1

Introduction

Plasma instabilities are normal modes of a system that grow in space or time. Thus the word “instability” implies a well-defined relationship between wavevector $k$ and frequency $\omega$; this in turn implies that the associated plasma fluctuations are relatively weak so that linear theory is appropriate to describe the physics.

This book uses linear Vlasov theory to describe the propagation, damping and growth of plasma modes. Linear theory cannot describe the ultimate fate of a plasma instability, nor its interactions with other modes. Of course the questions of how an instability reaches maximum amplitude, whether and how it contributes to plasma transport and whether such transport affects the overall flow of mass, momentum and energy at large scales are crucial for establishing the relevance of microphysics to large scale modelling of space plamas. But these questions must be addressed by nonlinear theory and computer simulation, which are beyond the purview of this book. Our relatively modest goal is to use computer solutions of the unapproximated Vlasov dispersion equation to firmly establish the properties of plasma normal modes; our hope is that this information will provide a useful foundation for the interpretation of computer simulations and spacecraft observations under conditions of relatively weak fluctuation amplitudes.

1.1 Micro- vs macro-

The most general classification of growing modes in a plasma divides them into two broad categories: macroinstabilities at relatively long wavelengths and microinstabilities at shorter wavelengths. Macroinstabilities depend on the configuration-space properties of the plasma and are well described by fluid equations; microinstabilities are driven by the departure from thermo-dynamic equilibrium of the plasma velocity distributions and therefore must
be described by the Vlasov equation or other kinetic equation. In a magnetized plasma with the gyroradius of a characteristic ion \( a_i \), macroinstabilities generally grow most rapidly at \( k a_i \ll 1 \), whereas microinstabilities generally have maximum growth rates at \( k a_i \gg 1 \).

The distinction between the two categories is not clear cut; macromodes may have appreciable growths at short wavelength and micromodes may persist to small wavenumbers. The distinction is further blurred by the fact that the wavenumber at maximum growth rate of some instabilities depends on the plasma parameters and may slide from the micro- to the macro- regime as the parameters change. Nevertheless, the macro- vs micro-distinction is a convenient one to begin any general discussion of unstable plasma modes.

1.2 The kinetic equation

Most plasma microphenomena are thought to be well described by Maxwell’s equations coupled with a kinetic equation; this equation determines the time development of \( f_j(x, v, t) \), the \( j \)th species distribution function. On the other hand, plasma macrophenomena often are described adequately by the field equations coupled with a set of fluid equations that may, under suitable assumptions, be derived from a kinetic equation. Fluid equation variables are functions only of space and time (\( x \) and \( t \)).

The velocity-space information of a kinetic equation implies that the physics of microphenomena is much more diverse than that associated with macrophenomena. The most important new plasma property associated with kinetic plasma physics is wave-particle interactions. This is the capability of waves with appropriate phase speeds or group velocities to exchange energy with plasma particles moving with the same velocities. Wave-particle interactions are exemplified by Landau and cyclotron damping (discussed in Chapter 2) and by plasma instabilities driven by non-Maxwellian properties of the velocity distribution (Chapters 3, 4, 7 and 8).

For the nonrelativistic plasmas considered in this book, the general form of the kinetic equation is (Montgomery and Tidman, 1964; Nicholson, 1983)

\[
\frac{\partial f_j}{\partial t} + v \cdot \frac{\partial f_j}{\partial x} + \frac{e_j}{m_j} \left( E + \frac{v \times B}{c} \right) \cdot \frac{\partial f_j}{\partial v} = \left( \frac{\partial f_j}{\partial t} \right)_{\text{collision}} \tag{1.2.1}
\]

where \( e_j \) is the charge of the \( j \)th species particle, \( m_j \) is the mass of the \( j \)th species particle, and the right-hand side represents the effect of some unspecified collision term. We use Gaussian cgs units throughout this book because they are the most widely used units in the research literature.
1.2 The kinetic equation

A concise but thorough discussion of the relationship between cgs and rationalized mks units is given as an Appendix in Boyd and Sanderson [1969].

In many of the models used in this book, we assume that the plasma consists of two species, electrons (denoted by subscript \(e\)) and ions (subscript \(i\)); the latter will usually be taken to be protons (subscript \(p\)). At times we will consider one of these species to consist of two distinct components; we will assume that the distribution function of each component satisfies Equation (1.2.1). In such cases the subscript notation will be introduced as appropriate. However, the \(e\) and \(i\) subscripts will denote overall electron and ion properties throughout the book. The notation \(\sum_j\) will denote a summation over all components and species in the plasma.

The distribution function is related to the macroscopic plasma quantities through velocity moment integrals. The particle density of the \(j\)th species or component is obtained from the zeroth velocity moment:

\[
n_j = \int d^3v \, f_j. \tag{1.2.2}
\]

From the first velocity moment we define the particle flux density of the \(j\)th species or component

\[
\Gamma_j = \int d^3v \, v f_j, \tag{1.2.3}
\]

the momentum density \(P_j = m_j \Gamma_j\), and the drift velocity \(v_{dj} = \Gamma_j/n_j\). If there is a background magnetic field in the \(z\)-direction, the field-aligned component of the drift velocity will be denoted by \(v_{ozj} = 2\Gamma_{zj}/n_j\), and the cross-field drift by \(v_{xdj}\).

From the second moments of the velocity we define the kinetic energy density tensor of the \(j\)th species or component

\[
W_j = \frac{m_j}{2} \int d^3v \, v^2 f_j
\]

with

\[
W_j = \frac{m_j}{2} \int d^3v \, v^2 f_j
\]

and the temperature of the \(j\)th species or component

\[
T_j = \frac{m_j}{3n_j} \int d^3v \, (v - v_{dj})^2 f_j
\]

with separate temperatures parallel and perpendicular to the background magnetic field

\[
T_{ij} = \frac{m_j}{n_j} \int d^3v \, (v_z - v_{oz})^2 f_j
\]
Introduction

and

\[ T_{\perp j} = \frac{m_j}{2n_j} \int d^3v (v_\perp - v_{\perp j})^2 f_j. \]

The third velocity moment yields the kinetic energy flux density, or, as it is more commonly known, the heat flux density of the \( j \)th species or component:

\[ q_j = \frac{m_j}{2} \int d^3v \, v^2 f_j. \]

Maxwell’s equations for the electric and magnetic fields are

\[ \nabla \cdot \mathbf{E} = 4\pi \rho \] (1.2.4)

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \] (1.2.5)

\[ \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J} \] (1.2.6)

\[ \nabla \cdot \mathbf{B} = 0 \] (1.2.7)

where the charge density \( \rho = \Sigma \rho_j n_j \) and the current density \( \mathbf{J} = \Sigma \rho_j \mathbf{J}_j \) are defined in terms of \( f_j \) by Equations (1.2.2) and (1.2.3). Equations (1.2.1) and (1.2.4) through (1.2.7) are the basic equations for the kinetic theory of microinstabilities.

**Problem 1.2.1.** Prove that, if the collision term conserves the particle number of each species, velocity integration of the kinetic equation (1.2.1) yields the equation of continuity

\[ \frac{\partial n_j}{\partial t} + \nabla \cdot \mathbf{J}_j = 0. \] (1.2.8)

### 1.3 The Vlasov equation

Detailed derivations of the kinetic equation and various forms of the collision term of Equation (1.2.1) in a fully ionized plasma are given in Montgomery and Tidman (1964), Clemmow and Dougherty (1969) and Nicholson (1983), for example. In this book we are concerned with plasmas that are sufficiently hot and/or tenuous so that the number of particles within a sphere of Debye length radius is large:

\[ n \lambda_D^3 = n_e \left( \frac{T_e}{4\pi n_e e^2} \right)^{3/2} \gg 1. \] (1.3.1)

This is called a collisionless plasma. In this case, the right hand side of
1.4 Definitions

Equation (1.2.1) is negligible and one has the Vlasov equation

\[ \frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \frac{\partial f_j}{\partial \mathbf{x}} + \frac{e_j}{m_j} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_j}{\partial \mathbf{v}} = 0. \tag{1.3.2} \]

This is the kinetic equation used throughout this book.

**Problem 1.3.1.** By integrating appropriate velocity moments of the Vlasov equation, derive the following momentum and energy equations:

\[ \frac{\partial \Gamma_j}{\partial t} + \nabla \cdot \mathbf{W}_j = \frac{e_j}{m_j} \left( n_j \mathbf{E} + \frac{\Gamma_j \times \mathbf{B}}{c} \right), \tag{1.3.3} \]

\[ \frac{\partial W_j}{\partial t} + \nabla \cdot \mathbf{q}_j = \frac{e_j}{m_j} \Gamma_j \cdot \mathbf{E}. \tag{1.3.4} \]

Why does the magnetic field not appear in the energy equation?

1.4 Definitions

Whenever it is present, the background magnetic field is assumed to be constant, uniform, and pointing in the z-direction: \( \mathbf{B}_0 = \hat{z}B_0 \), with \( B_0 > 0 \). The subscripts \( \parallel \) and \( \perp \) refer to directions parallel and perpendicular to \( \mathbf{B}_0 \), respectively; thus, for example, \( \mathbf{v}_\perp = \mathbf{v}_x \hat{x} + \mathbf{v}_y \hat{y} \). The unperturbed plasma is assumed to be charge neutral with a total electron density \( n_e \). Throughout this book the following notation will be used:

- Thermal speed of the \( j \)th component: \( \nu_j = \left( \frac{T_{\parallel j}}{m_j} \right)^{1/2} \)
- Plasma frequency of the \( j \)th component: \( \omega_j = \left( \frac{4\pi e^2 n_j}{m_j} \right)^{1/2} \)
- Cyclotron frequency of the \( j \)th species: \( \Omega_j = \frac{e_j B_0}{m_j c} \)
- Debye wavenumber of the \( j \)th component: \( k_j = \left( \frac{4\pi e^2 n_j}{T_{\parallel j}} \right)^{1/2} \)
- (Signed) gyroradius of the \( j \)th component: \( a_j = \frac{\nu_j}{\Omega_j} \left( \frac{T_{\perp j}}{T_{\parallel j}} \right)^{1/2} \)
- Beta of the \( j \)th component: \( \beta_j = \frac{8\pi n_j T_{\parallel j}}{B_0^2} \)
- Inertial length of the \( j \)th component: \( \xi_j \)
- Plasma beta: \( \beta = \frac{8\pi n_e \mu_0 T_{\parallel}}{B_0^2} \)
- Alfvén speed: \( v_A = \left( \frac{B_0^2}{4\pi n_e \mu_0} \right)^{1/2} \)
6

Introduction

The Boltzmann factor $k_B$ is always understood to multiply the temperatures $T_j$. Note that $\beta_j$ and $v_A$ are defined in terms of the total electron density, $n_e$, not the $j$th component density $n_j$.

**Problem 1.4.1.** Assume representative solar wind parameters $n_e \approx 10^3 \text{ cm}^{-3}$, $T_e \approx T_p \approx 10^5 \text{ K}$, and $B_0 \approx 10^{-4} \text{ G}$. Calculate the electron and proton thermal speeds and the Alfvén speed, and compare them with a representative solar wind speed of 400 km/sec. Similarly, calculate and compare representative electron and ion plasma and cyclotron frequencies in the solar wind.

A reduced distribution function, which is useful for one-dimensional pictures, is defined as an integral over the velocity components perpendicular to some direction:

$$f_j(v_z) = \int dv_x dv_y f_j(v).$$

Charge neutrality, which is assumed in zeroth order for all situations, requires

$$\rho = \sum_j e_j n_j = 0. \quad (1.4.1)$$

In addition, for all configurations except those explicitly associated with a current, we will assume that an electric field acts to maintain zero current and in zeroth order

$$\mathbf{J} = \sum_j e_j n_j \mathbf{v}_{dj} = 0. \quad (1.4.2)$$

Throughout this book we consider weak plasma fluctuations. This means that the fluctuating fields are sufficiently small in amplitude that both the fields and the distribution functions may be expanded

$$f_j(x,v,t) = f_j^{(0)}(x,v) + f_j^{(1)}(x,v,t) + f_j^{(2)}(x,v,t) + \ldots \quad (1.4.3)$$

$$\mathbf{E}(x,t) = \mathbf{E}_0(x) + \mathbf{E}^{(1)}(x,t) + \mathbf{E}^{(2)}(x,t) + \ldots$$

$$\mathbf{B}(x,t) = \mathbf{B}_0(x) + \mathbf{B}^{(1)}(x,t) + \mathbf{B}^{(2)}(x,t) + \ldots$$

where superscript $(j)$ represents a quantity proportional to $|\mathbf{E}^{(j)}|$.

1.5 Fluctuations, waves and instabilities

The traditional development of the linear theory of instabilities in collisionless plasmas follows a well-established procedure: The linear Vlasov
1.5 Fluctuations, waves and instabilities

The dispersion equation is subjected to a Fourier/Laplace analysis in space/time, yielding fluctuating particle densities and particle flux densities that are inserted into Maxwell’s equations (1.2.4) through (1.2.7) to yield a dispersion equation. The solution of this dispersion equation relates frequency $\omega$ and wavevector $k$ and thereby determines the normal modes of the plasma.

The dispersion equation may be solved either as a boundary value problem ($\omega$ is given as real, and one solves for a complex component of $k$) or as an initial value problem ($k$ is given as real, and one solves for a complex $\omega$). The latter approach is subject to fewer mathematical ambiguities and is the approach more often followed in the literature; we follow it exclusively.

Thus throughout this book the complex frequency will be $\omega = \omega_r + i\gamma$ where $\gamma$ is the growth or damping rate. We regard as a heavily damped oscillation any solution of the linear dispersion equation that satisfies $\gamma < -|\omega_r|/2\pi$. We use the term waves to describe those weakly damped solutions that satisfy $-|\omega_r|/2\pi < \gamma \leq 0$, and describe as instabilities growing solutions with $\gamma > 0$. And fluctuations will denote both stable waves and instabilities. The phase speed of a fluctuation, the speed at which a point of constant phase of a single mode propagates through the plasma, is $\omega_r/k$.

Although the observed frequency of a plasma fluctuation is a function of the relative motion between the observer and the medium bearing the wave, the damping or growth rates $\gamma$ calculated from homogeneous plasma theory are independent of the frame in which the calculation is performed.

Problem 1.5.1. Show that an observer moving with velocity $v_o$ with respect to a plane wave of frequency $\omega_r$ and wavevector $k$ observes a frequency $\omega_r - k \cdot v_o$. This change in the observed frequency is called the Doppler shift.

Unless stated otherwise, the wavevector will be taken to lie in the $(y, z)$-plane, so that

$$k = \hat{y}k_y + \hat{z}k_z.$$  \hfill (1.5.1)

Subscripted wavenumbers represent components and in analytic expressions may assume either positive or negative values. The wavenumber magnitude $k = (k_y^2 + k_z^2)^{1/2}$ will be understood to be always positive. Our numerical evaluations will consistently use $k_y \geq 0$ and $k_z \geq 0$; we will reverse wave propagation direction by reversing the sign of $\omega_r$. The angle between $k$ and $B_0$ will be denoted by $\theta$; thus $\hat{k} \cdot \hat{B}_0 = \cos \theta$. The maximum growth rate over the full range of wavevectors associated with an instability will be denoted by $\gamma_m$; the wavevector corresponding to maximum growth will be $k_m$.

If the distribution functions of each plasma species are Maxwellian and
no external electric fields are present, the dispersion equation typically yields non-growing roots. In order to yield one or more plasma instabilities, the dispersion equation must be based on distribution functions involving free energy; that is, some non-Maxwellian property corresponding, for example, to an anisotropy or an inhomogeneity.

As the free energy (say a relative drift speed between two components) is increased, the imaginary part of the frequency, $\gamma$, of a damped mode becomes less negative until $\gamma = 0$ is reached at some wavevector. We term this condition the threshold of the associated instability because a further increase of free energy leads, at some wavevectors, to $\gamma > 0$, that is, wave growth. At and somewhat above threshold, it is often true that at least one component ($j$) is resonant with the instability; i.e. $|\xi_j| \lesssim 1$ where $\xi_j$ is the argument of some plasma dispersion function $Z(\xi_j)$ used in the linear dispersion equation. In this regime, wave growth depends on velocity-space details of the $j$th component distribution function and the instability is termed kinetic.

If, as the free energy is further increased (e.g. the relative drift speeds of the components become much greater than the component thermal speeds), the maximum growth rate also continues to increase, all plasma components often become nonresonant ($|\xi_j| \gg 1$), and the dispersion equation can be reduced to a cold plasma form. In this regime, the growing mode is usually termed a fluid instability.

Given a particular source of free energy, a plasma may be unstable to several different modes. So the classification of any microinstability requires identification of both the free energy and the dispersion properties. Although at times it will be necessary to defer to historical precedent, we will, as much as possible, identify microinstabilities described in this book by both their free energy and dispersion. Thus, for example, the kinetic instability with ion acoustic dispersion and driven by the electron/ion relative drift speed will be called the electron/ion acoustic instability. As the electron/ion relative drift speed increases and the unstable mode becomes fluid-like, the instability is more appropriately called the electron/ion two-stream instability.

When discussing plasma waves and instabilities, it is convenient to separate their fluctuating electric fields into two types: longitudinal ($\mathbf{k} \times \mathbf{E}^{(1)} = 0$) and transverse ($\mathbf{k} \cdot \mathbf{E}^{(1)} = 0$). The complete solution of the general dispersion equation will typically have contributions from both types of fields; we define these as $E_{L}^{(1)} = k k \cdot \mathbf{E}^{(1)}/k^2$ and $E_{T}^{(1)} = \mathbf{k} \times \mathbf{E}^{(1)}/k$, respectively.

Plasma fluctuations that have only a longitudinal electric field may be derived through the use of a kinetic equation such as (1.2.1) and a single Maxwell equation: Poisson’s equation (1.2.4). Such waves and instabilities
have $\mathbf{B}^{(l)} = 0$ and are usually called “electrostatic.” In contrast, waves and instabilities with fluctuating electric and magnetic fields perpendicular to the wavevector and with no longitudinal electric field can be described through the use of an appropriate kinetic equation, Faraday’s equation (1.2.5) and the Ampère–Maxwell equation (1.2.6). These fluctuations are sometimes called “electromagnetic.”

Most fluctuations in space plasmas of nonzero $\beta$ have both transverse and longitudinal components. To provide a consistent terminology throughout this book, we will use the following nomenclature: fluctuations with only a longitudinal component will be called “electrostatic;” fluctuations with only transverse electric fields will be termed “strictly electromagnetic.” If the fluctuating field energy of a mode satisfies

$$0 < |\mathbf{E}_T^{(l)}|^2 + |\mathbf{B}^{(l)}|^2 < |\mathbf{E}_L^{(l)}|^2$$

we will term the wave “primarily electrostatic,” and if

$$0 < |\mathbf{E}_L^{(l)}|^2 < |\mathbf{E}_T^{(l)}|^2 + |\mathbf{B}^{(l)}|^2$$

we will term the mode “primarily electromagnetic.” Finally, the term “electromagnetic” will encompass fluctuations with arbitrary ratios of longitudinal and transverse fluctuating fields.
10

Introduction

In each chapter of this book, the same basic procedure is followed. We define a configuration, indicate the derivation of the linear dispersion equation, exhibit results from the computer solution of this dispersion equation and derive approximate analytic solutions where appropriate. In those chapters that treat instabilities, we finally discuss space plasma physics applications, primarily drawing on illustrations from our own areas of expertise. Figure 1.1 is a diagram illustrating some of the space plasma regimes from which we consider applications in this book.

References


