6

Impulse Control of Jump Diffusions

6.1 A General Formulation and a Verification Theorem

Suppose that – if there are no interventions – the state $Y(t) \in \mathbb{R}^k$ of the system we consider is a jump diffusion of the form

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) + \int_{\mathbb{R}^\ell} \gamma(Y(t^-), z)\widetilde{N}(dt, dz), \qquad (6.1.1)$$

$$Y(0) = y \in \mathbb{R}^k,$$

where $b : \mathbb{R}^k \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \to \mathbb{R}^{k\times m}$, and $\gamma : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^{k\times \ell}$ are given functions satisfying the conditions for the existence and uniqueness of a solution $Y(t)$ (see Theorem 1.19).

The generator A of $Y(t)$ is

$$A\phi(y) = \sum_{i=1}^k b_i(y)\frac{\partial \phi}{\partial y_i} + \frac{1}{2}\sum_{i,j=1}^k (\sigma\sigma^T)_{ij}(y)\frac{\partial^2 \phi}{\partial y_i \partial y_j}$$
$$+ \sum_{j=1}^\ell \int_{\mathbb{R}} \{\phi(y + \gamma^{(j)}(y, z_j)) - \phi(y) - \nabla\phi(y)\cdot\gamma^{(j)}(y, z_j)\}\nu_j(dz_j).$$

Now suppose that at any time t and any state y we are free to intervene and give the system an impulse $\zeta \in \mathcal{Z} \subset \mathbb{R}^p$, where $\mathcal{Z}$ is a given set (the set of admissible impulse values). Suppose the result of giving the impulse ζ when the state is y is that the state jumps immediately from $y = Y(t^-)$ to $Y(t) = \Gamma(y, \zeta) \in \mathbb{R}^k$, where $\Gamma : \mathbb{R}^k \times \mathcal{Z} \to \mathbb{R}^k$ is a given function.

An *impulse control* for this system is a double (possibly finite) sequence

$$v = (\tau_1, \tau_2, \ldots, \tau_j, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_j, \ldots)_{j\le M}, \quad M \le \infty,$$

where $0 \le \tau_1 \le \tau_2 \le \cdots$ are $\mathcal{F}_t$-stopping times (the *intervention times*) and $\zeta_1, \zeta_2, \ldots$ are the corresponding *impulses* at these times. We assume that ζ_j is $\mathcal{F}_{\tau_j}$-measurable for all j.

If $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ is an impulse control, the corresponding state process $Y^{(v)}(t)$ is defined by

$$Y^{(v)}(0^-) = y \quad \text{and} \quad Y^{(v)}(t) = Y(t), \quad 0 < t \leq \tau_1, \tag{6.1.2}$$

$$Y^{(v)}(\tau_j) = \Gamma(\check{Y}^{(v)}(\tau_j^-), \zeta_j), \quad j = 1, 2, \ldots \tag{6.1.3}$$

(If $\tau_1 = 0$ we put $Y^{(v)}(\tau_1) = \Gamma(Y^{(v)}(0^-), \zeta_1) = \Gamma(y, \zeta_1)$.)

$$\begin{aligned} \mathrm{d}Y^{(v)}(t) &= b(Y^{(v)}(t))\mathrm{d}t + \sigma(Y^{(v)}(t))\mathrm{d}B(t) \\ &\quad + \int_{\mathbb{R}^\ell} \gamma(Y^{(v)}(t), z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z) \quad \text{for } \tau_j < t < \tau_{j+1} \wedge \tau^*. \end{aligned} \tag{6.1.4}$$

If $\tau_{j+1} = \tau_j$, then $Y^{(v)}(t)$ jumps from $\check{Y}^{(v)}(\tau_j^-)$ to $\Gamma(\Gamma(\check{Y}^{(v)}(\tau_j^-), \zeta_j), \zeta_{j+1})$,

where

$$\check{Y}^{(v)}(\tau_j^-) = Y^{(v)}(\tau_j^-) + \Delta_N Y(\tau_j), \tag{6.1.5}$$

$\Delta_N Y^{(v)}(t)$ is as in (5.2.2) the jump of $Y^{(v)}$ stemming from the jump of the random measure $N(t, \cdot)$ only and

$$\tau^* = \tau^*(\omega) = \lim_{R \to \infty} (\inf\{t > 0; |Y^{(v)}(t)| \geq R\}) \leq \infty \tag{6.1.6}$$

is the explosion time of $Y^{(v)}(t)$. Note that here we must distinguish between the (possible) jump of $Y^{(v)}(\tau_j)$ stemming from the random measure N, denoted by $\Delta_N Y^{(v)}(\tau_j)$ and the jump caused by the intervention v, given by

$$\Delta_v Y^{(v)}(\tau_j) := \Gamma(\check{Y}^{(v)}(\tau_j^-), \zeta) - \check{Y}^{(v)}(\tau_j^-). \tag{6.1.7}$$

Let $\mathcal{S} \subset \mathbb{R}^k$ be a fixed open set (the solvency region). Define

$$\tau_{\mathcal{S}} = \inf\{t \in (0, \tau^*); Y^{(v)}(t) \notin \mathcal{S}\}. \tag{6.1.8}$$

Suppose we are given a continuous *profit function* $f : \mathcal{S} \to \mathbb{R}$ and a continuous *bequest function* $g : \mathbb{R}^k \to \mathbb{R}$. Moreover, suppose the profit/utility of making an intervention with impulse $\zeta \in \mathcal{Z}$ when the state is y is $K(y, \zeta)$, where $K : \mathcal{S} \times \mathcal{Z} \to \mathbb{R}$ is a given continuous function.

We assume we are given a set $\mathcal{V}$ of *admissible impulse controls* which is included in the set of $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ such that a unique solution $Y^{(v)}$ of (6.1.2)–(6.1.4) exists and

$$\tau^* = \infty \quad \text{a.s.} \tag{6.1.9}$$

and (if $M = \infty$)

$$\lim_{j \to \infty} \tau_j = \tau_{\mathcal{S}}. \tag{6.1.10}$$

We also assume that

$$E^y \left[\int_0^{\tau_{\mathcal{S}}} \left| f(Y^{(v)}(t)) \right| \mathrm{d}t \right] < \infty \quad \text{for all } y \in \mathbb{R}^k, v \in \mathcal{V}, \tag{6.1.11}$$

$$E\left[g^-(Y^{(v)}(\tau_{\mathcal{S}}))\mathcal{X}_{\{\tau_{\mathcal{S}}<\infty\}}\right] < \infty \quad \text{for all } y \in \mathbb{R}^k,\ v \in \mathcal{V}, \tag{6.1.12}$$

and

$$E\left[\sum_{\tau_j \le \tau_{\mathcal{S}}} K^-(\check{Y}^{(v)}(\tau_j^-), \zeta_j)\right] < \infty \quad \text{for all } y \in \mathbb{R}^k,\ v \in \mathcal{V}. \tag{6.1.13}$$

Now define the performance criterion

$$J^{(v)}(y) = E^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y^{(v)}(t))\mathrm{d}t + g(Y^{(v)}(\tau_{\mathcal{S}}))\mathcal{X}_{\{\tau_{\mathcal{S}}<\infty\}} + \sum_{\tau_j \le \tau_{\mathcal{S}}} K(\check{Y}^{(v)}(\tau_j^-), \zeta_j)\right].$$

The *impulse control* problem is the following.

Find $\Phi(y)$ and $v^* \in \mathcal{V}$ such that

$$\Phi(y) = \sup\{J^{(v)}(y); v \in \mathcal{V}\} = J^{(v^*)}(y). \tag{6.1.14}$$

The following concept is crucial.

Definition 6.1. *Let $\mathcal{H}$ be the space of all measurable functions $h : \mathcal{S} \to \mathbb{R}$. The* intervention operator *$\mathcal{M} : \mathcal{H} \to \mathcal{H}$ is defined by*

$$\mathcal{M}h(y) = \sup\{h(\Gamma(y,\zeta)) + K(y,\zeta); \zeta \in \mathcal{Z} \quad \textit{and} \quad \Gamma(y,\zeta) \in \mathcal{S}\}. \tag{6.1.15}$$

As in Chap. 2 we put

$$\mathcal{T} = \{\tau; \tau \text{ stopping time},\ 0 \le \tau \le \tau_{\mathcal{S}} \text{ a.s.}\}.$$

We can now state the main result of this chapter, a verification theorem for impulse control problems.

Theorem 6.2 (Quasi-Integrovariational Inequalities for Impulse Control).

(a) *Suppose we can find $\phi : \bar{\mathcal{S}} \to \mathbb{R}$ such that*
 (i) $\phi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}})$.
 (ii) $\phi \ge \mathcal{M}\phi$ *on* $\mathcal{S}$.
 Define

$$D = \{y \in \mathcal{S}; \phi(y) > \mathcal{M}\phi(y)\} \qquad \textit{(the continuation region).}$$

 Assume
 (iii) $E^y\left[\int_0^{\tau_{\mathcal{S}}} \mathcal{X}_{\partial D}(Y^{(v)}(t))\mathrm{d}t\right] = 0$ *for all* $y \in \mathcal{S},\ v \in \mathcal{V}$.
 (iv) ∂D *is a Lipschitz surface.*

(v) $\phi \in C^2(\mathcal{S} \setminus \partial D)$ *with locally bounded derivatives near* ∂D.

(vi) $A\phi + f \leq 0$ *on* $\mathcal{S} \setminus \partial D$.

(vii) $Y^{(v)}(\tau_{\mathcal{S}}) \in \partial \mathcal{S}$ *a.s. on* $\{\tau_{\mathcal{S}} < \infty\}$ *and* $\phi(Y^{(v)}(t)) \to g(Y^{(v)}(\tau_{\mathcal{S}})) \cdot \chi_{\{\tau_{\mathcal{S}} < \infty\}}$ *as* $t \to \tau_{\mathcal{S}}^-$ *a.s., for all* $y \in \mathcal{S}$, $v \in \mathcal{V}$.

(viii) $\{\phi^-(Y^{(v)}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable, for all* $y \in \mathcal{S}$, $v \in \mathcal{V}$.

(ix) $E^y \left[|\phi(Y^{(v)}(\tau))| + \int_0^{\tau_{\mathcal{S}}} |A\phi(Y^{(v)}(t))| \mathrm{d}t \right] < \infty$ *for all* $\tau \in \mathcal{T}, v \in \mathcal{V}$, $y \in \mathcal{S}$.

Then

$$\phi(y) \geq \Phi(y) \quad \textit{for all } y \in \mathcal{S}. \tag{6.1.16}$$

(b) *Suppose in addition that*

(x) $A\phi + f = 0$ *in* D.

(xi) $\hat{\zeta}(y) \in \mathrm{Argmax}\{\phi(\Gamma(y,\cdot)) + K(y,\cdot)\} \in \mathcal{Z}$ *exists for all* $y \in \mathcal{S}$ *and* $\hat{\zeta}(\cdot)$ *is a Borel measurable selection.*

Put $\hat{\tau}_0 = 0$ *and define* $\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \ldots; \hat{\zeta}_1, \hat{\zeta}_2, \ldots)$ *inductively by* $\hat{\tau}_{j+1} = \inf\{t > \hat{\tau}_j; Y^{(\hat{v}_j)}(t) \notin D\} \wedge \tau_{\mathcal{S}}$ *and* $\hat{\zeta}_{j+1} = \hat{\zeta}(Y^{(\hat{v}_j)}(\hat{\tau}_{j+1}^-))$ *if* $\hat{\tau}_{j+1} < \tau_{\mathcal{S}}$, *where* $Y^{(\hat{v}_j)}$ *is the result of applying* $\hat{v}_j := (\hat{\tau}_1, \ldots, \hat{\tau}_j; \hat{\zeta}_1, \ldots, \hat{\zeta}_j)$ *to* Y.

Suppose

(xii) $\hat{v} \in \mathcal{V}$ *and* $\{\phi(Y^{(\hat{v})}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable.*

Then

$$\phi(y) = \Phi(y) \quad \textit{and } \hat{v} \textit{ is an optimal impulse control.} \tag{6.1.17}$$

Sketch of Proof. (a) By Theorem 2.1 and (iii)–(v), we may assume that $\phi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$. Choose $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots) \in \mathcal{V}$ and set $\tau_0 = 0$. By another approximation argument we may assume that we can apply the Dynkin formula to the stopping times τ_j. Then for $j = 0, 1, 2, \ldots$, with $Y = Y^{(v)}$

$$E^y[\phi(Y(\tau_j))] - E^y[\phi(\check{Y}(\tau_{j+1}^-))] = -E^y \left[\int_{\tau_j}^{\tau_{j+1}} A\phi(Y(t)) \mathrm{d}t \right], \tag{6.1.18}$$

where $\check{Y}(\tau_{j+1}^-) = Y(\tau_{j+1}^-) + \Delta_N Y(\tau_{j+1})$, as before. Summing this from $j = 0$ to $j = m$ we get

$$\begin{aligned} \phi(y) + \sum_{j=1}^{m} E^y[\phi(Y(\tau_j)) - \phi(\check{Y}(\tau_j^-))] - E^y[\phi(\check{Y}(\tau_{m+1}^-))] \\ = -E^y \left[\int_0^{\tau_{m+1}} A\phi(Y(t)) \mathrm{d}t \right] \geq E^y \left[\int_0^{\tau_{m+1}} f(Y(t)) \mathrm{d}t \right]. \end{aligned} \tag{6.1.19}$$

Now

$$\begin{aligned}\phi(Y(\tau_j)) &= \phi(\Gamma(\check{Y}(\tau_j^-),\zeta_j)) \\ &\le \mathcal{M}\phi(\check{Y}(\tau_j^-)) - K(\check{Y}(\tau_j^-),\zeta_j) \quad \text{if } \tau_j < \tau_{\mathcal{S}} \text{ by (6.1.15)}\end{aligned}$$

and

$$\phi(Y(\tau_j)) = \phi(\check{Y}(\tau_j^-)) \quad \text{if } \tau_j = \tau_{\mathcal{S}} \text{ by (vii).}$$

Therefore

$$\mathcal{M}\phi(\check{Y}(\tau_j^-)) - \phi(\check{Y}(\tau_j^-)) \ge \phi(Y(\tau_j)) - \phi(\check{Y}(\tau_j^-)) + K(\check{Y}(\tau_j^-),\zeta_j)$$

and

$$\begin{aligned}&\phi(y) + \sum_{j=1}^{m} E^y[\{\mathcal{M}\phi(\check{Y}(\tau_j^-)) - \phi(\check{Y}(\tau_j^-))\}\cdot \mathcal{X}_{\{\tau_j<\tau_{\mathcal{S}}\}}] \\ &\quad \ge E^y\left[\int_0^{\tau_{m+1}} f(Y(t))\mathrm{d}t + \phi(\check{Y}(\tau_{m+1}^-)) + \sum_{j=1}^{m} K(\check{Y}(\tau_j^-),\zeta_j)\right].\end{aligned}$$

Letting $m \to M$ we get

$$\phi(y) \ge E^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y(t))\mathrm{d}t + g(Y(\tau_{\mathcal{S}}))\mathcal{X}_{\{\tau_{\mathcal{S}}<\infty\}} + \sum_{j=1}^{M} K(\check{Y}(\tau_j^-),\zeta_j)\right] = J^{(v)}(y). \tag{6.1.20}$$

Hence $\phi(y) \ge \Phi(y)$.

(b) Next assume (x)–(xii) also hold. Apply the above argument to $\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \ldots; \hat{\zeta}_1, \hat{\zeta}_2, \ldots)$. Then by (x) we get *equality* in (6.1.19) and by our choice of $\zeta_j = \hat{\zeta}_j$ we have *equality* in (6.1.20). Hence

$$\phi(y) = J^{(\hat{v})}(y),$$

which combined with (a) completes the proof. □

Remark 6.3. In the case of a pure diffusion process, the same verification theorem holds, just skip condition (ix).

6.2 Examples

Example 6.4 (Optimal Stream of Dividends Under Transaction Costs). This example is an extension to the jump diffusion case of a problem studied in [J-PS]. Suppose that if we make no interventions the amount $X(t)$ available (cash flow) is given by

$$\mathrm{d}X(t) = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B(t) + \theta\int_{\mathbb{R}} z\widetilde{N}(\mathrm{d}t,\mathrm{d}z), \quad X(0) = x > 0, \tag{6.2.1}$$

where $\mu, \sigma > 0$, $\theta \geq 0$ are constants and we assume that $z \leq 0$ a.s. ν. Suppose that at any time t we are free to take out an amount $\zeta > 0$ from $X(t)$ by applying the transaction cost

$$k(\zeta) = c + \lambda\zeta, \tag{6.2.2}$$

where $c > 0$, $\lambda \geq 0$ are constants. The constant c is called the *fixed* part and the quantity $\lambda\zeta$ is called the *proportional* part, respectively, of the transaction cost. The resulting cash flow $X^{(v)}(t)$ is given by

$$X^{(v)}(t) = X(t) \quad \text{if } 0 \leq t < \tau_1, \tag{6.2.3}$$

$$X^{(v)}(\tau_j) = \Gamma(X^{(v)}(\tau_j^-) + \Delta_N X(\tau_j), \zeta_j) = \check{X}^{(v)}(\tau_j^-) - (1+\lambda)\zeta_j - c, \tag{6.2.4}$$

and

$$\mathrm{d}X^{(v)}(t) = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B(t) + \theta \int_{\mathbb{R}} z\widetilde{N}(\mathrm{d}t, \mathrm{d}z) \text{ if } \tau_j \leq t < \tau_{j+1}. \tag{6.2.5}$$

Put

$$\tau_{\mathcal{S}} = \inf\{t > 0; X^{(v)}(t) \leq 0\} \quad \text{(time of bankruptcy)} \tag{6.2.6}$$

and

$$J^{(v)}(s,x) = E^{s,x}\left[\sum_{\tau_j \leq \tau_{\mathcal{S}}} \mathrm{e}^{-\rho(s+\tau_j)}\zeta_j\right], \tag{6.2.7}$$

where $\rho > 0$ is constant (the discounting exponent).

We seek $\Phi(s,x)$ and $v^* = (\tau_1^*, \tau_2^*, \ldots; \zeta_1^*, \zeta_2^*, \ldots) \in \mathcal{V}$ such that

$$\Phi(s,x) = \sup_{v \in \mathcal{V}} J^{(v)}(s,x) = J^{(v^*)}(s,x), \tag{6.2.8}$$

where $\mathcal{V}$ is the set of impulse controls s.t. $X^{(v)}(t) \geq 0$ for all $t \leq \tau_{\mathcal{S}}$. This is a problem of the type (6.1.14), with

$$Y^{(v)}(t) = \begin{bmatrix} s+t \\ X^{(v)}(t) \end{bmatrix}, \quad t \geq 0, \quad Y^{(v)}(0^-) = \begin{bmatrix} s \\ x \end{bmatrix} = y,$$

$$\Gamma(y,\zeta) = \Gamma(s,x,\zeta) = \begin{bmatrix} s \\ x - c - (1+\lambda)\zeta \end{bmatrix},$$

$$K(y,\zeta) = K(s,x,\zeta) = \mathrm{e}^{-\rho s}\zeta, \quad f = g = 0,$$

and

$$\mathcal{S} = \{(s,x); x > 0\}.$$

As a candidate for the value function Φ we try

$$\phi(s,x) = \mathrm{e}^{-\rho s}\psi(x). \tag{6.2.9}$$

Then

$$\mathcal{M}\psi(x) = \sup\left\{\psi(x - c - (1+\lambda)\zeta) + \zeta,\ 0 < \zeta < \frac{x-c}{1+\lambda}\right\}.$$

We now guess that the continuation region has the form

$$D = \{(s,x); 0 < x < x^*\} \quad \text{for some } x^* > 0. \tag{6.2.10}$$

Then (x) of Theorem 6.2 gives

$$-\rho\psi(x) + \mu\psi'(x) + \frac{1}{2}\sigma^2\psi''(x) + \int_{\mathbb{R}}\{\psi(x+\theta z) - \psi(x) - \psi'(x)\theta z\}\nu(\mathrm{d}z) = 0.$$

To solve this equation we try a function of the form

$$\psi(x) = \mathrm{e}^{rx}$$

for some constant $r \in \mathbb{R}$. Then r must solve the equation

$$h(r) := -\rho + \mu r + \frac{1}{2}\sigma^2 r^2 + \int_{\mathbb{R}}\{\mathrm{e}^{r\theta z} - 1 - r\theta z\}\nu(\mathrm{d}z) = 0. \tag{6.2.11}$$

Since $h(0) = -\rho < 0$ and $\lim_{|r|\to\infty} h(r) = \infty$, we see that there exist two solutions r_1, r_2 of $h(r) = 0$ such that

$$r_2 < 0 < r_1.$$

Moreover, since $\mathrm{e}^{r\theta z} - 1 - r\theta z \geq 0$ for all r, z we have

$$|r_2| > r_1.$$

With such a choice of r_1, r_2 we try

$$\psi(x) = A_1\mathrm{e}^{r_1 x} + A_2\mathrm{e}^{r_2 x}, \quad A_i \text{ constants.}$$

Since

$$\psi(0) = 0 \quad \text{we have} \quad A_1 + A_2 = 0$$

so we write $A_1 = A = -A_2 > 0$ and

$$\psi(x) = A\big(\mathrm{e}^{r_1 x} - \mathrm{e}^{r_2 x}\big), \quad 0 < x < x^*.$$

Define

$$\psi_0(x) = A\big(\mathrm{e}^{r_1 x} - \mathrm{e}^{r_2 x}\big) \quad \text{for } \textit{all } x > 0. \tag{6.2.12}$$

To study $\mathcal{M}\psi$ we first consider

$$g(\zeta) := \psi_0(x - c - (1+\lambda)\zeta) + \zeta, \quad \zeta > 0.$$

The first-order condition for a maximum point $\hat{\zeta} = \hat{\zeta}(x)$ for $g(\zeta)$ is that

$$\psi_0'(x - c - (1+\lambda)\hat{\zeta}) = \frac{1}{1+\lambda}.$$

Now

$$\psi_0'(x) > 0 \quad \text{for all } x \text{ and}$$
$$\psi_0''(x) < 0 \quad \text{iff} \quad x < \tilde{x} := \tfrac{2(\ln|r_2| - \ln r_1)}{r_1 - r_2}.$$

Therefore the equation $\psi_0'(x) = 1/(1+\lambda)$ has exactly two solutions $x = \underline{x}$ and $x = \bar{x}$ where

$$0 < \underline{x} < \tilde{x} < \bar{x}$$

(provided that $\psi_0'(\tilde{x}) < 1/(1+\lambda) < \psi_0'(0)$). See Fig. 6.1.

Choose

$$x^* = \bar{x} \quad \text{and put} \quad \hat{x} = \underline{x}. \tag{6.2.13}$$

If we require that $\psi(x) = \mathcal{M}\psi_0(x)$ for $x \geq x^*$ we get

$$\psi(x) = \psi_0(\hat{x}) + \hat{\zeta}(x) \quad \text{for} \quad x \geq x^*,$$

where

$$x - c - (1+\lambda)\hat{\zeta}(x) = \hat{x},$$

i.e.,

$$\hat{\zeta}(x) = \frac{x - \hat{x} - c}{1+\lambda} \quad \text{for} \quad x \geq x^*. \tag{6.2.14}$$

Hence we propose that ψ has the form

$$\psi(x) = \begin{cases} \psi_0(x) = A(\mathrm{e}^{r_1 x} - \mathrm{e}^{r_2 x}), & 0 < x < x^*, \\ \psi_0(\hat{x}) + \frac{x - \hat{x} - c}{1+\lambda}, & x \geq x^*. \end{cases} \tag{6.2.15}$$

Now choose A such that ψ is continuous at $x = x^*$. This gives

$$A = (1+\lambda)^{-1}\left[\mathrm{e}^{r_1 x^*} - \mathrm{e}^{r_2 x^*} - \mathrm{e}^{r_1 \hat{x}} + \mathrm{e}^{r_2 \hat{x}}\right]^{-1}(x^* - \hat{x} - c). \tag{6.2.16}$$

By our choice of x^* we then have that ψ is also differentiable at $x = x^*$.

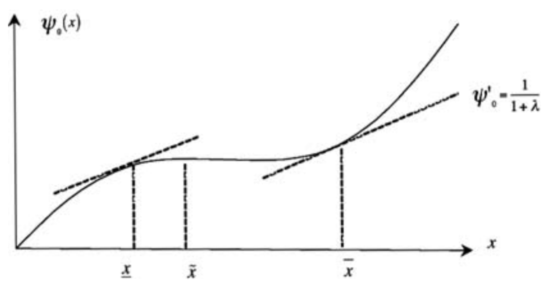

Fig. 6.1. The function $\psi_0(x)$

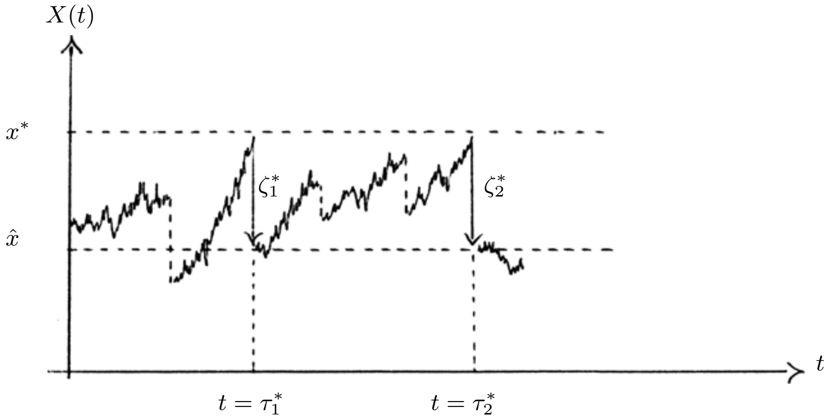

Fig. 6.2. The optimal impulse control of Example 6.4

We can now check that, with these values of $x^*, \hat{x}$, and A, our choice of $\phi(s,x) = \mathrm{e}^{-\rho s}\psi(x)$ satisfies all the requirements of Theorem 6.2, provided that some conditions on the parameters are satisfied. We leave this verification to the reader.

Thus the solution of the impulse control problem (6.2.8) can be described as follows. As long as $X(t) < x^*$ we do nothing. If $X(t)$ reaches the value x^* then immediately we make an intervention to bring $X(t)$ down to the value $\hat{x}$. See Fig. 6.2.

Example 6.5. As another illustration of how to apply Theorem 6.2 we consider the following example, which is a jump diffusion version of the example in [Ø2] studied in connection with questions involving vanishing fixed costs. Variations of this problem have been studied by many authors (see, e.g., [HST, J-P, MØ, ØS, ØUZ, V]). One possible economic interpretation is that the given process represents the exchange rate of a given currency and the impulses represent the interventions taken in order to keep the exchange rate in a given "target zone." See, e.g., [J-P, MØ].

Suppose that without interventions the system has the form

$$Y(t) = \begin{bmatrix} s+t \\ X(t) \end{bmatrix} \in \mathbb{R}^2, \quad Y(0) = y = (s,x), \tag{6.2.17}$$

where $X(t) = x + B(t) + \int_0^t \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z)$ and $B(0) = 0$. We assume that $z \le 0$ a.s. z. Suppose that we are only allowed to give the system impulses ζ with values in $\mathcal{Z} := (0,\infty)$ and that if we apply an impulse control $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ to $Y(t)$ it gets the form

$$Y^{(v)}(t) = \begin{bmatrix} s+t \\ X(t) - \sum_{\tau_k \le t} \zeta_k \end{bmatrix} = \begin{bmatrix} s+t \\ X^{(v)}(t) \end{bmatrix}. \tag{6.2.18}$$

Suppose that the *cost rate* $f(t,\xi)$ if $X^{(v)}(t)=\xi$ at time t is given by

$$f(t,\xi)=\mathrm{e}^{-\rho t}\xi^2, \tag{6.2.19}$$

where $\rho>0$ is constant. In an effort to reduce the cost one can apply the impulse control v in order to reduce the value of $X^{(v)}(t)$. However, suppose the cost of an intervention of size $\zeta>0$ at time t is

$$K(t,\xi,\zeta)=K(\zeta)=c+\lambda\zeta, \tag{6.2.20}$$

where $c>0$ and $\lambda\geq 0$ are constants. Then the expected total discounted cost associated to a given impulse control is

$$J^{(v)}(s,x)=E^x\left[\int_0^\infty \mathrm{e}^{-\rho(s+t)}(X^{(v)}(t))^2\mathrm{d}t+\sum_{k=1}^{N}\mathrm{e}^{-\rho(s+\tau_k)}(c+\lambda\zeta_k)\right]. \tag{6.2.21}$$

We seek $\Phi(s,x)$ and $v^*=(\tau_1^*,\tau_2^*,\ldots;\zeta_1^*,\zeta_2^*,\ldots)$ such that

$$\Phi(s,x)=\inf_v J^{(v)}(s,x)=J^{(v^*)}(s,x). \tag{6.2.22}$$

This is an impulse control problem of the type described above, except that it is a minimum problem rather than a maximum problem. Theorem 6.2 still applies, with the corresponding changes.

Note that it is not optimal to move $X(t)$ downward if $X(t)$ is already below 0. Hence we may restrict ourselves to consider impulse controls $v=(\tau_1,\tau_2,\ldots;\zeta_1,\zeta_2,\ldots)$ such that

$$\sum_{j=1}^{\tau_k}\zeta_j\leq X(\tau_k)\quad\text{for all } k. \tag{6.2.23}$$

We let $\mathcal{V}$ denote the set of such impulse controls.

We guess that the optimal strategy is to wait until the level of $X(t)$ reaches an (unknown) value $x^*>0$. At this time, τ_1, we intervene and give $X(t)$ an impulse ζ_1, which brings it down to a lower value $\hat{x}>0$. Then we do nothing until the next time, τ_2, that $X(t)$ reaches the level x^*, etc. This suggests that the continuation region D in Theorem 6.2 has the form

$$D=\{(s,x),\ x<x^*\}. \tag{6.2.24}$$

See Fig. 6.3.

Let us try a value function φ of the form

$$\varphi(s,x)=\mathrm{e}^{-\rho s}\psi(x), \tag{6.2.25}$$

where ψ remains to be determined.

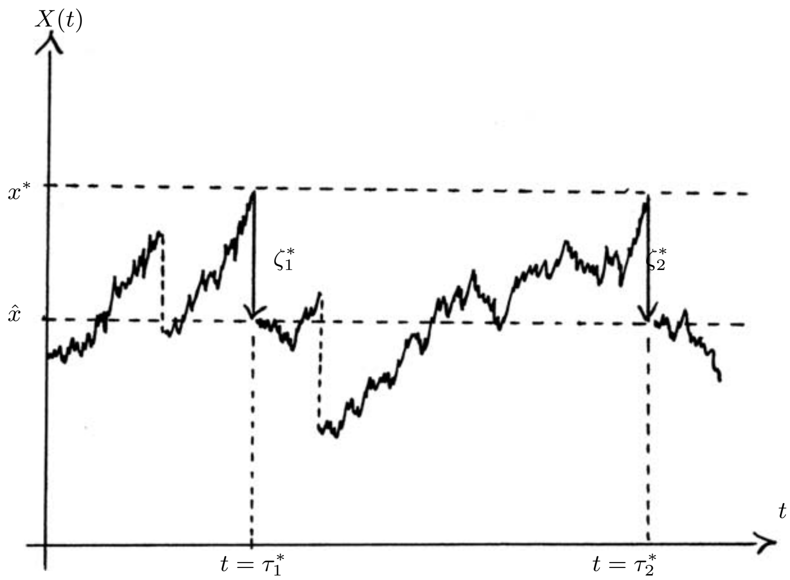

Fig. 6.3. The optimal impulse control of Example 6.5

Condition (x) of Theorem 6.2 gives that for $x < x^*$ we should have

$$A\varphi + f = \mathrm{e}^{-\rho s}\left(-\rho\psi(x) + \frac{1}{2}\psi''(x) + \int_{\mathbb{R}}\{\psi(x+z) - \psi(x) - z\psi'(x)\}\nu(\mathrm{d}z)\right) + \mathrm{e}^{-\rho s}x^2 = 0.$$

So for $x < x^*$ we let ψ be a solution $h(x)$ of the equation

$$\int_{\mathbb{R}}\{h(x+z) - h(x) - zh'(x)\}\nu(\mathrm{d}z) + \frac{1}{2}h''(x) - \rho h(x) + x^2 = 0. \qquad (6.2.26)$$

We see that any function $h(x)$ of the form

$$h(x) = C_1\mathrm{e}^{r_1 x} + C_2\mathrm{e}^{r_2 x} + \frac{1}{\rho}x^2 + \frac{1 + \int_{\mathbb{R}} z^2\nu(\mathrm{d}z)}{\rho^2}, \qquad (6.2.27)$$

where C_1, C_2 are arbitrary constants, is a solution of (6.2.26), provided that $r_1 > 0, r_2 < 0$ are roots of the equation

$$K(r) := \int_{\mathbb{R}}\{\mathrm{e}^{rz} - 1 - rz\}\nu(\mathrm{d}z) + \frac{1}{2}r^2 - \rho = 0.$$

Note that if we make no interventions at all, then the cost is

$$J^{(v)}(s,x) = \mathrm{e}^{-\rho s}E^x\left[\int_0^\infty \mathrm{e}^{-\rho t}(X(t))^2\mathrm{d}t\right] = \mathrm{e}^{-\rho s}\int_0^\infty \mathrm{e}^{-\rho t}(x^2 + tb)\mathrm{d}t = \mathrm{e}^{-\rho s}\left(\frac{1}{\rho}x^2 + \frac{b}{\rho^2}\right), \qquad (6.2.28)$$

where $b = 1 + \int_{\mathbb{R}} z^2 \nu(\mathrm{d}z)$. Hence we must have

$$0 \le \psi(x) \le \frac{1}{\rho} x^2 + \frac{b}{\rho^2} \quad \text{for all } x. \tag{6.2.29}$$

Comparing this with (6.2.27) we see that we must have $C_2 = 0$. Hence $C_1 \le 0$. So we put

$$\psi(x) = \psi_0(x) := \frac{1}{\rho} x^2 + \frac{b}{\rho^2} - a\mathrm{e}^{r_1 x} \quad \text{for } x \le x^*, \tag{6.2.30}$$

where $a = -C_1$ remains to be determined.

We guess that $a > 0$.

To determine a we first find ψ for $x > x^*$ and then require ψ to be C^1 at $x = x^*$.

By (ii) and (6.2.24) we know that for $x > x^*$ we have

$$\psi(x) = \mathcal{M}\psi(x) := \inf\{\psi(x - \zeta) + c + \lambda\zeta, \zeta > 0\}. \tag{6.2.31}$$

The first-order condition for a minimum $\hat{\zeta} = \hat{\zeta}(x)$ of the function

$$G(\zeta) := \psi(x - \zeta) + c + \lambda\zeta, \quad \zeta > 0$$

is

$$\psi'(x - \hat{\zeta}) = \lambda.$$

Suppose there is a unique point $\hat{x} \in (0, x^*)$ such that

$$\psi'(\hat{x}) = \lambda. \tag{6.2.32}$$

Then

$$\hat{x} = x - \hat{\zeta}(x), \text{ i.e., } \hat{\zeta}(x) = x - \hat{x}$$

and from (6.2.31) we deduce that

$$\psi(x) = \psi_0(\hat{x}) + c + \lambda(x - \hat{x}) \quad \text{for } x \ge x^*.$$

In particular,

$$\psi'(x^*) = \lambda \tag{6.2.33}$$

and

$$\psi(x^*) = \psi_0(\hat{x}) + c + \lambda(x^* - \hat{x}). \tag{6.2.34}$$

To summarize we put

$$\psi(x) = \begin{cases} \dfrac{1}{\rho} x^2 + \dfrac{b}{\rho^2} - a\mathrm{e}^{r_1 x}, & \text{for } x \le x^*, \\ \psi_0(\hat{x}) + c + \lambda(x - \hat{x}), & \text{for } x > x^*, \end{cases} \tag{6.2.35}$$

where $\hat{x}, x^*$, and a are determined by (6.2.32)–(6.2.34), i.e.,

$$ar_1 e^{r_1 \hat{x}} = \frac{2}{\rho}\hat{x} - \lambda \quad (\text{i.e., } \psi'(\hat{x}) = \lambda), \tag{6.2.36}$$

$$ar_1 e^{r_1 x^*} = \frac{2}{\rho}x^* - \lambda \quad (\text{i.e., } \psi'(x^*) = \lambda), \tag{6.2.37}$$

$$ae^{r_1 x^*} - ae^{r_1 \hat{x}} = \frac{1}{\rho}((x^*)^2 - (\hat{x})^2) - c - \lambda(x^* - \hat{x}). \tag{6.2.38}$$

One can now prove (see [Ø2, Theorem 2.5]).

For each $c > 0$ there exists $a = a^*(c) > 0$, $\hat{x} = \hat{x}(c) > 0$, and $x^* = x^*(c) > \hat{x}$ such that (6.2.36)–(6.2.38) hold. With this choice of a, $\hat{x}$, and x^*, the function $\varphi(s,x) = e^{-\rho s}\psi(x)$ with ψ given by (6.2.35) coincides with the value function Φ defined in (6.2.22). Moreover, the optimal impulse control $v^* = (\tau_1^*, \tau_2^*, \ldots; \zeta_1^*, \zeta_2^*, \ldots)$ is to do nothing while $X(t) < x^*$, then move $X(t)$ from x^* down to $\hat{x}$ (i.e., apply $\zeta_1^* = x^* - \hat{x}$) at the first time τ_1^* when $X(t)$ reaches a value $\geq x^*$, then wait until the next time, τ_2^*, $X(t)$ again reaches the value x^*, etc.

Remark 6.6. In [Ø2] this result is used to study how the value function $\Phi(s,x) = \Phi_c(s,x)$ depends on the fixed part $c > 0$ of the intervention cost. It is proved that the function

$$c \to \Phi_c(s,x)$$

is continuous but not differentiable at $c = 0$. In fact, we have

$$\frac{\partial}{\partial c}\Phi_c(s,x) \to \infty \text{ as } c \to 0^+.$$

Subsequently this high c-sensitivity of the value function for c close to 0 was proved for other processes as well. See [ØUZ].

Remark 6.7. For applications of impulse control theory in inventory control see, e.g., [S, S2] and the references therein.

6.3 Exercises

Exercise* 6.1. Solve the impulse control problem

$$\Phi(s,x) = \inf_v J^{(v)}(s,x) = J^{(v^*)}(s,x),$$

where

$$J^{(v)}(s,x) = E\left[\int_0^\infty e^{-\rho(s+t)}(X^{(v)}(t))^2 dt + \sum_{k=1}^N e^{-\rho(s+\tau_k)}(c + \lambda|\zeta_k|)\right].$$

The inf is taken over all impulse controls $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ with $\zeta_i \in \mathbb{R}$ and the corresponding process $X^{(v)}(t)$ is given by

$$\begin{aligned} \mathrm{d}X^{(v)}(t) &= \mathrm{d}B(t) + \int_{\mathbb{R}} \theta(X^{(v)}(t), z)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad \tau_k < t < \tau_{k+1}, \\ X^{(v)}(\tau_{k+1}) &= \check{X}^{(v)}(\tau_{k+1}^-) + \zeta_{k+1}, \quad k = 0, 1, 2, \ldots, \end{aligned}$$

where $B(0) = 0$, $x \in \mathbb{R}$, and we assume that there exists $\xi > 0$ such that

$$\theta(x, z) = 0 \ \text{ for a.a. } z \text{ if } |x| < \xi, \tag{6.3.1}$$

$$\theta(x, z) \geq 0 \text{ if } x \geq \xi \text{ and } \theta(x, z) \leq 0 \ \text{ if } x \leq -\xi \quad \text{for a.a. } z, \tag{6.3.2}$$

Exercise* 6.2 (Optimal Stream of Dividends with Transaction Costs from a Geometric Lévy Process).

For $v = (\tau_1, \tau_2, \ldots; \zeta_1, \zeta_2, \ldots)$ with $\zeta_i \in \mathbb{R}_+$ define $X^{(v)}(t)$ by

$$\begin{aligned} \mathrm{d}X^{(v)}(t) &= \mu X^{(v)}(t)\mathrm{d}t + \sigma X^{(v)}(t)\mathrm{d}B(t) \\ &\quad + \theta X^{(v)}(t^-) \int_{\mathbb{R}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z), \quad \tau_i \leq t \leq \tau_{i+1}, \\ X^{(v)}(\tau_{i+1}) &= \check{X}^{(v)}(\tau_{i+1}^-) - (1+\lambda)\zeta_{i+1} - c, \quad i = 0, 1, 2, \ldots, \\ X^{(v)}(0^-) &= x > 0, \end{aligned}$$

where $\mu, \sigma \neq 0$, $\theta, \lambda \geq 0$, and $c > 0$ are constants (see (6.1.5)), $-1 \leq \theta z \leq 0$ a.s. ν.

Find Φ and v^* such that

$$\Phi(s, x) = \sup_v J^{(v)}(s, x) = J^{(v^*)}(s, x).$$

Here

$$J^{(v)}(s, x) = E^x \left[\sum_{\tau_k < \tau_{\mathcal{S}}} \mathrm{e}^{-\rho(s+\tau_k)} \zeta_k \right] \qquad (\rho > 0 \text{ constant})$$

is the expected discounted total dividend up to time $\tau_{\mathcal{S}}$, where

$$\tau_{\mathcal{S}} = \tau_{\mathcal{S}}(\omega) = \inf\{t > 0; X^{(v)}(t) \leq 0\}$$

is the time of bankruptcy. (See also Exercise 7.2.)

Exercise* 6.3 (Optimal Forest Management (Inspired by Y. Willassen [W])).

Suppose the biomass of a forest at time t is given by

$$X(t) = x + \mu t + \sigma B(t) + \theta \int_{\mathbb{R}} z\widetilde{N}(t, \mathrm{d}z),$$

where $\mu > 0$, $\sigma > 0$, $\theta > 0$ are constants and we assume that $z \leq 0$ a.s. ν. At times $0 \leq \tau_1 < \tau_2 < \cdots$ we decide to cut down the forest and replant it, with the cost

$$c + \lambda \check{X}(\tau_k^-), \quad \text{with} \quad \check{X}(\tau_k^-) = X(\tau_k^-) + \Delta_N X(\tau_k),$$

where $c > 0$ and $\lambda \in [0, 1)$ are constants and $\Delta_N X(t)$ is the (possible) jump in X at t coming from the jump in $N(t, \cdot)$ only, not from the intervention.

Find the sequence of stopping times $v = (\tau_1, \tau_2, \ldots)$ which maximizes the expected total discounted net profit $J^{(v)}(s, x)$ given by

$$J^{(v)}(s, x) = E^x \left[\sum_{k=1}^{\infty} \mathrm{e}^{-\rho(s+\tau_k)} (\check{X}(\tau_k^-) - c - \lambda \check{X}(\tau_k^-)) \right],$$

where $\rho > 0$ is a given discounting exponent.