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978-0-521-33927-8 - Van der Corput's Method of Exponential Sums
S. W. Graham and G. Kolesnik
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1. INTRODUCTION

1.1 BASIC DEFINITIONS

In this monograph, we will give an account of van der Corput's method in one and two dimensions. The purpose of this method is to obtain bounds for exponential sums, particularly those exponential sums that arise in number-theoretic problems.

An exponential sum is a sum of the form

$$\sum_{a<n\leq b} e(f(n)), \tag{1.1.1}$$

where $f(n)$ is a real-valued function, and the notation $e(x)$ is used to denote $e^{2\pi ix}$. Such sums arise in several different contexts in number theory. One of the simplest is in estimates for the Riemann-zeta function. The problem of obtaining an upper bound for $\zeta(\sigma+it)$ can be reduced to the problem of bounding

$$\sum_{N<n\leq 2N} n^{-\sigma+it}; \tag{1.1.2}$$

the details of the reduction will be given in Section 2.5. The $n^{-\sigma}$ can be removed via partial summation, and we are then left with a sum of the (1.1.1) with $f(n) = -\frac{t}{2\pi}\log n$.

Another simple example where exponential sums arise is in the Dirichlet divisor problem. Let $d(n)$ denote the numbers of divisors of the natural number n, and let

$$\Delta(x) = \sum_{n\leq x} d(n) - x\log x - (2\gamma-1)x, \tag{1.1.3}$$

where γ is Euler's constant. An elementary argument (see, for example, Chapter 18.2 of Hardy and Wright (1979)) can be used to show that $\Delta(x) \ll x^{1/2}$. Voronoï(1903) showed that the error term could be improved to $O(x^{1/3})$. As we shall see in Chapter 4,

$$\Delta(x) = -2\sum_{n\leq x^{1/2}} \psi(x/n) + O(1), \tag{1.1.4}$$

where $\psi(w) = w - [w] - 1/2$. Now ψ is a periodic function of period 1, and it can be expressed as a Fourier series. This gives a way of writing (1.1.4) in terms of exponential sums.

There are other problems that lead to consideration of sums of the form $\sum_n \psi(f(n))$, and we will discuss some of these in Chapter 4.

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1.2 HISTORICAL OVERVIEW

The application of exponential sums to number theory began with the Weyl's paper, (Weyl 1916) "Über die Gleichverteilung von Zahlen mod. Eins." One of several new ideas that Weyl introduced was a useful transformation that arises upon squaring an exponential sum. Let S denote the sum in (1.1.1). Then

$$|S|^2 = \sum_{a<m,n\le b} e(f(m)-f(n)) = \sum_{|h|<b-a} \sum_{n\in I_h} e(f(n+h)-f(n)) \tag{1.2.1}$$

where $I_r = \{n : a < n, n+r \le b\}$. This is useful because the differenced function $f(n+r)-f(n)$ occurring in the inner sum is easier to handle than the original function $f(n)$. For example, if $f(n)$ is a polynomial of degree k, then $f(n+r)-f(n)$ is a polynomial of degree $k-1$. After $k-1$ applications of (1.2.1), the problem reduces to consideration of exponential sums of linear functions. The latter are easily handled because they are geometric series.

Van der Corput (1922) modified and improved Weyl's method as follows. Let H be an arbitrary positive integer. Then

$$HS = \sum_{h=1}^{H} \sum_{a-h<n\le b-h} e(f(n+h)).$$

By squaring both sides of the above and applying Cauchy's inequality, one obtains

$$|S|^2 \le \frac{|b-a|+H}{H} \sum_{|h|<H} \left(1-\frac{|h|}{H}\right) \sum_{n\in I_h} e(f(n+h)-f(n)). \tag{1.2.2}$$

The details of the derivation of (1.2.2) will be given in Section 2.3. Note that (1.2.2) is very similar to (1.2.1) when $H = b-a$.

Another innovation of van der Corput was a combination of the Poisson summation formula and the method of stationary phase. Under suitable conditions on f, it can be shown that

$$\sum_{n\in I} e(f(n)) = \sum_{\alpha\le\nu\le\beta} \frac{e(-\phi(\nu)-1/8)}{|f''(x_\nu)|^{1/2}} + \text{ error terms}, \tag{1.2.3}$$

where α and β are defined in terms of f', x_ν is defined by the relation $f'(x_\nu) = \nu$, and $\phi(\nu) = -f(x_\nu) + \nu x_\nu$. A more complete discussion of (1.2.3) will be given in Chapter 3.

Van der Corput used his method to prove that

$$\Delta(x) \ll x^{33/100}. \tag{1.2.4}$$

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(Actually he proved (1.2.4) with the smaller exponent 163/494.) Walfisz (1924) used the same method to show that

$$\zeta(1/2+it) \ll t^{163/998}. \tag{1.2.5}$$

Van der Corput (1928) improved the exponent in the divisor problem to 27/82. Titchmarsh (1931) sharpened the exponent in (1.2.5) to 27/164, and Phillips (1933) further reduced the exponent to 229/1392.

A more important aspect of Phillips' 1933 paper was his refinement of exponent systems. Van der Corput (1922) defined an exponent system to be a set of ordered pairs $\{(k_1,l_1),\dots,(k_r,l_r)\}$ such that, if $|f'| \approx y$ and $N_1 \le 2N$ then

$$\sum_{N<n\le N_1} e(f(n)) \ll y^{k_1}N^{l_1} + \dots + y^{k_r}N^{l_r},$$

provided f satisfies a certain set of hypotheses. (A complete statement of the hypotheses is lengthy; see the discussion in Chapter 3.) Phillips gave a form of this theory in which only one pair (k,l) is required. Thus he refers to exponent pairs, rather than exponent systems. The trivial bound $\sum_{N<n\le N_1} e(f(n)) \ll N$ shows that $(0,1)$ is an exponent pair. Phillips showed that if (k,l) is an exponent pair then so are

$$A(k,l) = \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2}\right) \text{ and } B(k,l) = (l-1/2, k+1/2).$$

The A-result follows from (1.2.2), and the B-result follows from (1.2.3). The theory of exponent pairs will be discussed in Chapter 3.

It can be shown that if (k,l) is an exponent pair and $\theta(k,l) = (k+l-1/2)/2$, then

$$\zeta(1/2+it) \ll t^{\theta(k,l)}\log t.$$

For example, Phillips' exponent 229/1392 is $\theta(k,l)$ when

$$(k,l) = ABA^3BA^2BA^2B(0,1) = (97/696, 480/696).$$

This naturally leads to the question of finding the minimum of $\theta(k,l)$ for all exponent pairs that are obtainable from $(0,1)$ by application of the A and B processes. Rankin (1955) found this minimum. Graham (1985) gave further details and considered the corresponding problem when θ is a rational function of k and l. Graham's method will be presented in Chapter 5.

1.3 TWO DIMENSIONAL SUMS

In some applications, sums of the form

$$\sum_{(m,n)\in\mathbf{D}} e(f(m,n))$$

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arise, where $\mathbf{D}$ is some simple region in $\mathbf{R}^2$. These sums may be reduced to one-dimensional sums by noting that

$$\Big|\sum_{(m,n)\in\mathbf{D}} e(f(m,n))\Big| \le \sum_{m}\Big|\sum_{\substack{n\\(m,n)\in\mathbf{D}}} e(f(m,n))\Big|,$$

but such a simplification can be wasteful. A more efficient approach is to develop two-dimensional analogs of Process A and Process B. The first steps in this direction were taken by Titchmarsh (1934a) in an application to Epstein's zeta-function. Other authors extended the two dimensional method to bounds for $\zeta(1/2+it)$ and $\Delta(x)$. The best results obtained by this method to date are due to Kolesnik (1985), who shows that

$$\zeta(1/2+it) \ll t^{139/858+\epsilon}$$

and

$$\Delta(x) \ll x^{139/429+\epsilon}. \tag{1.3.1}$$

In Chapter 6, we give an overview of two-dimensional sums. The algorithm of Chapter 5 can be easily adapted to show the limitations of what can be accomplished with these sums. We will show for example, that with a two-dimensional method, the exponent in (1.3.1) cannot be made smaller than

$$0.32392\ldots.$$

(Note that $139/429 = 0.324009\ldots$.) We also prove two-dimensional analogs of the B and AB theorems.

1.4 THE METHOD OF BOMBIERI AND IWANIEC

The latest advance in estimation of exponential sums began with the work of Bombieri and Iwaniec (1985a,b). They proved that for any $\epsilon > 0$,

$$\zeta(1/2+it) \ll t^{9/56+\epsilon}.$$

This is better than any result that can be obtained from either the one-dimensional or two-dimensional methods. Their work uses the Weyl-van der Corput inequality (1.2.2) and the method of stationary phase. They also use an elaborate form of the large sieve inequality and some mean value theorems that are reminiscent of Vinogradov's method.

Huxley and Watt (1988) generalize the Bombieri-Iwaniec result to show that for every $\epsilon > 0$,

$$\left(\frac{9}{56}+\epsilon, \frac{37}{56}+\epsilon\right)$$

is an exponent pair. We will prove this result in Chapter 7.

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The line of research opened up by Bombieri and Iwaniec has proved promising, and we expect further progress along the same lines in the future. Iwaniec and Mozzochi (1988) proved that

$$\Delta(x) \ll x^{7/22+\epsilon}.$$

Watt (1989) has shown that

$$\zeta(1/2+it) \ll t^{89/560+\epsilon}.$$

1.5 NOTATION

As mentioned earlier, $e(x)$ denotes $e^{2\pi ix}$, $[x]$ denotes the greatest integer $\leq x$, and $\psi(x) = x - [x] - 1/2$.

We will make extensive use of the Vinogradov notations $\ll$ and $\gg$. For functions f and g with g taking non-negative real values, $f \ll g$ means $|f| \leq C|g|$ for some unspecified constant C. If f is also non-negative, $f \gg g$ means $g \ll f$. We also use the Landau notation $f = O(g)$; this is equivalent to $f \ll g$. The notation $f \approx g$ means that $f \ll g$ and $g \ll f$.

The notation $||x||$ denotes the distance to the nearest integer of x; i.e.

$$||x|| = \min_{n \in \mathbf{Z}} |x - n|.$$

We also use the generalized factorial $(\alpha)_n$. This is defined for non-negative integers n by the recursive relations

$$(\alpha)_0 = 1\,,\ (\alpha)_{n+1} = (\alpha + n)(\alpha)_n.$$

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2. THE SIMPLEST VAN DER CORPUT ESTIMATES

2.1 ESTIMATES USING FIRST AND SECOND DERIVATIVES

In the next two chapters, we will obtain upper bounds for sums of the form

$$S = \sum_{n \in I} e(f(n)), \tag{2.1.1}$$

where $I = (a, b]$. (We assume throughout this chapter that a and b are integers.) In this chapter, we obtain the simplest such upper bounds.

From the triangle inequality, we see that

$$|S| \leq \sum_{n \in I} |e(f(n)| = |I| = b - a. \tag{2.1.2}$$

This estimate (henceforth called the *trivial estimate*) holds with equality if $f(n) = xn + y$ and x is an integer. Thus any improvement on (2.1.2) will require some hypothesis on f.

It is instructive to examine the case $f(n) = xn + y$ more closely. If x is not an integer, then

$$|\sum_{n \in I} e(xn + y)| = \frac{|e(bx) - e(ax)|}{|e(x) - 1|} \leq \frac{1}{|\sin \pi x|}. \tag{2.1.3}$$

An examination of the graphs $y = \sin \pi x$ and $y = 2x$ shows that $\sin \pi x \geq 2x$ for $0 \leq x \leq 1/2$. Now define

$$||x|| = \min\{|x - n| : n \in \mathbf{Z}\}.$$

Since $|\sin \pi x|$ is periodic with period 1 and $\sin \pi(1 - x) = \sin \pi x$, we have

$$|\sin \pi x| \geq 2||x||$$

for all real x. Consequently,

$$|\sum_{n \in I} e(xn + y)| \ll ||x||^{-1}.$$

Our first theorem generalizes this result.

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THEOREM 2.1. *(Kusmin-Landau) If f is continuously differentiable, f' is monotonic, and $||f'|| \geq \lambda > 0$ on I then*

$$\sum_{n\in I} e(f(n)) \ll \lambda^{-1}.$$

Proof. Since

$$\Big|\sum_{n\in I} e(f(n))\Big| = \Big|\sum_{n\in I} e(-f(n))\Big|,$$

we may assume that f' is increasing. Our hypotheses imply that for some integer k,

$$k + \lambda \leq f'(n) \leq k + 1 - \lambda.$$

Now

$$\sum_{n\in I} e(f(n)) = \sum_{n\in I} e(f(n) - kn),$$

so we may assume that $\lambda \leq f'(n) \leq 1 - \lambda$.

Define $g(n) = f(n+1) - f(n)$. From the mean value theorem, we see that there is some x_n such that $n \leq x_n \leq n+1$ and $g(n) = f'(x_n)$. Consequently, g is increasing and $\lambda \leq g(n) \leq 1 - \lambda$. We write

$$e(f(n)) = \frac{e(f(n)) - e(f(n+1))}{1 - e(g(n))} = \{e(f(n)) - e(f(n+1))\}C_n,$$

where $C_n = \frac{1}{2}(1 + i\cot \pi g(n))$. Therefore

$$\begin{aligned}\sum_{n\in I} e(f(n)) &= \sum_{n=a+1}^{b-1} \{e(f(n)) - e(f(n+1)\}C_n + e(f(b)) \\ &= \sum_{n=a+2}^{b-1} e(f(n))(C_n - C_{n-1}) + e(f(a+1))C_{a+1} + e(f(b))(1 - C_{b-1}),\end{aligned}$$

and thus

$$\Big|\sum_{n\in I} e(f(n))\Big| \leq \frac{1}{2} \sum_{n=a+2}^{b-1} |\cot(\pi g(n-1)) - \cot(\pi g(n))| + |C_{a+1}| + |1 - C_{b-1}|.$$

Since $\cot \pi g(n)$ is a decreasing function, the absolute value bars in the right-hand sum may be removed. The remaining sum telescopes, so

$$\Big|\sum_{n\in I} e(f(n))\Big| \leq \frac{1}{2}\{\cot(\pi g(a+1)) - \cot(\pi g(b-1))\} + |C_{a+1}| + |1 - C_{b-1}|.$$

The theorem follows by using the bound $|\cot \pi x| \ll ||x||^{-1}$.

It is clear from (2.1.3) that Theorem 2.1 is sharp. On the other hand, the hypothesis $||f'|| \geq \lambda$ is rather restrictive. Our next theorem relaxes this hypothesis.

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THEOREM 2.2. *(van der Corput) Suppose that f is a real valued function with two continuous derivatives on I. Suppose also that there is some $\lambda > 0$ and some $\alpha \geq 1$ such that*

$$\lambda \leq |f''(x)| \leq \alpha\lambda$$

on I. Then

$$\sum_{n\in I} e(f(n)) \ll \alpha|I|\lambda^{1/2} + \lambda^{-1/2}.$$

Proof. Let $\delta(< 1/2)$ be a parameter to be chosen later. From the hypothesis, it follows that I can be split into $\leq \alpha|I|\lambda+2$ intervals on which $||f'|| \geq \delta$, and $\leq \alpha|I|\lambda+1$ other intervals, each of length $\leq 2\delta/\lambda$. We apply Theorem 2.1 to the former set of intervals, and the trivial estimate to the latter. We get

$$\sum_{n\in I} e(f(n)) \ll (\alpha|I|\lambda + 1)(1/\delta + \delta/\lambda + 1).$$

We choose $\delta = \lambda^{1/2}$; this proves the desired result if $\lambda \leq 1/4$. If $\lambda > 1/4$, the result follows from the trivial estimate.

2.2 SOME SIMPLE INEQUALITIES

In this section, we will derive a couple of results that we shall be using throughout this book. Both are simple but, nonetheless, useful.

LEMMA 2.3. *If $X_1, \ldots, X_k$ are positive numbers and $a_1, \ldots, a_k$ are non-negative numbers such that $a_1 + \ldots + a_k = 1$, then*

$$\min(X_1, \ldots, X_k) \leq X_1^{a_1} \ldots X_k^{a_k} \leq \max(X_1, \ldots, X_k).$$

The proof is obvious. We shall often refer to this as the *convexity principle.*

Our second Lemma generalizes the following well known principle. Suppose we have an estimate of the form

$$U \ll AH^a + BH^{-b},$$

where A, B, a, and b are positive constants and H is at our disposal. We obviously want to take H so as to minimize the right hand side. By choosing H to satisfy $AH^a = BH^{-b}$, we get

$$U \ll (A^b B^a)^{1/(a+b)},$$

and this is best possible apart from the value of the implied constant. To generalize this principle, we prove

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LEMMA 2.4. *Suppose that*

$$L(H) = \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j},$$

where A_i, B_j, a_i, and b_j are positive. Assume that $H_1 \le H_2$. Then there is some H with $H_1 \le H \le H_2$ and

$$L(H) \ll \sum_{i=1}^{m}\sum_{j=1}^{n}(A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^{m} A_i H_1^{a_i} + \sum_{j=1}^{n} B_j H_2^{-b_j}.$$

The implied constants depend only on m and n.

Proof. Define

$$L_+(H) = \max(A_1 H^{a_1}, \dots, A_m H^{a_m})$$

and

$$L_-(H) = \max(B_1 H^{-b_1}, \dots, B_n H^{-b_n}).$$

Now $L \le mL_+ + nL_-$, so it suffices to bound L_+ and L_-. We observe that L_+ is a strictly increasing continuous function, $L_+(0) = 0$, and $L_+(\infty) = \infty$. Similarly L_- is a strictly decreasing continuous function, $L_-(0) = \infty$, and $L_-(\infty) = 0$. Therefore there is a unique H_0 such that $L_+(H_0) = L_-(H_0)$. We distinguish three cases: (a) $H_1 \le H_0 \le H_2$, (b) $H_0 < H_1$, (c) $H_0 > H_2$.

If $H_1 \le H_0 \le H_2$ then there is some i and some j such that

$$L_+(H_0) = A_i H_0^{a_i} = L_-(H_0) = B_j H_0^{-b_j}.$$

Consequently

$$L_+(H_0) = L_-(H_0) = (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)}.$$

If $H_0 < H_1$ then

$$L_-(H_1) < L_+(H_1) \le \sum_{i=1}^{m} A_i H_1^{a_i}.$$

If $H_0 > H_2$ then

$$L_+(H_2) < L_-(H_2) \le \sum_{j=1}^{n} B_j H_2^{-b_j}.$$

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2.3 THE WEYL-VAN DER CORPUT INEQUALITY

Theorem 2.2 has the disadvantage of being worse than the trivial estimate if $\lambda > 1$. This limitation can be overcome if we replace f by a function with smaller derivatives; the next lemma allows us to make such a replacement.

LEMMA 2.5. *(Weyl-van der Corput) Suppose $\xi(n)$ is a complex valued function such that $\xi(n) = 0$ if $n \notin I$. If H is a positive integer then*

$$|\sum_n \xi(n)|^2 \le \frac{|I|+H}{H} \sum_{|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_n \xi(n)\overline{\xi(n-h)}.$$

Proof. Observe that

$$H \sum_n \xi(n) = \sum_{k=1}^{H} \sum_n \xi(n+k) = \sum_n \sum_{k=1}^{H} \xi(n+k).$$

The inner sum is empty unless $a - H < n \le b - 1$. By Cauchy's inequality,

$$\begin{aligned} H^2 |\sum_n \xi(n)|^2 &\le (|I|+H) \sum_n |\sum_{k=1}^{H} \xi(n+k)|^2 \\ &= (|I|+H) \sum_{k=1}^{H} \sum_{l=1}^{H} \sum_n \overline{\xi(n+k)}\xi(n+l). \end{aligned}$$

We collect together the terms with $l - k = h$ to get the result.

For applications of Lemma 2.5 in this chapter, we will take

$$\xi(n) = \begin{cases} e(f(n)) & \text{if } n \in I; \\ 0 & \text{otherwise.} \end{cases}$$

Let S be as in (2.1.1) and define

$$S_1(h) = \sum_{n \in I(h)} e(f(n+h) - f(n)), \tag{2.3.1}$$

where $I(h) = \{n : n \in I \text{ and } n + h \in I\}$. We see that

$$|S|^2 \le \frac{(|I|+H)}{H} \sum_{|h|<H} |S_1(h)|.$$

If we also assume that $H \le |I|$ then

$$|S|^2 \le \frac{2|I|}{H} \sum_{|h|<H} |S_1(h)|. \tag{2.3.2}$$

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