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Transmission Problems for Elliptic Second-Order Equations in Non-Smooth Domains

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Chapter 1

Preliminaries

1.1 List of symbols

Let $G \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near the point $\mathcal{O}$ it is a conical surface with vertex at $\mathcal{O}$. We assume that $G = G_+ \cup G_- \cup \Sigma_0$ is divided into two subdomains G_+ and G_- by a $\Sigma_0 = G \cap \{x_n = 0\}$, where $\mathcal{O} \in \overline{\Sigma_0}$. Let us fix some notation used throughout the work:

- $[l]$ – the integral part of l (if l is not integer);
- $\mathbb{R}$ – the set of real numbers;
- $\mathbb{R}_+$ – the set of positive numbers;
- $\mathbb{R}^n$ – the n-dimensional Euclidean space, $n \geq 2$;
- $\mathbb{N}$ – the set of natural numbers;
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ – the set of non-negative integers;
- S^{n-1} – a unit sphere in $\mathbb{R}^n$ centered at $\mathcal{O}$;
- $(r, \omega), \omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$ – spherical coordinates of $x \in \mathbb{R}^n$ with pole $\mathcal{O}$: $x_1 = r \cos \omega_1,\ x_2 = r \cos \omega_2 \sin \omega_1, \ldots, x_{n-1} = r \cos \omega_{n-1} \sin \omega_{n-2} \ldots \sin \omega_1$, $x_n = r \sin \omega_{n-1} \sin \omega_{n-2} \ldots \sin \omega_1$;
- $\mathcal{C}$ – a rotational cone $\{x_1 > r \cos \frac{\omega_0}{2}\}$ with the vertex at $\mathcal{O}$;
- $\partial\mathcal{C}$ – the lateral surface of $\mathcal{C}$: $\{x_1 = r \cos \frac{\omega_0}{2}\}$;
- Ω – a domain on the unit sphere S^{n-1} with smooth boundary $\partial\Omega$ obtained by the intersection of the cone $\mathcal{C}$ with the sphere S^{n-1};
- $\partial\Omega = \partial\mathcal{C} \cap S^{n-1}$;
- $G_a^b = \{(r, \omega) \mid 0 \leq a < r < b; \omega \in \Omega\} \cap G$ – a layer in $\mathbb{R}^n$;

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- $\Gamma_a^b = \{(r,\omega) \mid 0 \le a < r < b; \omega \in \partial\Omega\} \cap \partial G$ – the lateral surface of layer G_a^b;
- $G_d = G \setminus G_0^d$; $\quad \Gamma_d = \partial G \setminus \Gamma_0^d$, $d > 0$;
- $\Sigma_a^b = G_a^b \cap \{x_n = 0\} \subset \Sigma_0$; $\quad \Sigma_d = \Sigma_0 \setminus \Sigma_0^d$, $d > 0$;
- $\Omega_\rho = G_0^d \cap \{|x| = \rho\}$; $\quad 0 < \rho < d$;
- $\Omega_+ = \Omega \cap \{x_n > 0\}$, $\Omega_- = \Omega \cap \{x_n < 0\} \Longrightarrow \Omega = \Omega_+ \cup \Omega_- \cup \sigma_0$;
- $\sigma_0 = \Sigma_0 \cap \Omega$; $\partial_\pm\Omega = \overline{\Omega_\pm} \cap \partial\mathcal{C}$; $\partial\Omega_\pm = \partial_\pm\Omega \cup \sigma_0$;
- $G^{(k)} := G_{2^{-(k+1)}d}^{2^{-k}d}$, $\quad k = 0, 1, 2, \ldots$;
- $u(x) = \begin{cases} u_+(x), & x \in G_+, \\ u_-(x), & x \in G_-; \end{cases} \quad f(x) = \begin{cases} f_+(x), & x \in G_+, \\ f_-(x), & x \in G_- \end{cases} \quad$ etc.;
- $[u]_{\Sigma_0}$ denotes the saltus of the function $u(x)$ on crossing Σ_0, i.e., $[u]_{\Sigma_0} = u_+(x)\Big|_{\Sigma_0} - u_-(x)\Big|_{\Sigma_0}$, where $u_\pm(x)\Big|_{\Sigma_0} = \lim\limits_{G_\pm \ni y \to x \in \Sigma_0} u_\pm(y)$;
- $n_i = \cos(\overrightarrow{n}, x_i)$, $i = 1, \ldots, n$, where $\overrightarrow{n}$ denotes the unit outward vector with respect to G_+ (or G) normal to Σ_0 (respectively $\partial G \setminus \mathcal{O}$).

1.2 Operators and formulae related to spherical coordinates

Let us recall first some well-known formulae related to the spherical coordinates $(r, \omega_1, \ldots, \omega_{n-1})$ centered at the conical point $\mathcal{O}$:

$$dx = r^{n-1} dr d\Omega, \quad d\Omega_\rho = \rho^{n-1} d\Omega, \tag{1.2.1}$$

$d\Omega = J(\omega) d\omega$ denotes the $(n-1)$-dimensional area element of the unit sphere, where

$$J(\omega) = \sin^{n-2}\omega_1 \sin^{n-3}\omega_2 \ldots \sin\omega_{n-2}, \tag{1.2.2}$$
$$d\omega = d\omega_1 \ldots d\omega_{n-1}, \tag{1.2.3}$$

$ds = r^{n-2} dr d\sigma$ denotes the $(n-1)$-dimensional area element on $\partial\mathcal{C}$ and $d\sigma$ denotes the $(n-2)$-dimensional area element on $\partial\Omega$;

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} |\nabla_\omega u|^2, \tag{1.2.4}$$

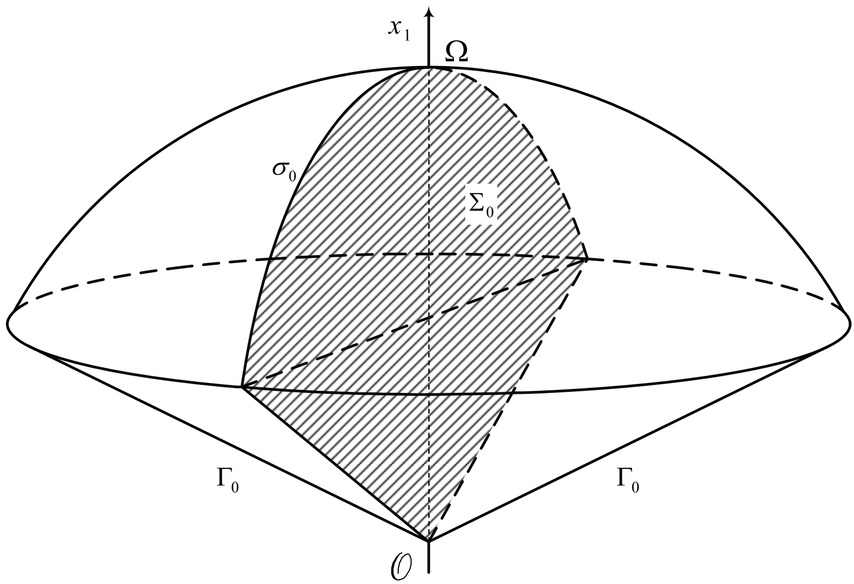

Figure 1

where $|\nabla_\omega u|$ denotes the projection of the vector ∇u onto the tangent plane to the unit sphere at the point ω:

$$\nabla_\omega u = \left\{ \frac{1}{\sqrt{q_1}} \frac{\partial u}{\partial \omega_1}, \dots, \frac{1}{\sqrt{q_{n-1}}} \frac{\partial u}{\partial \omega_{n-1}} \right\}, \tag{1.2.5}$$

$$|\nabla_\omega u|^2 = \sum_{i=1}^{n-1} \frac{1}{q_i} \left(\frac{\partial u}{\partial \omega_i} \right)^2, \quad \text{where } q_1 = 1,\ q_i = (\sin\omega_1 \cdots \sin\omega_{i-1})^2,\ i \geq 2,$$

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_\omega u, \text{ denotes the Laplace operator,}$$

$$\Delta_\omega u = \frac{1}{J(\omega)} \sum_{i=1}^{n-1} \frac{\partial}{\partial \omega_i} \left(\frac{J(\omega)}{q_i} \cdot \frac{\partial u}{\partial \omega_i} \right) = \sum_{i=1}^{n-1} \frac{1}{q_j \sin^{n-i-1} \omega_i} \frac{\partial}{\partial \omega_i} \left(\sin^{n-i-1} \omega_i \frac{\partial u}{\partial \omega_i} \right) \tag{1.2.6}$$

denotes the Beltrami-Laplace operator,

$$\operatorname{div}_\omega u = \frac{1}{J(\omega)} \sum_{i=1}^{n-1} \frac{\partial}{\partial \omega_i} \left(\frac{J(\omega)}{\sqrt{q_i}} u_i \right), \ u = (u_1, \dots, u_{n-1}). \tag{1.2.7}$$

Lemma 1.1. *Assume for some $d > 0$ that G_0^d is the rotational cone with the vertex at $\mathcal{O}$ and the aperture ω_0, and let*

$$\Gamma_0^d = \left\{ (r,\omega) \middle| x_1^2 = \cot^2 \frac{\omega_0}{2} \sum_{i=2}^{n} x_i^2;\ |\omega_1| = \frac{\omega_0}{2},\ \omega_0 \in (0, 2\pi) \right\}. \tag{1.2.8}$$

Then

$$x_i \cos(\vec{n}, x_i)|_{\Gamma_0^d} = 0,\ \textit{and}\ \cos(\vec{n}, x_1)|_{\Gamma_0^d} = -\sin \frac{\omega_0}{2}. \tag{1.2.9}$$

Proof. By virtue of (1.2.8) we can rewrite the equation of Γ_0^d in the form $F(x) \equiv x_1^2 - \cot^2 \frac{\omega_0}{2} \sum\limits_{i=2}^{n} x_i^2 = 0$. We use the formula $\cos(\vec{n}, x_i) = \frac{\frac{\partial F}{\partial x_i}}{|\nabla F|}$, $\forall i = 1, \ldots, n$. Because of $\frac{\partial F}{\partial x_1} = 2x_1$, $\frac{\partial F}{\partial x_i} = -2\cot^2 \frac{\omega_0}{2} x_i$, $\forall i = 2, \ldots, n$, we obtain

$$x_i \cos(\vec{n}, x_i)\Big|_{\Gamma_0^d} = \frac{1}{|\nabla F|} x_i \frac{\partial F}{\partial x_i}\Big|_{\Gamma_0^d} = \frac{2}{|\nabla F|}\left(x_1^2 - \cot^2 \frac{\omega_0}{2} \sum_{i=2}^{n} x_i^2 \right)\Big|_{\Gamma_0^d} = 0.$$

Furthermore from

$$|\nabla F|^2 = \left(\frac{\partial F}{\partial x_1}\right)^2 + \sum_{i=2}^{n} \left(\frac{\partial F}{\partial x_i}\right)^2 = 4\left(x_1^2 + \cot^4 \frac{\omega_0}{2} \sum_{i=2}^{n} x_i^2 \right)$$

$$\Rightarrow |\nabla F|^2\Big|_{\Gamma_0^d} = 4x_1^2 \left(1 + \frac{\cos^2 \frac{\omega_0}{2}}{\sin^2 \frac{\omega_0}{2}}\right) = \frac{4x_1^2}{\sin^2 \frac{\omega_0}{2}},$$

we get

$$\cos(\vec{n}, x_1)\Big|_{\Gamma_0^d} = -2x_1 \frac{\sin \frac{\omega_0}{2}}{2x_1} = -\sin \frac{\omega_0}{2}, \qquad \text{since} \quad \angle(\vec{n}, x_1) > \frac{\pi}{2}. \qquad \square$$

1.3 The quasi-distance function r_ε and its properties

Let us assume that the cone K is contained in a rotational cone $\mathcal{C}$ with the opening angle ω_0. Furthermore, let us suppose that the axis of $\mathcal{C}$ coincides with $\{(x_1, 0, \ldots, 0) : x_1 > 0\}$. In this case we define **the quasi-distance** $r_\varepsilon(x)$ as follows. We fix the point $Q = (-1, 0, \ldots, 0) \in S^{n-1} \setminus \bar{\Omega}$ and consider the unit radius-vector $\vec{l} = \mathcal{O}Q = \{-1, 0, \ldots, 0\}$. We denote by $\vec{r}$ the radius-vector of the point $x \in \bar{G}$ and introduce the vector $\vec{r_\varepsilon} = \vec{r} - \varepsilon\vec{l}$ for each $\varepsilon > 0$. Since $\varepsilon\vec{l} \notin G_0^d$ for all $\varepsilon \in]0, d[$, it follows that $r_\varepsilon(x) = |\vec{r} - \varepsilon\vec{l}| \neq 0$ for all $x \in \bar{G}$. It is easy to verify that $r_\varepsilon(x)$ has the following properties (see in detail §1.4 [14]):

1. *there exists $h > 0$ such that: $r_\varepsilon(x) \geq hr$ and $r_\varepsilon(x) \geq h\varepsilon$, $\forall x \in \overline{G}$, where* $h = \begin{cases} 1, & \textit{if } x_1 \geq 0, \\ \sin \frac{\omega_0}{2}, & \textit{if } x_1 < 0; \end{cases}$

2. *if* $x \in G_d$, *then* $r_\varepsilon(x) \geq \frac{d}{2}$ *for all* $\varepsilon \in]0, \frac{d}{2}[$;
3. $\lim\limits_{\varepsilon \to 0^+} r_\varepsilon(x) = r$, *for all* $x \in \bar{G}$;
4. $|\nabla r_\varepsilon|^2 = 1$, *and* $\triangle r_\varepsilon = \frac{n-1}{r_\varepsilon}$.

1.4 Function spaces

We use the standard function spaces:

- $C^k(\overline{G_\pm})$ with the norm $|u_\pm|_{k,G_\pm}$;
- the Lebesgue space $L_p(G_\pm), p \geq 1$ with the norm $\|u_\pm\|_{p,G_\pm}$;
- the Sobolev space $W^{k,p}(G_\pm)$ with the norm $\|u_\pm\|_{p,k;G_\pm}$

and introduce their direct sums

- $\mathbf{C}^k(\overline{G}) = C^k(\overline{G_+}) \dot{+} C^k(\overline{G_-})$ with the norm $|u|_{k,G} = |u_+|_{k,G_+} + |u_-|_{k,G_-}$;
- $\mathbf{L}_p(G) = L_p(G_+) \dot{+} L_p(G_-)$ with the norm

$$\|u\|_{\mathbf{L}_p(G)} = \left(\int\limits_{G_+} |u_+|^p dx \right)^{\frac{1}{p}} + \left(\int\limits_{G_-} |u_-|^p dx \right)^{\frac{1}{p}};$$

- $\mathbf{W}^{k,p}(G) = W^{k,p}(G_+) \dot{+} W^{k,p}(G_-)$ with the norm

$$\|u\|_{p,k;G} = \left(\int\limits_{G_+} \sum_{|\beta|=0}^{k} |D^\beta u_+|^p dx \right)^{\frac{1}{p}} + \left(\int\limits_{G_-} \sum_{|\beta|=0}^{k} |D^\beta u_-|^p dx \right)^{\frac{1}{p}}.$$

For any integer $k \geq 0$ and real α we define the weighted Sobolev space $\mathbf{V}^k_{p,\alpha}(G)$ as the space of distributions $u \in \mathcal{D}'(G)$ with the finite norm

$$\|u\|_{\mathbf{V}^k_{p,\alpha}(G)} = \left(\int\limits_{G_+} \sum_{|\beta|=0}^{k} r^{\alpha+p(|\beta|-k)} |D^\beta u_+|^p \, dx \right)^{\frac{1}{p}} + \left(\int\limits_{G_-} \sum_{|\beta|=0}^{k} r^{\alpha+p(|\beta|-k)} |D^\beta u_-|^p \, dx \right)^{\frac{1}{p}}$$

and $\mathbf{V}^{k-\frac{1}{p}}_{p,\alpha}(\partial G)$ as the space of functions φ, given on ∂G, with the norm $\|\varphi\|_{\mathbf{V}^{k-\frac{1}{p}}_{p,\alpha}(\partial G)} = \inf \|\Phi\|_{\mathbf{V}^k_{p,\alpha}(G)}$, where the infimum is taken over all functions Φ such that $\Phi\Big|_{\partial G} = \varphi$ in the sense of traces. We write

$$\mathbf{W}^k(G) \equiv \mathbf{W}^{k,2}(G), \quad \overset{\circ}{\mathbf{W}}{}^k_\alpha(G) \equiv \mathbf{V}^k_{2,\alpha}(G), \quad \overset{\circ}{\mathbf{W}}{}^{k-\frac{1}{2}}_\alpha(\partial G) \equiv \mathbf{V}^{k-\frac{1}{2}}_{2,\alpha}(\partial G).$$

1.5 Some inequalities

In this section we recall some elementary inequalities (see e.g. [2, 30]) which will be frequently used throughout this book.

Lemma 1.2 (Cauchy's Inequality). *For any $a, b \geq 0$ and $\varepsilon > 0$, we have*

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2. \tag{1.5.1}$$

Lemma 1.3 (Young's Inequality). *For any $a, b \geq 0$, $\varepsilon > 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$ab \leq \frac{1}{p}(\varepsilon a)^p + \frac{1}{q}\left(\frac{b}{\varepsilon}\right)^q. \tag{1.5.2}$$

Lemma 1.4 (Hölder's Inequality). *For any non-negative real numbers a_i, b_i, $i = 1, \dots, n$, and $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}. \tag{1.5.3}$$

Lemma 1.5. (Theorem 41 [30]). *For any non-negative real numbers a, b and $m \geq 1$ we have*

$$ma^{m-1}(a-b) \geq a^m - b^m \geq mb^{m-1}(a-b). \tag{1.5.4}$$

Lemma 1.6 (Jensen's Inequality (Theorem 65 [30])**).** *Let a_i, $i = 1, \dots, n$, be any non-negative real numbers and $p > 0$. Then*

$$\lambda \sum_{i=1}^{n} a_i^p \leq \left(\sum_{i=1}^{n} a_i\right)^p \leq \Lambda \sum_{i=1}^{n} a_i^p, \tag{1.5.5}$$

where $\lambda = \min(1, n^{p-1})$ and $\Lambda = \max(1, n^{p-1})$.

Lemma 1.7. *For any a, $b \in \mathbb{R}$ and $m > 1$ we have*

$$|b|^m \geq |a|^m + m|a|^{m-2}a(b-a). \tag{1.5.6}$$

Proof. By the Young inequality (1.5.2) with $\varepsilon = 1$, $p = m$, $q = \frac{m}{m-1}$, we obtain

$$m|a|^{m-2}ab \leq m|b| \cdot |a|^{m-1} \leq |b|^m + (m-1)|a|^m \Longrightarrow (1.5.6). \qquad \square$$

Theorem 1.8 (Hölder's Inequality, see Theorem 189 [30]**).**
Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in L^p(G)$, $v \in L^q(G)$. Then

$$\int\limits_G |uv|dx \leq \|u\|_{L^p(G)}\|v\|_{L^q(G)}. \tag{1.5.7}$$

If $p = 1$, then (1.5.7) is valid with $q = \infty$.

Corollary 1.9. *Let $1 \le p \le q$ and $u \in L^q(G)$. Then*

$$\|u\|_{L^p(G)} \le (\operatorname{meas} G)^{1/p-1/q}\|u\|_{L^q(G)}. \tag{1.5.8}$$

Corollary 1.10 (Interpolation inequality).
Let $1 < p \le q \le r$ and $1/q = \lambda/p + (1-\lambda)/r$. Then the inequality

$$\|u\|_{L^q(G)} \le \|u\|^{\lambda}_{L^p(G)}\|u\|^{1-\lambda}_{L^r(G)} \tag{1.5.9}$$

holds for all $u \in L^r(G)$.

Theorem 1.11 (Clarkson's Inequality, see §3.2, Chapter I [68]**).** *Let $u, v \in L^p(G)$. Then*

$$\left\|\frac{u+v}{2}\right\|^p_{L^p(G)} + \left\|\frac{u-v}{2}\right\|^p_{L^p(G)} \le \frac{1}{2}\Big(\|u\|^p_{L^p(G)} + \|v\|^p_{L^p(G)}\Big),\ 2 \le p < \infty;$$

$$\left\|\frac{u+v}{2}\right\|^{\frac{p}{p-1}}_{L^p(G)} + \left\|\frac{u-v}{2}\right\|^{\frac{p}{p-1}}_{L^p(G)} \le \Big(\frac{1}{2}\|u\|^p_{L^p(G)} + \frac{1}{2}\|v\|^p_{L^p(G)}\Big)^{\frac{1}{p-1}},\ 1 \le p \le 2.$$

Theorem 1.12 (Fatou's Theorem, see Theorem 19 §6, Chapter III [25]**).** *Let $\{f_k\} \in L^1(G)$, $k \in \mathbb{N}$, be a sequence of non-negative functions which is convergent almost everywhere in G to the function f. Then*

$$\int\limits_G f dx \le \sup \int\limits_G f_k dx. \tag{1.5.10}$$

We need also the following well-known inequalities:

Theorem 1.13. (See e.g. (6.23), (6.24) Chapter I [42] or Lemma 6.36 [49]). *Let ∂G be piecewise smooth and $u \in W^{1,1}(G)$. Then there is a constant $c > 0$ that depends only on G such that*

$$\int\limits_\Gamma |u| ds \le c \int\limits_G (|u| + |\nabla u|)\, dx \quad \textit{for each } \Gamma \subseteq \partial\Omega; \tag{1.5.11}$$

$$\int\limits_{\partial G} v^2 ds \le \int\limits_G (\delta|\nabla v|^2 + c_\delta v^2) dx,\ \forall v(x) \in W^{1,2}(G), \forall \delta > 0. \tag{1.5.12}$$

1.6 Sobolev embedding theorems

We now recall the well-known *Sobolev inequalities* and *Kondrashov compactness results* which are frequently referred as the *embedding theorems* (see [68], §§1.4.5–1.4.6 [52], §7.7[29]).

Theorem 1.14 (Sobolev inequalities (see e.g. [70, Theorem 2.4.1], [29, Theorem 7.10])**).** *Let G be a bounded open domain in $\mathbb{R}^n$ and $p > 1$. Then*

$$W_0^{1,p}(G) \hookrightarrow \begin{cases} L_{\frac{np}{n-p}}(G) & \text{for } p < n, \\ C^0(\overline{G}) & \text{for } p > n. \end{cases} \tag{1.6.1}$$

Furthermore, there exists a constant $c = c(n,p)$ such that for all $u \in W_0^{1,p}(G)$ we have

$$\|u\|_{L_{np/(n-p)}(G)} \le c\|\nabla u\|_{L_p(G)} \tag{1.6.2}$$

for $p < n$ and

$$\sup_G |u| \le c(\operatorname{meas} G)^{1/n-1/p}\|\nabla u\|_{L_p(G)} \tag{1.6.3}$$

for $p > n$.

The following Embedding Theorems 1.15–1.20 were proved first by Sobolev [68] and can be found with complete proofs in [52, Section 1.4]. Let G be a $C^{0,1}$ bounded domain in $\mathbb{R}^n$.

Theorem 1.15. *Let $k \in \mathbb{N}$ and $p \in \mathbb{R}$ with $p \ge 1$, $kp < n$. Then the embedding*

$$W^{k,p}(G) \hookrightarrow L_q(G) \tag{1.6.4}$$

is continuous for $1 \le q \le np/(n-kp)$ and compact for $1 \le q < np/(n-kp)$. If $kp = n$, then the embedding (1.6.4) *is continuous and compact for any $q \ge 1$.*

Theorem 1.16. (For the proof see, for example, (2.19) §2, chapter II [43]). *Let $u \in W^1(G)$. Then*

$$\|u\|^2_{\mathbf{L}_{\frac{2p}{p-2}}(G)} \le \delta\|\nabla u\|^2_{\mathbf{L}_2(G)} + c(\delta, p, n, meas\, G)\|u\|^2_{\mathbf{L}_2(G)}, \quad p > n,\ \forall\delta > 0. \tag{1.6.5}$$

Theorem 1.17. *Let $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ and let $p, q \in \mathbb{R}$ with $p, q \ge 1$. If $kp < n$, then the embedding*

$$W^{m+k,p}(G) \hookrightarrow W^{m,q}(G) \tag{1.6.6}$$

is continuous for any $q \in \mathbb{R}$ satisfying $1 \le q \le np/(n-kp)$. If $k = np$, then the embedding (1.6.6) *is continuous for any $q \ge 1$.*

Theorem 1.18. *Let $k, m \in \mathbb{N}_0$ and $p > 1$. Then the embedding $W^{k,p}(G) \hookrightarrow C^{m+\beta}(G)$ is continuous if*

$$(k-m-1)p < n < (k-m)p \quad \text{and} \quad 0 < \beta \le k - m - n/p, \tag{1.6.7}$$

and compact if the inequality for β in (1.6.7) *is sharp. If $(k-m-1)p = n$, then the embedding is continuous for any $\beta \in (0,1)$.*

Theorem 1.19. *Let $u \in W^{k,p}(G)$ with $k \in \mathbb{N}$, $p \in \mathbb{R}$, $kp > n$ and $p > 1$. Then $u \in C^m(G)$ for $0 \le m < k - n/p$ and there exists a constant c, independent of u, such that*

$$\sup_{x \in G} |D^\alpha u(x)| \le c\|u\|_{W^{k,p}(G)}$$

for all $|\alpha| < k - n/p$.

Theorem 1.20. *Let G be a lipschitzian domain and $T_s \subset \overline{G}$ be a piecewise C^k-smooth s-dimensional manifold. Let $k \ge 1$, $p > 1$, $kp \le n$, $n - kp < s \le n$, $1 \le q \le q* = sp/(n - kp)$. Then the embedding $W^{k,p}(G) \hookrightarrow L_q(T_s)$ is continuous and the inequality*

$$\|u\|_{L_q(T_s)} \le c\|u\|_{W^{k,p}(G)} \tag{1.6.8}$$

holds. If $q < q*$, then the above embedding is compact.

1.7 The Cauchy problem for a differential inequality

Theorem 1.21. (See Theorem 1.57 [14]). *Let $V(\varrho)$ be a monotonically increasing, non-negative differentiable function defined on $[0, 2d]$ and satisfying the problem*

$$\begin{cases} V'(\rho) - \mathcal{P}(\varrho)V(\varrho) + \mathcal{N}(\rho)V(2\rho) + \mathcal{Q}(\rho) \ge 0, \quad 0 < \rho < d, \\ V(d) \le V_0, \end{cases} \tag{CP}$$

where $\mathcal{P}(\varrho), \mathcal{N}(\varrho), \mathcal{Q}(\varrho)$ are non-negative continuous functions defined on $[0, 2d]$ and V_0 is a constant. Then we have

$$V(\varrho) \le \exp\Big(\int_\varrho^d \mathcal{B}(\tau)d\tau\Big)\Big\{V_0 \exp\Big(-\int_\varrho^d \mathcal{P}(\tau)d\tau\Big) + \int_\varrho^d \mathcal{Q}(\tau)\exp\Big(-\int_\varrho^\tau \mathcal{P}(\sigma)d\sigma\Big)d\tau\Big\} \tag{1.7.1}$$

with

$$\mathcal{B}(\varrho) = \mathcal{N}(\varrho)\exp\Big(\int_\varrho^{2\varrho} \mathcal{P}(\sigma)d\sigma\Big). \tag{1.7.2}$$

1.8 Additional auxiliary results

1.8.1 Stampacchia's Lemma

Lemma 1.22. (See Lemma 3.11 of [57]). *Let $\varphi : [k_0, \infty) \to \mathbb{R}$ be a non-negative and non-increasing function which satisfies*

$$\varphi(h) \le \frac{C}{(h-k)^\alpha}[\varphi(k)]^\beta \quad \text{for} \quad h > k > k_0, \tag{1.8.1}$$

where C, α, β are positive constants with $\beta > 1$. Then $\varphi(k_0 + d) = 0$, where $d^\alpha = C\,|\varphi(k_0)|^{\beta-1}\,2^{\alpha\beta/(\beta-1)}$.

Proof. For each $s = 1, 2, \dots$ we let $k_s = k_0 + d - \frac{d}{2^s}$ and consider the sequence $\{\varphi(k_s)\}$. From (1.8.1) we obtain

$$\varphi(k_{s+1}) \le \frac{C2^{(s+1)\alpha}}{d^\alpha}[\varphi(k_s)]^\beta, \quad s = 1, 2, \dots. \tag{1.8.2}$$

We now prove by induction that

$$\varphi(k_s) \le \frac{\varphi(k_0)}{2^{-s\mu}}, \text{ where } \mu = \frac{\alpha}{1-\beta} < 0. \tag{1.8.3}$$

For $s = 0$ the claim is evident. Let us suppose that (1.8.3) is valid up to s. By (1.8.2) and the definition of d^α if follows that

$$\varphi(k_{s+1}) \le C\frac{2^{(s+1)\alpha}}{d^\alpha}\frac{[\varphi(k_0]^\beta}{2^{-s\beta\mu}} \le \frac{\varphi(k_0)}{2^{-(s+1)\mu}}. \tag{1.8.4}$$

Since the right-hand side of (1.8.4) tends to zero as $s \to \infty$, we obtain $0 \le \varphi(k_0 + d) \le \varphi(k_s) \to 0$. □

1.8.2 Other assertions

Lemma 1.23. (See Lemma 2.1 [20]). *Let us consider the function*

$$\eta(x) = \begin{cases} e^{\varkappa x} - 1, & x \ge 0, \\ -e^{-\varkappa x} + 1, & x \le 0, \end{cases}$$

where $\varkappa > 0$. Let a, b be positive constants, $m > 1$. If $\varkappa > (2b/a) + m$, then we have

$$a\eta'(x) - b|\eta(x)| \ge \frac{a}{2}e^{\varkappa x}, \quad \forall x \ge 0; \tag{1.8.5}$$

$$\eta(x) \ge \left[\eta\left(\frac{x}{m}\right)\right]^m, \quad \forall x \ge 0. \tag{1.8.6}$$

Moreover, there exist some $d \ge 0$ and $M > 0$ such that

$$\eta(x) \le M\left[\eta\left(\frac{x}{m}\right)\right]^m, \quad \eta'(x) \le M\left[\eta\left(\frac{x}{m}\right)\right]^m, \quad \forall x \ge d, \tag{1.8.7}$$

$$|\eta(x)| \ge x, \quad \forall x \in \mathbb{R}. \tag{1.8.8}$$

Proof. We refer to Lemma 1.60 [14] for the proof. □

Lemma 1.24. (See Lemma 4.1 in Chapter 2 [18]). *Let $\psi(s)$ be a bounded non-negative function defined on the interval $[0, \varrho]$. Suppose that for any $0 \le \sigma < s \le \varrho$ the function $\psi(s)$ satisfies*

$$\psi(\sigma) \le \delta\psi(s) + \frac{A}{(s-\sigma)^\alpha} + B,$$

where $\delta \in (0,1)$, A, B and α are non-negative constants. Then

$$\psi(r) \le C\left[\frac{A}{(R-r)^\alpha} + B\right], \quad 0 \le r < R \le \varrho, \tag{1.8.9}$$

where C depends only on α, δ.

Theorem 1.25 (S. Bernstein [4]). *If y is a bounded analytic solution of the equation*

$$y'' = f(x, y, y'),\ a_0 < x < b_0, \quad \text{where } |f(x,y,y')| < Ay'^2 + B,$$

then y' is also bounded:

$$|y'| < \sqrt{\frac{B}{2A}} e^{4AM}, \quad M = \sup_{x \in (a_0, b_0)} |y(x)|.$$