Chapter 2
Lekhnitskii Formalism

From Section 1.1, we know that the basic equations for anisotropic elasticity consist of the equilibrium equations for static loading conditions (1.4), the strain–displacement relations for small deformations (1.7) as well as the stress–strain laws for linear anisotropic elastic solids (1.12). That is,

$$\sigma_{ij,j} + f_i = 0, \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad i,j,k,l = 1,2,3. \tag{2.1}$$

These three equation sets (2.1) constitute 15 partial differential equations with 15 unknown functions $u_i, \varepsilon_{ij}, \sigma_{ij},\ i,j = 1,2,3$, in terms of three coordinate variables $x_i, i = 1,2,3$. If only the two-dimensional deformation is considered, the complex variable formulation can be used to establish the general solution for these 15 unknown functions satisfying 15 basic equations. In the literature, there are two different complex variable formulations for two-dimensional linear anisotropic elasticity. One is the *Lekhnitskii formalism* (Lekhnitskii, 1963, 1968) which starts with the equilibrated stress functions followed by compatibility equations, and the other is the *Stroh formalism* (Stroh, 1958, 1962; Ting, 1996) which starts with the compatible displacements followed by equilibrium equations. The Lekhnitskii formalism excerpted from the two well-known books (Lekhnitskii, 1963, 1968) will be presented in this chapter, whereas the Stroh formalism excerpted from the other well-known book (Ting, 1996) will be presented in the next chapter.

2.1 Governing Differential Equations

In the Lekhnitskii formalism, the two-dimensional problem is considered as a body bounded by a cylindrical surface. The region of the cross section can be either finite or infinite. The body possesses rectilinear anisotropy of the form shown in $(2.1)_3$ and is under the influence of body forces and tractions distributed along the surface. In order to have the fields in which the stresses depend only on two coordinates, the body forces and surface tractions are assumed to act in planes normal to the generator of the cylindrical surface and do not vary along the generator. In the case of a body of finite length and finite cross section, the stresses are assumed to reduce to an equivalent axial force and moment which act on the ends.

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Note that the rectilinear anisotropy stated above is different from curvilinear anisotropy. Since the curvilinear anisotropy is characterized by the fact that at different points of the body there exist directions which are not parallel but are equivalent in the sense of the elastic properties, a homogeneous curvilinear anisotropic body may be a nonhomogeneous rectilinear anisotropic body, and vice versa. In the following, we only consider the case of *homogeneous rectilinear anisotropic body*. For those who are interested in the case of curvilinear anisotropic body, refer to Lekhnitskii (1963) for further discussions.

In order to conform with Lekhnitskii's presentation (Lekhnitskii, 1963), in this chapter most of the equations in tensor notation will be written out one by one and the associated notation will also be changed slightly according to the conventional replacement and the contracted notation stated in Section 1.2 such as $u_1 \to u, u_2 \to v, u_3 \to w$, $2\varepsilon_{12} \to \gamma_{xy}, \sigma_{12} \to \tau_{xy}, C_{1112} \to C_{16}, 4S_{2331} \to S_{45}$. With this understanding, the basic equations (2.1) can now be rewritten as follows.

Strain–Displacement:

$$\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = \frac{\partial w}{\partial z}, \\ &\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},\ \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \end{aligned} \tag{2.2a}$$

Stress–Strain:

$$\begin{aligned} \varepsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z + S_{14}\tau_{yz} + S_{15}\tau_{xz} + S_{16}\tau_{xy}, \\ \varepsilon_y &= S_{12}\sigma_x + S_{22}\sigma_y + S_{23}\sigma_z + S_{24}\tau_{yz} + S_{25}\tau_{xz} + S_{26}\tau_{xy}, \\ \varepsilon_z &= S_{13}\sigma_x + S_{23}\sigma_y + S_{33}\sigma_z + S_{34}\tau_{yz} + S_{35}\tau_{xz} + S_{36}\tau_{xy}, \\ \gamma_{yz} &= S_{14}\sigma_x + S_{24}\sigma_y + S_{34}\sigma_z + S_{44}\tau_{yz} + S_{45}\tau_{xz} + S_{46}\tau_{xy}, \\ \gamma_{xz} &= S_{15}\sigma_x + S_{25}\sigma_y + S_{35}\sigma_z + S_{45}\tau_{yz} + S_{55}\tau_{xz} + S_{56}\tau_{xy}, \\ \gamma_{xy} &= S_{16}\sigma_x + S_{26}\sigma_y + S_{36}\sigma_z + S_{46}\tau_{yz} + S_{56}\tau_{xz} + S_{66}\tau_{xy}. \end{aligned} \tag{2.2b}$$

Equilibrium Equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \hat{F}}{\partial x} = 0,\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} - \frac{\partial \hat{F}}{\partial y} = 0,\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0, \tag{2.2c}$$

where $\hat{F}$ is the potential of the body forces $\hat{f}_x, \hat{f}_y$, i.e.,

$$\hat{f}_x = -\frac{\partial \hat{F}}{\partial x}, \quad \hat{f}_y = -\frac{\partial \hat{F}}{\partial y}. \tag{2.2d}$$

Note that in (2.2c) we have employed the assumptions that the stresses depend only on two coordinates and the body forces act in planes normal to the generator of the cylindrical surface and do not vary along the generator.

With the relations given in the third, fourth, and fifth equations of (2.2a), integration of the third, fourth, and fifth equations of (2.2b) with respect to z will then lead to

$$
\begin{aligned}
w &= zD(x,y) + W_0(x,y),\\
v &= -\tfrac{z^2}{2}\tfrac{\partial D}{\partial y} + z\left(S_{14}\sigma_x + S_{24}\sigma_y + S_{34}\sigma_z + S_{44}\tau_{yz} + S_{45}\tau_{xz} + S_{46}\tau_{xy} - \tfrac{\partial W_0}{\partial y}\right)\\
&\quad + V_0(x,y),\\
u &= -\tfrac{z^2}{2}\tfrac{\partial D}{\partial x} + z\left(S_{15}\sigma_x + S_{25}\sigma_y + S_{35}\sigma_z + S_{45}\tau_{yz} + S_{55}\tau_{xz} + S_{56}\tau_{xy} - \tfrac{\partial W_0}{\partial x}\right)\\
&\quad + U_0(x,y),
\end{aligned}
\tag{2.3a}
$$

where

$$
D(x,y) = S_{13}\sigma_x + S_{23}\sigma_y + S_{33}\sigma_z + S_{34}\tau_{yz} + S_{35}\tau_{xz} + S_{36}\tau_{xy}, \tag{2.3b}
$$

and U_0, V_0, W_0 are arbitrary functions of x and y which appear as a result of integration with respect to z.

Employing the results of (2.3a) into the first, second, and sixth equations of (2.2a) as well as (2.2b), and comparing the coefficients of z^2, z, and the free terms, we get

$$
\frac{\partial^2 D}{\partial x^2} = 0,\ \frac{\partial^2 D}{\partial y^2} = 0,\ \frac{\partial^2 D}{\partial x \partial y} = 0, \tag{2.4a}
$$

and

$$
\begin{aligned}
&\tfrac{\partial}{\partial x}\left(S_{15}\sigma_x + S_{25}\sigma_y + \cdots + S_{56}\tau_{xy} - \tfrac{\partial W_0}{\partial x}\right) = 0,\\
&\tfrac{\partial}{\partial y}\left(S_{14}\sigma_x + S_{24}\sigma_y + \cdots + S_{46}\tau_{xy} - \tfrac{\partial W_0}{\partial y}\right) = 0,\\
&\tfrac{\partial}{\partial y}\left(S_{15}\sigma_x + S_{25}\sigma_y + \cdots + S_{56}\tau_{xy} - \tfrac{\partial W_0}{\partial x}\right)\\
&\quad + \tfrac{\partial}{\partial x}\left(S_{14}\sigma_x + S_{24}\sigma_y + \cdots + S_{46}\tau_{xy} - \tfrac{\partial W_0}{\partial y}\right) = 0
\end{aligned}
\tag{2.4b}
$$

and

$$
\begin{aligned}
\frac{\partial U_0}{\partial x} &= S_{11}\sigma_x + S_{12}\sigma_y + \cdots + S_{16}\tau_{xy},\\
\frac{\partial V_0}{\partial y} &= S_{12}\sigma_x + S_{22}\sigma_y + \cdots + S_{26}\tau_{xy},\\
\frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} &= S_{16}\sigma_x + S_{26}\sigma_y + \cdots + S_{66}\tau_{xy}.
\end{aligned}
\tag{2.4c}
$$

From (2.4a), it follows that D is a linear function of x and y,

$$
D = S_{33}(Ax + By + C), \tag{2.5}
$$

where A, B, and C are the arbitrary constants; then by (2.3b) we have

$$
\sigma_z = Ax + By + C - \frac{1}{S_{33}}(S_{13}\sigma_x + S_{23}\sigma_y + S_{34}\tau_{yz} + S_{35}\tau_{xz} + S_{36}\tau_{xy}). \tag{2.6}
$$

Integration of (2.4b) gives

$$\begin{aligned} S_{15}\sigma_x + S_{25}\sigma_y + \cdots + S_{56}\tau_{xy} - \frac{\partial W_0}{\partial x} &= -\alpha y + \omega_2, \\ S_{14}\sigma_x + S_{24}\sigma_y + \cdots + S_{46}\tau_{xy} - \frac{\partial W_0}{\partial y} &= \alpha x - \omega_1, \end{aligned} \tag{2.7}$$

where α,ω_1,ω_2 are the new arbitrary constants. Substituting the results of (2.7) and (2.5) into (2.3a), the general expressions for the displacements can now be written as

$$\begin{aligned} u &= -\frac{AS_{33}}{2}z^2 - \alpha yz + U(x,y) + \omega_2 z - \omega_3 y + u_0, \\ v &= -\frac{BS_{33}}{2}z^2 + \alpha xz + V(x,y) + \omega_3 x - \omega_1 z + v_0, \\ w &= (Ax + By + C)S_{33}z + W(x,y) + \omega_1 y - \omega_2 x + w_0, \end{aligned} \tag{2.8a}$$

where the new functions U, V, W are related to U_0, V_0, W_0 by

$$\begin{aligned} U_0 &= U - \omega_3 y + u_0, \\ V_0 &= V + \omega_3 x + v_0, \\ W_0 &= W + \omega_1 y - \omega_2 x + \omega_0. \end{aligned} \tag{2.8b}$$

In the general expression (2.8a), the constants u_0, v_0, w_0 and $\omega_1, \omega_2, \omega_3$ obviously characterize the *rigid body translations* and *rotations* with respect to the x, y, and z axes; α is the relative angle of rotation about the z-axis associated with the *torsion* problems, i.e., the angle of twist per unit length; A and B characterize the *bending* of the body in the x–z and y–z planes.

With the relations (2.8b), substitution of expression (2.6) for σ_z into (2.4c) and (2.7) will lead to the following equations for determining the unknown functions U, V, and W:

$$\begin{aligned} \frac{\partial U}{\partial x} &= \hat{S}_{11}\sigma_x + \hat{S}_{12}\sigma_y + \hat{S}_{14}\tau_{yz} + \hat{S}_{15}\tau_{xz} + \hat{S}_{16}\tau_{xy} + S_{13}(Ax + By + C), \\ \frac{\partial V}{\partial y} &= \hat{S}_{12}\sigma_x + \hat{S}_{22}\sigma_y + \hat{S}_{24}\tau_{yz} + \hat{S}_{25}\tau_{xz} + \hat{S}_{26}\tau_{xy} + S_{23}(Ax + By + C), \\ \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} &= \hat{S}_{16}\sigma_x + \hat{S}_{26}\sigma_y + \hat{S}_{46}\tau_{yz} + \hat{S}_{56}\tau_{xz} + \hat{S}_{66}\tau_{xy} + S_{36}(Ax + By + C), \end{aligned} \tag{2.9a}$$

$$\begin{aligned} \frac{\partial W}{\partial x} &= \hat{S}_{15}\sigma_x + \hat{S}_{25}\sigma_y + \hat{S}_{45}\tau_{yz} + \hat{S}_{55}\tau_{xz} + \hat{S}_{56}\tau_{xy} + S_{35}(Ax + By + C) + \alpha y, \\ \frac{\partial W}{\partial y} &= \hat{S}_{14}\sigma_x + \hat{S}_{24}\sigma_y + \hat{S}_{44}\tau_{yz} + \hat{S}_{45}\tau_{xz} + \hat{S}_{46}\tau_{xy} + S_{34}(Ax + By + C) - \alpha x, \end{aligned} \tag{2.9b}$$

where $\hat{S}_{ij}$ are the reduced elastic compliance defined in (1.56).

Up to now only the strain–displacement and stress–strain relations, (2.2a) and (2.2b), and the assumption that $\sigma_{ij} = \sigma_{ij}(x, y)$ have been utilized for getting (2.9a) and (2.9b). To determine the unknown functions U, V, and W through (2.9a) and (2.9b), we now consider the equilibrium equations (2.2c), which will be satisfied by introducing two Airy stress functions $\phi\ (x, y)$ and $\psi(x, y)$, as

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} + \hat{F},\ \sigma_y = \frac{\partial^2\phi}{\partial x^2} + \hat{F},\ \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y},\ \tau_{xz} = \frac{\partial\psi}{\partial y},\ \tau_{yz} = -\frac{\partial\psi}{\partial x}. \quad (2.10)$$

With relations (2.10), eliminating U and V from (2.9a) and W from (2.9b), by differentiating, we obtain the following system of differential equations which the stress functions must satisfy:

$$\begin{aligned} L_4\phi + L_3\psi &= -(\hat{S}_{12} + \hat{S}_{22})\frac{\partial^2\hat{F}}{\partial x^2} + (\hat{S}_{16} + \hat{S}_{26})\frac{\partial^2\hat{F}}{\partial x\partial y} - (\hat{S}_{11} + \hat{S}_{12})\frac{\partial^2\hat{F}}{\partial y^2}, \\ L_3\phi + L_2\psi &= -2\alpha + AS_{34} - BS_{35} + (\hat{S}_{14} + \hat{S}_{24})\frac{\partial\hat{F}}{\partial x} + (\hat{S}_{15} + \hat{S}_{25})\frac{\partial\hat{F}}{\partial y}, \end{aligned} \quad (2.11a)$$

in which L_2, L_3, L_4 are the differential operators of the second, third, and fourth order which have the form

$$\begin{aligned} L_2 &= \hat{S}_{44}\frac{\partial^2}{\partial x^2} - 2\hat{S}_{45}\frac{\partial^2}{\partial x\partial y} + \hat{S}_{55}\frac{\partial^2}{\partial y^2}, \\ L_3 &= -\hat{S}_{24}\frac{\partial^3}{\partial x^3} + (\hat{S}_{25} + \hat{S}_{46})\frac{\partial^3}{\partial x^2\partial y} - (\hat{S}_{14} + \hat{S}_{56})\frac{\partial^3}{\partial x\partial y^2} + \hat{S}_{15}\frac{\partial^3}{\partial y^3}, \\ L_4 &= \hat{S}_{22}\frac{\partial^4}{\partial x^4} - 2\hat{S}_{26}\frac{\partial^4}{\partial x^3\partial y} + (2\hat{S}_{12} + \hat{S}_{66})\frac{\partial^4}{\partial x^2\partial y^2} - 2\hat{S}_{16}\frac{\partial^4}{\partial x\partial y^3} + \hat{S}_{11}\frac{\partial^4}{\partial y^4}. \end{aligned} \quad (2.11b)$$

Solving the stress functions from (2.11), the stresses, strains, and displacements can then be determined from (2.10), (2.6), (2.2b), (2.9), and (2.8). To uniquely determine all these values, the boundary conditions and the requirement of the single-valued displacement should all be satisfied. Since the system of differential equations obtained in (2.11) is the combined result of the 15 basic equations shown in (2.2a–c), before discussing its associated general solutions we like to summarize briefly the derivation procedure stated between (2.2) and (2.11).

Summary of the derivation for (2.11):

1. Introduce two Airy stress functions ϕ,ψ in (2.10).
2. Stresses σ_{ij} in terms of ϕ,ψ will satisfy equilibrium (2.2c) automatically.
3. By employing the stress–strain relation (2.2b) and the strain–displacement relation (2.2a), the strain and displacement components can also be expressed in terms of the Airy stress functions.
4. During the integration the compatibility of displacements should be satisfied, which will then lead to the system of differential equations (2.11).

2.2 General Solutions

The general solution of (2.11) can be written in the form

$$\phi = \phi^{(h)} + \phi^{(p)}, \quad \psi = \psi^{(h)} + \psi^{(p)}, \tag{2.12a}$$

where

$$\begin{aligned} L_4\phi^{(h)} + L_3\psi^{(h)} &= 0, \\ L_3\phi^{(h)} + L_2\psi^{(h)} &= 0, \end{aligned} \tag{2.12b}$$

and $\phi^{(p)}$,$\psi^{(p)}$ are particular solutions of the nonhomogeneous system (2.11a). The particular solutions depend on the form of the known functions of the right-hand sides of (2.11a) and are usually not difficult to find. Hence, in the following we only consider the general solutions of the homogeneous system (2.12b). Eliminating one of the functions, say $\psi^{(h)}$, from $(2.12b)_1$ and $(2.12b)_2$, we obtain an equation of the sixth order as

$$(L_4L_2 - {L_3}^2)\phi^{(h)} = 0. \tag{2.13}$$

The operator of the sixth homogeneous order $L_4L_2 - {L_3}^2$ can be decomposed into six linear operators of the first order. In other words, (2.13) can be represented in the following form:

$$D_6D_5D_4D_3D_2D_1\phi^{(h)} = 0, \tag{2.14a}$$

where

$$D_k = \frac{\partial}{\partial y} - \mu_k \frac{\partial}{\partial x}, \tag{2.14b}$$

and μ_k are the roots of the algebraic equation associated with the differential equation (2.13), i.e.,

$$l_4(\mu)l_2(\mu) - {l_3}^2(\mu) = 0, \tag{2.15a}$$

in which

$$\begin{aligned} l_2(\mu) &= \hat{S}_{55}\mu^2 - 2\hat{S}_{45}\mu + \hat{S}_{44}, \\ l_3(\mu) &= \hat{S}_{15}\mu^3 - (\hat{S}_{14} + \hat{S}_{56})\mu^2 + (\hat{S}_{25} + \hat{S}_{46})\mu - \hat{S}_{24}, \\ l_4(\mu) &= \hat{S}_{11}\mu^4 - 2\hat{S}_{16}\mu^3 + (2\hat{S}_{12} + \hat{S}_{66})\mu^2 - 2\hat{S}_{26}\mu + \hat{S}_{22}. \end{aligned} \tag{2.15b}$$

One can prove that μ_k cannot be real if the strain energy is positive (Lekhnitskii, 1963). In other words, μ_k are always complex or purely imaginary and consist of three pairs of complex conjugates since the characteristic equation (2.15a) is a sixth-order algebraic equation with real coefficients. Let

$$\mu_{k+3} = \overline{\mu}_k, \quad \mathrm{Im}\,\mu_k > 0,\ k = 1,2,3, \tag{2.16}$$

where Im denotes the imaginary part and the overbar denotes the complex conjugate. *Assume that the roots μ_k are distinct*; (2.14a) can be solved by considering the following six equations of the first order:

$$D_1\phi^{(h)} = \varphi_2,\ D_2\varphi_2 = \varphi_3,\ D_3\varphi_3 = \varphi_4,\ D_4\varphi_4 = \varphi_5,\ D_5\varphi_5 = \varphi_6,\ D_6\varphi_6 = 0. \tag{2.17}$$

Solving (2.17) successively in the order of φ_6, φ_5, φ_4, φ_3, φ_2, $\phi^{(h)}$, we obtain

$$\phi^{(h)} = 2\,\mathrm{Re}\sum_{k=1}^{3}\phi_k(z_k), \quad z_k = x + \mu_k y, \tag{2.18}$$

in which Re denotes the real part. Similarly, by expressing (2.13) in terms of $\psi^{(h)}$ and solving it in the same way as that shown in (2.14), (2.15), (2.16), (2.17), and (2.18), we can obtain

$$\psi^{(h)} = 2\,\mathrm{Re}\sum_{k=1}^{3}\psi_k(z_k). \tag{2.19}$$

Knowing that

$$\frac{\partial\phi_k}{\partial x} = \frac{\partial\phi_k}{\partial z_k}\frac{\partial z_k}{\partial x} = \frac{\partial\phi_k}{\partial z_k}, \quad \frac{\partial\phi_k}{\partial y} = \frac{\partial\phi_k}{\partial z_k}\frac{\partial z_k}{\partial y} = \mu_k\frac{\partial\phi_k}{\partial z_k}, \tag{2.20}$$

and similarly for ψ_k, substitution of (2.18) and (2.19) into (2.12b) will then lead to

$$\begin{aligned} &2\,\mathrm{Re}\sum_{k=1}^{3}\left\{l_4(\mu_k)\phi_k''''(z_k) + l_3(\mu_k)\psi_k'''(z_k)\right\} = 0,\\ &2\,\mathrm{Re}\sum_{k=1}^{3}\left\{l_3(\mu_k)\phi_k'''(z_k) + l_2(\mu_k)\psi_k''(z_k)\right\} = 0, \end{aligned} \tag{2.21}$$

in which the prime ($'$) denotes differentiation with respect to z_k. Integration of the first or second equation of (2.21) with respect to z_k can now provide the relation between ϕ_k and ψ_k, i.e.,

$$\psi_k(z_k) = \eta_k\phi_k'(z_k) + a_k z_k + b_k,\ k = 1,2,3, \tag{2.22a}$$

where

$$\eta_k = \frac{-l_3(\mu_k)}{l_2(\mu_k)} = \frac{-l_4(\mu_k)}{l_3(\mu_k)}, \tag{2.22b}$$

and a_k, b_k are the arbitrary constants. *For monoclinic materials with the symmetry plane at* $z = 0$, the coefficients of the polynomial $l_3(\mu)$ all vanish and the sextic equation (2.15a) reduces to two equations, $l_2(\mu) = 0$ and $l_4(\mu) = 0$. Let μ_1 and μ_2 be the roots of $l_4(\mu) = 0$ and μ_3 be the roots of $l_2(\mu) = 0$, i.e., $l_4(\mu_1) = l_4(\mu_2) = l_2(\mu_3) = 0$, which will then lead to the results that $l_2(\mu_1)$, $l_2(\mu_2)$, $l_4(\mu_3) \neq 0$ since μ_k are assumed to be distinct. With these values and $l_3(\mu_k) = 0$, we have $\eta_1 = \eta_2 = 0$ and $\eta_3 \to \infty$. To avoid using a coefficient that may approach infinity, we let

$$\lambda_1 = \eta_1 = \frac{-l_3(\mu_1)}{l_2(\mu_1)},\ \lambda_2 = \eta_2 = \frac{-l_3(\mu_2)}{l_2(\mu_2)},\ \lambda_3 = \frac{1}{\eta_3} = \frac{-l_3(\mu_3)}{l_4(\mu_3)}. \tag{2.23}$$

By choosing the arbitrary constants a_k, b_k to be zero, and using (2.23), (2.22a), (2.18), and (2.19), the general expressions for the stress functions now take the following form:

$$\begin{aligned} \phi &= 2\,\mathrm{Re}\{\phi_1(z_1) + \phi_2(z_2) + \phi_3(z_3)\} + \phi^{(p)}, \\ \psi &= 2\,\mathrm{Re}\left\{\lambda_1\phi_1'(z_1) + \lambda_2\phi_2'(z_2) + \frac{1}{\lambda_3}\phi_3'(z_3)\right\} + \psi^{(p)}. \end{aligned} \tag{2.24}$$

Knowing that λ_3 may approach zero for some materials, and $\phi_3(z_3)$ is an arbitrary function at this stage, to avoid having the infinite coefficient we may absorb the coefficient into the function and introduce the new stress functions $f_k(z_k)$, $k = 1, 2, 3$, as follows:

$$f_1(z_1) = \phi_1'(z_1), \quad f_2(z_2) = \phi_2'(z_2), \quad f_3(z_3) = \frac{1}{\lambda_3}\phi_3'(z_3). \tag{2.25}$$

With these new stress functions, substitution of (2.24) into (2.10) will then lead to

$$\begin{aligned} \sigma_x &= 2\,\mathrm{Re}\left\{\mu_1^2 f_1'(z_1) + \mu_2^2 f_2'(z_2) + \mu_3^2\lambda_3 f_3'(z_3)\right\} + \frac{\partial^2\phi^{(p)}}{\partial y^2} + \hat{F}, \\ \sigma_y &= 2\,\mathrm{Re}\left\{f_1'(z_1) + f_2'(z_2) + \lambda_3 f_3'(z_3)\right\} + \frac{\partial^2\phi^{(p)}}{\partial x^2} + \hat{F}, \\ \tau_{xy} &= -2\,\mathrm{Re}\left\{\mu_1 f_1'(z_1) + \mu_2 f_2'(z_2) + \mu_3\lambda_3 f_3'(z_3)\right\} - \frac{\partial^2\phi^{(p)}}{\partial x\partial y}, \\ \tau_{xz} &= 2\,\mathrm{Re}\left\{\mu_1\lambda_1 f_1'(z_1) + \mu_2\lambda_2 f_2'(z_2) + \mu_3 f_3'(z_3)\right\} + \frac{\partial\psi^{(p)}}{\partial y}, \\ \tau_{yz} &= -2\,\mathrm{Re}\left\{\lambda_1 f_1'(z_1) + \lambda_2 f_2'(z_2) + f_3'(z_3)\right\} - \frac{\partial\psi^{(p)}}{\partial x}. \end{aligned} \tag{2.26}$$

By substituting (2.26) into (2.9a) and (2.9b) and integrating the resulting equations, we can find the functions U, V, and W as

$$U = 2\,\mathrm{Re}\sum_{k=1}^{3} a_{1k} f_k(z_k) + U^{(p)},$$
$$V = 2\,\mathrm{Re}\sum_{k=1}^{3} a_{2k} f_k(z_k) + V^{(p)}, \tag{2.27a}$$
$$W = 2\,\mathrm{Re}\sum_{k=1}^{3} a_{3k} f_k(z_k) + W^{(p)},$$

where

$$\begin{aligned} a_{1k} &= p_1(\mu_k) + \lambda_k q_1(\mu_k), \\ a_{2k} &= [p_2(\mu_k) + \lambda_k q_2(\mu_k)]/\mu_k, \\ a_{3k} &= [p_4(\mu_k) + \lambda_k q_4(\mu_k)]/\mu_k, \ k = 1,2, \\ a_{13} &= \lambda_3 p_1(\mu_3) + q_1(\mu_3), \\ a_{23} &= [\lambda_3 p_2(\mu_3) + q_2(\mu_3)]/\mu_3, \\ a_{33} &= [\lambda_3 p_4(\mu_3) + q_4(\mu_3)]/\mu_3, \end{aligned} \tag{2.27b}$$

and

$$p_j(\mu_k) = \mu_k^2 \hat{S}_{j1} + \hat{S}_{j2} - \mu_k \hat{S}_{j6}, \ q_j(\mu_k) = \mu_k \hat{S}_{j5} - \hat{S}_{j4}, \quad j = 1,2,4,5,6. \tag{2.27c}$$

In (2.27a), $U^{(p)}$,$V^{(p)}$,$W^{(p)}$ are the solutions of (2.9a) and (2.9b) which correspond to the functions $\phi^{(p)}$,$\psi^{(p)}$,$\hat{F}$ and to the linear functions $S_{ij}(Ax + By + C)$, αy, $-\alpha x$ which contain the constants α, A, B, C.

2.3 Boundary Conditions

2.3.1 Lateral Surface Conditions

The general expressions for the stresses and displacements given in (2.26), (2.27), and (2.8a) contain the arbitrary complex functions $f_k(z_k)$ which should be determined through the satisfaction of the boundary conditions on the lateral surface. As stated in Chapter 1, the boundary conditions are usually described by prescribing the tractions or displacements. Because the general expressions for the stresses and displacements are written in terms of the unknown functions $f_k(z_k)$, it is now better to express the boundary conditions in terms of $f_k(z_k)$.

First Fundamental Problem:

$$\sigma_x n_1 + \tau_{xy} n_2 = \hat{t}_x, \ \tau_{xy} n_1 + \sigma_y n_2 = \hat{t}_y, \ \tau_{xz} n_1 + \tau_{yz} n_2 = 0, \tag{2.28}$$

where $(\hat{t}_x, \hat{t}_y, 0)$ are the tractions prescribed along the boundary. As shown in Fig. 2.1 the normal vector **n** of the boundary surface can be expressed by

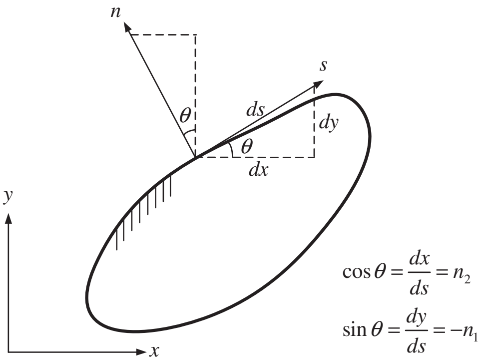

Fig. 2.1 Tangent and normal directions of boundary surfaces

$$n_1 = -\frac{dy}{ds},\ n_2 = \frac{dx}{ds}, \tag{2.29}$$

where tangential direction s is chosen such that when one faces the direction of increasing s the material lies on the right side. Substituting (2.10) and (2.29) into (2.28) and integrating the resulting equations with respect to s, we obtain

$$\frac{\partial\phi}{\partial y} = \tilde{t}_x(s) + c_1, \quad \frac{\partial\phi}{\partial x} = \tilde{t}_y(s) + c_2, \quad \psi = c_3, \tag{2.30a}$$

where c_1, c_2, c_3 are the integration constants and

$$\begin{aligned}\tilde{t}_x(s) &= -\int_0^s \left(\hat{t}_x + \hat{F}\frac{dy}{ds}\right) ds,\\ \tilde{t}_y(s) &= \int_0^s \left(\hat{t}_y - \hat{F}\frac{dx}{ds}\right) ds.\end{aligned} \tag{2.30b}$$

Employing (2.24) and (2.25) into (2.30a), we now have

$$\begin{aligned}&2\,\mathrm{Re}\,\{\mu_1 f_1 + \mu_2 f_2 + \mu_3\lambda_3 f_3\} = \tilde{t}_x(s) - \frac{\partial\phi^{(p)}}{\partial y} + c_1,\\ &2\,\mathrm{Re}\,\{f_1 + f_2 + \lambda_3 f_3\} = \tilde{t}_y(s) - \frac{\partial\phi^{(p)}}{\partial x} + c_2,\\ &2\,\mathrm{Re}\,\{\lambda_1 f_1 + \lambda_2 f_2 + f_3\} = -\psi^{(p)} + c_3.\end{aligned} \tag{2.31}$$

Second Fundamental Problem:

$$u = \hat{u},\ v = \hat{v},\ w = \hat{w}, \tag{2.32}$$

where $(\hat{u},\hat{v},\hat{w})$ are the displacements prescribed along the boundary. Substituting (2.8a), (2.27), into (2.32), we obtain

$$
\begin{aligned}
2\,\mathrm{Re}\sum_{k=1}^{3} a_{1k} f_k &= -U^{(p)} + \hat{U} + \omega_3 y - u_0, \\
2\,\mathrm{Re}\sum_{k=1}^{3} a_{2k} f_k &= -V^{(p)} + \hat{V} - \omega_3 x - v_0, \\
2\,\mathrm{Re}\sum_{k=1}^{3} a_{3k} f_k &= -W^{(p)} + \hat{W} - w_0,
\end{aligned}
\tag{2.33a}
$$

where $\hat{U}, \hat{V}, \hat{W}$ are given as

$$
\begin{aligned}
\hat{U} &= \hat{u} + \frac{AS_{33}}{2} z^2 + \alpha yz - \omega_2 z, \\
\hat{V} &= \hat{v} + \frac{BS_{33}}{2} z^2 - \alpha xz + \omega_1 z, \\
\hat{W} &= \hat{w} - (Ax + By + C)S_{33} z - \omega_1 y + \omega_2 x.
\end{aligned}
\tag{2.33b}
$$

2.3.2 End Conditions

As stated at the beginning of Section 2.1 that if a body of finite length and finite cross section is considered, the axial force and moment which act on the ends will be removed by imposing an equivalent stress distribution. Since the stresses do not depend on z, these conditions exist not only at the ends but also in any cross section. The conditions at the ends have the form:

$$
\begin{aligned}
&\iint \tau_{xz} dx\, dy = 0, \qquad \iint \tau_{yz}\, dx\, dy = 0, \qquad \iint \sigma_z\, dx\, dy = P_z, \\
&\iint \sigma_z\, y\, dx\, dy = M_1, \quad \iint \sigma_z\, x\, dx\, dy = M_2, \quad \iint (\tau_{yz} x - \tau_{xz}\, y) dx\, dy = M_t,
\end{aligned}
\tag{2.34}
$$

where P_z, M_1, M_2, and M_t are, respectively, the axial force, bending moments about x- and y-axes, and twisting moment. The integrals are taken over the entire area of the cross section. In the Lekhnitskii formulation, the net forces in the x and y directions should be zero under the assumption of two-dimensional equilibrated stress fields, and the first two conditions of (2.34) should then be satisfied identically. A direct proof is given as follows:

$$
\begin{aligned}
\iint \tau_{xz}\, dx\, dy &= \iint \left[\tau_{xz} + x\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} \right) \right] dx\, dy \\
&= \iint \left[\frac{\partial (x\tau_{xz})}{\partial x} + \frac{\partial (x\tau_{yz})}{\partial y} \right] dx\, dy = \int_{\Gamma} [x\tau_{xz} n_1 + x\tau_{yz} n_2] ds = 0.
\end{aligned}
\tag{2.35}
$$

In the above derivation, the first equality is obtained by adding a zero quantity from the equilibrium equation (2.2c) and the third equality is obtained by transforming the double integral into a contour integral (contour Γ), and the last equality is due to the boundary conditions set on the lateral surface ($\hat{t}_z = 0$). It can be proved by the same way that the integral of τ_{yz} is also equal to zero. Substituting (2.6) into the remaining four equations of (2.34), we have

$$
\begin{aligned}
P_z &= CS - \frac{1}{S_{33}} \iint (S_{13}\sigma_x + S_{23}\sigma_y + S_{34}\tau_{yz} + S_{35}\tau_{xz} + S_{36}\tau_{xy})\, dx\, dy,\\
M_1 &= BI_1 - \frac{1}{S_{33}} \iint (S_{13}\sigma_x + S_{23}\sigma_y + S_{34}\tau_{yz} + S_{35}\tau_{xz} + S_{36}\tau_{xy})\, y\, dx\, dy,\\
M_2 &= AI_2 - \frac{1}{S_{33}} \iint (S_{13}\sigma_x + S_{23}\sigma_y + S_{34}\tau_{yz} + S_{35}\tau_{xz} + S_{36}\tau_{xy}) x\, dx\, dy,\\
M_t &= \iint (\tau_{yz}x - \tau_{xz}y) dx\, dy,
\end{aligned} \tag{2.36a}
$$

where S is the area of the cross section and I_1 and I_2 are the principal moments of inertia with respect to x- and y-axes defined by

$$
S = \iint dx\, dy, \quad I_1 = \iint y^2 dx\, dy, \quad I_2 = \iint x^2 dx\, dy. \tag{2.36b}
$$

Note that the origin of the x–y coordinate passes through the centroid, and the x- and y-axes are the *principal axes of inertia* of the cross section, and hence $\iint x\, dx\, dy = 0$, $\iint y\, dx\, dy = 0$, and $\iint xy\, dx\, dy = 0$.

By employing similar techniques as (2.35), it can be shown that each term of the integrals in (2.36a) can be expressed in terms of the surface traction $(\hat{t}_x, \hat{t}_y, 0)$ and body force $(\hat{f}_x, \hat{f}_y, 0)$ or the twisting moment M_t. For example,

$$
\begin{aligned}
\iint \sigma_x dx\, dy &= \iint \left[\sigma_x + x\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) + x\hat{f}_x\right] dx\, dy\\
&= \iint \left[\frac{\partial (x\sigma_x)}{\partial x} + \frac{\partial (x\tau_{xy})}{\partial y}\right] dx\, dy + \iint x\hat{f}_x\, dx\, dy\\
&= \int_\Gamma x(\sigma_x n_1 + \tau_{xy} n_2) ds + \iint x\hat{f}_x\, dx\, dy\\
&= \int_\Gamma x\hat{t}_x ds + \iint x\hat{f}_x\, dx\, dy,
\end{aligned} \tag{2.37a}
$$

$$\begin{aligned}
\iint \sigma_{xy}\, dx\, dy &= \iint \left[\sigma_x + x\left(\tfrac{\partial \sigma_x}{\partial x} + \tfrac{\partial \tau_{xy}}{\partial y}\right) + x\hat{f}_x\right] y\, dx\, dy \\
&= \iint \left[\tfrac{\partial (xy\sigma_x)}{\partial x} + \tfrac{\partial (xy\tau_{xy})}{\partial y}\right] dx\, dy - \iint x\tau_{xy} dx\, dy + \iint xy\hat{f}_x\, dx\, dy \\
&= \int_\Gamma xy(\sigma_x n_1 + \tau_{xy} n_2) ds + \iint xy\hat{f}_x\, dx\, dy \\
&\qquad - \iint x\left[\tau_{xy} + \tfrac{1}{2}\left(\tfrac{\partial \sigma_y}{\partial y} + \tfrac{\partial \tau_{xy}}{\partial x}\right) x + \tfrac{1}{2} x\hat{f}_y\right] dx\, dy \\
&= \int_\Gamma xy\hat{t}_x ds + \iint xy\hat{f}_x\, dx\, dy - \iint \tfrac{1}{2}\left[\tfrac{\partial (x^2\tau_{xy})}{\partial x} + \tfrac{\partial (x^2\sigma_y)}{\partial y}\right] dx\, dy \\
&\qquad - \iint \tfrac{1}{2} x^2 \hat{f}_y dx\, dy \\
&= \int_\Gamma \left(xy\hat{t}_x - \tfrac{1}{2}x^2\hat{t}_y\right) ds + \iint (xy\hat{f}_x - \tfrac{1}{2}x^2\hat{f}_y)\, dx\, dy,
\end{aligned} \tag{2.37b}$$

$$\begin{aligned}
\iint \tau_{xz} y\, dx\, dy &= \iint \left[-\frac{1}{2}(\tau_{yz}x - \tau_{xz}\, y) + \frac{1}{2}(\tau_{yz}x + \tau_{xz}y)\right] dx\, dy \\
&= -\frac{1}{2}M_t + \frac{1}{2}\iint \left[\tau_{yz}\, x + \tau_{xz} y + xy\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right)\right] dx\, dy \\
&= -\frac{1}{2}M_t + \frac{1}{2}\iint \left[\frac{\partial (xy\tau_{xz})}{\partial x} + \frac{\partial (xy\tau_{yz})}{\partial y}\right] dx\, dy \\
&= -\frac{1}{2}M_t + \frac{1}{2}\int_\Gamma xy(\tau_{xz} n_1 + \tau_{yz} n_2)\, ds \\
&= -\frac{1}{2}M_t,
\end{aligned} \tag{2.37c}$$

$$\begin{aligned}
M_t &= \iint (\tau_{yz}x - \tau_{xz}y)\, dx\, dy = -\iint \left(\frac{\partial \psi}{\partial x}x + \frac{\partial \psi}{\partial y}\, y\right) dx\, dy \\
&= -\iint \left[\frac{\partial (x\psi)}{\partial x} + \frac{\partial (y\psi)}{\partial y}\right] dx\, dy + 2\iint \psi\, dx\, dy \\
&= -\int \psi(xn_1 + yn_2)\, ds + 2\iint \psi\, dx\, dy \\
&= 2\iint \psi\, dx\, dy.
\end{aligned} \tag{2.37d}$$

Note that the last equality of (2.37d) comes from the fact that $\psi = 0$ along the boundary of the cross section, which can be proved by inserting $(2.10)_{4,5}$ and (2.29) into $(2.28)_3$ and knowing that if the region of the cross section is simply connected a constant value of ψ will not induce stresses and can be selected to be zero.

Similarly,

$$\iint \sigma_y \, dx\, dy = \int_\Gamma y\hat{t}_y \, ds + \iint y\hat{f}_y \, dx\, dy,$$
$$\iint \tau_{xy} \, dx\, dy = \int_\Gamma x\hat{t}_y \, ds + \iint x\hat{f}_y \, dx\, dy = \int_\Gamma y\hat{t}_x \, ds + \iint y\hat{f}_x dx\, dy, \tag{2.37e}$$

$$\iint \sigma_x \, x \, dx\, dy = \int_\Gamma \frac{x^2}{2}\hat{t}_x \, ds + \iint \frac{x^2}{2}\hat{f}_x \, dx\, dy,$$
$$\iint \sigma_y \, x \, dx\, dy = \int_\Gamma \left(xy\hat{t}_y - \frac{y^2}{2}\hat{t}_x\right) ds + \iint \left(xy\hat{f}_y - \frac{y^2}{2}\hat{f}_x\right) dx\, dy, \tag{2.37f}$$
$$\iint \sigma_y \, y \, dx\, dy = \int_\Gamma \frac{y^2}{2}\hat{t}_y \, ds + \iint \frac{y^2}{2}\hat{f}_y \, dx\, dy,$$

$$\iint \tau_{xy} x \, dx\, dy = \int_\Gamma \frac{x^2}{2}\hat{t}_y \, ds + \iint \frac{x^2}{2}\hat{f}_y \, dx\, dy,$$
$$\iint \tau_{xy} \, y \, dx\, dy = \int_\Gamma \frac{y^2}{2}\hat{t}_x \, ds + \iint \frac{y^2}{2}\hat{f}_y \, dx\, dy, \tag{2.37g}$$

$$\iint \tau_{xz} x \, dx\, dy = 0,$$
$$\iint \tau_{yz} y \, dx\, dy = 0, \tag{2.37h}$$
$$\iint \tau_{yz} x \, dx\, dy = \frac{1}{2} M_t.$$

With the results of (2.37a–h), the conditions (2.36a) take the form (Lekhnitskii, 1963)

$$\begin{aligned} P_z = CS & \\ & -\tfrac{1}{S_{33}} \int_\Gamma [(S_{13}x + S_{36}y)\hat{t}_x + S_{23}y\hat{t}_y] \, ds \\ & -\tfrac{1}{S_{33}} \iint [(S_{13}x + S_{36}y)\hat{f}_x + S_{23}y\hat{f}_y] \, dx\, dy, \end{aligned} \tag{2.38a}$$

$$\begin{aligned} M_1 = BI_1 + \frac{S_{35}}{2S_{33}} M_t & \\ & - \frac{1}{S_{33}} \int_\Gamma \left[\left(S_{13}xy + \frac{S_{36}}{2}y^2\right)\hat{t}_x + \frac{1}{2}(S_{23}y^2 - S_{13}x^2)\hat{t}_y\right] ds \\ & - \frac{1}{S_{33}} \iint \left[\left(S_{13}\, xy + \frac{S_{36}}{2}y^2\right)\hat{f}_x + \frac{1}{2}(S_{23}y^2 - S_{13}x^2)\hat{f}_y\right] dx\, dy, \end{aligned} \tag{2.38b}$$

$$M_2 = AI_2 - \frac{S_{34}}{2S_{33}}M_t - \frac{1}{S_{33}}\int_\Gamma \left[\frac{1}{2}(S_{13}x^2 - S_{23}y^2)\hat{t}_x + \left(S_{23}\,xy + \frac{S_{36}}{2}x^2\right)\hat{t}_y\right] ds - \frac{1}{S_{33}}\iint \left[\frac{1}{2}(S_{13}x^2 - S_{23}y^2)\hat{f}_x + \left(S_{23}xy + \frac{S_{36}}{2}x^2\right)\hat{f}_y\right] dx\, dy, \tag{2.38c}$$

$$M_t = 2\iint \psi\, dx\, dy. \tag{2.38d}$$

The stress functions ϕ and ψ which satisfy (2.11a) and the boundary conditions on the cross section will contain the four arbitrary constants α, A, B, C. If the axial force P_z and the moments M_1, M_2, M_t are given, these constants can be determined from (2.38a–d).

2.4 Special Cases

From the derivation shown in Section 2.1 we see that although the stresses depend only on two coordinates (x and y), through the integration the displacements may depend on the third coordinate (z) whose general expressions have been obtained in (2.8). In other words, the two-dimensional problems considered in Lekhnitskii formulation cover not only the pure 2D cases but also some special 3D cases. In this section, we like to discuss the special cases that all the physical responses such as the displacements and stresses depend only on two coordinates. In addition, the cases of finite length and finite cross section with equivalent axial forces and moments acting on the ends will also be discussed.

2.4.1 Generalized Plane Deformation

Besides the assumptions stated at the beginning of this section, we assume that the length of the body is infinite and the region of the cross section is arbitrary (it can be finite or infinite, simply connected or multiply connected), see Fig. 2.2. An isotropic body under such conditions would experience plane deformation (or plane strain), i.e., w=0. In a body with anisotropy of a general form, plane deformation is usually not possible. We can only assert that all components of stresses and displacements will not depend on z. The deformation of such body is called *generalized plane deformation* or called *generalized plane strain*. The expressions for the displacements of generalized plane deformation are obtained as a special case by setting

$$A = B = C = \alpha = 0, \quad \omega_1 = \omega_2 = 0 \tag{2.39}$$

in (2.8a), and

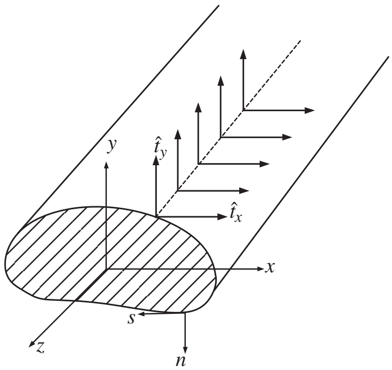

Fig. 2.2 Generalized plane deformation

$$\begin{aligned} u &= U(x, y) - \omega_3 y + u_0, \\ v &= V(x, y) + \omega_3 x + v_0, \\ w &= W(x, y) + w_0. \end{aligned} \tag{2.40}$$

The general solutions and boundary conditions for this kind of special case can also be expressed by (2.26), (2.27), (2.31), and (2.33).

2.4.2 Plane Deformation

Let us assume that the body considered in the case of generalized plane deformation has at each point a plane of elastic symmetry normal to the generator, that is, normal to the z-axis. In other words, the body is composed of the *monoclinic materials* with the symmetry plane at z=0 for which $C_{14} = C_{15} = C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0$. It is now possible to satisfy all the equations of the theory of elasticity by setting $w = w_0$ Then, instead of (2.40), we have

$$\begin{aligned} u &= U(x, y) - \omega_3 y + u_0, \\ v &= V(x, y) + \omega_3 x + v_0, \\ w &= w_0. \end{aligned} \tag{2.41}$$

Consequently, $\gamma_{yz} = \gamma_{xz} = \varepsilon_z = 0$, and on the basis of the equations of the generalized Hook's law, $\tau_{xz} = \tau_{yz} = 0$ and

$$\sigma_z = -\frac{1}{S_{33}}(S_{13}\sigma_x + S_{23}\sigma_y + S_{36}\tau_{xy}). \tag{2.42}$$

With the results of $\tau_{xz} = \tau_{yz} = 0$, by using the Airy stress function $\phi(x, y)$ introduced in (2.10) for the in-plane stresses $\sigma_x, \sigma_y, \tau_{xy}$, the differential equation that the stress function must satisfy can then be reduced from (2.11) to

$$L_4\phi = -(\hat{S}_{12} + \hat{S}_{22})\frac{\partial^2 \hat{F}}{\partial x^2} + (\hat{S}_{16} + \hat{S}_{26})\frac{\partial^2 \hat{F}}{\partial x \partial y} - (\hat{S}_{11} + \hat{S}_{12})\frac{\partial^2 \hat{F}}{\partial y^2}, \tag{2.43a}$$

in which L_4 is the differential operator of the fourth order which has the form

$$L_4 = \hat{S}_{22}\frac{\partial^4}{\partial x^4} - 2\hat{S}_{26}\frac{\partial^4}{\partial x^3 \partial y} + (2\hat{S}_{12} + \hat{S}_{66})\frac{\partial^4}{\partial x^2 \partial y^2} - 2\hat{S}_{16}\frac{\partial^4}{\partial x \partial y^3} + \hat{S}_{11}\frac{\partial^4}{\partial y^4}. \tag{2.43b}$$

The general solution of the Airy stress function ϕ to the differential equation (2.43) can also be obtained by reducing the expression (2.24) to

$$\phi = 2\,\mathrm{Re}\{\phi_1(z_1) + \phi_2(z_2)\} + \phi^{(p)}, \tag{2.44a}$$

in which

$$z_k = x + \mu_k y,\ k = 1, 2. \tag{2.44b}$$

In (2.44b) the complex parameters μ_k are the roots of the algebraic equation

$$l_4(\mu) = \hat{S}_{11}\mu^4 - 2\hat{S}_{16}\mu^3 + (2\hat{S}_{12} + \hat{S}_{66})\mu^2 - 2\hat{S}_{26}\mu + \hat{S}_{22} = 0, \tag{2.45}$$

and are assumed to be distinct. In the case of repeated root such as an isotropic body whose $\mu_1 = \mu_2 = i$, the general solution should be expressed as

$$\phi = 2\,\mathrm{Re}\{\phi_1(z_1) + \bar{z}_1\phi_2(z_1)\} + \phi^{(p)}. \tag{2.46}$$

Detailed discussion of the general solutions for the bodies with repeated complex parameters μ_k will be given in the next chapter for Stroh formalism.

With the result of (2.44), the general expressions for the stresses, displacements, and boundary conditions can then be obtained from (2.26), (2.27), (2.31), and (2.33) by deleting the terms associated with z_3. They are

Stresses and Displacements

$$\begin{aligned}
\sigma_x &= 2\,\mathrm{Re}\left\{\mu_1^2 f_1'(z_1) + \mu_2^2 f_2'(z_2)\right\} + \frac{\partial^2 \phi^{(p)}}{\partial y^2} + \hat{F},\\
\sigma_y &= 2\,\mathrm{Re}\left\{f_1'(z_1) + f_2'(z_2)\right\} + \frac{\partial^2 \phi^{(p)}}{\partial x^2} + \hat{F},\\
\tau_{xy} &= -2\,\mathrm{Re}\left\{\mu_1 f_1'(z_1) + \mu_2 f_2'(z_2)\right\} - \frac{\partial^2 \phi^{(p)}}{\partial x\, \partial y},
\end{aligned} \tag{2.47a}$$

$$\begin{aligned}
u &= 2\,\mathrm{Re}\sum_{k=1}^{2} a_{1k} f_k(z_k) - \omega_3 y + u_0 + u^{(p)},\\
v &= 2\,\mathrm{Re}\sum_{k=1}^{2} a_{2k} f_k(z_k) - \omega_3 x + v_0 + v^{(p)},
\end{aligned} \tag{2.47b}$$

where

$$a_{1k} = \mu_k^2 \hat{S}_{11} + \hat{S}_{12} - \mu_k \hat{S}_{16},$$
$$a_{2k} = (\mu_k^2 \hat{S}_{21} + \hat{S}_{22} - \mu_k \hat{S}_{26})/\mu_k. \tag{2.47c}$$

Boundary Conditions

$$2\,\mathrm{Re}\,\{\mu_1 f_1 + \mu_2 f_2\} = \tilde{t}_x(s) - \frac{\partial \phi^{(p)}}{\partial y} + c_1,$$
$$2\,\mathrm{Re}\,\{f_1 + f_2\} = \tilde{t}_y(s) - \frac{\partial \phi^{(p)}}{\partial x} + c_2, \tag{2.48a}$$

where

$$\tilde{t}_x(s) = -\int_0^s \left(\hat{t}_x + \hat{F}\frac{dy}{ds}\right) ds,$$
$$\tilde{t}_y(s) = \int_0^s \left(\hat{t}_y - \hat{F}\frac{dx}{ds}\right) ds, \tag{2.48b}$$

or

$$2\,\mathrm{Re}\sum_{k=1}^{2} a_{1k} f_k = -u^{(p)} + \hat{u} + \omega_3 y - u_0,$$
$$2\,\mathrm{Re}\sum_{k=1}^{2} a_{2k} f_k = -v^{(p)} + \hat{v} - \omega_3 x - v_0. \tag{2.49}$$

The problem then reduces to the determination of two complex functions f_1 and f_2 which should satisfy the boundary conditions on the contour of the cross-section region.

Using St. Venant's principle, we can apply all the formulas for an infinite cylinder to a body of finite length with fixed ends. In the case of free ends, the axial forces P_z and the bending moment M_1, M_2, which are given at the ends, can be removed by imposing the elementary distribution,

$$\sigma_z = -\frac{p_z}{S} - \frac{M_1}{I_1} y - \frac{M_2}{I_2} x,$$
$$\sigma_x = \sigma_y = \tau_{yz} = \tau_{xz} = \tau_{xy} = 0, \tag{2.50}$$

on the distribution of the stresses in the finite cylinder.

2.4.3 Generalized Plane Stress

Consider a thin plate of constant thickness, made from a homogeneous anisotropic material having at each point a plane of elastic symmetry parallel to the middle plane, i.e., *monoclinic materials*. Assume that the surface stresses are distributed along the edge symmetrically with respect to the middle plane, and that they vary negligibly with respect to the thickness of the plate; the body forces also are distributed symmetrically.

Taking the middle plane as the coordinate plane, the x–y plane, we introduce the mean values of the stresses and displacements with respect to the thickness (see Fig. 2.3)

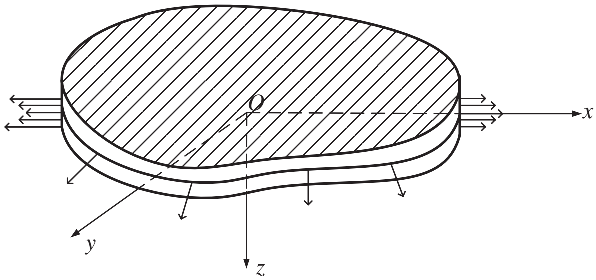

Fig. 2.3 Generalized plane stress

$$\tilde{\sigma}_x = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_x\, dz, \quad \tilde{\sigma}_y = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_y\, dz, \quad \tilde{\tau}_{xy} = \frac{1}{h}\int_{-h/2}^{h/2} \tau_{xy}\, dz,$$
$$\tilde{u} = \frac{1}{h}\int_{-h/2}^{h/2} u\, dz, \quad \tilde{v} = \frac{1}{h}\int_{-h/2}^{h/2} v\, dz. \tag{2.51}$$

As we discussed in Section 1.3 for the stress–strain relation of two-dimensional problems, by replacing everywhere the reduced elastic compliances $\hat{S}_{ij}$ to the elastic compliances S_{ij} the basic equations that hold in the case of plane deformation can now be applied to the case of generalized plane stress for these average quantities. It is therefore the problems of plane deformation and generalized plane stress are usually combined and known as "*the plane problem of the theory of elasticity*".

2.4.4 Anisotropic Rod by Bending and Twisting

Consider certain cases of the equilibrium of an anisotropic rod where twisting and bending moments are applied simultaneously. Unlike the isotropic rods where the twisting moments will only induce twisting and the bending moments will only induce bending, due to the anisotropy of the rods both of the twisting and bending moments will induce bending as well as twisting. Figure 2.4 shows the case when

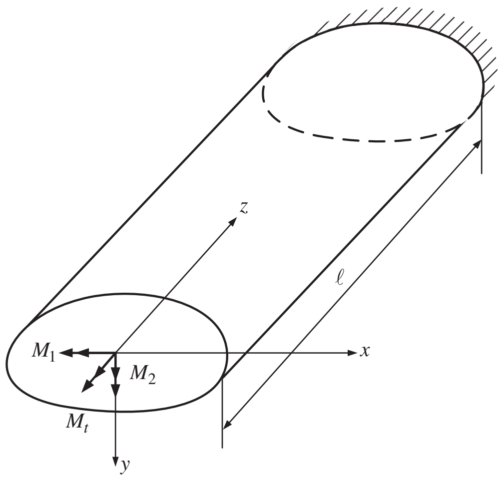

Fig. 2.4 Anisotropic rod by bending and twisting

the arbitrary twisting moment M_t and bending moments M_1, M_2 act on the end of a rod where the x- and y-axes coincide with the principal axes of inertia of the cross section. Since the lateral surface of the rod is free from external loads and no body forces are considered in this case, application of the end moments $M_t,\ M_1, M_2$ will then lead to

$$\hat{t}_x = \hat{t}_y = \hat{f}_x = \hat{f}_y = 0,\ P_z = 0,\ M_1, M_2, M_t \neq 0. \tag{2.52}$$

With (2.52), from (2.38a–c) we have

$$A = \frac{M_2}{I_2} + \frac{S_{34}}{2S_{33}}\frac{M_t}{I_2}, \quad B = \frac{M_1}{I_1} - \frac{S_{35}}{2S_{33}}\frac{M_t}{I_1}, \quad C = 0. \tag{2.53}$$

The displacements for this kind of problems can then be expressed by substituting (2.53) into (2.8a). They are

$$\begin{aligned}
u &= -\frac{1}{4I_2}(2S_{33}M_2 + S_{34}M_t)z^2 - \alpha yz + U(x,y) + \omega_2 z - \omega_3 y + u_0,\\
v &= -\frac{1}{4I_1}(2S_{33}M_1 - S_{35}M_t)z^2 + \alpha xz + V(x,y) + \omega_3 x - \omega_1 z + v_0,\\
w &= \frac{1}{2}\left[(2S_{33}M_2 + S_{34}M_t)\frac{x}{I_2} + (2S_{33}M_1 - S_{35}M_t)\frac{y}{I_1}\right]z + W(x,y) + \omega_1 y - \omega_2 x + w_0.
\end{aligned} \tag{2.54}$$

The constants u_0, v_0, w_0 and $\omega_1, \omega_2, \omega_3$ corresponding to the rigid body translation and rotation can be found from the conditions at the fixed end:

$$u = v = w = 0,\ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \quad \text{when } z = l, x = y = 0. \tag{2.55}$$

Substituting (2.54) into (2.55), we get

$$
\begin{aligned}
u_0 &= -\frac{1}{4I_2}(2S_{33}M_2 + S_{34}M_t)l^2 - U_0,\\
v_0 &= -\frac{1}{4I_1}(2S_{33}M_1 - S_{35}M_t)l^2 - V_0, \quad w_0 = -W_0,\\
\omega_1 &= -\frac{1}{2I_1}(2S_{33}M_1 - S_{35}M_t)l,\\
\omega_2 &= \frac{1}{2I_2}(2S_{33}M_2 + S_{34}M_t)l, \qquad \omega_3 = -\alpha l + \omega_3^0,
\end{aligned}
\tag{2.56}
$$

where U_0, V_0, W_0, and ω_3^0 are the values of U, V, W, and $(\partial U/\partial y - \partial V/\partial x)/2$ at the center of the cross section $x = y = 0$. In order to find U, V, W, and the associated stress fields, we can follow the steps described in (2.9)–(2.27) for the general cases.

With the prior conditions (2.52) and the results of (2.53), the system of differential equations (2.11) which the stress functions must satisfy reduce to

$$
\begin{aligned}
L_4\phi + L_3\psi &= 0,\\
L_3\phi + L_2\psi &= -2\alpha^*,
\end{aligned}
\tag{2.57a}
$$

where

$$
\alpha^* = \alpha - \frac{M_t}{4S_{33}}\left(\frac{S_{34}^2}{I_2} + \frac{S_{35}^2}{I_1}\right) + \frac{S_{35}M_1}{2I_1} - \frac{S_{34}M_2}{2I_2}. \tag{2.57b}
$$

A particular solution of the nonhomogeneous system (2.57) can be chosen as

$$
\phi^{(p)} = 0, \quad \psi^{(p)} = \frac{-\alpha^*}{2(\hat{S}_{44}\hat{S}_{55} - \hat{S}_{45}^2)}(\hat{S}_{55}x^2 + 2\hat{S}_{45}xy + \hat{S}_{44}y^2). \tag{2.58}
$$

With this particular solution, the unknowns remained in the general expressions for the stresses and displacements (2.26) and (2.27) are the complex functions $f_1(z_1), f_2(z_2), f_3(z_3)$ which should then be determined through the satisfaction of the lateral surface boundary conditions (2.31) or (2.33).

In order to relate the twisting angle α and the twisting moment M_t we rewrite (2.38d) by letting

$$
\psi = \alpha^*\psi^*, \tag{2.59}
$$

and hence

$$
M_t = 2\alpha^*\iint \psi^* dxdy, \ \alpha^* = \alpha - \frac{M_t}{4S_{33}}\left(\frac{S_{34}^2}{I_2} + \frac{S_{35}^2}{I_1}\right) + \frac{S_{35}M_1}{2I_1} - \frac{S_{34}M_2}{2I_2}. \tag{2.60}
$$

Re-organizing (2.60), we get

$$\alpha = \frac{M_t}{D} - \frac{S_{35}M_1}{2I_1} + \frac{S_{34}M_2}{2I_2}, \tag{2.61a}$$

where D is the *generalized torsional rigidity* defined by

$$\frac{1}{D} = \frac{1}{D^*} + \frac{1}{4S_{33}}\left(\frac{S_{34}^2}{I_2} + \frac{S_{35}^2}{I_1}\right), \quad \text{or} \quad D = \frac{4S_{33}D^*}{4S_{33} + \left(\frac{S_{34}^2}{I_2} + \frac{S_{35}^2}{I_1}\right)D^*}, \tag{2.61b}$$

and

$$D^* = 2\iint \psi^* dx\, dy. \tag{2.61c}$$

Based upon the derivation given in (2.52)–(2.61), several different rods deformed by twisting and bending moments have been analyzed and presented by Lekhnitskii (1963). The cross sections discussed in Lekhnitskii (1963) include elliptic, rectangular, aero-dynamic profile, elliptic ring, and elliptic sector. Following are some special cases of the joint action of twisting and bending moments.

Twisting Moment Only ($M_t \neq 0, P_z = M_1 = M_2 = 0$)

Substituting $M_1 = M_2 = 0$ into (2.53), (2.54), (2.55), (2.56), (2.57), and (2.58), we can get the results for a rod with anisotropy of a general kind under the influence of twisting moments. Since the constants A, B which are related to the bending deformation still exist, the twisting moments induce not only torsion but also bending. This situation is more complex than the usual pure torsion problem of isotropic rods and hence is called *generalized torsion*. If the rod has at each point a plane of elastic symmetry normal to its axis, which is one kind of monoclinic materials discussed in Section 1.2.2, $S_{34} = S_{35} = 0$ and hence $\hat{S}_{34} = \hat{S}_{35} = 0, \hat{S}_{44} = S_{44}, \hat{S}_{45} = S_{45}, \hat{S}_{55} = S_{55}$ by (1.56). From (2.53) we see that A=B=C=0. In other words, under twisting moment pure torsion deformation can be obtained not only for the isotropic materials but also for the monoclinic materials.

With $M_1 = M_2 = 0$, the relation between the twisting angle α and the twisting moment M_t, (2.61), now becomes

$$\alpha = \frac{M_t}{D}, \ \frac{1}{D} = \frac{1}{D^*} + \frac{1}{4S_{33}}\left(\frac{S_{34}^2}{I_2} + \frac{S_{35}^2}{I_1}\right). \tag{2.62}$$

Torsion Without Bending ($\alpha \neq 0, A = B = C = 0$)

From (2.53) we see that in some particular cases the bending deformation will not occur if the moments are selected in the following way:

$$M_1 = \frac{S_{35}}{2S_{33}} M_t,\ M_2 = -\frac{S_{34}}{2S_{33}} M_t. \tag{2.63}$$

Substituting (2.63) into (2.61), we obtain

$$\alpha = \frac{M_t}{D}, \quad D = D^*, \tag{2.64}$$

which is the same as the case that only twisting moment is applied on a rod with $S_{34} = S_{35} = 0$. Obviously, the torsional rigidity D becomes more rigid than that of the generalized torsion problems because $D^* \geq D$ in the general cases.

Bending Without Torsion ($B \neq 0, A = C = \alpha = 0$ or $A \neq 0, B = C = \alpha = 0$)

We seek moments such that the rod will be bent in the principal plane, the *y–z* plane or *x–z* plane, and the bending will not be accompanied by twisting. This situation can be made by letting A=0 or B=0 in (2.53) and then $\alpha = 0$ in (2.61a), which will lead to

$$M_2 = -\frac{S_{34}}{2S_{33}} M_t,\ M_t = \frac{S_{35} D^*}{2I_1 \left(1 + \frac{S_{35}^2}{4S_{33} I_1}\right)} M_1, \tag{2.65a}$$

or

$$M_1 = \frac{S_{35}}{2S_{33}} M_t,\ M_t = \frac{-S_{34} D^*}{2I_2 \left(1 + \frac{S_{34}^2}{4S_{33} I_1}\right)} M_2. \tag{2.65b}$$

When we apply the moments given in (2.65a) the rod will be bent in the *y–z* plane without torsion. The bent axis along the center of the cross section may be found from $(2.8a)_2$ with x=y=0. The *bending rigidity* may be represented by the inverse of $BS_{33}/2$, i.e., larger B means smaller bending rigidity which will induce larger deflection. When we substitute (2.65a) into (2.38b), we get

$$B = \frac{4S_{33} I_1}{4S_{33} I_1 + S_{35}^2 D^*} \frac{M_1}{I_1}. \tag{2.66}$$

For the same rod bent only by the moment M_1, from (2.38b) we have

$$B = \frac{M_1}{I_1}. \tag{2.67}$$

A comparison of (2.66) and (2.67) shows that the deflection of the rod undergoes bending without torsion due to the applied moments M_1, M_2 and M_t given in (2.65a) will be less than the deflection induced by the single moment M_1. In other words, if the moments which prevent twisting are applied to the rod, the rigidity of its bending increases, or say, the constant B decreases. Similar situation can be made for the applied moments given in (2.65b) for the bending in the *x–z* plane without torsion.

2.5 Anisotropic Cantilever Under Transverse Force

Consider an elastic body in the form of a cylinder or prism in equilibrium with one end fixed and a transverse force P acting on the other end (see Fig. 2.5). Assume that the cantilever is made from a homogeneous material with rectilinear anisotropy of the most general form. Under this condition, we may assume that the stresses $\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}$ are functions of x and y only and that

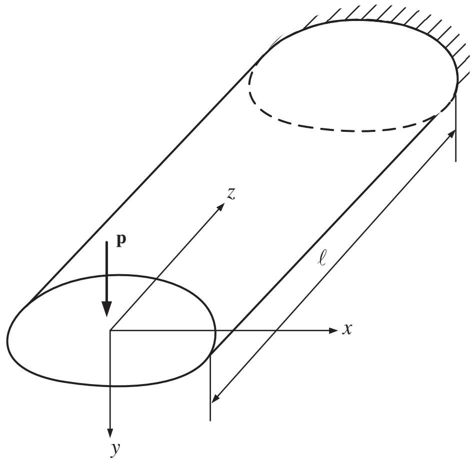

Fig. 2.5 Anisotropic cantilever under transverse force

$$\sigma_z = -\frac{M_1}{I_1}y + \sigma_z^0 = -\frac{Pz}{I_1}y + \sigma_z^0, \tag{2.68}$$

where σ_z^0 is a function of x and y only. Then, the equilibrium equations stated in (2.2c) should be modified as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \hat{F}}{\partial x} = 0,\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} - \frac{\partial \hat{F}}{\partial y} = 0,\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - \frac{P}{I_1}y = 0, \tag{2.69}$$

in which the only difference from the two-dimensional problems discussed in the previous sections comes from the addition of the third term on the left-hand side of the third equation of (2.69). By following the procedure described in the previous sections, one can find the general solutions for the anisotropic cantilever under transverse force. Detailed discussion of this kind of problems can be found in Lekhnitskii (1963).