Figure 2.7a is GAS but lacks IFP, *e.g.*, is IFP$(-\nu_1)$, $\nu_1 > 0$, then a feedforward of νI (where $\nu = \nu_1 + \varepsilon$ and ε is an arbitrarily small positive number) will render H_1 strictly input passive, as depicted in Figure 2.7b. To make the feedback system equivalent to the original system in Figure 2.7a, a positive feedback of νI is added to H_2. According to Theorem 2.44, the equilibrium $(x_1, x_2) = (0, 0)$ of the closed-loop system is GAS if H_2 with positive feedback is passive, *i.e.*, H_2 has excessive output feedback passivity of ν. Similarly, a shortage of output feedback passivity of H_2 can be compensated for by excessive input feedforward passivity of H_1 so that the closed-loop system is GAS. More rigorously, we have:

Theorem 2.48 ([110]). *Assume that in the feedback interconnection shown in Figure 2.7a, H_1 is GAS and IFP(ν) and the system H_2 is ZSD and OFP(ρ). Then $(x_1, x_2) = (0, 0)$ is AS if $\nu + \rho > 0$. If, in addition, the storage functions of H_1 and H_2 are radially unbounded, then $(x_1, x_2) = (0, 0)$ is GAS.*

If the systems are characterized by a more general supply rate as in (2.86), the above condition can be further extended:

Theorem 2.49 ([110]). *Assume that the systems H_1 and H_2 are dissipative with respect to the following supply rates:*

$$w_i(u_i, y_i) = u_i^T y_i - \rho_i^T(y_i)y_i - \nu_i^T(u_i)(u_i), \quad i = 1, 2, \tag{2.115}$$

where $u_i, y_i \in \mathbb{R}^m$, $i = 1, 2$. Furthermore assume that they are ZSD and that their respective storage functions $S_1(x_1)$ and $S_2(x_2)$ are C^1. Then the equilibrium $(x_1, x_2) = (0, 0)$ of the feedback interconnection in Figure 2.7a is

1. *stable, if $\nu_1^T(v)v + \rho_2^T(v)v \geq 0$ and $\nu_2^T(v)v + \rho_1^T(v)v \geq 0$, $\forall\ v \in \mathbb{R}^m$;*
2. *asymptotically stable, if $\nu_1^T(v)v + \rho_2^T(v)v > 0$ and $\nu_2^T(v)v + \rho_1^T(v)v > 0$, $\forall\ v \in \mathbb{R}^m$ and $v \neq 0$.*

One special case of the supply rates is $\nu_i(u_i) = \bar{\nu}_i u_i$ and $\rho_i(y_i) = \bar{\rho}_i y_i$, where $\bar{\nu}_i$ and $\bar{\rho}_i$ are scalar constants. In this case,

$$\nu_1^T(v)v + \rho_2^T(v)v = \bar{\nu}_1 v^T v + \bar{\rho}_2 v^T v = (\bar{\nu}_1 + \bar{\rho}_2)\, v^T v, \tag{2.116}$$

$$\nu_2^T(v)v + \rho_1^T(v)v = \bar{\nu}_2 v^T v + \bar{\rho}_1 v^T v = (\bar{\nu}_2 + \bar{\rho}_1)\, v^T v. \tag{2.117}$$

Then, the equilibrium $(x_1, x_2) = (0, 0)$ of the feedback interconnection is

1. stable if $\bar{\nu}_1 + \bar{\rho}_2 \geq 0$ and $\bar{\nu}_2 + \bar{\rho}_1 \geq 0$;
2. asymptotically stable if $\bar{\nu}_1 + \bar{\rho}_2 > 0$ and $\bar{\nu}_2 + \bar{\rho}_1 > 0$.

Another special case is

$$\nu_1(v) = \rho_1(v) = 0, \quad v_2(v) = \nu v \ \text{ and } \rho_2(v) = \rho v. \tag{2.118}$$

This leads to the following stability condition:

Proposition 2.50. *Assume that H_1 is passive (i.e., dissipative with respect to the supply rate $w_1 = u_1^T y_1$) and H_2 is dissipative with respect to the supply rate of $w_2 = u_2^T y_2 - \rho y_2^T y_2 - \nu u_2^T u_2$. Assume that systems H_1 and H_2 are ZSD and their respective storage functions $S_1(x_1)$ and $S_2(x_2)$ are C^1. Then, the equilibrium $(x_1, x_2) = (0, 0)$ of the feedback interconnection in Figure 2.7a is asymptotically stable if $\rho > 0$ and $\nu > 0$.*

This condition does not require system H_1 to be AS.

2.7 Heat Exchanger Example

Some process systems are inherently passive (after proper rescaling of the inputs and/or outputs). One of the examples is the heat exchanger, a device built for efficient heat transfer from one fluid to another. The fluids are separated by a solid wall so that they never mix. Heat exchangers are widely used in air conditioning, refrigeration, space heating, power production, and in virtually every chemical plant.

Consider a single tube-in-shell heat exchanger as depicted in Figure 2.8, where cooling water is used to remove heat from a process stream. The volumetric flow rates of the process (hot) and service (cold) streams are v_h and v_c. The inlet and outlet temperatures of the hot and cold streams are T_{hi}, T_{ho}, T_{ci} and T_{co}, respectively. Strictly speaking, a tube-in-shell heat exchanger is a distributed parameter system (which can be represented by partial differential equations), because the temperatures of the hot and cold streams in the tube are functions of the location in the tube. To simplify our discussion, an approximate lumped parameter model given by Hangos *et al.* [54] is adopted. The model was built under the following assumptions:

1. Constant volume of the hot and cold streams in the heat exchanger (V_h and V_c);
2. Constant physicochemical properties, including density of the hot and cold streams (ρ_h and ρ_c) and their specific heat (c_{Ph} and c_{Pc});
3. Constant heat transfer coefficient U and area A;
4. Both hot and cold streams are well mixed and the temperatures of the hot and cold streams inside the tube are approximated by the outlet temperatures T_{ho} and T_{co}.

The state equations of the heat exchanger can be developed based on energy balance [54]:

$$\dot{T}_{co}(t) = \frac{v_c(t)}{V_c}\left[T_{ci}(t) - T_{co}(t)\right] + \frac{UA}{c_{Pc}\rho_c V_c}\left[T_{ho}(t) - T_{co}(t)\right], \tag{2.119}$$

$$\dot{T}_{ho}(t) = \frac{v_h(t)}{V_h}\left[T_{hi}(t) - T_{ho}(t)\right] + \frac{UA}{c_{Ph}\rho_h V_h}\left[T_{co}(t) - T_{ho}(t)\right]. \tag{2.120}$$

The inputs of the above process are the inlet temperatures and flow rates of the hot and cold streams. The outputs and states are the outlet temperatures. Depending on the choices of the manipulated variables, different models can be derived.

Example 2.51 (Linear model). If the inlet temperatures are manipulated to control the outlet temperatures, with the assumption that the flow rates of the cold and hot streams are constant, a linear model can be derived:

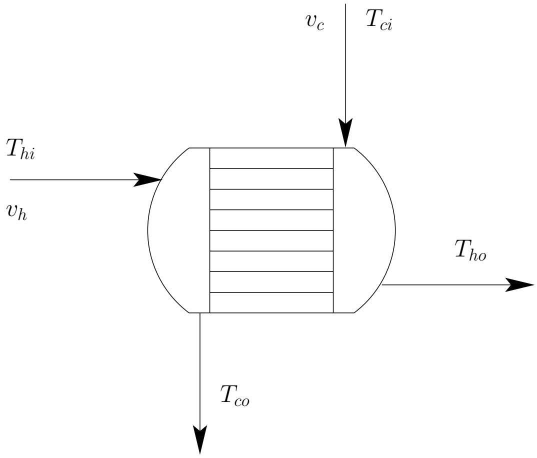


Fig. 2.8. A heat exchanger

$$\dot{x}(t) = \begin{bmatrix} -\frac{v_c}{V_c} - \frac{UA}{c_{Pc}\rho_c V_c} & \frac{UA}{c_{Pc}\rho_c V_c} \\ \frac{UA}{c_{Ph}\rho_h V_h} & -\frac{v_h}{V_h} - \frac{UA}{c_{Ph}\rho_h V_h} \end{bmatrix} x(t) + \begin{bmatrix} \frac{v_c}{V_c} & 0 \\ 0 & \frac{v_h}{V_h} \end{bmatrix} u(t), \quad (2.121)$$

$$y(t) = x(t), \quad (2.122)$$

where $x = [x_1, x_2]^T = [T_{co}, T_{ho}]^T$ and $u = [u_1, u_2]^T = [T_{ci}, T_{hi}]^T$. Define the following constants $k_1 = \frac{UA}{c_{Pc}\rho_c V_c}$, $k_2 = \frac{UA}{c_{Ph}\rho_h V_h}$, $a_1 = \frac{v_c}{V_c}$ and $a_2 = \frac{v_h}{V_h}$. Clearly, these constants are positive for any design and operating conditions. Then, the state equation becomes

$$\dot{x} = \begin{bmatrix} -a_1 - k_1 & k_1 \\ k_2 & -a_2 - k_2 \end{bmatrix} x + \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} u. \quad (2.123)$$

To study the passivity of the above system, we define the following storage function:

$$S(x) = \frac{1}{2} x^T \begin{bmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{bmatrix} x > 0, \quad \forall\, x \neq 0. \quad (2.124)$$

Therefore,

$$\begin{aligned} \dot{S}(x) &= x^T \left\{ \begin{bmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{bmatrix} \begin{bmatrix} -a_1 - k_1 & k_1 \\ k_2 & -a_2 - k_2 \end{bmatrix} x + \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} u \right\} \\ &= x^T \begin{bmatrix} -\frac{a_1}{k_1} - 1 & 1 \\ 1 & -\frac{a_2}{k_2} - 1 \end{bmatrix} x + x^T \begin{bmatrix} \frac{a_1}{k_1} & 0 \\ 0 & \frac{a_2}{k_2} \end{bmatrix} u \\ &= -\frac{a_1}{k_1} x_1^2 - \frac{a_2}{k_2} x_2^2 - (x_1 - x_2)^2 + \frac{a_1}{k_1} x_1 u_1 + \frac{a_2}{k_2} x_2 u_2 \\ &\leq \frac{a_1}{k_1} y_1 u_1 + \frac{a_2}{k_2} y_2 u_2. \end{aligned} \quad (2.125)$$

Note that the coefficients $k_1, k_2, a_1, a_2 > 0$. If the outputs are rescaled as $y^* = [y_1^*, y_2^*]^T = \left[\frac{a_1}{k_1} y_1, \frac{a2}{k_2} y_2\right]^T$,

$$\dot{S}(x) < u^T y^*, \quad \forall\, x \neq 0, \tag{2.126}$$

leading to the conclusion that the heat exchanger is passive, regardless of design parameters (such as U, V_c, V_h, A), types of fluid (such as c_{Pc} and ρ_c) and operating conditions (such as v_c and v_h). If the heat exchanger parameters given in [65] are adopted, then $v_c = 2.29 \times 10^3$ ft^3/h, $v_h = 6.24 \times 10^3$ ft^3/h, $V_c = 5.57$ ft^3, $V_h = 20.40$ ft^3, $A = 521.5$ ft^2, $c_{Ph} = 0.58$ Btu/(lb·F), $c_{Pc} = 0.56$ Btu/(lb·F), $U = 75$ Btu/(h· ft^2·F), $\rho_h = 47.74$ lb/ft^3 and $\rho_c = 44.93$ lb/ft^3. In this case, (2.121) and (2.122) become

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -690.87 & 279.17 \\ 69.254 & -375.29 \end{bmatrix} x + \begin{bmatrix} 411.7 & 0 \\ 0 & 306.03 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x. \end{aligned} \tag{2.127}$$

It is easy to verify that the above process is passive, because matrices

$$P = \begin{bmatrix} 0.0024 & 0 \\ 0 & 0.0030 \end{bmatrix} \text{ and } L = \begin{bmatrix} 3.3562 & -0.9044 \\ -0.9044 & 2.4526 \end{bmatrix} \tag{2.128}$$

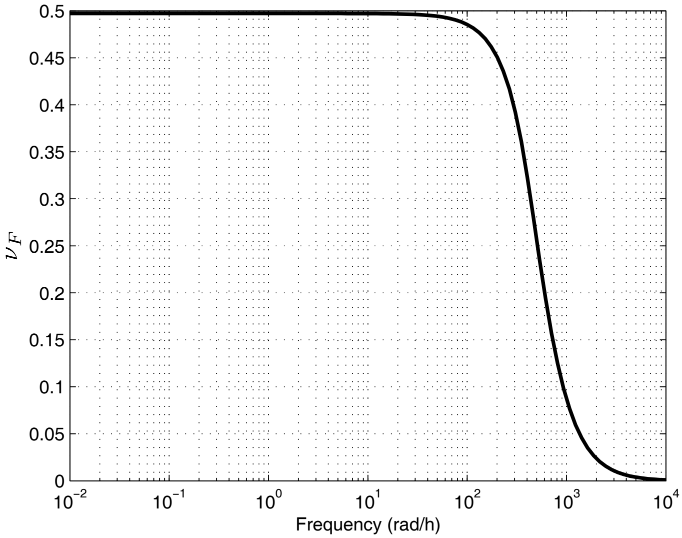
$$Q = W = 0 \tag{2.129}$$

are found to satisfy the conditions given in (2.41). The IFP index plot of this process is shown in Figure 2.9a. Its phase plot is given in Figure 2.9b, from which it can be seen that the phase shift is within $(-90°, 90°)$ at all frequencies.

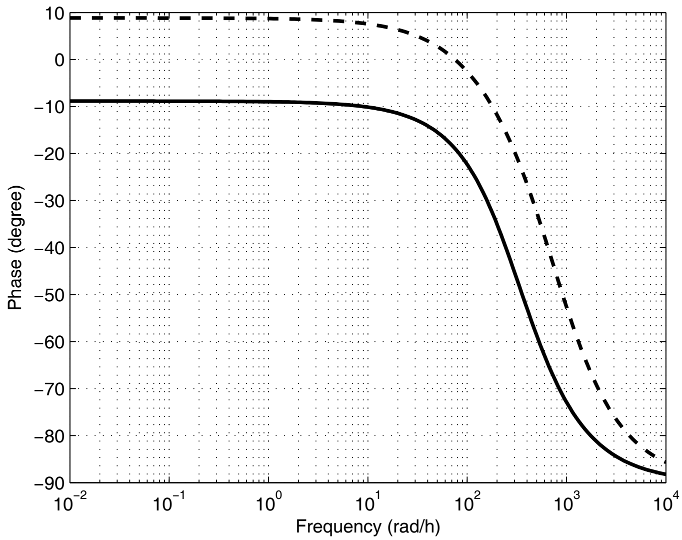
Example 2.52 (Nonlinear model). A more realistic choice of manipulated variables is the flow rates of hot and cold streams, *i.e.*, $u = [u_1, u_2]^T = [v_h, v_c]^T$. In this case, we assume that the inlet temperatures T_{ci} and T_{hi} are constant. This leads to a nonlinear model. To study the passivity of the process with respect to an equilibrium point $x_0 = [x_{10}, x_{20}]^T = [T_{co0}, T_{hi0}]^T$, we define the following deviation variables: $x' = [x_1', x_2']^T = x - x_0$ and $u' = [u_1', u_2']^T = u - u_0$, where $u_0 = [v_{h0}, v_{c0}]^T$. (Note: The deviation variables can have negative values.) Therefore,

$$\begin{aligned} \dot{x}_1' &= -k_1 (x_1' + x_{10}) + k_1 (x_2' + x_{20}) + \left[\frac{T_{ci}}{V_c} - \frac{1}{V_c}(x_1' + x_{10})\right](u_1' + u_{10}), \\ \dot{x}_2' &= k_2 (x_1' + x_{10}) - k_2 (x_2' + x_{20}) + \left[\frac{T_{hi}}{V_h} - \frac{1}{V_h}(x_2' + x_{20})\right](u_2' + u_{20}). \end{aligned} \tag{2.130}$$

Assume that (x_0, u_0) is at steady state, *i.e.*,



(a) IFP index plot



(b) Phase plot

Fig. 2.9. Linear heat exchanger model

$$\begin{aligned} 0 &= -k_1 x_{10} + k_1 x_{20} + \frac{T_{ci}}{V_c} u_{10} - \frac{1}{V_c} x_{10} u_{10}, \\ 0 &= k_2 x_{10} - k_2 x_{20} + \frac{T_{hi}}{V_h} u_{20} - \frac{1}{V_h} x_{20} u_{20}. \end{aligned} \tag{2.131}$$

Therefore,

$$\begin{aligned} \dot{x}_1' &= -k_1 x_1' + k_1 x_2' + \frac{T_{ci}}{V_c} u_1' - \frac{1}{V_c} x_1' u_1' - \frac{1}{V_c} x_{10} u_1' - \frac{1}{V_c} x_1' u_{10}, \\ \dot{x}_2' &= k_2 x_1' - k_2 x_2' + \frac{T_{hi}}{V_h} u_2' - \frac{1}{V_h} x_2' u_2' - \frac{1}{V_h} x_{20} u_2' - \frac{1}{V_h} x_2' u_{20}. \end{aligned} \tag{2.132}$$

Define a storage function

$$S(x') = \frac{1}{2} x'^T \begin{bmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{bmatrix} x' > 0, \quad \forall\ x' \neq 0; \tag{2.133}$$

then,

$$\begin{aligned} \dot{S}(x') = &-(x_1' - x_2')^2 - \frac{1}{V_c k_1}(u_1' + u_{10}) x_1'^2 - \frac{1}{V_h k_2}(u_2' + u_{20}) x_2'^2 \\ &+ \left(\frac{T_{ci} - x_{10}}{V_c k_1}\right) x_1' u_1' + \left(\frac{T_{hi} - x_{20}}{V_h k_2}\right) x_2' u_2'. \end{aligned} \tag{2.134}$$

Define a rescaled output $y^* = [y_1^*, y_2^*]^T = \left[\left(\frac{T_{ci}-x_{10}}{V_c k_1}\right) x_1', \left(\frac{T_{hi}-x_{20}}{V_h k_2}\right) x_2'\right]^T$. Also note $u_1 = u_1' + u_{10} \geq 0$ and $u_2 = u_2' + u_{20} \geq 0$ because u_1 and u_2 are physical flow rates. Then,

$$\begin{aligned} \dot{S}(x') &\leq -(x_1' - x_2')^2 + y_1^* u_1' + y_2^* u_2' \\ &\leq y^{*T} u'. \end{aligned} \tag{2.135}$$

Therefore, the process is passive with respect to the equilibrium $x' = [0, 0]^T$. It is interesting to point out that

1. Similar to the linear case, the heat exchanger is inherently passive because the passivity condition is valid for *any* design parameters, types of fluid and operating conditions (different T_{ci} and T_{hi}).
2. The system is passive with respect to any physical equilibrium point $[x_{10}, x_{20}]^T$ because (2.135) holds for any x_0.
3. The equilibrium point x_0 is GS but not GAS. If $x_1' = x_2' \neq 0$, the unforced system does not converge to $x' = 0$.
4. Output rescaling is equivalent to sensor calibration. Because T_{ci} is never greater than T_{co}, the rescaling coefficient for y_1^* is non-positive. A higher inlet cold stream flow rate will lead to a lower outlet temperatures (T_{co} and T_{ho}). This implies that the direction of x' movement has to be reversed to obtain a minimum phase condition.

In addition, as the system outputs are simply rescaled states, the above system is ZSD. As a result, the heat exchanger is very easy to control. According to Proposition 2.50, any output feedback controller (a mapping from y^* to u') which is dissipative with respect to a supply rate of $w = u'^T y^* - \nu y^{*T} y^* - \rho u'^T u'$, $\rho > 0$ and $\nu > 0$ (*i.e.*, with simultaneous excessive IFP and OFP) will asymptotically stabilize the equilibrium $x' = [0, 0]^T$. A special case is a proportional only controller $u' = -ky^*$ for any $k > 0$.

2.8 Summary

In this chapter, the basic concepts of dissipative systems and passive systems are introduced. The input-output properties of passive systems are discussed. These properties lead to useful stability conditions for interconnected systems, on which the developments described in later chapters build. At first glance, it seems that the stability conditions based on passivity could be conservative compared to those based on dissipativity, because passive systems are a special case of dissipative systems. With the notions of IFP and OFP, the conservativeness vanishes because dissipative systems with respect to different supply rates can be represented by passive systems with certain IFP and OFP. Excess and shortage of IFP and OFP are also used to characterize processes in terms of their passivity. In the next few chapters, passivity-based system analysis and control design are developed for linear processes. These approaches can be implemented numerically and applied directly in routine process control practice.