# Mathematics for Engineers II

Calculus and Linear Algebra

Bearbeitet von
Gerd Baumann

1. Auflage 2009. Buch. XIII, 310 S.
ISBN 978 3 486 59040 1
Format (B x L): 17 x 24 cm
Gewicht: 700 g

Weitere Fachgebiete > Mathematik > Mathematische Analysis > Moderne Anwendungen der Analysis

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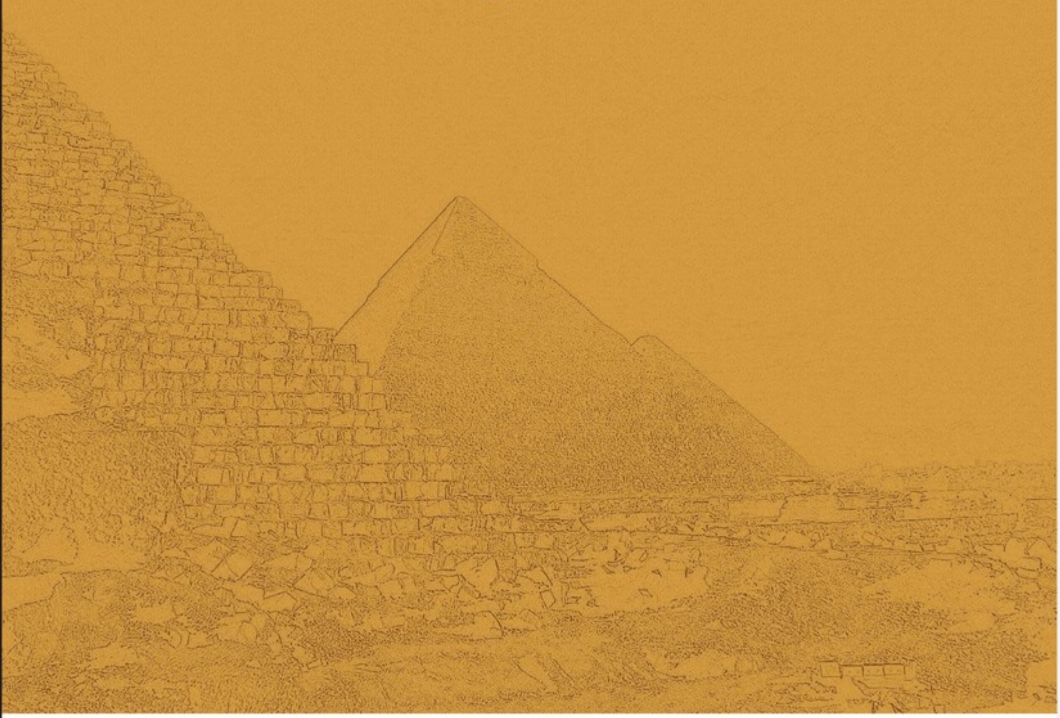

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# Mathematics for Engineers II

## Calculus and Linear Algebra

by
Gerd Baumann

Oldenbourg Verlag München

**Prof. Dr. Gerd Baumann** is head of Mathematics Department at German University Cairo (GUC). Before he was Professor at the Department of Mathematical Physics at the University of Ulm.


Rosenheimer Straße 145, D-81671 München
Telefon: (089) 45051-0
oldenbourg.de



Editor: Kathrin Mönch
Producer: Anna Grosser
Cover design: Kochan & Partner, München
Printed on acid-free and chlorine-free paper
Printing: Druckhaus „Thomas Müntzer“ GmbH, Bad Langensalza

ISBN 978-3-486-59040-1

# Preface

*Theory without Practice is empty,*
*Practice without Theory is blind.*

The current text *Mathematics for Engineers* is a collection of four volumes covering the first three up to the fifth terms in undergraduate education. The text is mainly written for engineers but might be useful for students of applied mathematics and mathematical physics, too.

Students and lecturers will find more material in the volumes than a traditional lecture will be able to cover. The organization of each of the volumes is done in a systematic way so that students will find an approach to mathematics. Lecturers will select their own material for their needs and purposes to conduct their lecture to students.

For students the volumes are helpful for their studies at home and for their preparation for exams. In addition the books may be also useful for private study and continuing education in mathematics. The large number of examples, applications, and comments should help the students to strengthen their knowledge.

The volumes are organized as follows: Volume I treats basic calculus with differential and integral calculus of single valued functions. We use a systematic approach following a bottom-up strategy to introduce the different terms needed. Volume II covers series and sequences and first order differential equations as a calculus part. The second part of the volume is related to linear algebra. Volume III treats vector calculus and differential equations of higher order. In Volume IV we use the material of the previous volumes in numerical applications; it is related to numerical methods and practical calculations. Each of the volumes is accompanied by a CD containing the *Mathematica* notebooks of the book.

As prerequisites we assume that students had the basic high school education in algebra and geometry. However, the presentation of the material starts with the very elementary subjects like numbers and introduces in a systematic way step by step the concepts for functions. This allows us to repeat most of the material known from high school in a systematic way, and in a broader frame. This way the reader will be able to use and categorize his knowledge and extend his old frame work to a new one.

The numerous examples from engineering and science stress on the applications in engineering. The idea behind the text concept is summarized in a three step process:

Theory → Examples → Applications

When examples are discussed in connection with the theory then it turns out that the theory is not only valid for this specific example but useful for a broader application. In fact, usually theorems or a collection of theorems can even handle whole classes of problems. These classes are sometimes completely separated from this introductory example; e.g. the calculation of areas to motivate integration or the calculation of the power of an engine, the maximal height of a satellite in space, the moment of inertia of a wheel, or the probability of failure of an electronic component. All these problems are solvable by one and the same method, integration.

However, the three-step process is not a feature which is always used. Some times we have to introduce mathematical terms which are used later on to extend our mathematical frame. This means that the text is not organized in a historic sequence of facts as traditional mathematics texts. We introduce definitions, theorems, and corollaries in a way which is useful to create progress in the understanding of relations. This way of organizing the material allows us to use the complete set of volumes as a reference book for further studies.

The present text uses *Mathematica* as a tool to discuss and to solve examples from mathematics. The intention of this book is to demonstrate the usefulness of *Mathematica* in everyday applications and calculations. We will not give a complete description of its syntax but demonstrate by examples the use of its language. In particular, we show how this modern tool is used to solve classical problems and to represent mathematical terms.

We hope that we have created a coherent way of a first approach to mathematics for engineers.

**Acknowledgments** Since the first version of this text, many students made valuable suggestions. Because the number of responses are numerous, I give my thanks to all who contributed by remarks and enhancements to the text. Concerning the historical pictures used in the text, I acknowledge the support of the http://www-gapdcs.st-and.ac.uk/~history/ webserver of the University of St Andrews, Scotland. The author deeply appreciates the understanding and support of his wife, Carin, and daughter, Andrea, during the preparation of the books.

Cairo

Gerd Baumann

*To Carin and Andrea*

# Contents

# 1

# Outline

## 1.1. Introduction

During the years in circles of engineering students the opinion grew that calculus and higher mathematics is a simple collection of recipes to solve standard problems in engineering. Also students believed that a lecturer is responsible to convey these recipes to them in a nice and smooth way so that they can use it as it is done in cooking books. This approach of thinking has the common short coming that with such kind of approach only standard problems are solvable of the same type which do not occur in real applications.

We believe that calculus for engineers offers a great wealth of concepts and methods which are useful in modelling  engineering problems. The reader should be aware that this collection of definitions, theorems, and corollaries is not the final answer of mathematics but a first approach to organize knowledge in a systematic way. The idea is to organize methods and knowledge in a systematic way. This text was compiled with the emphasis on understanding concepts. We think that nearly everybody agrees that this should be the primary goal of calculus instruction.

This first course of Engineering Mathematics will start with the basic foundation of mathematics. The basis are numbers, relations, functions, and properties of functions. This first chapter will give you tools to attack simple engineering problems in different fields. As an engineer you first have to understand the problem you are going to tackle  and after that you will apply mathematical tools to solve the problem. These two steps are common to any kind of problem solving in engineering as well as in science. To understand a problem in engineering needs to be able to use and apply engineering knowledge and engineering procedures. To solve the related mathematical problem needs the knowledge of the basic steps how mathematics is working. Mathematics gives you the frame to handle a problem in a systematic way and use the procedure and knowledge mathematical methods to derive a solution. Since mathematics sets up the frame for the solution of a problem you should be able to use it efficiently. It is not to apply recipes to solve a problem but to use the appropriate concepts to solve it.

Mathematics by itself is for engineers a tool. As for all other engineering applications working with tools you must know how they act and react in applications. The same is true for mathematics. If you know how a mathematical procedure (tool) works and how the components of this tool are connected by each other you will understand its application. Mathematical tools consist as engineering tools of components. Each component is usually divisible into other components until the basic component (elements) are found. The same idea is used in mathematics. There are basic elements from mathematics you should know as an engineer. Combining these basic elements we are able to set up a mathematical frame which incorporates all those elements which are needed to solve a problem. In other words, we use always basic ideas to derive advanced structures. All mathematical thinking follows a simple track which tries to apply fundamental ideas used to handle more complicated situations. If you remember this simple concept you will be able to understand advanced concepts in mathematics as well as in engineering.

## 1.2. Concept of the Text

Concepts and conclusions are collected in definitions and theorems. The theorems are applied in examples to demonstrate their meaning. Every concept in the text is illustrated by examples, and we included more than 1,000 tested exercises for assignments, class work and home work ranging from elementary applications of methods and algorithms to generalizations and extensions of the theory. In addition, we included many applied problems from diverse areas of engineering. The applications chosen demonstrate concisely how basic calculus mathematics can be, and often must be, applied in real life situations.

During the last 25 years a number of symbolic software packages have been developed to provide symbolic mathematical computations on a computer. The standard packages widely used in academic applications are *Mathematica*®, Maple® and Derive®. The last one is a package which is used for basic calculus while the two other programs are able to handle high sophisticated calculations. Both *Mathematica* and Maple have almost the same mathematical functionality and are very useful in symbolic calculations. However the author's preference is *Mathematica* because the experience over the last 25 years showed that *Mathematica*'s concepts are more stable than Maple's one. The author used both of the programs and it turned out during the years that programs written in *Mathematica* some years ago still work with the latest version of *Mathematica* but not with Maple. Therefore the book and its calculations are based on a package which is sustainable for the future.

Having a symbolic computer algebra program available can be very useful in the study of techniques used in calculus. The results in most of our examples and exercises have been generated using problems for which exact values can be determined, since this permits the performance of the calculus method to be monitored. Exact solutions can often be obtained quite easily using symbolic computation. In addition, for many techniques the analysis of a problem requires a high amount of laborious steps, which can be a tedious task and one that is not particularly instructive once the techniques of calculus have been mastered. Derivatives and integrals can be quickly obtained symbolically with computer algebra systems, and a little insight often permits a symbolic computation to aid in understanding the process as well.

We have chosen *Mathematica* as our standard package because of its wide distribution and reliability. Examples and exercises have been added whenever we felt that a computer algebra system would be of significant benefit.

## 1.3. Organization of the Text

The book is organized in chapters which continues to cover the basics of calculus. We examine first series and their basic properties. We use series as a basis for discussing infinite series and the related convergence tests. Sequences are introduced and the relation and conditions for their convergence is examined. This first part of the book completes the calculus elements of the first volume. The next chapter discusses applications of calculus in the field of differential equations. The simplest kind of differential equations are examined and tools for their solution are introduced. Different symbolic solution methods for first order differential equations are discussed and applied. The second part of this volume deals with linear algebra. In this part we discus the basic elements of linear algebra such as vectors and matrices. Operations on these elements are introduced and the properties of a vector space are examined. The main subject of linear algebra is to deal with solutions of linear systems of equations. Strategies for solving linear systems of equations are discussed and applied to engineering applications. After each section there will be a test and exercise subsection divided into two parts. The first part consists of a few test questions which examines the main topics of the previous section. The second part contains exercises related to applications and advanced problems of the material discussed in the previous section. The level of the exercises ranges from simple to advanced.

The whole material is organized in four chapters where the first of this chapter is the current introduction. In Chapter 2 we deal with series and sequences and their convergence. The series and sequences are discussed for the finite and infinite case. In Chapter 3 we examine first order ordinary differential equations. Different solution approaches and classifications of differential equations are discussed. In Chapter 4 we switch to linear algebra. The subsections of this chapter cover vectors, matrices, vector spaces, linear systems of equations, and linear transformations.

## 1.4. Presentation of the Material

Throughout the book we will use the traditional presentation of mathematical terms using symbols, formulas, definitions, theorems, etc. to set up the working frame. This representation is the classical mathematical part. In addition to these traditional presentation tools we will use *Mathematica* as a symbolic, numeric, and graphic tool. *Mathematica* is a computer algebra system (CAS) allowing hybrid calculations. This means calculations on a computer are either symbolic or/and numeric. *Mathematica* is a tool allowing us in addition to write programs and do automatic calculations. Before you use such kind of tool it is important to understand the mathematical concepts which are used by *Mathematica* to derive symbolic or numeric results. The use of *Mathematica* allows you to minimize the calculations but you should be aware that you will only understand the concept if you do your own calculations by pencil and paper. Once you have understood the way how to avoid errors in calculations and concepts you are ready to use the symbolic calculations offered by *Mathematica*. It is important for your understanding that you make errors and derive an improved understanding from these errors. You will never reach a higher level of understanding if you apply the functionality of *Mathematica* as a black box solver of your problems. Therefore I recommend to you first try to understand by using pencil and paper calculations and then switch to the computer algebra system if you have understood the concepts.

You can get a test version of *Mathematica* directly from Wolfram Research by requesting a download address from where you can download the trial version of *Mathematica*. The corresponding web address to get *Mathematica* for free is:

http://www.wolfram.com/products/mathematica/experience/request.cgi

# 2

# Power Series

## 2.1 Introduction

In this chapter we will be concerned with infinite series, which are sums that involve infinitely many terms. Infinite series play a fundamental role in both mathematics and engineering — they are used, for example, to approximate trigonometric functions and logarithms, to solve differential equations, to evaluate difficult integrals, to create new functions, and to construct mathematical models of physical laws. Since it is important to add up infinitely many numbers directly, one goal will be to define exactly what we mean by the sum of infinite series. However, unlike finite sums, it turns out that not all infinite series actually have a finite value or in short a sum, so we will need to develop tools for determining which infinite series have sums and which do not. Once the basic ideas have been developed we will begin to apply our work.

## 2.2 Approximations

In Vol. I Section 3.5 we used a tangent line to the graph of a function to obtain a linear approximation to the function near the point of tangency. In the current section we will see how to improve such local approximations by using polynomials. We conclude the section by obtaining a bound on the error in these approximations.

### 2.2.1 Local Quadratic Approximation

Recall that the local linear approximation of a function $f$ at $x_0$ is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0). \tag{2.1}$$

In this formula, the approximating function

$$p(x) = f(x_0) + f'(x_0)(x - x_0) \tag{2.2}$$

is a first-degree polynomial satisfying $p(x_0) = f(x_0)$ and $p'(x_0) = f'(x_0)$. Thus the local linear approximation of $f$ at $x_0$ has the property that its value and the values of its first derivatives match those of $f$ at $x_0$. This kind of approach will lead us later on to interpolation of data points (see Vol. IV)

If the graph of a function $f$ has a pronounced bend at $x_0$, then we can expect that the accuracy of the local linear approximation of $f$ at $x_0$ will decrease rapidly as we progress away from $x_0$ (Figure 2.1)

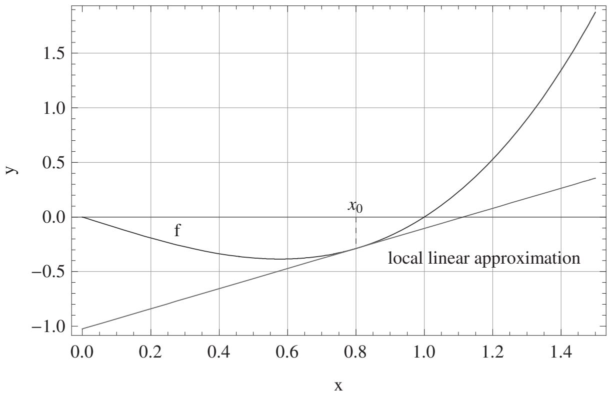


**Figure 2.1.** Graph of the function $f(x) = x^3 - x$ and its linear approximation at $x_0$.▲

One way to deal with this problem is to approximate the function $f$ at $x_0$ by a polynomial of degree 2 with the property that the value of $p$ and the values of its first two derivatives match those of $f$ at $x_0$. This ensures that the graphs of $f$ and $p$ not only have the same tangent line at $x_0$, but they also bend in the same direction at $x_0$. As a result, we can expect that the graph of $p$ will remain close to the graph of $f$ over a larger interval around $x_0$ than the graph of the local linear approximation. Such a polynomial $p$ is called the local quadratic approximation of $f$ at $x = x_0$.

To illustrate this idea, let us try to find a formula for the local quadratic approximation of a function $f$ at $x = 0$. This approximation has the form

$$f(x) \approx c_0 + c_1 x + c_2 x^2 \tag{2.3}$$

which reads in *Mathematica* as

$$\mathbf{eq11 = f(x) == c2\,x^2 + c1\,x + c0}$$

$$f(x) = \mathrm{c2}\,x^2 + \mathrm{c1}\,x + \mathrm{c0}$$

where $c_0$, $c_1$, and $c_2$ must be chosen such that the values of

$$p(x) = c_0 + c_1 x + c_2 x^2 \tag{2.4}$$

and its first two derivatives match those of $f$ at 0. Thus, we want

$$p(0) = f(0),\ \ p'(0) = f'(0),\ \ p''(0) = f''(0). \tag{2.5}$$

In *Mathematica* notation this reads

**eq1 = {p0 == f(0), pp0 == f′(0), ppp0 == f″(0)}; TableForm[eq1]**

p0 == 0
pp0 == −1
ppp0 == 0

where p0, pp0, and ppp0 is used to represent the polynomial, its first and second order derivative at $x = 0$. But the values of $p(0)$, $p'(0)$, and $p''(0)$ are as follows:

**p(x_) := c0 + c1 x + c2 x^2**

**p(x)**

c0 + c1 x + c2 x^2

The determining equation for the term p0 is

**eqh1 = p(0) == p0**

c0 == p0

The first order derivative allows us to find a relation for the second coefficient

**pd = $\frac{\partial p(x)}{\partial x}$**

c1 + 2 c2 x

This relation is valid for $x_0 = 0$ so we replace $x$ by 0 in *Mathematica* notation this is /. x −> 0

**eqh2 = (pd /. x → 0) == pp0**

c1 == pp0

The second order derivative in addition determines the higher order coefficient

**pdd = $\frac{\partial^2 p(x)}{\partial x \partial x}$**

2 c2

**eqh3 = (pdd /. x → 0) == ppp0**

2 c2 == ppp0

Knowing the relations among the coefficients allows us to eliminate the initial conditions for the $p$ coefficients which results to

**sol = Flatten[Solve[Eliminate[Flatten[{eq1, eqh1, eqh2, eqh3}], {p0, pp0, ppp0}], {c0, c1, c2}]]**

$$\left\{c0 \to f(0),\ c1 \to f'(0),\ c2 \to \frac{f''(0)}{2}\right\}$$

and substituting these in the representation of the approximation of the function yields the following formula for the local quadratic approximation of $f$ at $x = 0$.

**eq11 /. sol**

$$f(x) = \frac{1}{2} x^2 f''(0) + x f'(0) + f(0)$$

***Remark 2.1.*** Observe that with $x_0 = 0$. Formula (2.1) becomes $f(x) \approx f(0) + f'(0)\, x$ and hence the linear part of the local quadratic approximation if $f$ at 0 is the local linear approximation of $f$ at 0.

***Example 2.1.*** Approximation

Find the local linear and quadratic approximation of $e^x$ at $x = 0$, and graph $e^x$ and the two approximations together.

***Solution 2.1.*** If we let $f(x) = e^x$, then $f'(x) = f''(x) = e^x$; and hence $f(0) = f'(0) = f''(0) = e^0 = 1$

Thus, the local quadratic approximation of $e^x$ at $x = 0$ is

$$e^x \approx 1 + x + \frac{1}{2} x^2 \tag{2.6}$$

and the actual linear approximation (which is the linear part of the quadratic approximation) is

$$e^x \approx 1 + x. \tag{2.7}$$

The graph of $e^x$ and the two approximations are shown in the following Figure 2.2. As expected, the local quadratic approximation is more accurate than the local linear approximation near $x = 0$.

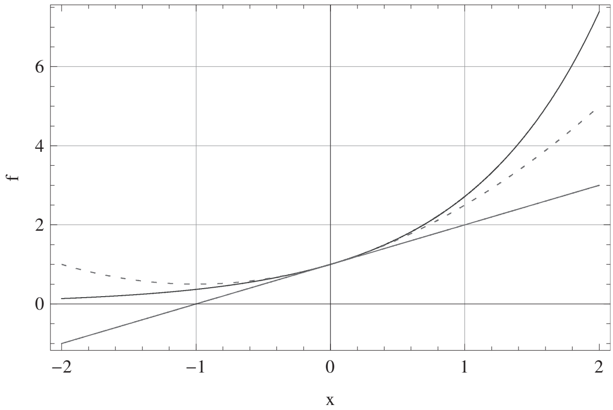


**Figure 2.2.** Linear and quadratic approximation of the function $f(x) = e^x$.The quadratic approximation is plotted as a dashed line.▲

### 2.2.2 Maclaurin Polynomial

It is natural to ask whether one can improve on the accuracy of a local quadratic approximation by using a polynomial of order 3. Specifically, one might look for a polynomial of degree 3 with the property that its value and values of its first three derivatives match those of $f$ at a point; and if this provides an improvement in accuracy, why not go on polynomials of even higher degree? Thus, we are led to consider the following general problem.

Given a function $f$ that can be differentiated $n$ times at $x = x_0$, find a polynomial $p$ of degree $n$ with the property that the value of $p$ and the values of its first $n$ derivatives match those of $f$ at $x_0$.

We will begin by solving this problem in the case where $x_0 = 0$. Thus, we want a polynomial

$$p(x) = c_0 + c_1\, x + c_2\, x^2 + c_3\, x^3 + \ldots + c_n\, x^n \tag{2.8}$$

such that

$$f(0) = p(0),\;\; f'(0) = p'(0),\;\; \ldots,\;\; f^{(n)}(0) = p^{(n)}(0). \tag{2.9}$$

But

$$p(x) = c_0 + c_1\, x + c_2\, x^2 + c_3\, x^3 + \ldots + c_n\, x^n$$

$$p'(x) = c_1 + 2\, c_2\, x + 3\, c_3\, x^2 + \ldots + n\, c_n\, x^{n-1}$$

$$p''(x) = 2\, c_2 + 3 \times 2\, c_3\, x + \ldots + n\,(n-1)\, c_n\, x^{n-2}$$

$$\vdots$$

$$p^{(n)}(x) = n\,(n-1)\,(n-2)\ldots(1)\,c_n$$

Thus, to satisfy (2.9), we must have

$$f(0) = p(0) = c_0$$

$$f'(0) = p'(0) = c_1$$

$$f''(0) = p''(0) = 2\,c_2 = 2!\,c_2$$

$$f'''(0) = p'''(0) = 2\times 3\,c_3 = 3!\,c_3$$

$$\vdots$$

$$f^{(n)}(0) = p^{(n)}(0) = n\,(n-1)\,(n-2)\ldots(1)\,c_n = n!\,c_n$$

which yields the following values for the coefficients of $p(x)$:

$$c_0 = f(0),\;\; c_1 = f'(0),\;\; c_2 = \frac{f''(0)}{2!},\;\ldots,\;\; c_n = \frac{f^{(n)}(0)}{n!}.$$

The polynomial that results by using these coefficients in (2.8) is called the nth Maclaurin polynomial for $f$.

**Definition 2.1.** *Maclaurin Polynomial*

*If $f$ can be differentiated n times at $x = 0$, then we define the nth Maclaurin polynomial for $f$ to be*

$$p_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \ldots + \frac{f^{(n)}(0)}{n!}\,x^n.$$

*The polynomial has the property that its value and the values of its first n derivatives match the values of $f$ and its first n derivatives at $x = 0$.* ■

***Remark 2.2.*** Observe that $p_1(x)$ is the local linear approximation of $f$ at 0 and $p_2(x)$ is the local quadratic approximation of $f$ at $x_0 = 0$.

***Example 2.2.*** Maclaurin Polynomial

Find the Maclaurin polynomials $p_0$, $p_1$, $p_2$, $p_3$, and $p_n$ for $e^x$.

***Solution 2.2.*** For the exponential function, we know that the higher order derivatives are equal to the exponential function. We generate this by the following table

$$\textbf{Table}\left[\frac{\partial^n e^x}{\partial x^n}, \{n, 1, 8\}\right]$$

$$\{e^x, e^x, e^x, e^x, e^x, e^x, e^x, e^x\}$$

and thus the expansion coefficients of the polynomial defined as $f^{(n)}(0)$ follow by replacing $x$ with 0 ( /. $x \to 0$)

$$\textbf{Table}\left[\frac{\partial^n e^x}{\partial x^n}, \{n, 1, 8\}\right] /. \, x \to 0$$

{1, 1, 1, 1, 1, 1, 1, 1}

Therefore, the polynomials of order 0 up to 5 are generated by summation which is defined in the following line

$$p(\text{n\_}, \text{x\_}, \text{f\_}) := \textbf{Fold}\left[\textbf{Plus}, 0, \textbf{Table}\left[\frac{x^m \left(\frac{\partial^m f}{\partial x^m} /. \, x \to 0\right)}{m!}, \{m, 0, n\}\right]\right]$$

The different polynomial approximations are generated in the next line by using this function.

**TableForm[Table[$p(n, x, e^x)$, {$n$, 0, 5}]]**

$\{1\}$

$\{x+1\}$

$\left\{\frac{x^2}{2}+x+1\right\}$

$\left\{\frac{x^3}{6}+\frac{x^2}{2}+x+1\right\}$

$\left\{\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1\right\}$

$\left\{\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1\right\}$

Figure 2.3 shows the graph of $e^x$ and the graphs of the first four Maclaurin polynomials. Note that the graphs of $p_1(x)$, $p_2(x)$, and $p_3(x)$ are virtually indistinguishable from the graph of $e^x$ near $x = 0$, so that these polynomials are good approximations of $e^x$ for $x$ near 0.

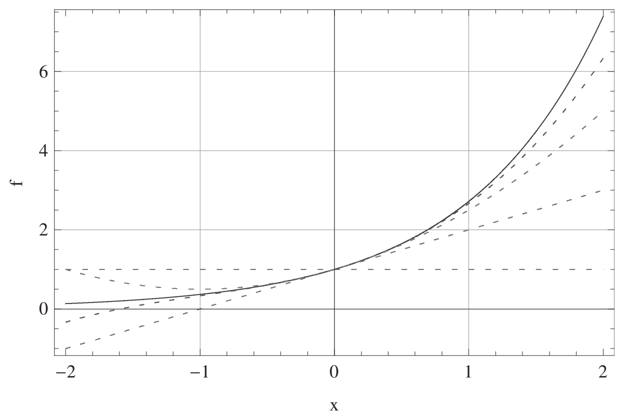


**Figure 2.3.** Maclaurin approximations of the function $f(x) = e^x$.The approximations are shown by dashed curves.▲

However, the farther $x$ is from 0, the poorer these approximations become. This is typical of the Maclaurin polynomials for a function $f(x)$; they provide good approximations of $f(x)$ near 0, but the accuracy diminishes as $x$ progresses away from 0. However, it is usually the case that the higher the degree of the polynomial, the larger the interval on which it provides a specified accuracy.

### 2.2.3 Taylor Polynomial

Up to now we have focused on approximating a function $f$ in the vicinity of $x = 0$. Now we will consider the more general case of approximating $f$ in the vicinity of an arbitrary domain value $x_0$. The basic idea is the same as before; we want to find an nth-degree polynomial $p$ with the property that its first $n$ derivatives match those of $f$ at $x_0$. However, rather than expressing $p(x)$ in powers of $x$, it will simplify the computations if we express it in powers of $x - x_0$; that is,

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots + c_n(x - x_0)^n. \tag{2.10}$$

We will leave it as an exercise for you to imitate the computations used in the case where $x_0 = 0$ to show that

$$c_0 = f(x_0),\ \ c_1 = f'(x_0),\ \ c_2 = \frac{f''(x_0)}{2!},\ \ldots,\ \ c_n = \frac{f^{(n)}(x_0)}{n!}. \tag{2.11}$$

Substituting these values in (2.10), we obtain a polynomial called the nth Taylor polynomial about $x = x_0$ for $f$.

**Definition 2.2.** *Taylor Polynomial*

*If $f$ can be differentiated n times at $x_0$, then we define the nth Taylor polynomial for $f$ about $x = x_0$ to be*

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \blacksquare$$

***Remark 2.3.*** Observe that the Maclaurin polynomials are special cases of the Taylor polynomials; that is, the nth-order Maclaurin polynomial is the nth-order Taylor polynomial about $x_0 = 0$. Observe also that $p_1(x)$ is the local linear approximation of $f$ at $x = x_0$ and $p_2(x)$ is the local quadratic approximation of $f$ at $x = x_0$.

***Example 2.3.*** Taylor Polynomial

Find the five Taylor polynomials for $\ln(x)$ about $x = 2$.

***Solution 2.3.*** In *Mathematica* Taylor series are generated with the function Series[]. Series[] uses the formula given in the definition to derive the approximation. The following line generates a table for the first five Taylor polynomials at $x = 2$

**TableForm[Table[Normal[Series[ln($x$), {$x$, 2, $m$}]], {$m$, 0, 4}]]**

$\ln(2)$

$\frac{x-2}{2} + \ln(2)$

$-\frac{1}{8}(x-2)^2 + \frac{x-2}{2} + \ln(2)$

$\frac{1}{24}(x-2)^3 - \frac{1}{8}(x-2)^2 + \frac{x-2}{2} + \ln(2)$

$-\frac{1}{64}(x-2)^4 + \frac{1}{24}(x-2)^3 - \frac{1}{8}(x-2)^2 + \frac{x-2}{2} + \ln(2)$

The graph of $\ln(x)$ and its first four Taylor polynomials about $x = 2$ are shown in Figure 2.4. As expected these polynomials produce the best approximations to $\ln(x)$ near 2.

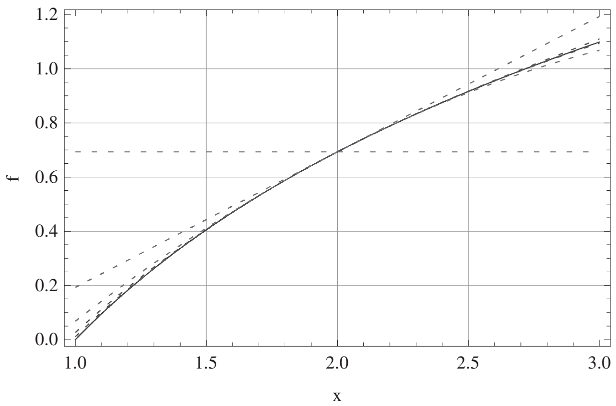


**Figure 2.4.** Taylor approximations of the function $f(x) = \ln(x)$.The approximations are shown by dashed curves.▲

## 2.2.4 Σ-Notation

Frequently, we will want to express the sums in the definitions given in sigma notation. To do this, we use the notation $f^{(k)}(x_0)$ to denote the kth derivative of $f$ at $x = x_0$, and we make the convention that $f^{(0)}(x_0)$ denotes $f(x_0)$. This enables us to write

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \quad (2.12)$$

In particular, we can write the nth-order Maclaurin polynomial for $f(x)$ as

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n. \quad (2.13)$$

***Example 2.4.*** Approximation with Polynomials

Find the nth Maclaurin polynomials for sin($x$), cos($x$) and 1 / (1 – $x$).

***Solution 2.4.*** We know that Maclaurin polynomials are generated by the function

$$\textbf{MaclaurinPolynomial(n\_, f\_, x\_) := Fold}\left[\textbf{Plus, 0, Table}\left[\frac{x^m \left(\frac{\partial^m f}{\partial x^m} \textbf{ /. } x \to 0\right)}{m!}, \{m, 0, n\}\right]\right]$$

A table of Maclaurin polynomials can thus be generated by

**TableForm[Table[MaclaurinPolynomial($m$, sin($x$), $x$), {$m$, 0, 5}]]**

$0$
$x$
$x$
$x - \frac{x^3}{6}$
$x - \frac{x^3}{6}$
$\frac{x^5}{120} - \frac{x^3}{6} + x$

In the Maclaurin polynomials for sin($x$), only the odd powers of $x$ appear explicitly. Due to the zero terms, each even-order Maclaurin polynomial is the same as the preceding odd-order Maclaurin polynomial. That is

$$p_{2k+1}(x) = p_{2k+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad k = 0, 1, 2, 3, \ldots. \tag{2.14}$$

The graph and the related Maclaurin approximations are shown in Figure 2.5

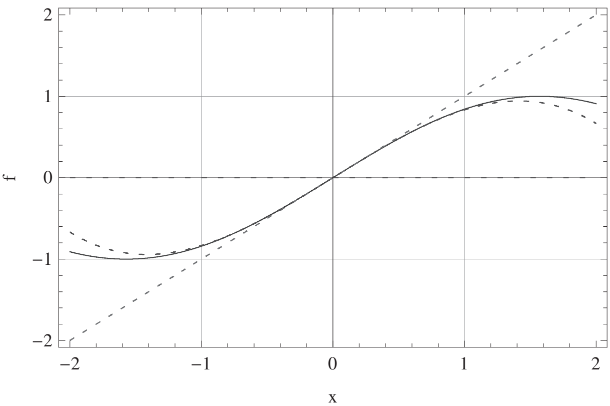


**Figure 2.5.** Maclaurin approximations of the function $f(x) = \sin(x)$.The approximations are shown by dashed curves.

In the Maclaurin polynomial for cos($x$), only the even powers of $x$ appear explicitly; the computation is similar to those for the sin($x$). The reader should be able to show the following:

**TableForm[Table[MaclaurinPolynomial($m$, cos($x$), $x$), {$m$, 0, 5}]]**

$1$

$1$

$1-\frac{x^2}{2}$

$1-\frac{x^2}{2}$

$\frac{x^4}{24}-\frac{x^2}{2}+1$

$\frac{x^4}{24}-\frac{x^2}{2}+1$

The graph of $\cos(x)$ and the approximations are shown in Figure 2.6.

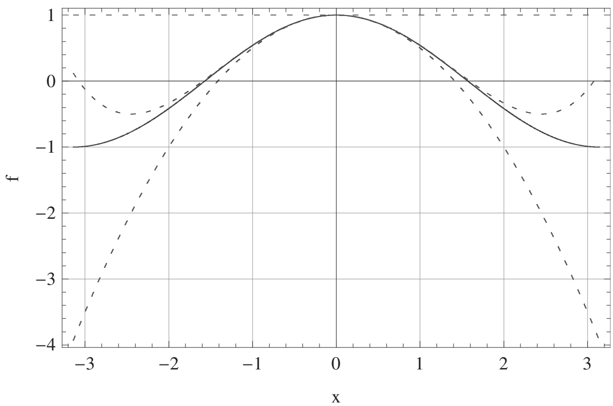


**Figure 2.6.** Maclaurin approximations of the function $f(x) = \cos(x)$.The approximations are shown by dashed curves.

Let $f(x) = 1/(1-x)$. The values of $f$ and its derivatives at $x = 0$ are collected in the results

**TableForm$\left[\text{Table}\left[\text{MaclaurinPolynomial}\left(m, \frac{1}{1-x}, x\right), \{m, 0, 5\}\right]\right]$**

$1$

$x+1$

$x^2+x+1$

$x^3+x^2+x+1$

$x^4+x^3+x^2+x+1$

$x^5+x^4+x^3+x^2+x+1$

From this sequence the Maclaurin polynomial for $1/(1-x)$ is

$$p_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n, \qquad n = 0, 1, 2, \ldots . \blacktriangle$$

***Example 2.5.*** Approximation with Taylor Polynomials

Find the nth Taylor polynomial for $1/x$ about $x = 1$.

***Solution 2.5.*** Let $f(x) = 1/x$. The computations are similar to those done in the last example. The results are

$$\mathbf{Series}\left[\frac{\mathbf{1}}{\boldsymbol{x}}, \{\boldsymbol{x}, \mathbf{1}, \mathbf{5}\}\right]$$

$$1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5 + O\left((x-1)^6\right)$$

This relation suggest the general formula

$$\sum_{k=0}^{n} (-1)^k (x-1)^k = 1 - (x-1) + (x-1)^2 + \ldots + (-1)^n (x-1)^n . \blacktriangle$$

### 2.2.5 $n^{th}$-Remainder

The nth Taylor polynomial $p_n$ for a function $f$ about $x = x_0$ has been introduced as a tool to obtain good approximations to values of $f(x)$ for $x$ near $x_0$. We now develop a method to forecast how good these approximations will be.

It is convenient to develop a notation for the error in using $p_n(x)$ to approximate $f(x)$, so we define $R_n(x)$ to be the difference between $f(x)$ and its nth Taylor polynomial. That is

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k. \tag{2.15}$$

This can also be written as

$$f(x) = p_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_n(x) \tag{2.16}$$

which is called Taylor's formula with remainder.

Finding a bound for $R_n(x)$ gives an indication of the accuracy of the approximation $p_n(x) \approx f(x)$. The following Theorem 2.1 given without proof states

**Theorem 2.1.** *Remainder Estimation*

*If the function $f$ can be differentiated $n+1$ times on an interval $I$ containing the number $x_0$, and if $M$ is an upper bound for $\left|f^{(n+1)}(x)\right|$ on $I$, that is $\left|f^{(n+1)}(x)\right| \le M$ for all $x$ in $I$, then*

$$\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\left|x - x_0\right|^{n+1}$$

*for all x in I.*■

***Example 2.6.*** Remainder of an Approximation

Use the nth Maclaurin polynomial for $e^x$ to approximate $e$ to five decimal-place accuracy.

***Solution 2.6.*** We note first that the exponential function $e^x$ has derivatives of all orders for every real number $x$. The Maclaurin polynomial is

$$\sum_{k=0}^{n} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

from which we have

$$e = e^1 \approx \sum_{k=0}^{n} \frac{1^k}{k!} = 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}.$$

Thus, our problem is to determine how many terms to include in a Maclaurin polynomial for $e^x$ to achieve five decimal-place accuracy; that is, we want to choose $n$ so that the absolute value of nth remainder at $x = 1$ satisfies

$$|R_n(1)| \leq 0.000005.$$

To determine $n$ we use the Remainder Estimation Theorem 2.1 with $f(x) = e^x$, $x = 1$, $x_0 = 0$, and $I$ being the interval [0,1]. In this case it follows from the Theorem that

$$\left|R_n(1)\right| \leq \frac{M}{(n+1)!} \tag{2.17}$$

where $M$ is an upper bound on the value of $f^{(n+1)}(x) = e^x$ for $x$ in the interval [0, 1]. However, $e^x$ is an increasing function, so its maximum value on the interval [0, 1] occurs at $x = 1$; that is, $e^x \leq e$ on this interval. Thus, we can take $M = e$ in (2.17) to obtain

$$\left|R_n(1)\right| \leq \frac{e}{(n+1)!}. \tag{2.18}$$

Unfortunately, this inequality is not very useful because it involves $e$ which is the very quantity we are trying to approximate. However, if we accept that $e < 3$, then we can replace (2.18) with the following less precise, but more easily applied, inequality:

$$\left|R_n(1)\right| \leq \frac{3}{(n+1)!}. \tag{2.19}$$

Thus, we can achieve five decimal-place accuracy by choosing $n$ such that

$$\frac{3}{(n+1)!} \leq 0.000005 \qquad \text{or} \qquad (n+1)! \geq 600\,000.$$

Since 9! = 362880 and 10!=3628800, the smallest value of $n$ that meets this criterion is $n$ = 9. Thus, by five decimal-place accuracy

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \approx 2.71828$$

As a check, a calculator's twelve-digits representation of $e$ is $e \approx 2.71828182846$, which agrees with the preceding approximation when rounded to five decimal places.▲

## 2.2.6 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

### 2.2.6.1 Test Problems

**T1.** Describe the term approximation in mathematical terms.

**T2.** What kind of approximation polynomials do you know?

**T3.** How are Maclaurin polynomials defined?

**T4.** Describe the difference between Maclaurin and Taylor polynomials.

**T5.** How can we estimate the error in an approximation?

**T6.** What do we mean by truncation error?

### 2.2.6.2 Exercises

**E1.** Find the Maclaurin polynomials up to degree 6 for $f(x) = \cos(x)$. Graph $f$ and these polynomials on a common screen. Evaluate $f$ and these polynomials at $x = \pi/4$, $\pi/2$, and $\pi$. Comment on how the Maclaurin polynomials converge to $f$(x).

**E2.** Find the Taylor polynomials up to degree 3 for $f(x) = 1/x$. Graph $f$ and these polynomials on a common screen. Evaluate $f$ and these polynomials at $x = 0.8$ and $x = 1.4$. Comment on how the Taylor polynomials converge to $f(x)$.

**E3.** Find the Taylor polynomial $T_n(x)$ for the function $f$ at the number $x_0$. Graph $f$ and $T_3(x)$ on the same screen for the following functions:

- **a.** $f(x) = x + e^{-x}$, at $x_0 = 0$,
- **b.** $f(x) = 1/x$, at $x_0 = 2$,
- **c.** $f(x) = e^{-x}\cos(x)$, at $x_0 = 0$,
- **d.** $f(x) = \arcsin(x)$, at $x_0 = 0$,
- **e.** $f(x) = \ln(x)/x$, at $x_0 = 1$,
- **f.** $f(x) = x\, e^{-x^2}$, at $x_0 = 0$,
- **g.** $f(x) = \sin(x)$, at $x_0 = \pi/2$.

**E4.** Use a computer algebra system to find the Taylor polynomials $T_n(x)$ centered at $x_0$ for $n$ = 2, 3, 5, 6. Then graph these polynomials and on the same screen the following functions:

- **a.** $f(x) = \cot(x)$, at $x_0 = \pi/4$,
- **b.** $f(x) = \sqrt[3]{1 + x^2}$ , at $x_0 = 0$,
- **c.** $f(x) = 1/\cosh(x)^2$, at $x_0 = 0$.

Approximate $f$ by a Taylor polynomial with degree $n$ at the number $x_0$. Use Taylor's Remainder formula to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when $x$ lies in the given interval. Check your result by graphing $|R_n(x)|$.

**a.** $f(x) = x^{-2}$, $x_0 = 1$, $n = 2$, $0.9 \leq x \leq 1.1$,

**b.** $f(x) = \sqrt{x}$, $x_0 = 4$, $n = 2$, $4.1 \leq x \leq 4.3$,

**c.** $f(x) = \sin(x)$, $x_0 = \pi/3$, $n = 3$, $0 \leq x \leq \pi$,

**d.** $f(x) = e^{-x^2}$, $x_0 = 0$, $n = 3$, $-1 \leq x \leq 1$,

**e.** $f(x) = \ln(1 + x^2)$, $x_0 = 1$, $n = 3$, $0.5 \leq x \leq 1.5$,

**f.** $f(x) = x \sinh(4x)$, $x_0 = 2$, $n = 4$, $1 \leq x \leq 2.5$.

**E6.** Use Taylor's Remainder formula to determine the number of terms of the Maclaurin series for $e^x$ that should be used to estimate $e^{0.1}$ to within 0.000001.

**E7.** Let $f$ (x) have derivatives through order $n$ at $x = x_0$. Show that theTaylor polynomial of order $n$ and its first $n$ derivatives have the same values that $f$ and its first $n$ derivatives have at $x = x_0$.

**E8.** For approximately what values of $x$ can you replace $\sin(x)$ by $x - (x^3/6)$ with an error of magnitude no greater than $5 \times 10^{-4}$? Give reasons for your answer.

**E9.** Show that if the graph of a twice-differentiable function $f(x)$ has an inflection point at $x = x_0$ then the linearization of $f$ at $x = x_0$ is also the quadratic approximation of $f$ at $x = x_0$. This explains why tangent lines fit so well at inflection points.

**E10** Graph a curve $y = 1/3 - x^2/5$ and $y = (x - \arctan(x))/x^3$ together with the line $y = 1/3$. Use a Taylor series to explain what you see. What is

$$\lim_{x\to 0} \frac{x - \arctan(x)}{x^3} \tag{1}$$

## 2.3 Sequences

In everyday language, the term sequence means a succession of things in a definite order — chronological order, size order, or logical order, for example. In mathematics, the term sequence is commonly used to denote a succession of numbers whose order is determined by a rule or a function. In this section, we will develop some of the basic ideas concerning sequences of numbers.

### 2.3.1 Definition of a Sequence

An infinite sequence, or more simply a sequence, is an unending succession of numbers, called terms. It is understood that the terms have a definite order; that is, there is a first term $a_1$, a second term $a_2$, a third term $a_3$, a fourth term $a_4$, and so forth. Such a sequence would typically be written as

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots$$

where the dots are used to indicate that the sequence continues indefinitely. Some specific examples are

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$$

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \ldots$$

$$2,\ 4,\ 6,\ 8,\ \ldots$$

$$1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \ldots.$$

Each of these sequences has a definite pattern that makes it easy to generate additional terms if we assume that those terms follow the same pattern as the displayed terms. However, such patterns can be deceiving, so it is better to have a rule of formula for generating the terms. One way of doing this is to look for a function that relates each term in the sequence to its term number. For example, in the sequence

$$2,\ 4,\ 6,\ 8,\ \ldots$$

each term is twice the term number; that is, the nth term in the sequence is given by the formula $2\,n$. We denote this by writing the sequence as

$$2,\ 4,\ 6,\ 8,\ \ldots,\ 2\,n,\ \ldots.$$

We call the function $f(n) = 2\,n$ the general term of this sequence. Now, if we want to know a specific term in the sequence, we just need to substitute its term number into the formula for the general term. For example, the 38th term in the sequence is $2 \cdot 38 = 76$.

***Example 2.7.*** Sequences

In each part find the general term of the sequence.

$a)\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots$

$b)\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots$

$c)\ \frac{1}{2},\ -\frac{2}{3},\ \frac{3}{4},\ -\frac{4}{5},\ \ldots$

$d)\ 1,\ 3,\ 5,\ 7,\ \ldots$

***Solution 2.7.*** a) To find the general formula of the sequence, we create a table containing the term numbers and the sequence terms themselves.

| n | f(n) |
|---|---|
| 1 | $\frac{1}{2}$ |
| 2 | $\frac{2}{3}$ |
| 3 | $\frac{3}{4}$ |
| 4 | $\frac{4}{5}$ |

On the left we start counting the term number and on the right there are the results of the counting. We see that the numerator is the same as the term number and denominator is one greater than the term number. This suggests that the $n$th term has numerator $n$ and denominator $n+1$. Thus the sequence can be generated by the following general expression

$$f(\mathrm{n\_}) := \frac{n}{n+1}$$

We introduce this *Mathematica* expression for further use. The application of this function to a sequence of numbers shows the equivalence of the sequences

**Table[$f(n)$, {$n$, 1, 6}]**

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$$

b) The same procedure is used to find the general term expression for the sequence

| n | f(n) |
|---|---|
| 1 | $\frac{1}{2}$ |
| 2 | $\frac{1}{4}$ |
| 3 | $\frac{1}{8}$ |
| 4 | $\frac{1}{16}$ |

Here the numerator is always the same and equals to 1. The denominator for the four known terms can be expressed as powers of 2. From the table we see that the exponent in the denominator is the same as the term number. This suggests that the denominator of the $n$th term is $2^n$. Thus the sequence can be expressed by

$$f(\mathrm{n\_}) := \frac{1}{2^n}$$

The generation of a table shows the agreement of the first four terms with the given numbers

**Table[$f(n)$, {$n$, 1, 6}]**

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$$

c) This sequence is identical to that in part a), except for the alternating signs.

| n | f(n) |
|---|---|
| 1 | $\frac{1}{2}$ |
| 2 | $-\frac{2}{3}$ |
| 3 | $\frac{3}{4}$ |
| 4 | $-\frac{4}{5}$ |

Thus the $n$th term in the sequence can be obtained by multiplying the $n$th term in part a) by $(-1)^{n+1}$. This factor produces the correct alternating signs, since its successive values, starting with $n = 1$ are $1, -1, 1, -1, \ldots$. Thus, the sequence can be written as

$$\boldsymbol{f(\mathrm{n\_}) := \frac{(-1)^{n+1}\, n}{n+1}}$$

and the verification shows agreement within the given numbers

**Table[*f*(*n*), {*n*, 1, 6}]**

$$\left\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}\right\}$$

d) For this sequence, we have the table

| n | f(n) |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 7 |

from which we see that each term is one less than twice its term number. This suggests that the $n$th term in the sequence is $2\,n - 1$. Thus we can generate the sequence by

$$\boldsymbol{f(\mathrm{n\_}) := 2\,n - 1}$$

and show the congruence by

**Table[*f*(*n*), {*n*, 1, 6}]**

{1, 3, 5, 7, 9, 11}

▲

When the general term of a sequence

$$a_1,\, a_2,\, a_3,\, \ldots,\, a_n,\, \ldots \tag{2.20}$$

is known, there is no need to write out the initial terms, and it is common to write only the general term enclosed in braces. Thus (2.20) might be written as

$$\{a_n\}_{n=1}^{+\infty}. \tag{2.21}$$

For example here are the four sequences from Example 2.7 expressed in brace notation and the corresponding results. Fo the first sequence we write

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{6}$$

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$$

Here we introduced a definition in *Mathematica* (s. Appendix) which allows us to use the same notation as introduced in (2.21). For the second sequence we write

$$\left\{\frac{1}{2^n}\right\}_{n=1}^{6}$$

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$$

The third sequence is generated by

$$\left\{\frac{(-1)^{n+1}\, n}{n+1}\right\}_{n=1}^{6}$$

$$\left\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}\right\}$$

and finally the last sequence of Example 2.7 is derived by

$$\{2\, n-1\}_{n=1}^{6}$$

$$\{1, 3, 5, 7, 9, 11\}$$

The letter $n$ in (2.21) is called the index of the sequence. It is not essential to use $n$ for the index; any letter not reserved for another purpose can be used. For example, we might view the general term of the sequence $a_1$, $a_2$, $a_3$ … to be the $k$th term, in which case we would denote this sequence as $\{a_k\}_{k=1}^{+\infty}$. Moreover, it is not essential to start the index at 1; sometimes it is more convenient to start at 0. For example, consider the sequence

$$1,\ \frac{1}{2},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \ldots$$

One way to write this sequence is

$$\left\{\frac{\mathbf{1}}{\mathbf{2^{n-1}}}\right\}_{n=1}^{5}$$

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\right\}$$

However, the general term will be simpler if we think of the initial term in the sequence as the zeroth term, in which case we write the sequence as

$$\left\{\frac{\mathbf{1}}{\mathbf{2^{n}}}\right\}_{n=0}^{6}$$

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$$

***Remark 2.4.*** In general discussions that involve sequences in which the specific terms and the starting point for the index are not important, it is common to write $\{a_n\}$ rather than $\{a_n\}_{n=1}^{\infty}$. Moreover, we can distinguish between different sequences by using different letters for their general terms; thus $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ denote three different sequences.

We began this section by describing a sequence as an unending succession of numbers. Although this conveys the general idea, it is not a satisfactory mathematical definition because it relies on the term *succession*, which is itself an undefined term. To motivate a precise definition, consider the sequence

$$2,\ 4,\ 6,\ 8,\ \ldots,\ 2\,n,\ \ldots$$

If we denote the general term by $f(n) = 2\,n$, then we can write this sequence as

$$f(1),\ f(2),\ f(3),\ \ldots,\ f(n),\ \ldots.$$

which is a list of values of the function

$$f(n) = 2\,n \qquad n = 1,\ 2,\ 3,\ \ldots.$$

whose domain is the set of positive integer. This suggests the following Definition 2.3.

**Definition 2.3.** *Sequence*

*A sequence is a function whose domain is a set of integers. Specifically, we will regard the expression $\{a_n\}_{n=1}^{\infty}$ to be an alternative notation for the function $f(n) = a_n$, $n = 1, 2, 3, \ldots$.* ■

### 2.3.2 Graphs of a Sequence

Since sequences are functions, it makes sense to talk about graphs of a sequence. For example, the graph of a sequence $\{1/n\}_{n=1}^{\infty}$ is the graph of the equation

$$y = \frac{1}{n} \qquad \text{for } n = 1, 2, 3, 4, \ldots .$$

Because the right side of this equation is defined only for positive integer values of $n$, the graph consists of a succession of isolated points (Figure 2.7)

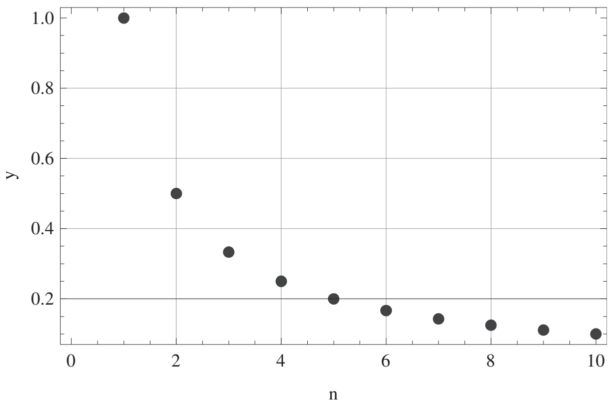


**Figure 2.7.** Graph of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{10}$.

This graph is resembling us to the graph of

$$y = \frac{1}{x} \qquad \text{for } x > 1$$

which is a continuous curve in the $(x, y)$-plane.

### 2.3.3 Limit of a Sequence

Since Sequences are functions, we can inquire about their limits. However, because a sequence $\{a_n\}$ is only defined for integer values of $n$, the only limit that makes sense is the limit of $a_n$ as $n \to \infty$. In Figure 2.8 we have shown the graph of four sequences, each of which behave differently as $n \to \infty$

- The terms in the sequence $\{n+1\}$ increases without bound.
- The terms in the sequence $\left\{(-1)^{n+1}\right\}$ oscillate between -1 and 1.
- The terms in the sequence $\left\{\frac{n}{n+1}\right\}$ increases toward a limiting value of 1.
- The terms in the sequence $\left\{1 + \left(-\frac{1}{2}\right)^n\right\}$ also tend toward a limiting value of 1, but do so in an oscillatory fashion.

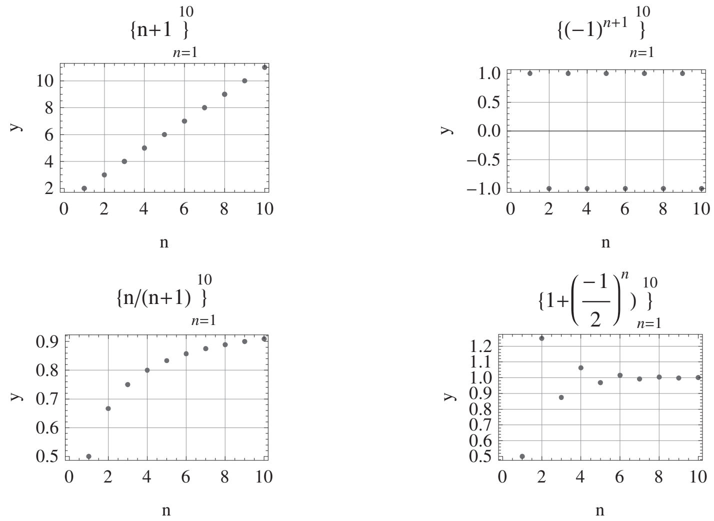


**Figure 2.8.** Graph of four different sequences.

Informally speaking, the limit of a sequence $\{a_n\}$ is intended to describe how $a_n$ behaves as $n \to \infty$. To be more specific, we will say that a sequence $\{a_n\}$ approaches a limit $L$ if the terms in the sequence eventually become arbitrarily close to $L$. Geometrically, this means that for any positive number $\epsilon$ there is a point in the sequence after which all terms lie between the lines $y = L - \epsilon$ and $y = L + \epsilon$

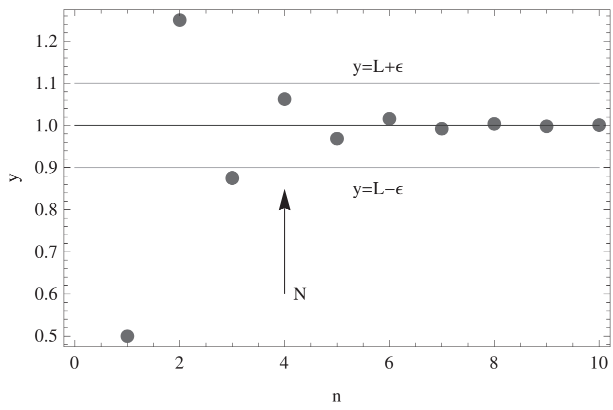


**Figure 2.9.** Limit process of a sequence.

The following Definition 2.4 makes these ideas precise.

**Definition 2.4.** *Limit of a Sequence*

*A sequence $\{a_n\}$ is said to converge to the limit $L$ if given any $\epsilon > 0$, there is a positive integer $N$ such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write*

$$\lim_{n\to\infty} a_n = L.$$

*A sequence that does not converge to some finite limit is said to diverge.* ■

***Example 2.8.*** Limit of a Sequence

The first two sequences in Figure 2.8 diverge, and the second two converge to 1; that is

$$\lim_{n\to\infty} \frac{n}{n+1}$$

```
1
```

and

$$\lim_{n\to\infty} \left(\left(-\frac{1}{2}\right)^n + 1\right)$$

```
1
```

The following Theorem 2.2, which we state without proof, shows that the familiar properties of limits apply to sequences. This theorem ensures that the algebraic techniques used to find limits of the form $\lim_{x\to\infty}$ can be used for limits of the form $\lim_{n\to\infty}$.

**Theorem 2.2.** *Rules for Limits of Sequences*

*Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to the limits $L_1$ and $L_2$, respectively, and $c$ is a constant. Then*

*a*) $\lim_{n\to\infty} c = c$

*b*) $\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n = c\, L_1$

*c*) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L_1 + L_2$

*d*) $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L_1 - L_2$

*e*) $\lim_{n\to\infty} (a_n\, b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n = L_1\, L_2$

*f*) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L_1}{L_2}$ if $L_2 \neq 0$. ■

***Example 2.9.*** Calculating Limits of Sequences

In each part, determine whether the sequence converges or diverges. If it converges, find the limit

$a)\left\{\frac{n}{2\,n+1}\right\}_{n=1}^{\infty},$

$b)\left\{(-1)^{n}\,\frac{1}{n}\right\}_{n=1}^{\infty},$

$c)\,\{8-2\,n\}_{n=1}^{\infty}.$

***Solution 2.9.*** a) Dividing numerator and denominator by $n$ yields

$$\lim_{n\to\infty}\frac{n}{2\,n+1}=\lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}=\frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}2+\lim_{n\to\infty}\frac{1}{n}}=\frac{1}{2+0}=\frac{1}{2}$$

which agrees with the calculation done by *Mathematica*

$$\lim_{n\to\infty}\frac{n}{2\,n+1}$$

$$\frac{1}{2}$$

b) Since the $\lim_{n\to\infty}\frac{1}{n}=0$, the product $(-1)^{n+1}\,(1/n)$ oscillates between positive and negative values, with the odd-numbered terms approaching 0 through positive values and the even-numbered terms approaching 0 through negative values. Thus,

$$\lim_{n\to\infty}(-1)^{n+1}\,\frac{1}{n}=0$$

which is also derived via

$$\lim_{n\to\infty}\frac{(-1)^{n+1}}{n}$$

$$0$$

so the sequence converges to 0.

c) $\lim_{n\to\infty}(8-2\,n)=-\infty$, so the sequence $\{8-2\,n\}_{n=1}^{\infty}$ diverges.▲

If the general term of a sequence is $f(n)$, and if we replace $n$ by $x$, where $x$ can vary over the entire interval $[1,\infty)$, then the values of $f(n)$ can be viewed as sample values of $f(x)$ taken at the positive integers. Thus, if $f(x)\to L$ as $x\to\infty$, then it must also be true that $f(n)\to L$ as $n\to\infty$ (see Figure 2.10). However, the converse is not true; that is, one cannot infer that $f(x)\to L$ as $x\to\infty$ from the fact that $f(n)\to L$ as $n\to\infty$ (see Figure 2.11).

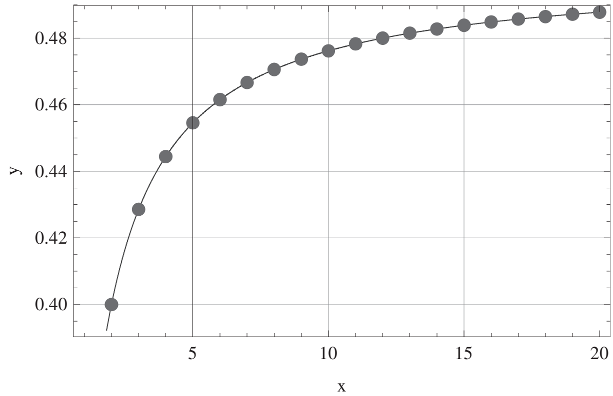


**Figure 2.10.** Replacement of a sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{20}$ by a function $f(x) = \frac{x}{2x+1}$.

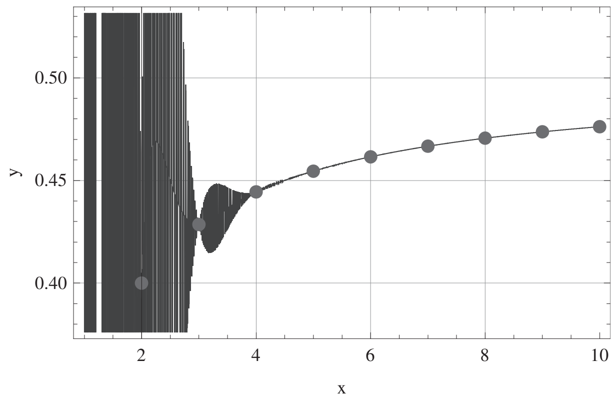


**Figure 2.11.** Replacement of a function $f(x) = \frac{x}{2x+1}$ by a sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{10}$.

***Example 2.10.*** L'Hopital's Rule

Find the limit of the sequence $\{n/e^n\}_{n=1}^{\infty}$.

***Solution 2.10.*** The expression $n/e^n$ is an indeterminate form of type $\infty/\infty$ as $n \to \infty$, so L'Hopital's rule is indicated. However, we cannot apply this rule directly to $n/e^n$ because the functions $n$ and $e^n$ have been defined here only at the positive integers, and hence are not differentiable functions. To circumvent this problem we extend the domains of these functions to all real numbers, here implied by replacing $n$ by $x$, and apply L'Hopital's rule to the limit of the quotient $x/e^x$. This yields

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

from which we can conclude that

$$\lim_{n\to\infty} \frac{n}{e^n} = 0$$

which can be verified by

$$\lim_{n\to\infty} \frac{n}{e^n}$$

0

▲

Sometimes the even-numbered and odd-numbered terms of a sequence behave sufficiently differently; so it is desirable to investigate their convergence separately. The following Theorem 2.3, whose proof is omitted, is helpful for that purpose.

**Theorem 2.3.** *Even and Odd Sequences*

*A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L.*■

***Example 2.11.*** Even and Odd Sequences

The sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \ldots$$

converges to 0, since the odd-numbered terms and the even-numbered terms both converge to 0, and the sequence

$$1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \ldots$$

diverges, since the odd-numbered terms converge to 1 and the even-numbered terms converge to 0.▲

### 2.3.4 Squeezing of a Sequence

The following Theorem 2.4, which we state without proof, is an adoption of the Squeezing Theorem to sequences. This theorem will be useful for finding limits of sequences that cannot be obtained directly.

**Theorem 2.4.** *The Squeezing Theorem for Sequences*

*Let* $\{a_n\}$, $\{b_n\}$, *and* $\{c_n\}$ *be sequences such that*

$$a_n \le b_n \le c_n \qquad \text{for all values of } n \text{ beyond some index } N.$$

*If the sequence* $\{a_n\}$ *and* $\{c_n\}$ *have a common limit* $L$ *as* $n \to \infty$*, then* $\{b_n\}$ *also has the limit* $L$ *as* $n \to \infty$. ■

***Example 2.12.*** Squeezing

Use numerical evidence to make a conjecture about the limit of the sequence

$$\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$$

and then confirm that your conjecture is correct.

***Solution 2.12.*** The following table shows the sequence for $n$ from 1 to 12.

**TableForm**$\left[N\left[\textbf{Table}\left[\frac{n!}{n^n}, \{n, 1, 12\}\right]\right]\right]$

1.
0.5
0.222222
0.09375
0.0384
0.0154321
0.0061199
0.00240326
0.000936657
0.00036288
0.000139906
0.0000537232

The values suggest that the limit of the sequence may be 0. To confirm this we have to examine the limit of

$$a_n = \frac{n!}{n^n} \qquad \text{as } n \to \infty.$$

Although this is an indeterminate form of type $\infty/\infty$, L'Hopital's rule is not helpful because we have no definition of $x!$ for values of $x$ that are not integers. However, let us write out some of the initial terms and the general term in the sequence

$$a_1 = 1, \qquad a_2 = \frac{1\times 2}{2\times 2}, \qquad a_3 = \frac{1\times 2\times 3}{3\times 3\times 3},$$

$$\ldots, \qquad a_n = \frac{1\times 2\times 3\times \ldots\times n}{n\times n\times n\times \ldots\times n}, \ \ldots.$$

We can rewrite the general term as

$$a_n = \frac{1}{n}\,\frac{(2\times 3\times \ldots \text{n})}{n\times n\times \ldots \times n}$$

from which it is evident that

$$0 \le a_n \le \frac{1}{n}.$$

However, the two outside expressions have a limit of 0 as $n \to \infty$; thus, the Squeezing Theorem 2.4 for sequences implies that $a_n \to 0$ as $n \to \infty$, which confirms our conjecture.▲

The following Theorem 2.5 is often useful for finding the limit of a sequence with both positive and negative terms—it states that the sequence $\{\,|\,a_n\,|\}$ that is obtained by taking the absolute value of each term in the sequence $\{a_n\}$ converges to 0, then $\{a_n\}$ also converges to 0.

**Theorem 2.5.** *Magnitude Sequences*

*If* $\lim_{n\to\infty} |\,a_n\,| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.■

***Example 2.13.*** Magnitude Sequence

Consider the sequence

$$1,\ -\frac{1}{2},\ \frac{1}{2^2},\ -\frac{1}{2^3},\ \ldots,\ (-1)^n\,\frac{1}{2^n},\ \ldots.$$

If we take the absolute value of each term, we obtain the sequence

$$1,\ \frac{1}{2},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \ldots,\ \frac{1}{2^n},\ \ldots.$$

which converges to 0. Thus from Theorem 2.5 we have

$$\lim_{n\to\infty}\left((-1)^n\,\frac{1}{2^n}\right) = 0.\ ▲$$

### 2.3.5 Recursion of a Sequence

Some sequences did not arise from a formula for the general term, but rather form a formula or set of formulas that specify how to generate each term in the sequence from terms that precede it; such sequences are said to be defined recursively, and the defining formulas are called recursion formulas. A good example is the mechanic's rule of approximating square roots. This can be done by

$$x_1 = 1, \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \tag{2.22}$$

describing the sequence produced by Newton's Method to approximate $\sqrt{a}$ as a root of the function $f(x) = x^2 - a$. The following table shows the first few terms of the approximation of $\sqrt{2}$

```
tab1 = N[Transpose[{{i}_{i=1}^{9}, NestList[1/2 (#1 + 2/#1) &, 1, 8]}]];
(TableForm[#1, TableHeadings → {{}, {"n", "x_n"}}] &)[N[tab1]]
```

| n | $x_n$ |
|---|---|
| 1. | 1. |
| 2. | 1.5 |
| 3. | 1.41667 |
| 4. | 1.41422 |
| 5. | 1.41421 |
| 6. | 1.41421 |
| 7. | 1.41421 |
| 8. | 1.41421 |
| 9. | 1.41421 |

It would take us to far afield to investigate the convergence of sequences defined recursively. Thus we let this subject as a challenge for the reader. However, we note that sequences are important in numerical procedures like Newton's root finding method (see Vol. IV).

## 2.3.6 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

### 2.3.6.1 Test Problems

**T1.** What is an infinite sequence? What does it mean for such a sequence to converge? To diverge? Give examples.

**T2.** What theorems are available for calculating limits of sequences? Give examples.

**T3.** What is a non decreasing sequence? Under what circumstances does such a sequence have a limit? Give examples.

**T4.** What is the limit of a sequence? Give examples.

**T5.** How does squeezing work with sequences? Give examples.

### 2.3.6.2 Exercises

**E1.** Each of the following examples gives a formula for the $n$th term $a_n$ of a sequence $\{a_n\}$. Find the values of $a_1$, $a_3$, $a_6$.

**a.** $a_n = \frac{1}{n!}$,

**b.** $a_n = \frac{2^n - 1}{2^n}$,

**c.** $a_n = \frac{2^n}{2^{n+1}}$,

**d.** $a_n = \frac{(-1)^{2n+1}}{3n-1}$.

**E2.** List the first five terms of the sequence.

**a.** $a_n = 1 - \left(\frac{2}{10}\right)^n$,

**b.** $a_n = 5\,\frac{(-1)^n}{n!}$,

$a_n = \frac{n+1}{5\,n-2}$,

**d.** $\{2, 4, 6, \ldots, (2\,n)\}$,

**e.** $a_1 = -1$, $a_{n+1} = a_n(2 - 2^n)$.

**E3.** List the first six terms of the sequence defined by

$$a_n = \frac{n}{3\,n+1}. \tag{1}$$

Does the sequence appear to have a limit? If so, find it.

**E4.** Each of the following examples gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

**a.** $a_1 = 1$, $a_{n+1} = a_n + 2^{-n}$,

**b.** $a_1 = 2$, $a_{n+1} = a_n/(2\,n+1)$,

**c.** $a_1 = -2$, $a_{n+1} = (-1)^{n+2}\,a_n$,

**d.** $a_1 = 1$, $a_{n+1} = n\,a_n - n^2\,a_n$,

**e.** $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} - a_n$,

**f.** $a_1 = 1$, $a_2 = -1$, $a_{n+2} = a_n + 2^{-n}\,a_{n+1}$.

**E5.** Find a formula for the $n$th term of the sequences:

**a.** $\{1, -1, 1, -1, 1, -1, \ldots\}$,

**b.** $\{-1, 1, -1, 1, -1, 1, \ldots\}$,

**c.** $\{1, -4, 9, -16, 25, \ldots\}$,

**d.** $\left\{1, \frac{-1}{4}, \frac{1}{9}, \frac{-1}{16}, \frac{1}{25}, \ldots\right\}$,

**e.** $\{1, 5, 9, 13, 17, \ldots\}$,

**f.** $\{1, 0, 1, 0, 1, 0, \ldots\}$,

**g.** $\{2, 5, 11, 17, 23, \ldots\}$.

Do the sequences converge? If so, to what value? In each case, begin by identifying the function $f$ that generates the sequence.

**E6.** The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

Do the sequences converge? If so, to what value? In each case, begin by identifying the function $f$ that generates the sequence.

**a.** $x_0 = 1$, $x_{n+1} = x_n - \frac{x_n^2-2}{2\,x_n} = \frac{x_n}{2} + \frac{1}{x_n}$,

**b.** $x_0 = 1$, $x_{n+1} = x_n - 1$,

**c.** $x_0 = 1$, $x_{n+1} = x_n - \frac{\tan(x_n)-1}{\sec(x_n)^2}$.

**E7.** Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.

**E8.** Which of the following sequences converge, and which diverge? Give reasons for your answers.

**a.** $a_n = 1 - \frac{1}{n}$,

**b.** $a_n = n - \frac{1}{n}$,

**c.** $a_n = \frac{2^n-1}{5^n}$,

**d.** $a_n = \frac{2^n-3}{2^n}$,

$a_n = ((-1)^n + 1)\left(\frac{n+1}{n^2}\right)$.

**E9.** Determine if the sequence is non decreasing and if it is bounded from above.

**a.** $a_n = 2 - \frac{2}{n} - 2^{-n}$,

**b.** $a_n = \frac{2^n\, 3^n}{n!}$,

**c.** $a_n = \frac{(2\,n+3)!}{(n+1)!}$,

**d.** $a_n = \frac{3\,n+1}{n+1}$.

**E10** Find the limit of the sequence

$$\left\{\sqrt{5}\,,\ \sqrt{5\sqrt{5}}\,,\ \sqrt{5\sqrt{5\sqrt{5}}}\ ,\ldots\right\} \tag{3}$$

## 2.4 Infinite Series

The purpose of this section is to discuss sums that contain infinitely many terms. The most familiar example of such sums occur in the decimal representation of real numbers. For example, when we write $\frac{1}{3}$ in the decimal form $\frac{1}{3} = 0.3333333\ldots$, we mean

$$\frac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \ldots$$

which suggests that the decimal representation of $\frac{1}{3}$ can be viewed as a sum of infinitely many real numbers.

### 2.4.1 Sums of Infinite Series

Our first objective is to define what is meant by the sum of infinite many real numbers. We begin with some terminology.

**Definition 2.5.** *Infinite Sum*

*An infinite series is an expression that can be written in the form*

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \ldots + u_k + \ldots. \tag{2.23}$$

*The numbers $u_1$, $u_2$, $u_3$,… are called the terms of the series.* ■

Since it is impossible to add infinitely many numbers together directly, sums of infinite series are defined and computed by an indirect limiting process. To motivate the basic idea, consider the decimal

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